Novel Triangular Number Unfolding in Rectangular Matrices

Jean-Yves BOULAY

Jean-Yves BOULAY independent researcher – FRANCE – jean-yvesboulay@orange.fr ORCID: 0000-0001-5636-2375
https://www.researchgate.net/profile/Jean-Yves-Boulay

Abstract: *In contrast to the classical geometric illustration in the form of a triangle, this work highlights a simple yet remarkable unfolding of the representation of triangular numbers into rectangular arrangements. These new matrices reveal a richer field of geometric representation for the sequence of integers, giving rise to numerous variants—the number of which grows exponentially—where parity directly governs the dynamics of their spatial expansion.*

Keywords: Triangular numbers, Figurate geometry, Rectangular matrices, Parity, Visual proofs, Mathematics education.

1 Introduction

Classical representations of triangular numbers—based on stacking discrete units into equilateral or right-angled triangles—effectively illustrate the cumulative arithmetic sum $T_n = 1 + 2 + \ldots + n$, yet they offer a static and singular spatial expression. Moreover, deriving the closed-form algebraic identity $T_n = \frac{n(n+1)}{2}$ through traditional figurate geometry typically requires an external operation: duplicating the triangle and interlocking two identical copies to form an $n \times (n + 1)$ rectangle.

In contrast, the rectangular unfolding framework proposed in this paper operates intrinsically upon a single triangular entity, opening a much richer field for geometric and combinatorial exploration. By treating consecutive integers as parity-governed gnomons—horizontal rows for odd numbers and vertical double-columns for even numbers—a sequence of consecutive integers can be arranged in numerous distinct ways while preserving total area, such as the 55 elementary units forming T_{10}. These space-filling tilings enable the simultaneous analysis of block dimensions, orthogonal orientations, symmetries, and internal growth trajectories that remain hidden in standard triangular layouts.

The contribution of this work is threefold:

- **Intrinsic Parity-Based Construction:** Establishing a gapless rectangular representation that directly visualizes the algebraic factors of T_n without duplicating the figure.

- **Classification of Expansion Dynamics:** Characterizing the four regular directional growth modes (diagonal, horizontal, vertical, and spiral) along with their 16 structural variants, and embedding them within the complete binary space of 2^{n-1} possible rectangular configurations.

- **Pedagogical Applicability:** Demonstrating how this visual and tactile framework supports active learning in mathematics education by connecting arithmetic parity, geometric symmetry, and combinatorial recursion.

The remainder of this paper is organized as follows. Section 2 introduces the parity-driven unfolding mechanism and its dimensional properties. Section 3 classifies the four regular expansion dynamics and their 16 variants. Section 4 generalizes the construction to the universal binary tree of 2^{n-1} configurations. Finally, Section 5 discusses the theoretical implications and educational applications of these rectangular matrices (see also the Appendix).

2. Rectangular Unfolding of Triangular Numbers

The classical geometric representation of triangular numbers relies on a pyramidal arrangement of units: $T_n = 1 + 2 + \ldots + n$.

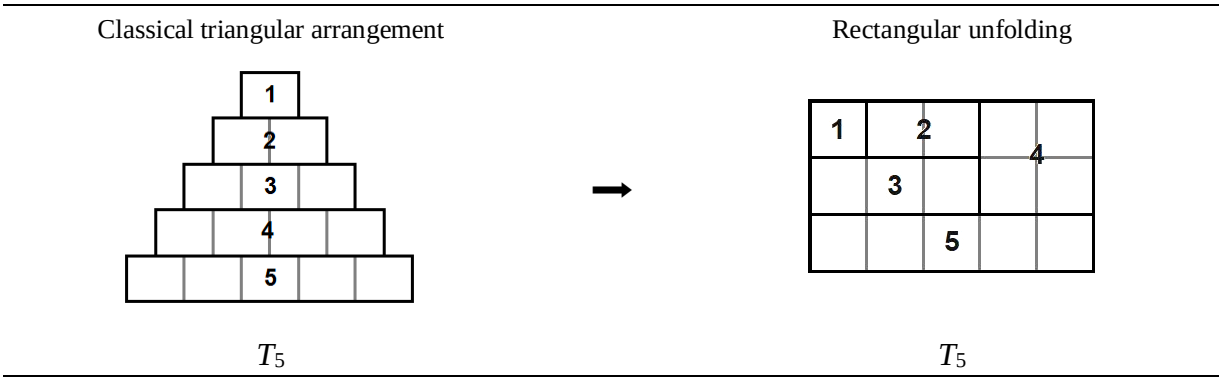


Figure 1: Dual geometric representation of triangular numbers, taking T_5 as example. Left: classical triangular arrangement illustrating the cumulative structure of $Tn = 1 + 2 + \cdots + n$. Right: rectangular unfolding in which the integers 1, 2 ,…, n form successive vertical and horizontal blocks of increasing width and height (see also Figure 2).

This visualization emphasizes the additive growth of successive layers. However, as shown in Figure 1, the same quantity admits an alternative and equally natural geometric form: a **rectangular unfolding** in which the integers 1, 2, … , n are no longer stacked evenly but aligned vertically and horizontally within a rectangle of width n (for odd n) or $n + 1$ (for even n).

The triangular and rectangular representations therefore appear as two projections of the same underlying structure. The triangular form expresses the cumulative nature of the sequence, while the rectangular form expresses its spatial generative law.

2.1 Triangular Number Unfolding in Rectangular Matrices

As illustrated in Figure 2, this geometric representation consists of alternately arranging each odd integer k in a horizontal row of length k, and each even integer in a vertical double column of height corresponding to $k/2$. The total area of the resulting rectangle is again $1 + 2 + \ldots + n = T_n$, but the geometry is no longer triangular: it becomes a **progressive rectangle** whose internal structure mirrors the arithmetic progression itself.

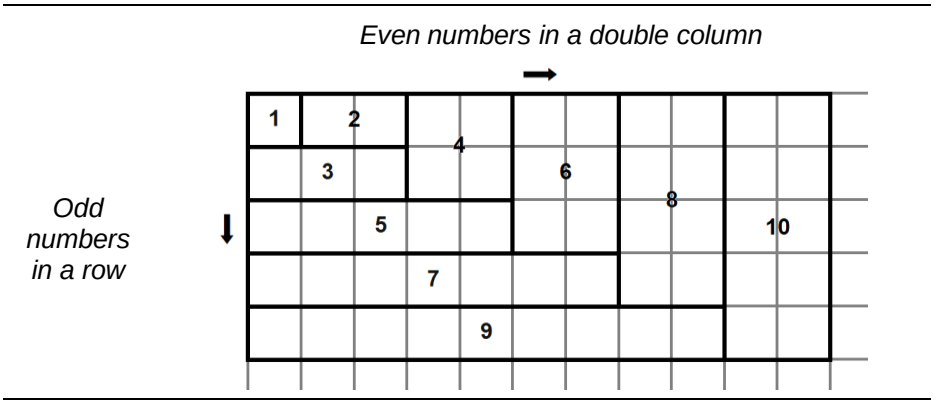


Figure 2: Mechanism for the Euclidean geometric construction of rectangular matrices illustrating triangular numbers. Shown from T_1 to T_{10} as example.

2.2 Geometric triangular structure in a unified spatial progression

In Figure 3, each grid illustrates the geometric structure of the corresponding triangular number, showing that the arithmetic progression generates a perfectly continuous sequence of figurate forms across the entire integer domain. The figure highlights the bilateral symmetry of the model, displaying odd-indexed triangular numbers T_{2m-1} as rectangles of dimensions $m \times (2m - 1)$ on the left, and even-indexed triangular numbers T_{2m} as rectangles of dimensions $m \times (2m + 1)$ on the right.

Unlike the classical triangular representation—where the sum of any two consecutive triangular numbers $T_{n-1} + T_n = n^2$ universally and continuously forms a geometric square, this rectangular framework introduces a parity-based structural rhythm: vertical stacking of consecutive pairs (T_{2m}, T_{2m+1}) constructs exact geometric squares of side $(2m + 1)$, whereas horizontal pairing (T_{2m-1}, T_{2m}) produces proportional 1×4 rectangles with an area strictly equivalent to even squares $(2m)^2$.

While classical geometric proofs of triangular numbers traditionally rely on duplicating the structure—interlocking two distinct copies of T_n to form a dual $n \times (n + 1)$ rectangle—the approach introduced here operates intrinsically upon a single triangular entity. By unrolling the successive integer layers directly within a single, unified rectangular matrix, this representation preserves the structural unity of the figure. It shifts the paradigm from a mere additive combination of two independent parts to an internal, self-contained generative law of space, revealing hidden symmetries and modular alignment properties native to the single triangular number itself.

Furthermore, this generative mechanism establishes a continuous tessellation of the two-dimensional Cartesian space, mapping the infinite sequence of natural numbers onto an unbroken geometric progression. Unlike arbitrary partitions, each successive integer n acts as a deterministic gnomon that perfectly completes and expands the pre-existing matrix.

By alternately distributing odd numbers along linear rows and even numbers within double-column blocks, the system fills the coordinate grid progressively and integrally. This spatial growth ensures that every discrete cell of the expanding rectangular boundary is entirely occupied, without gaps or overlaps, transforming the abstract sequence of integers into a perfectly dense, space-filling geometric matrix.

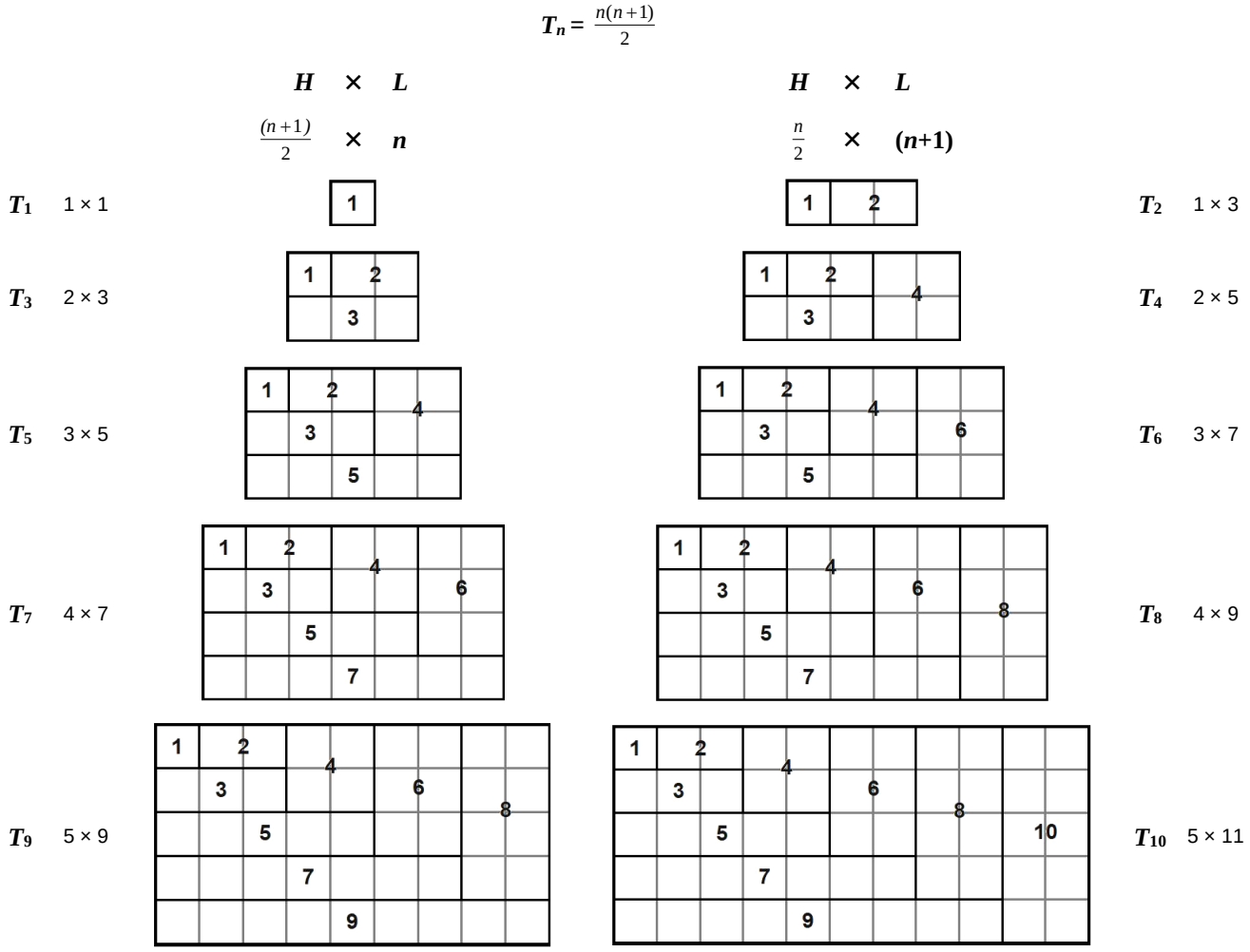


Figure 3: Parity-based rectangular decomposition of triangular numbers (T_1 to T_{10}), contrasting odd-indexed ($m \times 2m - 1$) grids (left) with even-indexed ($m \times 2m + 1$) grids (right).

3. The Four Rectangular Regular Configurations of Triangular Numbers

The geometric unfolding of a triangular number *Tn* in a compact rectangular matrix is governed by a directional algorithm. As the matrix grows from the initial central core (T_1, T_2), each subsequent gnomon—representing either an odd row or an even double-column—is appended along one of four cardinal directions: **Upwards** (↑), **Downwards** (↓), **Leftwards** (←), or **Rightwards** (→).

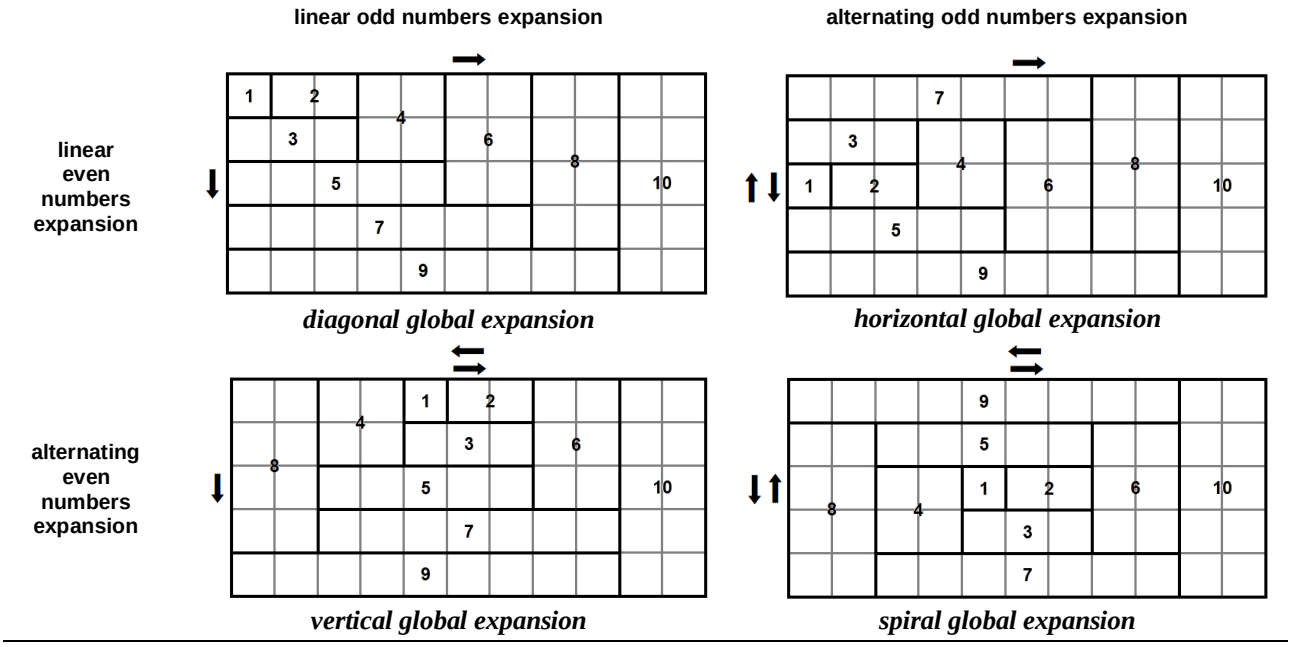


Figure 4: The four possible rectangular global regular expansion configurations (see also Figure 5).

Figure 4 illustrates the four possible modes of regular expansion for the rectangular unfolding of triangular numbers. As a reminder, the areas representing even numbers are always positioned either to the left or to the right of the blocks previously placed within the matrices, while the areas representing odd numbers are always positioned above or below the previously formed blocks.

The four rectangular configurations of triangular numbers can be characterized by progression rules that describe the spatial organization of their sequence. These four distinct modalities consist of a double linear progression, a global alternation, and two configurations that combine a linear progression with an alternating progression based on parity. In the latter two configurations, the roles of odd and even numbers are reversed: the horizontal linear progression of odd numbers is paired with a vertical alternating progression of even numbers, whereas the second configuration assigns the vertical alternating progression to the odd numbers and the horizontal linear progression to the even numbers. Synthetic directional diagrams make these rules immediately comparable, allowing for a distinction between the numerical structure and its geometric realization. This classification thus complements the one based on modes of global expansion by introducing a level of analysis focused on the directional mechanisms that organize the configurations.

Thus, these four regular possible configurations can be:

- **in diagonal global expansion;**
- **in horizontal global expansion;**
- **in vertical global expansion;**
- **in spiral global expansion.**

3.1 The Four Rectangular Global Expansion Dynamics

Figure 5 illustrates the geometric construction process within a rectangular matrix, tracing the successive movements from one number to the next. The numbers 4, 5, 6, and 7 serve as an illustrative example, where each arrow indicates the direction of growth associated with the placement of each successive block. Across the four configurations, the expansion vector from one block to the next can drive toward the left, right, upward, or downward—encompassing four possible orthogonal directions.

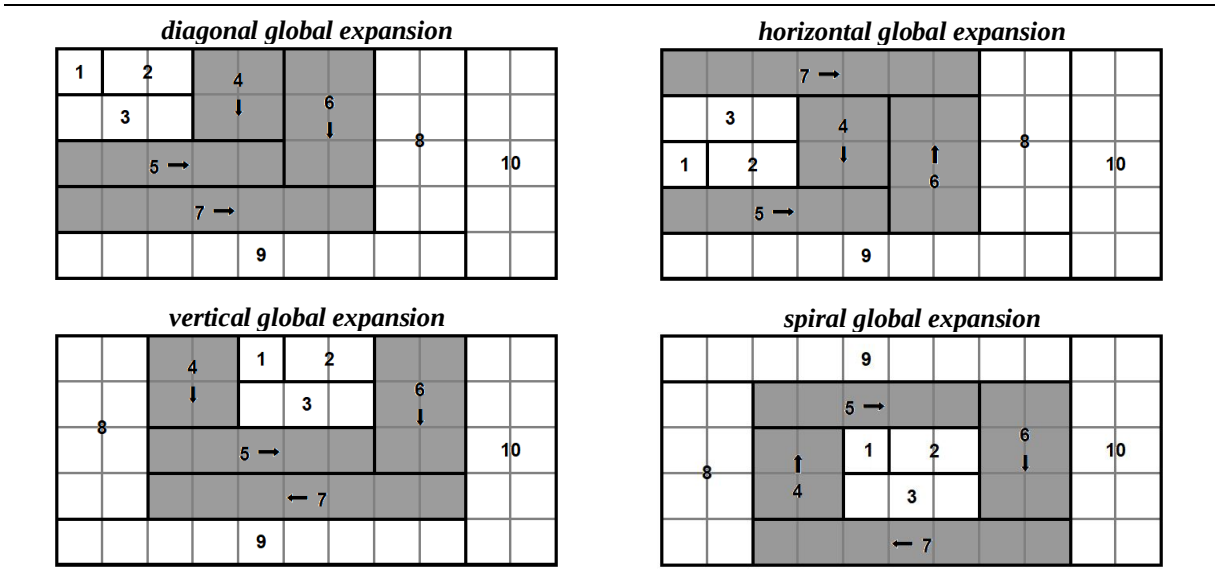


Figure 5: The four rectangular global expansion dynamics with areas 4→5, 5→6, 6→7 and 7→8 as an example.

In a first configuration, horizontal and vertical movements in the same direction (always downward or always to the right) alternate regularly. Therefore, the geometric dynamics evolve in a ***diagonal global expansion***.

In a second configuration, two vertical movements cancel each other out, while two horizontal ones combine. Therefore, the geometric dynamics evolve in ***a horizontal global expansion***.

In a third scenario, from one number to the next, two horizontal movements cancel each other out, while two vertical ones combine. Therefore, the geometric dynamics evolve in ***a vertical global expansion***.

Finally, in a fourth configuration, the four movements are all in different directions. Therefore, the geometric dynamics evolve in ***a spiral global expansion***.

3.2 Formal Description of the Four Global Expansion Dynamics and Their Variations

By analyzing the permutations of these growth steps, the system yields **4 fundamental expansion families**, each containing **4 chiral or structural variants**, totalling **16 distinct matrix configurations** for a given triangular number (illustrated below Figure 6 for T_{10}).

3.2.1 Diagonal Global Expansion

In the **Diagonal Expansion** family, the horizontal and vertical movements alternate regularly but do not cancel out; instead, they consistently choose the same directional signs (e.g., always combining downward ↓ and rightward → motions).

Mechanical Principle: The growth vector points along a diagonal axis of the Cartesian plane. The matrix exhibits a stepped, stair-like internal organization where the blocks climb or descend progressively from one corner of the global matrix to the opposite corner.

The 4 Variations: This family displays the most pronounced internal structural displacement. The four variations are defined by the four major diagonal trajectories of the Cartesian grid: Top-Left to Bottom-Right, Top-Right to Bottom-Left, and their respective inverse paths.

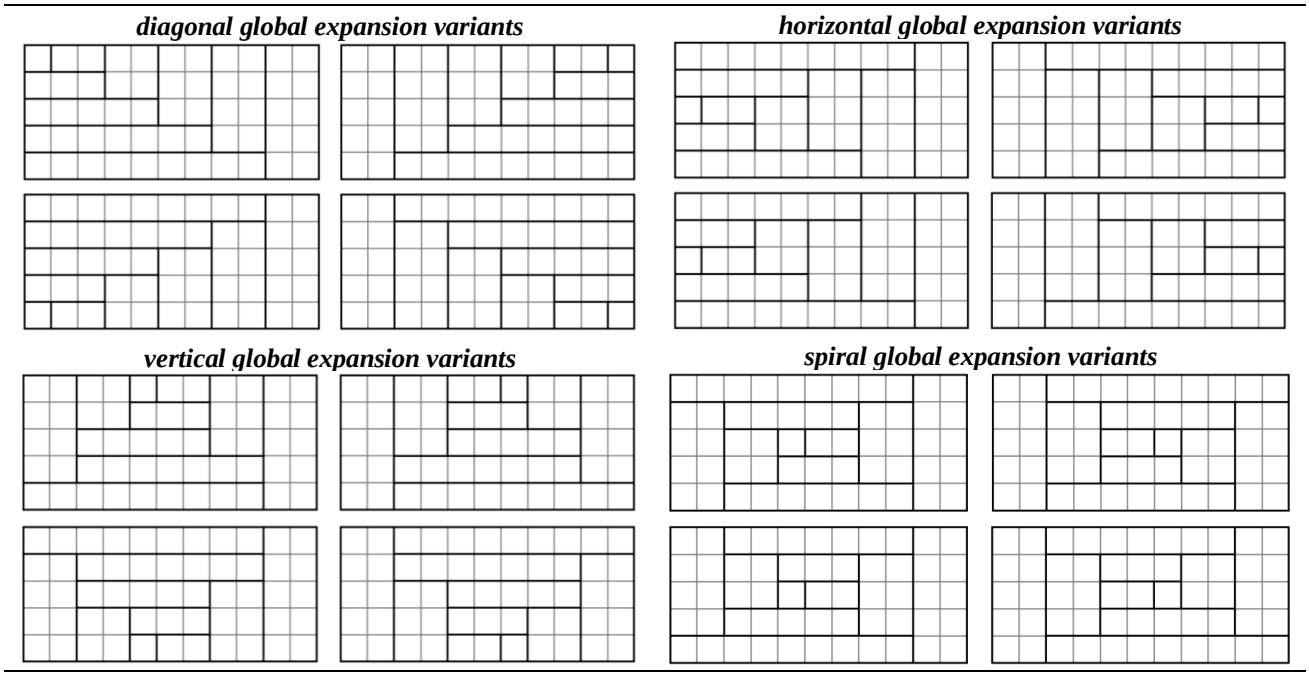


Figure 6: The sixteen global expansion variants of matrices for unfolding triangular numbers in rectangular configurations.

3.2.2 Horizontal Global Expansion

In the **Horizontal Expansion** family, the vertical growth vectors cancel each other out (↑ and ↓), while the horizontal movements combine to drive the primary geometric displacement along the horizontal axis.

Mechanical Principle: The matrix expands predominantly from left to right or right to left. The vertical blocks (the even double-columns) act as solid lateral walls or partitions that space out the horizontal lines, stretching the matrix along the Cartesian abscissa.

The 4 Variations: These configurations are characterized by central vertical columns flanking shifted horizontal sub-blocks. The four variations correspond to the left/right mirror symmetries of the initial core placement and whether the final, largest double-columns bind the matrix on its eastern or western boundaries.

3.2.3 Vertical Global Expansion

In the **Vertical Expansion** family, the horizontal growth vectors cancel each other out over a complete cycle (← followed immediately or closely by →), while the vertical movements combine to drive the matrix systematically along the vertical axis.

Mechanical Principle: The expansion dynamics privilege a vertical stack. The internal components alternate or align such that horizontal steps serve strictly to balance the width, whereas vertical steps continuously shift the boundaries upwards and downwards.

The 4 Variations: The four matrices in this group exhibit a stratified, layered horizontal bar structure inside the matrix. The variations depend on whether the longest odd horizontal rows are forced toward the upper and lower poles or clustered in alternating internal horizons, creating different vertical symmetries.

3.2.4 Spiral Global Expansion (The Concentric Envelope)

In the **Spiral Expansion** family, the generative dynamics follow a rotational sequence where the four directional movements are all distinct and alternate sequentially (e.g., ↓ → ↑ ←).

Mechanical Principle: Each integer layer acts as a concentric shell that wraps completely around the perimeter of the previous state. The geometric center remains highly balanced, and the matrix expands evenly outward in all four directions.

The 4 Variations: Depending on the initial orientation of the core (T_1, T_2) and the clockwise versus counter-clockwise rotation of the sequence, the spiral generates four distinct configurations. Visually, these variations manifest as nested rectangular rings, differing only by the absolute positioning of the outermost blocks (the final rows and double-columns).

4. Complete Space of Universal Expansion Configurations

When representing triangular numbers within a rectangular matrix, numerous other arrangements exist beyond the four regular expansion dynamics previously introduced (diagonal, spiral, horizontal, and vertical). Building on the established rule that odd integers n form horizontal rows of dimensions $1 \times n$ while even integers n form double columns of dimensions $n/2 \times 2$, this construction can be generalized to all possible directional combinations.

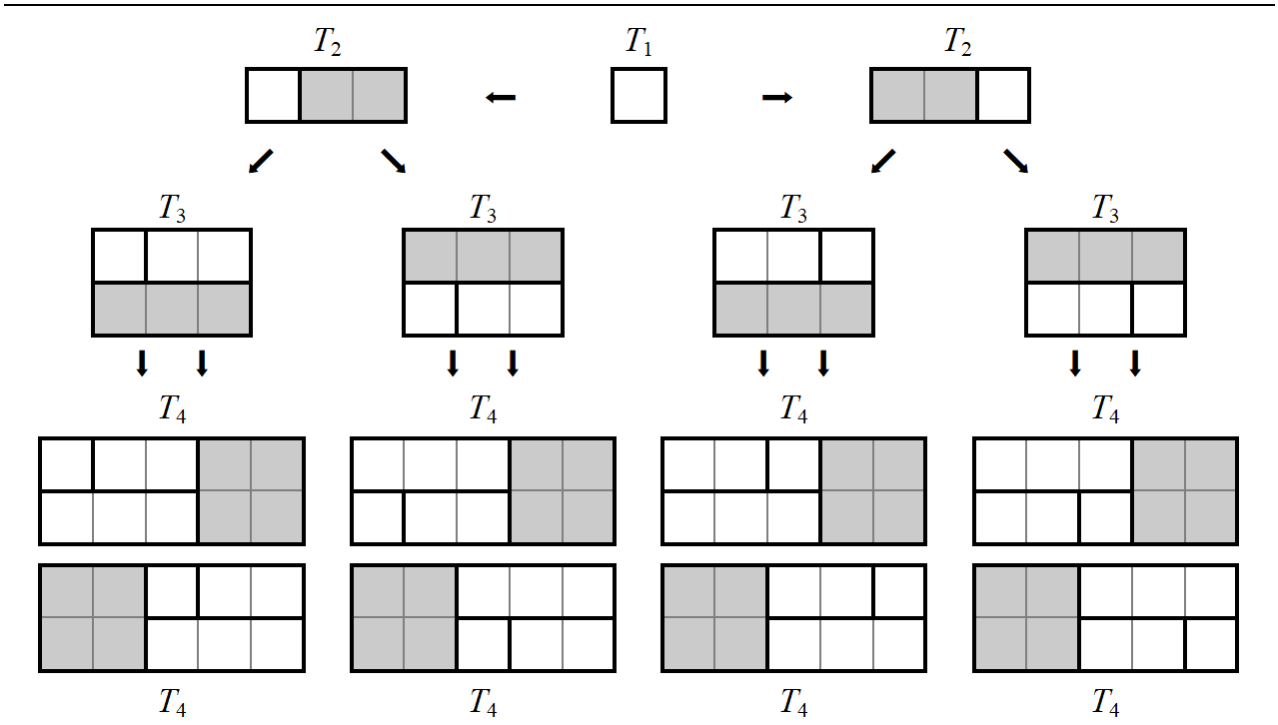


Figure 7: Complete binary tree of the universal expansion configurations **from T_1 to T_4 as example**. At each incremental step n, the newly added integer acts as a rectangular graft (shaded in grey) attaching to the previously constituted white matrix along two possible directions (left or right for even double-columns; top or bottom for odd rows), systematically generating $\mathbf{2^{n-1}}$ distinct matrix layouts.

As shown in **Figure 7,** this generative mechanism operates as a recursive binary branching process where each consecutive integer acts as a **rectangular graft** (highlighted in grey) attaching to the pre-existing white matrix:

- **From T_1 to T_2**: Starting from the initial single-unit core (T_1), the graft of value 2 (a 1×2 block) can be attached either to the left or to the right, yielding $2^1 = 2$ distinct rectangular configurations**;**

- **From T_2 to T_3**: For each of these two T_2 matrices, the graft of value 3 (a 1×3 horizontal row) can be appended either above or below the existing structure, generating $2^2 = 4$ valid arrangements**;**

- **From T_3 to T_4**: Similarly, onto each of the four T_3 configurations, the graft of value 4 (a 2×2 double-column block) attaches either to the left or right flank, producing $2^3 = 8$ distinct matrices.

Because every odd graft offers two vertical attachment poles (top or bottom) and every even graft offers two lateral attachment flanks (left or right), each incremental step doubles the number of possible layouts. Consequently, the complete universal

expansion of any triangular number T_n encompasses exactly $\mathbf{2^{n-1}}$ distinct, gapless rectangular configurations—yielding, for instance, $2^9 = 512$ unique matrices for T_{10}—among which the four regular expansion dynamics previously described represent the strictly periodic trajectories.

5. Discussions and Conclusion

The rectangular unfolding framework presented in this study offers a substantial shift in how triangular numbers are visualized and analyzed. Whereas classical figurate geometry relies on an external duplication of a triangle to establish the standard closed-form identity $T_n = \frac{n(n+1)}{2}$, the present method operates intrinsically on a single triangular entity. By partitioning consecutive integers according to their parity—assigning odd integers to $1 \times k$ horizontal rows and even integers to $\frac{k}{2} \times 2$ vertical double-columns—the sequence unfolds into a gapless, space-filling rectangular matrix. Furthermore, generalizing these directional attachments demonstrates that any triangular number T_n admits exactly 2^{n-1} distinct rectangular tilings, within which the four regular expansion dynamics (diagonal, horizontal, vertical, and spiral) emerge as highly structured, periodic trajectories.

This parity-driven spatial organization also complements recent structural investigations into the arithmetic properties of triangular sequences. In particular, previous work by Boulay (2026) established a direct structural link between triangular numbers and Sophie Germain numbers—where a prime p yields a safe prime $2p + 1$—highlighting how doubling and unit-shift transformations govern fundamental number-theoretic patterns. Notably, the rectangular dimensions derived in the present model—specifically $m \times (2m - 1)$ for odd-indexed triangular numbers T_{2m-1} and $m \times (2m + 1)$ for even-indexed triangular numbers T_{2m} —naturally embed the linear factors $(2m - 1)$ and $(2m + 1)$ into the geometric boundary of the matrix. Thus, rectangular unfolding provides a spatial bridge between elementary figurate geometry and deeper multiplicative or prime-related symmetries within the integer sequence.

Beyond its theoretical elegance, this geometric representation holds significant value for **mathematics education**:

- **Visual Proofs Without Duplication:** In secondary and undergraduate mathematics, students often struggle to connect the triangular dot pattern of $1 + 2 + \ldots + n$ with its algebraic formula without resorting to memorization. Rectangular matrices provide an immediate, self-contained visual proof: students can directly observe the height and width of the resulting grid— $m \times (2m - 1)$ for odd indices and $m \times (2m + 1)$ for even indices—making the algebraic factors n and $(n + 1)$ visually tangible.

- **Intuitive Grasp of Parity and Gnomonic Growth:** The strict alternation between horizontal rows and vertical double-columns gives parity an explicit spatial meaning. Learners can physically or digitally manipulate integer blocks as deterministic gnomons, reinforcing their understanding of how odd and even numbers interact to preserve rectangular symmetry and form exact squares when consecutive triangular numbers are combined.

- **Bridging Geometry, Combinatorics, and Algorithmic Thinking:** Exploring the 16 regular variants and the broader binary tree of 2^{n-1} configurations encourages active, inquiry-based learning. Tracing the internal growth vectors introduces students to foundational concepts in combinatorics, spatial transformations (reflections and rotations), and algorithmic recursion in an accessible visual environment.

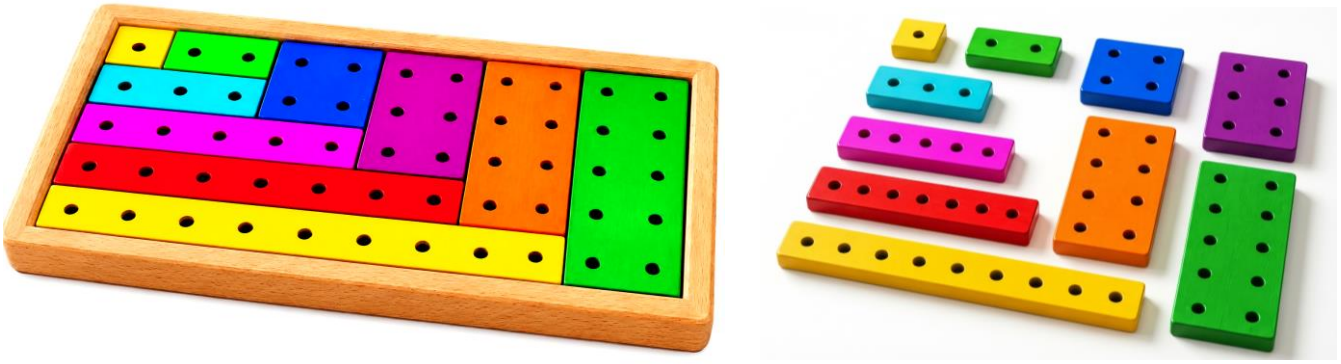
In conclusion, unfolding triangular numbers into rectangular matrices transforms a static arithmetic sum into a dynamic geometric system. Future research may extend this parity-driven matrix approach to other families of figurate numbers—such as pyramidal or polygonal sequences—as well as investigate its empirical effectiveness as a tactile and digital learning tool in the mathematics classroom (see also the Appendix).

References

Boulay, J.-Y. (2026). Structural Link Between Sophie Germain Numbers and Triangular Numbers https://www.researchgate.net/publication/404764110 DOI 10.13140/RG.2.2.23935.57769

APPENDIX

To bridge the theoretical framework developed in this study with hands-on classroom practice, this appendix presents a physical realization of the parity-driven rectangular unfolding model, designated as **Boulay's Rectangle©**. Designed as a tactile mathematical manipulative, the set translates the first ten consecutive integers (k = 1 to 10) into color-coded wooden pieces whose geometry directly reflects numerical parity—horizontal 1 × k strips for odd integers and vertical $\frac{k}{2}$ × 2 double-columns for even integers. By assembling these pieces inside a 5 × 11 wooden frame, learners can physically experience gnomonic growth, discover alternative tilings among the 2^{n-1} valid configurations, and visually verify the closed-form identity $T_n = \frac{n(n+1)}{2}$ without relying on external geometric duplication.



Boulay's Rectangle©

Figure A1. Tactile wooden manipulative illustrating the rectangular unfolding of the tenth triangular number T_{10} = 55 into a 5 × 11 grid. Each integer $k \in \{1, \ldots, 10\}$ is represented as a single contiguous piece according to its parity: odd integers form 1 × k horizontal rows, while even integers form $\frac{k}{2}$ × 2 vertical double-columns. Removing and rearranging the pieces outside the wooden tray enables learners to physically explore parity-driven gnomonic growth, verify the closed-form identity $T_n = \frac{n(n+1)}{2}$, without geometric duplication, and construct alternative tilings among the 2^{n-1} valid configurations.

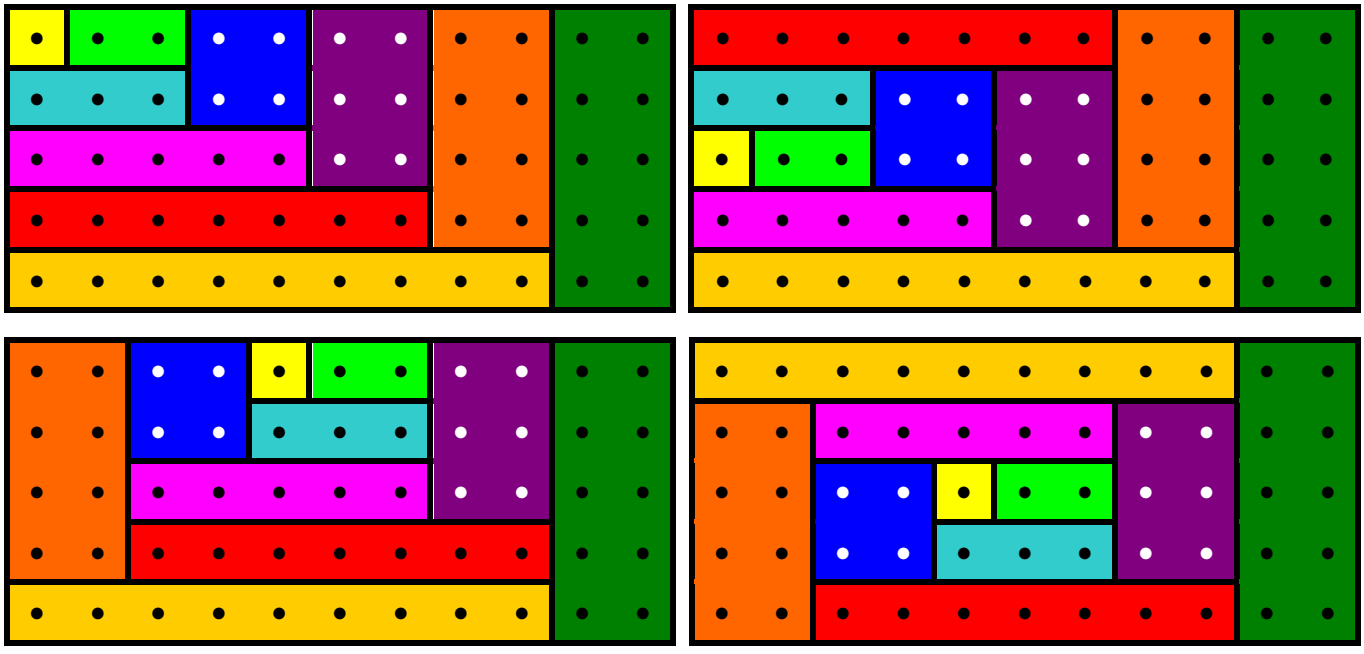


Figure A2. Some representative arrangements of a number rectangle illustrating the triangular number T_{10}. across the four regular expansion configurations (diagonal, horizontal, vertical, and spiral).