BUNDLE REPRESENTATIONS IN PROBLEM THEORY

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Abstract. In this paper, we introduce and study algebraic representations in problem and solution spaces. We develop the concept of problem and solution bundle in a space. The developed framework furnishes the internal profile of problems and the corresponding solution spaces, allowing us to read their internal structure by viewing the configuration of their subproblems.

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1. Introduction

A mathematical problem is rarely an isolated object. Its solution may depend on the resolution of a collection of subsidiary questions, and those subsidiary questions may themselves generate further problems. The resulting pattern is not necessarily linear: some subproblems may be nested, while others may be mutually independent or may overlap through common intermediate problems. A satisfactory structural account of problem solving should, therefore, describe not only a problem and one of its solutions but also the organization of the problems and solutions that are brought into play in passing from

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the former to the latter. The importance of hierarchical organization in the analysis of complex systems is well established; in particular, Simon emphasized that complex systems frequently admit descriptions in terms of hierarchies of interacting subsystems [4]. The present paper develops a mathematical framework for examining an analogous form of organization within problem theory.

The starting point is a relation of subproblemhood. Given problems V and Y, the relation $V \leq Y$ expresses that a solution to Y also provides a solution to V. Thus, the subproblem relation records a form of logical dependence: resolving a problem entails resolving the subproblems required by it. Mutual subproblemhood gives the corresponding notion of equivalence, while the distinction between a problem and an equivalent problem is retained at the level of the objects themselves. This order-oriented point of view places the framework in natural dialogue with the general theory of ordered sets and lattices, where chains, maximal and minimal elements, and structural decompositions provide standard tools for describing hierarchical organization [2, 3]. The merger operation introduced here, however, is defined intrinsically in terms of the maximal proper subproblems of a problem; no assumption is made that it is a lattice join or that the induced structure satisfies the axioms of a lattice.

More precisely, if Y is a problem and X is a solution to it, the paper considers two associated collections. The induced problem space $\mathcal{P}_Y(X)$ consists of the problems that arise as subproblems in the process of obtaining X as a solution to Y. Dually, the induced solution space $\mathcal{S}_Y(X)$ consists of the solutions to the problems occurring in that process. These spaces are coupled: the problems in the first space are precisely the subsidiary problems whose resolution is relevant to the original task, while the second records the corresponding solutions. In particular, the original problem and its given solution belong to their respective induced spaces. This paired construction is the basic object of study throughout the paper and continues the broader program of treating problems and their solutions as mathematical objects with an internal structure; see also the author's preceding work on problem theory [1].

The framework is equipped with numerical descriptors that record the sizes of the induced spaces. The complexity $\mathbb{C}[\mathcal{P}_Y(X)]$ is defined as the number of problems in the problem space, and the index $\mathbb{I}[\mathcal{S}_Y(X)]$ is the number of solutions in the solution space. The paper also assigns to a solution space the quantity

$$\mathcal{E}[\mathcal{S}_Y(X)] := \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]},$$

which it calls the entropy of that space. These are cardinality-based invariants internal to the present theory: in particular, the term *complexity* here refers to the size of an induced problem space, rather than to the running time or resource complexity of an algorithm, and the stated entropy is the reciprocal of the solution-space index, rather than Shannon entropy. The paper then develops structural classifications based on the subproblem and subsolution relations. Among these are reducibility and irreducibility, regularity and irregularity, and the existence of maximal or minimal proper subproblems. Regularity captures the situation in which the subproblems can be arranged in a single nested chain; irregularity occurs when incomparable subproblems are present. The induced spaces are further compared through notions of connectedness, compatibility, and merger. In the latter construction,

two compatible problems admit a merger when they occur as the only maximal proper subproblems of a common problem in an induced space; an analogous definition is made for compatible solutions.

A further part of the framework concerns the relation between the structure of a problem space and that of its corresponding solution space. Alternative solutions to the same problem are treated as interchangeable for the purposes of the induced solution space. The paper also introduces a notion of separability: a problem space is separable when it can be expressed as the union of two disjoint induced subspaces whose problems do not cross the division by equivalence; the corresponding definition for a solution space uses non-alternative solutions. In this setting, the terminology of topology is used in a structural sense, centered on the overlap, connection, and separation of induced spaces, rather than on a topology generated by open sets. One of the basic results establishes that a problem space is separable if and only if its associated solution space is separable. This equivalence is important for the subsequent bundle theory, since it allows structural decompositions on the problem side to be interpreted on the solution side, and conversely.

The principal construction of the paper is that of a *problem bundle* and its solution-space analogue. A problem bundle represents a problem Y as a merger

$$Y = Y_{i_1} \bowtie Y_{i_2} \bowtie \cdots,$$

where the constituents are proper subproblems that belong to the induced problem space $\mathcal{P}_Y(X)$. A solution bundle represents the given solution X as a merger of proper subsolutions in $\mathcal{S}_Y(X)$. A bundle is called *complete* when none of its constituent problems has a proper subproblem, or, respectively, none of its constituent solutions has a proper subsolution. A bundle is called *trivial* when the original problem or solution itself occurs among its constituents. These definitions are designed to expose the internal profile of an induced space: rather than merely listing all subproblems or subsolutions, a bundle identifies the maximal components through which the original object is assembled.

The first existence theorem shows that every problem space of finite complexity admits a problem bundle and every solution space of finite index admits a solution bundle. The resulting representation may be trivial or nontrivial. The paper next investigates how the existence of nontrivial bundles reflects the order structure of the underlying problem. In the finite setting considered in the regularity theorem, a regular problem admits only trivial bundles. Consequently, within the scope of those finiteness hypotheses, a nontrivial bundle is a certificate of non-chain-like structure: it signals that the subproblem hierarchy cannot be represented solely by a single nested sequence. A further criterion, formulated through finiteness and noncontainment conditions on the induced subproblem spaces, is given for the existence of a nontrivial complete bundle. The purpose of this criterion is to identify conditions under which a decomposition can be carried out all the way to constituents that have no proper subproblems.

Having established the existence and structural role of bundles, the paper introduces numerical invariants for their comparison. The *order* of a bundle is the number of its constituent problems, or constituent solutions, in the solution-space setting. When a nontrivial complete bundle exists, the *dimension* of the induced space is defined as the smallest order among all such complete bundles. This is a combinatorial dimension

attached to the bundle structure, not a vector-space dimension. For an arbitrary bundle, its *degree* is defined as the difference between its order and the dimension of the space. The resulting degree-order-dimension identity records the excess number of constituents in a given bundle over the minimum required by a complete bundle. In addition, the paper proves a separability criterion for bundles of order two: if a problem space admits a complete nontrivial bundle with exactly two constituents, then both the problem space and its corresponding solution space are separable. This result connects the size of a minimal decomposition with the possibility of separating the induced spaces into disjoint structural components.

The final part of the paper compares bundle representations arising from different but related problems and solutions. Problem bundles are compared through the equivalence of their constituent problems, while solution bundles are compared through the relation of being alternative solutions. The paper formulates the corresponding notions of equivalent problem bundles and alternative solution bundles, and establishes a uniqueness-by-type statement: under the stated non-equivalence and non-alternativity conditions on the constituents, the bundle representations are determined up to the relevant equivalence or alternative relation. This comparison is intended to distinguish the essential structural types in a bundle from differences that arise only when choosing equivalent formulations of a problem or alternative solutions to it.

1.1. **Structure and organization of the paper.** The organization of the paper is as follows. Section 2 develops the preliminary language of problems, solutions, and their induced spaces. It introduces the subproblem and subsolution relations, the associated classifications of problems, and the notions of compatibility, merger, connectedness, and separability; it also establishes the equivalence between separability of a problem space and separability of its solution space. Section 3 introduces problem and solution bundles and proves the finite-existence results, the restriction imposed by regularity, and the criterion for nontrivial complete bundles. Section 4 defines the order, dimension, and degree of a bundle and establishes the relation between a complete bundle of order two and separability. Finally, Section 5 studies the equivalence of problem bundles and alternativity of solution bundles, culminating in the comparison of bundle representations up to their constituent types.

2. Preliminaries and background terminologies

2.1. **Problems, solutions and induced spaces.**

Definition 2.1. Let X denote a solution (resp. answer) to the problem Y (resp. question). We call the collection of all problems to be solved to provide a solution X to the problem Y the problem space induced by providing a solution X to the problem Y. We denote this space by $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation by $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say that problem V is a subproblem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a subproblem of the problem Y, then we write $V \leq Y$. If V is a subproblem of the problem Y and $V \neq Y$, then we write $V < Y$ and call V a proper subproblem of Y.

Definition 2.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to the problem Y. We call the number of problems in the space (size) the *complexity* of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 2.3. Let X denote a solution (resp. answer) to the problem Y (resp. question). We call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing the solution X to problem Y. We denote this space by $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation by $K \subset \mathcal{S}_Y(X)$. We say that T is a subsolution of the solution X if it solves a subproblem of Y. We assign $T \in \mathcal{S}_Y(X)$ if the solution T is also a solution in this space. If T is a subsolution of the solution X, then we write $T \leq X$. If T is a subsolution of the solution X and $T \neq X$, then we write $T < X$ and call T a proper subsolution of X.

Proposition 2.4. *Let $\mathcal{S}_Y(X)$ be the solution space induced by providing solution X to problem Y. We have $X \in \mathcal{S}_Y(X)$.*

Proof. The solution space $\mathcal{S}_Y(X)$ contains all the subsolution of the solution X. Since $X \leq X$, we deduce the claim. □

Definition 2.5. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. We call the number of solutions in the space (size) the *index* of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.

Definition 2.6. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. By the *entropy* of the space, we mean the expression

$$\mathcal{E}[\mathcal{S}_Y(X)] := \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]}.$$

Proposition 2.7. *Let $\mathcal{P}_Y(X)$ be the induced problem space of providing the solution X to the problem Y. We have $Y \in \mathcal{P}_Y(X)$.*

Proof. The problem space $\mathcal{P}_Y(X)$ contains all the subproblems of problem Y. Since $Y \leq Y$, we deduce the claim. □

2.2. **Classifying problems, solutions and corresponding induced spaces.**

Definition 2.8. Let Y and V be any two problems. We say that problem Y is equivalent to problem V if providing a solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote equivalence by $V \equiv Y$.

Here, we present a simple criterion for creating a subspace of a problem space.

Proposition 2.9. *Let $X \in \mathcal{S}_V(U)$ and $Y \in \mathcal{P}_V(U)$. If X is a solution to the problem Y, then*

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Proof. Under the requirement $Y \in \mathcal{P}_V(U)$, it follows that Y is a subproblem to be solved to provide a solution U to problem V. Since $X \in \mathcal{S}_V(U)$, it follows that X is a solution obtained by providing a solution U to the problem V. Since X solves Y and $Y \in \mathcal{P}_Y(X)$, it follows that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

□

We use the following criterion to determine the solubility of a problem.

Proposition 2.10. *Let V be a problem with solution U. If $Y \in \mathcal{P}_V(U)$, then Y must have a solution.*

Proof. Clearly, problem V is solved by U with an induced problem space $\mathcal{P}_V(U)$. Since this space consists of all subproblems to be solved in order to provide the solution U to the problem V and $Y \in \mathcal{P}_V(U)$, it implies that Y has a solution. □

Definition 2.11. Let V be a problem. We say V is *reducible* if there exists a proper subproblem of V with no proper subproblem. On the other hand, we say that problem V is *irreducible* if every proper subproblem of V has a proper subproblem.

Definition 2.12. Let V be a problem and $\{Y_i\}_{i\geq 1}$ be the sequence of all the subproblems of V. We say that V is regular if

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

We say it is irregular if there exist subproblems Y_j and Y_k of V such that $Y_j \not\leq Y_k$ and $Y_k \not\leq Y_j$.

Definition 2.13. Let V be a problem and Y be a proper subproblem of V. We say that Y is the *maximal* subproblem of V if all other proper subproblems of V are subproblems of Y. We say that it is the *minimal* subproblem of V if it is a subproblem of all other subproblems of V.

Next we relate the notion of minimal subproblem to the notion of reducibility.

Proposition 2.14. *Let V be a problem. If there exists a minimal subproblem of V, then V must be reducible.*

Proof. Let Y be the minimal subproblem of problem V. Then Y has no proper subproblem. This implies that V must be reducible. □

In a similar fashion, we relate the notion of maximal subproblem with the notion of regularity.

Theorem 2.15. *Let V be a problem. If every subproblem of V has a maximal proper subproblem, then V must be regular.*

Proof. Let Y be the maximal proper subproblem of V, since $V \leq V$. Then we have the relation $Y < V$ and every other proper subproblem of V must be a subproblem of Y. Since every subproblem of V has a maximal subproblem, we let Z be the maximal proper subproblem of Y. Thus, $Z < Y$ and all other proper subproblems of Y are subproblems of Z. Since the proper subproblems of V excluding Y are proper subproblems of Y and the remaining (excluding Z) are subproblems of Z, we obtain the chain of subproblems

$$\cdots < Z < Y < V$$

and thus the chain contains all the subproblems of V. This shows that V must be a regular problem. □

Definition 2.16. Let V be a problem with solution U and Y a problem with solution X. We say that the induced problem spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are connected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) \neq \emptyset.$$

We say that the connection is high if

$$\frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_V(U)|} \geq \frac{1}{2} \quad \text{and} \quad \frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_Y(X)|} \geq \frac{1}{2}.$$

Otherwise, we say that the connection is low. On the other hand, we say that problem spaces are disconnected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) = \emptyset.$$

Proposition 2.17. *Let Y be a problem with solution X. If V is also a problem with a maximal proper subproblem Z such that $Z \in \mathcal{P}_Y(X)$ and V is regular, then V must be solvable and the induced problem space must be connected to $\mathcal{P}_Y(X)$.*

Proof. Since problem Y has solution X, each problem in space $\mathcal{P}_Y(X)$ has also been solved. The requirement $Z \in \mathcal{P}_Y(X)$ implies that the problem Z has been solved. Since V is regular, we have the chain of all subproblems of V as

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq Z$$

since Z is the maximal subproblem of V. Since Z is solved, it follows that all subproblems of V are solved and V must have a solution, say T, with induced problem space $\mathcal{P}_V(T)$. The latter claim follows by noting that $Z \in \mathcal{P}_V(T) \cap \mathcal{P}_Y(X)$. □

Definition 2.18. Let V and U be any two problems. We say that V and U are *compatible* if there exists a problem space $\mathcal{P}_X(Y)$ such that $V, U \in \mathcal{P}_X(Y)$. We denote this compatibility by $V \diamond U$ or $U \diamond V$. Similarly, we say that two solutions R, S to some (possibly) distinct problems are compatible if there exists a solution space $\mathcal{S}_X(Y)$ such that $R, S \in \mathcal{S}_X(Y)$. We denote this compatibility by $R \diamond S$ or $S \diamond R$.

Definition 2.19. Let U and V be compatible problems. We say V and U admit a *merger* in the space $\mathcal{P}_X(Y)$ if there exists a problem $S \in \mathcal{P}_X(Y)$ such that $V < S$ and $U < S$ and V, U are the only maximal subproblem of S. In notation, we write $V \bowtie U = S \in \mathcal{P}_X(Y)$ or $U \bowtie V = S \in \mathcal{P}_X(Y)$. Similarly, let R and T be compatible solutions. We say that R and T admit a *merger* in space $\mathcal{S}_X(Y)$ if there exists a solution $W \in \mathcal{S}_X(Y)$ such that $R < W$ and $T < W$ and R, T are the only maximal sub-solutions of W. In notation, we write $R \bowtie T = W \in \mathcal{S}_X(Y)$ or $R \bowtie T = W \in \mathcal{P}_X(Y)$

2.3. **The topology of problem and solution spaces.**

Definition 2.20. Let Y be a problem. We say that X and U are alternative solutions to Y if and only if U and X both solve Y. We denote this relation by $X \perp U$ or $U \perp X$.

Proposition 2.21. *The solution spaces remain invariant under replacement with alternative solutions.*

Proof. Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Suppose that $L \in \mathcal{S}_Y(X)$ with $L \perp K$, then there exists a problem $F \in \mathcal{P}_Y(X)$ that is solved by L. Since $L \perp K$, it follows that K also solves F. Thus, we can replace $L \in \mathcal{S}_Y(X)$ with K. □

Definition 2.22. Let $\mathcal{P}_Y(X)$ be a problem space. We say that $\mathcal{P}_Y(X)$ is *separable* if and only if there exist some $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. Otherwise, we say that the problem space is *inseparable*. Similarly, we say that a solution space $\mathcal{S}_Y(X)$ is separable if and only if there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Otherwise, we say that the solution space is inseparable.

Theorem 2.23. *Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Then $\mathcal{P}_Y(X)$ is separable if and only if $\mathcal{S}_Y(X)$ is separable.*

Proof. Suppose that $\mathcal{P}_Y(X)$ is separable. There exist $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. For any $F \in \mathcal{P}_V(U)$ there exists some $R \in \mathcal{S}_V(U)$ that solves F and some $W \in \mathcal{S}_K(L)$ that solves G. Since $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$ and the problems in both spaces are not equivalent, it follows that $R \not\perp W$ and $R \notin \mathcal{S}_K(L)$ and $W \notin \mathcal{S}_V(U)$. Since R and W are arbitrary, it follows that $\mathcal{S}_Y(X)$ must also be separable. Suppose without loss of generality that R solves some problem in the space $\mathcal{P}_K(L)$. In particular, there exists some $T \in \mathcal{P}_K(L)$ that is solved by R. Since R also solves F and there exists some $W \in \mathcal{S}_K(L)$ that solves T, it must be that $W \perp R$, a contradiction. In the case $R \perp W$, we obtain $R \in \mathcal{S}_K(L)$ and $W \in \mathcal{S}_V(U)$ by virtue of Proposition 2.21. Without loss of generality, we examine the case $R \perp W$ and $R \in \mathcal{S}_K(L)$ with $W \notin \mathcal{S}_V(U)$ then $W \in \mathcal{S}_V(U)$ by Proposition 2.21. This is also a contradiction. Conversely, suppose that the solution space $\mathcal{S}_Y(X)$ is separable. Then there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Clearly R solves some $G \in \mathcal{P}_V(U)$ and W solves some $T \in \mathcal{P}_K(L)$. We claim $T \not\equiv G$ with

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset.$$

Suppose that $T \equiv G$ for some $T \in \mathcal{P}_K(L)$ and $G \in \mathcal{P}_V(U)$, then $R \perp W$, leading to a contradiction. Since

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

it follows that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X).$$

Suppose to the contrary that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$$

then there exist a problem $A \in \mathcal{P}_Y(X)$ that has no solution in $\mathcal{S}_V(U) \cup \mathcal{S}_K(L)$ but has solution in $\mathcal{S}_Y(X)$. This assertion contradicts the equality

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X).$$

We note that $\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$ implies $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$. Suppose that $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) =/ \emptyset$. There exists a problem $J \in \mathcal{P}_V(U) \cap \mathcal{P}_K(L)$ so that there exists some $N \in \mathcal{S}_V(U) \cap \mathcal{S}_K(L)$ that solves J. This completes the proof. □

3. Problem and solution bundles in a space

Definition 3.1. Let Y be a problem with a solution X. Denote by $\mathcal{P}_Y(X)$ the induced problem space. We call the representation

$$Y = \bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j} := Y_{i_1} \bowtie Y_{i_2} \bowtie \cdots$$

a nontrivial bundle representation of the problem space $\mathcal{P}_Y(X)$. We say that the bundle representation is *complete* if each Y_{i_j} in the bundle does not have a proper subproblem. We say that the bundle representation is trivial if $Y = Y_{i_1} \bowtie Y_{i_2} \bowtie \cdots$ with $Y = Y_{i_j}$ for at least some $j \geq 1$.

Definition 3.2. Let Y be a problem with a solution X. Denote by $\mathcal{S}_Y(X)$ the induced solution space. We call the representation

$$X = \bowtie_{\substack{X_{i_j} \in \mathcal{S}_Y(X) \\ X_{i_j} \neq X \\ j \geq 1}} X_{i_j} := X_{i_1} \bowtie X_{i_2} \bowtie \cdots$$

a nontrivial bundle representation of the solution space $\mathcal{S}_Y(X)$. We say that the solution bundle is *complete* if each X_{i_j} in the bundle does not have a proper subsolution. We say that the bundle representation is trivial if $X = X_{i_1} \bowtie X_{i_2} \bowtie \cdots$ with $X = X_{i_j}$ for at least some $j \geq 1$. [1]

Theorem 3.3. *Every problem space $\mathcal{P}_Y(X)$ with complexity $\mathbb{C}[\mathcal{P}_Y(X)]$ satisfying $\mathbb{C}[\mathcal{P}_Y(X)] < \infty$ admits a problem bundle. Furthermore, every solution space with index $\mathbb{I}[\mathcal{S}_Y(X)]$ satisfying $\mathbb{I}[\mathcal{S}_Y(X)] < \infty$ admits a solution bundle.*

Proof. The inclusion $Y \in \mathcal{P}_Y(X)$ holds by proposition 2.7. If problem Y has no subproblem, then we have a bundle representation

$$Y = Y_i \bowtie Y_j \bowtie \cdots$$

with $Y = Y_i = Y_j \cdots$, since merging the same problem Y does not change the problem. Thus, the space admits a trivial bundle in this particular case. We may assume that the

[1] The concept of a problem and solution bundle is a framework for viewing the internal profile of problem and solution spaces.

problem Y has at least one proper subproblem Y_i and that Y_i is a maximal subproblem. Because $Y_i \in \mathcal{P}_Y(X)$, it must have a solution. We denote the solution of Y_i by X_i and construct the induced problem space $\mathcal{P}_{Y_i}(X_i)$. We deduce $\mathcal{P}_{Y_i}(X_i) \subset \mathcal{P}_Y(X)$. This implies that there exists a problem $Z \in \mathcal{P}_Y(X)$ such that $Z \notin \mathcal{P}_{Y_i}(X_i)$. We may also assume that $Z < Y$ and Z is maximal. [2] This implies $Z \in \mathcal{P}_Y(X)$ using the proposition 2.7. The containment $Z \in \mathcal{P}_Y(X)$ implies that the problem Z must have a solution. We denote the solution to Z by W and construct the induce problem space $\mathcal{P}_Z(W)$. Hence, we obtain

$$\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) \subseteq \mathcal{P}_Y(X).$$

If $\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) = \mathcal{P}_Y(X)$, then we obtain the nontrivial problem bundle representation

$$Y = Y_i \bowtie Z$$

since $Y_i, Z < Y$ and are all maximal subproblems of Y. On the other hand, if

$$\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) \subset \mathcal{P}_Y(X)$$

then there must exist a problem $T < Y$ such that $T \notin \mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W)$. Similarly, we assume $T \neq Y$ and that T is maximal. Because $T \in \mathcal{P}_Y(X)$, it must have a solution. We denote the solution to the problem T by U and construct the space $\mathcal{P}_T(U)$. Because $\mathcal{P}_T(U) \subset \mathcal{P}_Y(X)$, we deduce

$$\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) \cup \mathcal{P}_T(U) \subseteq \mathcal{P}_Y(X).$$

If

$$\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) \cup \mathcal{P}_T(U) = \mathcal{P}_Y(X)$$

then we obtain the bundle representation

$$Y = Y_i \bowtie Z \bowtie T$$

since $Y_i, T, Z < Y$ and are all maximal subproblems of Y. On the other hand, if

$$\mathcal{P}_{Y_i}(X_i) \cup \mathcal{P}_Z(W) \cup \mathcal{P}_T(U) \subset \mathcal{P}_Y(X)$$

we can repeat the argument. Due to the requirement $\mathbb{C}[\mathcal{P}_Y(X)] < \infty$, the argument is terminated by finite induction steps. This proves that every induced problem space with finite complexity must admit a bundle representation as mergers of subproblems in the space. A similar argument holds for the solution space $\mathcal{S}_Y(X)$. [3] □

Theorem 3.4. *Let Y be a regular problem with a solution X. The induced problem and solution spaces $\mathcal{P}_Y(X)$ with finite complexity $\mathbb{C}[\mathcal{P}_Y(X)] < \infty$ and $\mathcal{S}_Y(X)$ with finite index $\mathbb{I}[\mathcal{S}_Y(X)] < \infty$ admit only a trivial bundle.*

Proof. Let $\{Y_i\}_{i \geq 1}$ be the sequence of all subproblems of Y. The regularity of the problem Y implies that we may assume (without loss of generality) the chain of subspaces

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 < Y.$$

Because problem Y has solution X and each $Y_i \in \mathcal{P}_Y(X)$, there exists a solution $X_i \in \mathcal{S}_Y(X)$ to each Y_i. Now, we construct the space $\mathcal{P}_{Y_1}(X_1)$ and deduce $\mathcal{P}_{Y_1}(X_1) \subset \mathcal{P}_Y(X)$. Hence, there exists a problem $V \in \mathcal{P}_Y(X)$ such that $V \notin \mathcal{P}_{Y_1}(X_1)$. The regularity of the problem Y implies that $V = Y$ and

$$\cdots \subseteq \mathcal{P}_{Y_3}(X_3) \subseteq \mathcal{P}_{Y_2}(X_2) \subseteq \mathcal{P}_{Y_1}(X_1) \subset \mathcal{P}_Y(X).$$

[2] Assuming that $Z = Y$ will reconstruct the space $\mathcal{P}_Y(X)$ that recovers a trivial problem bundle representation.

[3] One can construct a similar argument for the solution space $\mathcal{S}_Y(X)$ replacing each subproblem space with their corresponding solution spaces.

We write

$$\mathcal{P}_{Y_1}(X_1) \cup \{Y\} = \mathcal{P}_Y(X).$$

By the maximality of Y_1, we deduce $Y = Y_1 \bowtie Y$. Thus, the space $\mathcal{P}_Y(X)$ admits a trivial bundle. A similar argument can be constructed for the solution space by replacing each $\mathcal{P}_{Y_i}(X_i)$ with $\mathcal{S}_{Y_i}(X_i)$ and $\mathcal{P}_Y(X)$ with $\mathcal{S}_Y(X)$. □

Proposition 3.5. *Let Y be a problem with a solution X. If the induced problem space $\mathcal{P}_Y(X)$ (resp. solution space $\mathcal{S}_Y(X)$) admits a nontrivial bundle, then Y must be irregular.*

Proof. Suppose that the problem space $\mathcal{P}_Y(X)$ (resp. solution space $\mathcal{S}_Y(X)$) admits a nontrivial bundle, and assume that Y is regular. By the regularity of problem Y, theorem 3.4 implies that the problem space $\mathcal{P}_Y(X)$ (resp. solution space $\mathcal{S}_Y(X)$) admit only a trivial bundle. This violates the requirement that the spaces $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ admit a nontrivial bundle. □

[4]

In the following, we prove a stronger version for the converse of proposition 3.5.

Theorem 3.6 (The complete nontrivial bundle criterion)**.** *Let Y be a problem with a solution X. Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the induced problem and solution spaces, respectively. Furthermore, assume that the complexity $\mathbb{C}[\mathcal{P}_Y(X)] < \infty$ and the index $\mathbb{I}[\mathcal{S}_Y(X)] < \infty$. Assume that the problem space $\mathcal{P}_Y(X)$ has the property that for any problem $V \in \mathcal{P}_Y(X)$ with $Y \neq V$ and a solution W the induced problem space $\mathcal{P}_V(W)$ satisfies*

$$\mathcal{P}_V(W) \not\subset \bigcup_{\substack{\mathcal{P}_S(T) \subset \mathcal{P}_Y(X) \\ S \neq V}} \mathcal{P}_S(T).$$

Furthermore, assume that whenever

$$\bigcup_{\substack{\mathcal{P}_S(T) \subset \mathcal{P}_Y(X) \\ S \neq V}} \mathcal{P}_S(T) \subset \mathcal{P}_Y(X)$$

then there exists some

$$R \in \mathcal{P}_Y(X) \setminus \bigcup_{\substack{\mathcal{P}_S(T) \subset \mathcal{P}_Y(X) \\ S \neq V}} \mathcal{P}_S(T)$$

with $R \neq Y$. The space $\mathcal{P}_Y(X)$ (resp. the space $\mathcal{S}_Y(X)$) admits a nontrivial complete bundle.

Proof. We arbitrarily choose a problem $V \in \mathcal{P}_Y(X)$ such that $V \neq Y$ with a solution W and construct the induced problem space $\mathcal{P}_V(W)$. We deduce $\mathcal{P}_V(W) \subset \mathcal{P}_Y(X)$. By virtue of the requirement that the space $\mathcal{P}_Y(X)$ does not have a maximal proper subproblem, there must exist a problem $K \in \mathcal{P}_Y(X) \setminus \mathcal{P}_V(W)$ with $K \neq Y, V$ and a solution L. Otherwise, $K \in \mathcal{P}_V(W)$ so that we have $\mathcal{P}_K(L) \subset \mathcal{P}_V(W)$ violating the requirement

$$\mathcal{P}_V(W) \not\subset \bigcup_{\substack{\mathcal{P}_S(T) \subset \mathcal{P}_Y(X) \\ S \neq V}} \mathcal{P}_S(T).$$

Hence, we deduce

$$\mathcal{P}_V(W) \cup \mathcal{P}_K(L) \subseteq \mathcal{P}_Y(X).$$

[4]The bundle representation of problem and solution spaces allows one to read the internal structure of problems by viewing the configuration of their subproblems.

If $\mathcal{P}_V(W) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$, then $X = V \bowtie K$ is a nontrivial problem bundle. We assume that it is not complete. This implies that at least one problem in the bundle has a proper subproblem. Without loss of generality, we let $U < V$ where U is minimal (due to the complexity of the space) with a solution R and construct the problem space $\mathcal{P}_U(R)$. We get $\mathcal{P}_U(R) \subset \mathcal{P}_V(W)$ violating the requirement that for any problem $V \in \mathcal{P}_Y(X)$ with $Y \neq V$ and a solution W the induced problem space $\mathcal{P}_V(W)$ satisfies

$$\mathcal{P}_V(W) \not\subset \bigcup_{\substack{\mathcal{P}_S(T) \subset \mathcal{P}_Y(X) \\ S \neq V}} \mathcal{P}_S(T).$$

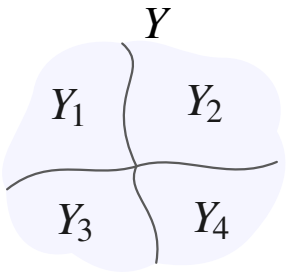
On the other hand, if

$$\mathcal{P}_V(W) \cup \mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$$

then by the property of the space $\mathcal{P}_Y(X)$ there exists a problem

$$R \in \mathcal{P}_Y(X) \setminus (\mathcal{P}_V(W) \cup \mathcal{P}_K(L))$$

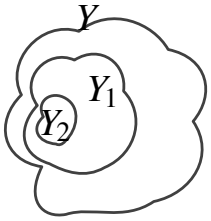
with $R \neq Y$. We let Q be a solution to the problem R and construct the induced problem space $\mathcal{P}_R(Q)$. Using $\mathcal{P}_R(Q) \subset \mathcal{P}_Y(X)$, we deduce $\mathcal{P}_V(W) \cup \mathcal{P}_K(L) \cup \mathcal{P}_R(Q) \subseteq \mathcal{P}_Y(X)$. In the case $\mathcal{P}_V(W) \cup \mathcal{P}_K(L) \cup \mathcal{P}_R(Q) = \mathcal{P}_Y(X)$, we deduce $Y = V \bowtie K \bowtie R$. A similar argument (as before) will imply that this bundle is complete. Due to $\mathbb{C}[\mathcal{P}_Y(X)] < \infty$ the argument is repeated in finite induction steps, yielding a nontrivial complete bundle representation of the space $\mathcal{P}_Y(X)$. □



$Y_1 < Y$
$Y_2 < Y$
$Y_3 < Y$
$Y_4 < Y$

$Y_1 \bowtie Y_2 \bowtie Y_3 \bowtie Y_4 \equiv Y$

FIGURE 1. A nontrivial problem bundle: the problem Y is represented as the bundle of four proper subproblems Y_1, Y_2, Y_3, Y_4.



Nested subproblem structure

$\cdots < Y_2 < Y_1 < Y$

Each inner problem is a proper subproblem of the problem containing it.

Trivial bundle

$Y_1 \bowtie Y_2 \bowtie Y = Y.$

FIGURE 2. A schematic representation of a trivial problem bundle for a regular problem. The nested regions illustrate the chain of proper subproblems, and the bundle identity is $Y_1 \bowtie Y_2 \bowtie Y = Y$.

4. The order of a bundle and the dimension of a problem and solution space

Definition 4.1. Let Y be a problem with a solution X. Denote by $\mathcal{P}_Y(X)$ the induced problem space with a bundle representation

$$Y = \underset{\substack{Y_{i_j}\in\mathcal{P}_Y(X)\\ Y_{i_j}\neq Y\\ j\geq 1}}{\bowtie} Y_{i_j} := Y_{i_1} \bowtie Y_{i_2} \bowtie \cdots .$$

We call the number of problems Y_{i_j} in the bundle the order of the bundle and denote the order by

$$O\left(\underset{\substack{Y_{i_j}\in\mathcal{P}_Y(X)\\ Y_{i_j}\neq Y\\ j\geq 1}}{\bowtie} Y_{i_j} \right).$$

Similarly, we call the number of solutions X_{i_j} in the solution bundle

$$X = \underset{\substack{X_{i_j}\in\mathcal{S}_Y(X)\\ X_{i_j}\neq X\\ j\geq 1}}{\bowtie} X_{i_j} := X_{i_1} \bowtie X_{i_2} \bowtie \cdots$$

the order of the solution bundle and denote the order by

$$O\left(\underset{\substack{X_{i_j}\in\mathcal{S}_Y(X)\\ X_{i_j}\neq X\\ j\geq 1}}{\bowtie} X_{i_j} \right).$$

Definition 4.2. Let Y be a problem with a solution X, and let

$$\mathcal{B}_Y = \left\{ \underset{\substack{Y_{i_j}\in\mathcal{P}_Y(X)\\ Y_{i_j}\neq Y\\ j\geq 1}}{\bowtie} Y_{i_j} \ : \ Y = \underset{\substack{Y_{i_j}\in\mathcal{P}_Y(X)\\ Y_{i_j}\neq Y\\ j\geq 1}}{\bowtie} Y_{i_j} \right\}$$

denote the collection of all nontrivial complete bundle representations of the problem space $\mathcal{P}_Y(X)$. We call a complete bundle in $\mathcal{B}_Y$ with a minimal number of problems Y_{i_j} a basis of space $\mathcal{P}_Y(X)$. We call this number the dimension of the space $\mathcal{P}_Y(X)$ and denote the dimension by

$$\dim[\mathcal{P}_Y(X)] = \min\# \left\{ Y_{i_j} \ : \underset{\substack{Y_{i_j}\in\mathcal{P}_Y(X)\\ Y_{i_j}\neq Y\\ j\geq 1}}{\bowtie} Y_{i_j} \in \mathcal{B}_Y \right\}$$

The dimension of a solution space is defined in a similar way.

Definition 4.3. Let Y be a problem with a solution X, and let

$$\mathcal{B}_X = \left\{ \bowtie_{\substack{X_{i_j} \in \mathcal{S}_Y(X) \\ X_{i_j} \neq X \\ j \geq 1}} X_{i_j} \ : \ X = \bowtie_{\substack{X_{i_j} \in \mathcal{S}_Y(X) \\ X_{i_j} \neq X \\ j \geq 1}} X_{i_j} \right\}$$

denote the collection of all nontrivial complete bundle representations of the solution space $\mathcal{S}_Y(X)$. We call a complete bundle in $\mathcal{B}_X$ with a minimal number of solutions X_{i_j} a basis of space $\mathcal{P}_Y(X)$. We call this number the dimension of the space $\mathcal{S}_Y(X)$ and denote the dimension by

$$\dim[\mathcal{S}_Y(X)] = \min\# \left\{ X_{i_j} \ : \bowtie_{\substack{X_{i_j} \in \mathcal{S}_Y(X) \\ X_{i_j} \neq X \\ j \geq 1}} X_{i_j} \in \mathcal{B}_X \right\}$$

The order of a problem bundle (resp. solution bundle) is related to the dimension of the problem spaces (resp. solution space) that admit the bundle. For an arbitrary problem bundle

$$Y = \bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j}$$

we define the degree of the problem bundle as the difference

$$O\left(\bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j} \right) - \dim[\mathcal{P}_Y(X)].$$

We denote the degree of the problem bundle by

$$\deg\left(\bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j} \right).$$

Hence, we deduce the equation

$$O\left(\bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j} \right) - \dim[\mathcal{P}_Y(X)] = \deg\left(\bowtie_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ Y_{i_j} \neq Y \\ j \geq 1}} Y_{i_j} \right).$$

We call this equation the *problem bundle degree-order-space dimension equation*. Similarly, we define the *solution bundle degree-order-space dimension equation* by

$$O\left(\bigbowtie_{\substack{X_{i_j}\in\mathcal{S}_Y(X)\\ X_{i_j}\neq X\\ j\geq 1}} X_{i_j}\right) - \dim[\mathcal{S}_Y(X)] = \deg\left(\bigbowtie_{\substack{X_{i_j}\in\mathcal{S}_Y(X)\\ X_{i_j}\neq X\\ j\geq 1}} X_{i_j}\right).$$

[5]

Theorem 4.4. *Let Y be a problem with a solution X. Denote the induced problem and solution spaces by $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$. If the space $\mathcal{P}_Y(X)$ admits a complete nontrivial bundle of order 2, then $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ are separable.*

[6]

Proof. We suppose that the space $\mathcal{P}_Y(X)$ admits a nontrivial complete bundle of order 2. Thus, we can write

$$Y = Y_1 \bowtie Y_2$$

for some $Y_1, Y_2 \in \mathcal{P}_Y(X)$ with $Y_1, Y_2 \neq Y$. We assume that the space $\mathcal{P}_Y(X)$ is not separable. Because $Y_1, Y_2 \in \mathcal{P}_Y(X)$, problem Y_1 and Y_2 must have a solution. We let X_1 and X_2 denote a solution to the problem Y_1 and Y_2, respectively, and construct the induced subproblem spaces $\mathcal{P}_{Y_1}(X_1)$ and $\mathcal{P}_{Y_2}(X_2)$. Hence $\mathcal{P}_{Y_1}(X_1) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_{Y_2}(X_2) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_{Y_1}(X_1) \cup \mathcal{P}_{Y_2}(X_2) = \mathcal{P}_Y(X)$ by virtue of the bundle representation. The assumption that the problem space $\mathcal{P}_Y(X)$ is not separable implies $\mathcal{P}_{Y_1}(X_1) \cap \mathcal{P}_{Y_2}(X_2) \neq \emptyset$. Assuming that $\mathcal{P}_{Y_1}(X_1) \cup \mathcal{P}_{Y_1}(X_1) \subset \mathcal{P}_Y(X)$ implies that

$$\mathcal{P}_Y(X) \setminus (\mathcal{P}_{Y_1}(X_1) \cup \mathcal{P}_{Y_2}(X_2)) \neq \{Y\}$$

since the space $\mathcal{P}_Y(X)$ admits the nontrivial problem bundle $Y = Y_1 \bowtie Y_2$. Therefore, there exists a problem $V \notin \mathcal{P}_{Y_1}(X_1) \cup \mathcal{P}_{Y_2}(X_2)$ and $V \in \mathcal{P}_Y(X)$ with $V \neq Y$. We let W be a solution to V and construct the induced problem space $\mathcal{P}_V(W)$. We obtain

$$\mathcal{P}_{Y_1}(X_1) \cup \mathcal{P}_{Y_2}(X_2) \cup \mathcal{P}_V(W) \subseteq \mathcal{P}_Y(X).$$

By induction, we deduce the nontrivial problem bundle

$$Y = Y_1 \bowtie Y_2 \bowtie W \bowtie \cdots.$$

Using the problem bundle $Y = Y_1 \bowtie Y_2$ of the space $\mathcal{P}_Y(X)$ in the above bundle, we get

$$Y_1 \bowtie Y_2 = Y = Y_1 \bowtie Y_2 \bowtie W \bowtie \cdots$$

which implies $W < Y_1$ or $W < Y_2$. This violates the requirement that the problem bundle $Y = Y_1 \bowtie Y_2$ is complete. Hence, there exists a problem $Y_3 \in \mathcal{P}_{Y_1}(X_1) \cap \mathcal{P}_{Y_2}(X_2)$. We deduce that $Y_3 < Y_1$ and $Y_3 < Y_2$. This violates the requirement that the bundle is complete. The separability of problem space $\mathcal{P}_Y(X)$ implies the separability of solution space $\mathcal{S}_Y(X)$ by theorem 2.23. □

[5]The bundle degree-order-space dimension equation formulated may be viewed as a principal equation for studying the dimension of problems and their corresponding solution spaces.

[6]The bundle representation serves as an important framework to determine problem spaces that can be merged using exactly two subproblems.

5. Equivalent and alternative bundles

In this section, we study the bundle representations of induced spaces for equivalent problems and alternative solutions.

Definition 5.1. Let Y_1, Y_2 be problems with solutions X_1 and X_2, respectively. Denote the induced problem spaces by $\mathcal{P}_{Y_1}(X_1)$ and $\mathcal{P}_{Y_2}(X_2)$. Also, let

$$Y_1 = \bigbowtie_{\substack{Y_{1_j} \in \mathcal{P}_{Y_1}(X_1) \\ Y_{1_j} \neq Y_1 \\ j \geq 1}} Y_{1_j}$$

and

$$Y_2 = \bigbowtie_{\substack{Y_{2_j} \in \mathcal{P}_{Y_2}(X_2) \\ Y_{2_j} \neq Y_2 \\ j \geq 1}} Y_{2_j}$$

be a bundle representation of the spaces $\mathcal{P}_{Y_1}(X_1)$ and $\mathcal{P}_{Y_2}(X_2)$, respectively. We say that the problem bundles are equivalent if there exist at least some Y_{1_j} and Y_{2_k} in each bundle so that $Y_{1_j} \equiv Y_{2_k}$. For those problems Y_{1_j} in the bundle for Y_1 such that there is no problem Y_{2_k} in the bundle for Y_2 so that $Y_{1_j} \equiv Y_{2_j}$, we have $Y_{1_j} = Y_{2_k}$. We denote the equivalence of the bundles by the notation

$$\bigbowtie_{\substack{Y_{1_j} \in \mathcal{P}_{Y_1}(X_1) \\ Y_{1_j} \neq Y_1 \\ j \geq 1}} Y_{1_j} \equiv \bigbowtie_{\substack{Y_{2_j} \in \mathcal{P}_{Y_2}(X_2) \\ Y_{2_j} \neq Y_2 \\ j \geq 1}} Y_{2_j}.$$

Definition 5.2. Let Y_1, Y_2 be problems with solutions X_1 and X_2, respectively. Denote the induced solution spaces by $\mathcal{S}_{Y_1}(X_1)$ and $\mathcal{S}_{Y_2}(X_2)$. Also, let

$$X_1 = \bigbowtie_{\substack{X_{1_j} \in \mathcal{S}_{Y_1}(X_1) \\ X_{1_j} \neq X_1 \\ j \geq 1}} X_{1_j}$$

and

$$X_2 = \bigbowtie_{\substack{X_{2_j} \in \mathcal{S}_{Y_2}(X_2) \\ X_{2_j} \neq X_2 \\ j \geq 1}} X_{2_j}$$

be a bundle representation of the spaces $\mathcal{S}_{Y_1}(X_1)$ and $\mathcal{S}_{Y_2}(X_2)$, respectively. We say that solution bundles are alternatives if there exist at least some solutions X_{1_j} and X_{2_k} in each bundle so that $X_{1_j} \perp X_{2_k}$. For those solutions X_{1_j} in the bundle for X_1 such that there is no X_{2_k} in the bundle for X_2 so that $X_{1_j} \perp X_{2_j}$, we have $X_{1_j} = X_{2_k}$. We denote this relation of the bundles by the notation

$$\bigbowtie_{\substack{X_{1_j} \in \mathcal{S}_{Y_1}(X_1) \\ X_{1_j} \neq X_1 \\ j \geq 1}} X_{1_j} \perp \bigbowtie_{\substack{X_{2_j} \in \mathcal{S}_{Y_2}(X_2) \\ X_{2_j} \neq X_2 \\ j \geq 1}} X_{2_j}.$$

Proposition 5.3. *Problem bundles containing problems Y_{i_j}, Y_{i_k} with $Y_{i_j} \not\equiv Y_{i_k}$ in a problem space are unique by their types up to equivalence. Furthermore, solution bundles containing solutions X_{i_j}, X_{i_k} with $X_{i_j} \not\perp X_{i_k}$ are unique by their types up to alternatives.*

Proof. We consider the problem space $\mathcal{P}_Y(X)$ induced by providing a solution X to the problem Y. We let

$$\overline{\mathcal{B}}_Y = \left\{ \begin{array}{c} \\ \\ \\ \bowtie\limits_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{i_j} \not\equiv Y_{i_k} \\ j \neq k}} Y_{i_j} \end{array} \; : \; Y = \bowtie\limits_{\substack{Y_{i_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{i_j} \not\equiv Y_{i_k} \\ j \neq k}} Y_{i_j} \right\}$$

denote the collection of all bundles in the space $\mathcal{P}_Y(X)$ with subproblems that are not pairwise equivalent. We arbitrarily choose the bundles

$$\bowtie\limits_{\substack{Y_{1_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{1_j} \not\equiv Y_{1_k} \\ j \neq k}} Y_{1_j}, \bowtie\limits_{\substack{Y_{2_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{2_j} \not\equiv Y_{2_k} \\ j \neq k}} Y_{2_j} \in \overline{\mathcal{B}}_Y$$

with

$$\bowtie\limits_{\substack{Y_{1_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{1_j} \not\equiv Y_{1_k} \\ j \neq k}} Y_{1_j} \equiv \bowtie\limits_{\substack{Y_{2_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{2_j} \not\equiv Y_{2_k} \\ j \neq k}} Y_{2_j}.$$

Without loss of generality, we let the bundle

$$\bowtie\limits_{\substack{Y_{1_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{1_j} \not\equiv Y_{1_k} \\ j \neq k}} Y_{1_j}$$

be trivial and assume that the second bundle

$$\bowtie\limits_{\substack{Y_{2_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{2_j} \not\equiv Y_{2_k} \\ j \neq k}} Y_{2_j}$$

is nontrivial. The triviality of the first bundle implies that there exists a problem Y_{1_k} in the bundle such that $Y_{1_k} = Y$. The equivalence of the bundles implies that there exists some problem Y_{2_l} in the second bundle such that $Y_{2_l} = Y$ or $Y_{2_l} \equiv Y$. The assumption that the bundle

$$\bowtie\limits_{\substack{Y_{2_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{2_j} \not\equiv Y_{2_k} \\ j \neq k}} Y_{2_j}$$

is nontrivial yields $Y_{2_l} \equiv Y$, i.e, problem Y_{2_l} is equivalent to problem Y and (hence) a solution to Y_{2_l} is also a solution to Y and vise-versa. Because the bundle

$$\bowtie\limits_{\substack{Y_{2_j} \in \mathcal{P}_Y(X) \\ j \geq 1 \\ Y_{2_j} \not\equiv Y_{2_k} \\ j \neq k}} Y_{2_j}$$

is nontrivial, problem Y_{2_l} is a proper subproblem of Y (i.e $Y_{2_l} < Y$) and a solution cannot solve Y without solving problems Y_{2_j} ($j \geq 1$) in the bundle with $j \neq l$, a deduction inconsistent with the equivalence of problem Y_{2_l} and problem Y. [7] □

6. Concluding remarks, conjectures, and open problems

This paper has developed a structural framework for the study of problems and solutions through the spaces induced by solving them. The central objects are the problem space $\mathcal{P}_Y(X)$, consisting of the subproblems involved in providing a solution X to a problem Y, and the corresponding solution space $\mathcal{S}_Y(X)$. The subproblem and subsolution relations provide these spaces with an internal order structure, while equivalence of problems and alternativity of solutions allow different but related descriptions to be compared within the same framework.

The merger operation $\bowtie$ provides a way of recording how compatible subproblems, or compatible subsolutions, combine to represent a larger problem or solution. This leads to the notions of trivial, nontrivial, and complete bundles. In particular, the bundle representation is intended to retain information about the internal organization of a problem space that is not visible from the original problem alone. The distinction between regular and irregular problems is relevant here: in the finite setting considered in the paper, regularity forces bundle representations to be trivial, whereas the existence of a nontrivial bundle forces irregularity.

Several consequences of this viewpoint have been established. First, every induced problem space of finite complexity, and every induced solution space of finite index, admits a bundle representation. Second, the paper gives a criterion for the existence of nontrivial complete bundles under additional structural hypotheses. Third, separability is shown to be equivalent for the problem space and its corresponding solution space. A complete nontrivial problem bundle of order two is also shown to imply separability of both spaces. Finally, the order of a bundle, the dimension defined through minimum-order complete nontrivial bundles, and the degree of a bundle provide numerical parameters with which to compare different representations. The notions of equivalent problem bundles and alternative solution bundles further indicate how such representations may be compared across equivalent problems and alternative solutions.

These results constitute a first step towards a systematic theory of problem–solution structure. They also expose several questions that require further investigation. We record some of them below. Throughout, any statement involving dimension is understood to require that the relevant collection of nontrivial complete bundles is nonempty, as in the definition of dimension above.

Open problems.

Question 6.1 (Existence of nontrivial complete bundles). Find necessary and sufficient conditions, stated directly in terms of the subproblem order and the merger structure, for an induced problem space $\mathcal{P}_Y(X)$ to admit a nontrivial complete bundle. The finite existence theorem proved above guarantees a bundle representation, but the representation it supplies may be trivial. Thus, finite complexity alone does not, under the definitions used here,

[7] The argument also holds for bundles in a solution space by replacing the problem spaces with their corresponding solution spaces.

establish that the collection $\mathcal{B}_Y$ of nontrivial complete bundles is nonempty. A sharper existence criterion would in particular clarify the role of the additional hypotheses in the complete nontrivial bundle criterion.

Question 6.2 (Higher-order separability). The paper relates complete bundles of order two to the separability of the induced problem and solution spaces. For an integer $k \geq 3$, one may define k-separability by requiring a decomposition into k induced subspaces whose union is the whole space, which are pairwise disjoint, and whose components in distinct parts are not equivalent (or, for solution spaces, are not alternative). Determine precise conditions under which a complete bundle of order k yields such a k-fold decomposition. It is also natural to ask whether an appropriate converse holds.

Question 6.3 (Problem-space and solution-space dimensions). The equivalence of separability for $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ does not by itself determine whether their dimensions coincide. Identify hypotheses under which complete problem bundles and complete solution bundles correspond in a way that preserves the number of components. In particular, determine whether the equality of dimensions holds in general whenever both dimensions are defined, or whether additional conditions on the problem–solution correspondence are necessary.

Question 6.4 (Infinite induced spaces). The bundle-existence arguments in this paper use finiteness of the complexity or index. Extend the construction to infinite problem and solution spaces. A natural direction is to determine which order-theoretic hypotheses, such as well-foundedness or suitable chain conditions, permit transfinite bundle constructions, and whether an ordinal, or cardinal-valued analogue of bundle order and dimension is appropriate.

Question 6.5 (Effective construction and computation). For finite spaces presented by explicit subproblem and subsolution relations, develop procedures for determining whether a nontrivial complete bundle exists, enumerating such bundles, and computing the dimension and degree when these invariants are defined. What bounds can be obtained for the computational cost of these procedures in terms of the sizes and order structure of the induced spaces?

Conjectures. The preceding questions suggest several precise statements that may serve as targets for further work.

Conjecture 6.6 (Problem–solution dimension correspondence)**.** *Let Y be a problem with a solution X, and suppose that the induced problem space $\mathcal{P}_Y(X)$ and solution space $\mathcal{S}_Y(X)$ are finite. If both spaces admit nontrivial complete bundle representations, then*

$$\dim\big[\mathcal{P}_Y(X)\big] = \dim\big[\mathcal{S}_Y(X)\big].$$

In other words, once nontrivial complete bundle representations exist on both sides, the minimum number of components needed to represent the problem should agree with the minimum number needed to represent its solution.

Conjecture 6.7 (Higher-order bundle–separability principle)**.** *Let Y be a problem with a solution X, and let $k \geq 3$. If the induced problem space $\mathcal{P}_Y(X)$ admits a complete nontrivial bundle of order k, then, under the compatibility conditions required for the corresponding induced subspaces to be defined, the problem space admits a k-fold separable decomposition. The corresponding solution space $\mathcal{S}_Y(X)$ admits a k-fold separable decomposition as well.*

The first conjecture proposes numerical compatibility between the problem and solution sides of the theory; the second proposes an extension of the order-two separability result to

bundles with more components. Both require further investigation and are not asserted as consequences of the results proved in this paper.

A further direction concerns the quantitative invariants introduced here. The complexity $\mathbb{C}[\mathcal{P}_Y(X)]$, the index $\mathbb{I}[\mathcal{S}_Y(X)]$, and the entropy $\mathcal{E}[\mathcal{S}_Y(X)]$ record the sizes of the induced spaces, whereas bundle order, dimension, and degree describe aspects of their representation. Establishing inequalities or structural relations between these two types of data would give the bundle degree-order-dimension identities additional quantitative significance.

We expect that further development of the merger operation, together with a more complete understanding of the order structure of induced spaces, will provide a foundation for addressing these questions. In this way, the theory of problem and solution bundles can develop into a systematic means of classifying, comparing, and measuring the internal structure of problems and their solutions.

References

1. T. Agama, *Problem theory and application to the P vs NP problem*, Hal-05763022, 2026.

2. G. Birkhoff, *Lattice Theory*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, RI, 1967.

3. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd ed., Cambridge University Press, Cambridge, 2002.

4. H. A. Simon, The architecture of complexity, *Proceedings of the American Philosophical Society* **106** (1962), no. 6, 467–482.

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