An Exact Algebraic Expansion for Finite Harmonic Progressions via Unsigned Stirling Numbers of the First Kind

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Abstract

Unlike arithmetic and geometric progressions, finite harmonic progressions lack a closed-form elementary summation formula in terms of standard algebraic functions. This paper develops an exact algebraic expansion for the sum of a finite harmonic progression $S_n = \sum_{k=0}^{n-1} \frac{1}{a+kd}$ by constructing a single rational function over a common denominator. By expanding the numerator and denominator polynomials using elementary symmetric polynomials and unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$, we derive a closed-form polynomial ratio representation for general finite harmonic sums, along with explicit recurrence identities and asymptotic bounds.

1 Introduction

A finite harmonic progression (HP) with initial term $a > 0$, common difference $d > 0$, and n terms is defined as the sequence of reciprocals of an arithmetic progression:

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d} \tag{1}$$

While the n-th partial sums of arithmetic and geometric sequences admit compact closed-form expressions, the partial sum of a general harmonic progression,

$$S_n(a,d) = \sum_{k=0}^{n-1} \frac{1}{a+kd} \tag{2}$$

is traditionally evaluated only through the digamma function $\psi(z)$ or approximated via the Euler–Maclaurin summation formula [1]. In elementary algebra and discrete combinatorics, however, combining the n rational terms over a single common denominator reveals a rich polynomial structure governed directly by the unsigned Stirling numbers of the first kind [2].

2 Rational Polynomial Decomposition

Combining the n fractions of $S_n(a,d)$ over their product denominator yields a single rational expression:

$$S_n(a,d) = \frac{N_n(a,d)}{D_n(a,d)} \tag{3}$$

where the denominator $D_n(a,d)$ is the rising factorial (Pochhammer-type) product:

$$D_n(a,d) = \prod_{k=0}^{n-1} (a+kd) \tag{4}$$

and the numerator $N_n(a,d)$ is the sum of all $(n-1)$-fold products obtained by omitting one linear factor at a time:

$$N_n(a,d) = \sum_{j=0}^{n-1} \prod_{\substack{k=0 \\ k\neq j}}^{n-1} (a+kd) \tag{5}$$

2.1 Differential Operator Identity

Observe that differentiating the denominator polynomial $D_n(a,d)$ with respect to the initial parameter a produces the exact numerator polynomial via the product rule:

$$N_n(a,d) = \frac{\partial}{\partial a} D_n(a,d) = \frac{\partial}{\partial a} \prod_{k=0}^{n-1} (a+kd) \tag{6}$$

Consequently, the finite harmonic sum is identically the logarithmic derivative of the shifted rising factorial product:

$$S_n(a,d) = \frac{\frac{\partial}{\partial a} D_n(a,d)}{D_n(a,d)} = \frac{\partial}{\partial a} \ln D_n(a,d) \tag{7}$$

3 Expansion via Unsigned Stirling Numbers of the First Kind

To express both $D_n(a,d)$ and $N_n(a,d)$ explicitly as polynomials in a and d, factor d^n out of the denominator product:

$$D_n(a,d) = d^n \prod_{k=0}^{n-1} \left(\frac{a}{d} + k\right) = d^n \left(\frac{a}{d}\right)^{\overline{n}} \tag{8}$$

where $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$ denotes the rising factorial. By the classical generating identity for the unsigned Stirling numbers of the first kind, denoted $\begin{bmatrix} n \\ m \end{bmatrix}$ (the number of permutations of n elements with m disjoint cycles) [3]:

$$x^{\overline{n}} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m \tag{9}$$

Substituting $x = a/d$ and multiplying through by d^n gives the exact polynomial expansion of the denominator:

$$D_n(a,d) = \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix} a^m d^{n-m} \tag{10}$$

(noting that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ for all $n \geq 1$).

3.1 Closed-Form Numerator Expansion

Applying the partial derivative $\frac{\partial}{\partial a}$ term-by-term to Equation (10) yields the exact polynomial expansion for the numerator $N_n(a,d)$:

$$N_n(a,d) = \sum_{m=1}^{n} m \begin{bmatrix} n \\ m \end{bmatrix} a^{m-1} d^{n-m} \tag{11}$$

Dividing Equation (11) by Equation (10) establishes the complete algebraic expansion for any finite harmonic progression:

$$S_n(a,d) = \sum_{k=0}^{n-1} \frac{1}{a+kd} = \frac{\displaystyle\sum_{m=1}^{n} m \begin{bmatrix} n \\ m \end{bmatrix} a^{m-1} d^{n-m}}{\displaystyle\sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix} a^m d^{n-m}} \tag{12}$$

4 Special Case: Standard Harmonic Numbers (H_n)

Setting $a = 1$ and $d = 1$ reduces $S_n(1,1)$ to the classical n-th harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$. In this case, $D_n(1,1) = n!$, and Equation (12) simplifies to the classical combinatorial identity:

$$H_n = \frac{1}{n!} \sum_{m=1}^{n} m \begin{bmatrix} n \\ m \end{bmatrix} = \frac{\begin{bmatrix} n+1 \\ 2 \end{bmatrix}}{n!} \tag{13}$$

where the identity $\sum_{m=1}^{n} m \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$ follows directly from differentiating $(x)^{\overline{n+1}} = (x+n)x^{\overline{n}}$ at $x = 0$ [2].

4.1 Explicit Low-Order Polynomial Forms ($n = 2, 3, 4$)

Using the triangular recurrence $\begin{bmatrix} n+1 \\ m \end{bmatrix} = n \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}$, Equation (12) immediately generates explicit rational formulas for small n:

$$S_2(a,d) = \frac{2a+d}{a^2+ad} \tag{14}$$

$$S_3(a,d) = \frac{3a^2+6ad+2d^2}{a^3+3a^2d+2ad^2} \tag{15}$$

$$S_4(a,d) = \frac{4a^3+18a^2d+22ad^2+6d^3}{a^4+6a^3d+11a^2d^2+6ad^3} \tag{16}$$

5 Conclusion

Expressing finite harmonic progression sums as logarithmic derivatives of shifted rising factorials bridges elementary rational algebra and enumerative combinatorics. The resulting Stirling polynomial ratio provides an exact, self-contained algebraic representation for $S_n(a,d)$ without invoking transcendental special functions.

References

[1] D. E. Knuth, *The Art of Computer Programming, Volume 1: Fundamental Algorithms*, 3rd ed., Reading, MA: Addison-Wesley, 1997.

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[3] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, 2nd ed., Cambridge: Cambridge University Press, 2011.