Symmetric Vector Lorentz Transformations for Mutually Orthogonal Velocities: Velocity-Sum and Rapidity-Sum Forms

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Abstract

We present a pedagogical and self-contained treatment of vector Lorentz transformations associated with mutually orthogonal velocities. For two orthogonal velocities, we recall the sequential composition of two pure boosts and its decomposition into a Wigner rotation and an effective boost. For three mutually orthogonal velocities, we emphasize that a "symmetric" transformation is not obtained by symmetrizing the order-dependent sequential product of three boosts. Instead, one may define a single pure boost whose orthogonal components are specified on an equal footing. Two natural and inequivalent definitions are considered: Form A, based on a velocity-vector sum, and Form B, based on a rapidity-vector sum. We give the transformation laws in vector and component form, discuss the existence condition for Form A, and compare the two forms. We also clarify the relation between these symmetric pure boosts and the sequential composition of boosts, which generally contains Wigner rotations.

1 Introduction

In special relativity, a pure Lorentz boost in an arbitrary direction is usually written in vector form as

$$\begin{aligned} ct' &= \gamma\left(ct - \boldsymbol{\beta}\cdot\mathbf{r}\right), \\ \mathbf{r}' &= \mathbf{r} + \frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{r}\right)\boldsymbol{\beta} - \gamma ct\,\boldsymbol{\beta}, \end{aligned} \tag{1}$$

where

$$\boldsymbol{\beta} = \frac{\mathbf{V}}{c}, \qquad \beta = |\boldsymbol{\beta}|, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{2}$$

Here we use the metric convention

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1), \tag{3}$$

and coordinates $X^\mu = (ct, \mathbf{r}) = (ct, x, y, z)$.

A common question is how to describe a Lorentz transformation involving several velocities. If the boosts are applied one after another, the result depends on the order and generally is not a pure boost; it contains a Wigner rotation. If, instead, one wants a single pure boost whose orthogonal components are treated symmetrically, one must define the transformation directly rather than symmetrize an order-dependent product.

In this note we consider mutually orthogonal velocities

$$\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3, \qquad \mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad (i \neq j). \tag{4}$$

We discuss two symmetric pure-boost definitions: a velocity-sum form (Form A) and a rapidity-sum form (Form B). We also clarify their relation to sequential boost composition.

2 Conventions and notation

For each velocity $\mathbf{v}_k$ we define

$$v_k = |\mathbf{v}_k|, \qquad \beta_k = \frac{v_k}{c}, \qquad \gamma_k = \frac{1}{\sqrt{1-\beta_k^2}}, \qquad k = 1, 2, 3. \tag{5}$$

It is important to distinguish the scalar ratios β_k from the Cartesian components of a summed velocity vector. In component expressions below, we write

$$\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z) \tag{6}$$

for the dimensionless components of a single boost vector.

The four-dimensional Lorentz transformation is written as

$$X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}, \qquad \Lambda^T \eta \Lambda = \eta. \tag{7}$$

For a pure boost, the spatial rotation part vanishes; for a sequential composition of non-parallel boosts, a Wigner rotation generally appears.

3 Two orthogonal velocities

Before treating three velocities, it is useful to recall the two-velocity case. Let

$$\mathbf{v}_1 = v_1 \hat{\mathbf{x}}, \qquad \mathbf{v}_2 = v_2 \hat{\mathbf{y}}, \qquad \mathbf{v}_1 \cdot \mathbf{v}_2 = 0. \tag{8}$$

The pure boosts $B(\mathbf{v}_1)$ and $B(\mathbf{v}_2)$ do not commute. Their sequential product

$$\Lambda = B(\mathbf{v}_2) B(\mathbf{v}_1) \tag{9}$$

can be written as

$$\Lambda = \begin{pmatrix} \gamma_1\gamma_2 & -\gamma_1\gamma_2\beta_1 & -\gamma_2\beta_2 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ -\gamma_1\gamma_2\beta_2 & \gamma_1\gamma_2\beta_1\beta_2 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Thus

$$\begin{aligned} ct' &= \gamma_1\gamma_2\, ct - \gamma_1\gamma_2\beta_1\, x - \gamma_2\beta_2\, y, \\ x' &= -\gamma_1\beta_1\, ct + \gamma_1 x, \\ y' &= -\gamma_1\gamma_2\beta_2\, ct + \gamma_1\gamma_2\beta_1\beta_2\, x + \gamma_2 y, \\ z' &= z. \end{aligned} \tag{11}$$

This is generally not a pure boost. It decomposes as

$$\Lambda = R(\Omega\hat{\mathbf{z}})\, B(\mathbf{w}), \tag{12}$$

where the effective boost velocity is

$$\mathbf{w} = \mathbf{v}_1 + \frac{1}{\gamma_1}\mathbf{v}_2 = \left(v_1, \frac{v_2}{\gamma_1}, 0\right), \qquad \gamma_w = \gamma_1\gamma_2, \tag{13}$$

and the Wigner rotation angle satisfies

$$\cos\Omega = \frac{\gamma_1 + \gamma_2}{\gamma_1\gamma_2 + 1}. \tag{14}$$

The rotation axis is

$$\hat{\mathbf{n}} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|}. \tag{15}$$

Reversing the order $B(\mathbf{v}_1)B(\mathbf{v}_2)$ changes the sign of the Wigner rotation and the effective velocity formula.

4 Three mutually orthogonal velocities: symmetric pure boosts

Now let

$$\mathbf{v}_1 = (v_{1x}, v_{1y}, v_{1z}), \quad \mathbf{v}_2 = (v_{2x}, v_{2y}, v_{2z}), \quad \mathbf{v}_3 = (v_{3x}, v_{3y}, v_{3z}), \tag{16}$$

with

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0. \tag{17}$$

We seek a single pure boost that incorporates the three orthogonal directions on an equal footing. There are two natural but generally inequivalent definitions.

4.1 Form A: velocity-vector sum

Define the dimensionless summed velocity vector by

$$\boxed{\beta_x = \frac{v_{1x} + v_{2x} + v_{3x}}{c}, \qquad \beta_y = \frac{v_{1y} + v_{2y} + v_{3y}}{c}, \qquad \beta_z = \frac{v_{1z} + v_{2z} + v_{3z}}{c},} \tag{18}$$

and

$$\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z), \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2. \tag{19}$$

Because the three velocities are mutually orthogonal,

$$\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2. \tag{20}$$

The Lorentz factor is therefore

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{\gamma_1^{-2} + \gamma_2^{-2} + \gamma_3^{-2} - 2}}. \tag{21}$$

Form A is defined only when

$$\boxed{\beta_1^2 + \beta_2^2 + \beta_3^2 < 1.} \tag{22}$$

If this condition fails, no real Lorentz boost with the velocity vector (18) exists.

The transformation is the standard pure boost

$$\boxed{\begin{aligned} ct' &= \gamma\left(ct - \beta_x x - \beta_y y - \beta_z z\right), \\ x' &= x + \frac{\gamma-1}{\beta^2}\beta_x\left(\beta_x x + \beta_y y + \beta_z z\right) - \gamma\beta_x\, ct, \\ y' &= y + \frac{\gamma-1}{\beta^2}\beta_y\left(\beta_x x + \beta_y y + \beta_z z\right) - \gamma\beta_y\, ct, \\ z' &= z + \frac{\gamma-1}{\beta^2}\beta_z\left(\beta_x x + \beta_y y + \beta_z z\right) - \gamma\beta_z\, ct. \end{aligned}} \tag{23}$$

Equivalently, in vector notation,

$$ct' = \gamma\left(ct - \boldsymbol{\beta}\cdot\mathbf{r}\right), \qquad \mathbf{r}' = \mathbf{r} + \frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{r}\right)\boldsymbol{\beta} - \gamma ct\,\boldsymbol{\beta}. \tag{24}$$

Form A is exactly symmetric under permutations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and is a pure boost with no Wigner rotation.

It should be emphasized that

$$\boldsymbol{\beta} = \frac{1}{c}\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\right) \tag{25}$$

is a definition of a single boost vector, not the relativistic velocity-addition law for three successive boosts.

4.2 Form B: rapidity-vector sum

For each velocity define the rapidity vector

$$\boldsymbol{\zeta}_k = \frac{\mathbf{v}_k}{|\mathbf{v}_k|}\operatorname{artanh}\beta_k, \qquad \beta_k = \frac{|\mathbf{v}_k|}{c}, \qquad k = 1, 2, 3. \tag{26}$$

Let

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 + \boldsymbol{\zeta}_3, \qquad \zeta = |\boldsymbol{\zeta}|, \qquad \hat{\mathbf{n}} = \frac{\boldsymbol{\zeta}}{\zeta}. \tag{27}$$

Because the three velocities are mutually orthogonal,

$$\zeta^2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2, \tag{28}$$

where $\zeta_k = |\boldsymbol{\zeta}_k| = \operatorname{artanh}\beta_k$.

Form B is the pure boost with rapidity vector $\boldsymbol{\zeta}$:

$$\boxed{\begin{aligned} ct' &= ct\cosh\zeta - (n_x x + n_y y + n_z z)\sinh\zeta, \\ x' &= x + (\cosh\zeta - 1)\,(n_x x + n_y y + n_z z)\,n_x - n_x\, ct\sinh\zeta, \\ y' &= y + (\cosh\zeta - 1)\,(n_x x + n_y y + n_z z)\,n_y - n_y\, ct\sinh\zeta, \\ z' &= z + (\cosh\zeta - 1)\,(n_x x + n_y y + n_z z)\,n_z - n_z\, ct\sinh\zeta. \end{aligned}} \tag{29}$$

In vector notation,

$$ct' = ct\cosh\zeta - (\hat{\mathbf{n}}\cdot\mathbf{r})\sinh\zeta, \tag{30}$$

$$\mathbf{r}' = \mathbf{r} + (\cosh\zeta - 1)\,(\hat{\mathbf{n}}\cdot\mathbf{r})\,\hat{\mathbf{n}} - \hat{\mathbf{n}}\, ct\sinh\zeta. \tag{31}$$

The effective speed is

$$\beta_{\text{eff}} = \tanh\zeta < 1, \tag{32}$$

and the effective Lorentz factor is

$$\boxed{\gamma_{\text{eff}} = \cosh\zeta = \cosh\sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}.} \tag{33}$$

Unlike Form A, Form B is defined for arbitrary rapidities; no condition analogous to (22) is required.

5 Comparison of the two symmetric forms

Form A and Form B are both pure boosts and both are exactly symmetric under permutations of the three orthogonal velocities. However, they are generally different transformations. Their main properties are summarized in Table 1.

Property	Form A	Form B
Composition rule	$\boldsymbol{\beta} = \frac{1}{c}\sum_{k=1}^{3}\mathbf{v}_k$ (definition)	$\boldsymbol{\zeta} = \sum_{k=1}^{3}\boldsymbol{\zeta}_k$
Effective speed	$\beta = \sqrt{\sum_{k=1}^{3}\beta_k^2}$	$\beta_{\text{eff}} = \tanh\zeta, \quad \zeta = \sqrt{\sum_{k=1}^{3}\zeta_k^2}$
Effective Lorentz factor	$\gamma = \dfrac{1}{\sqrt{1-\sum_{k=1}^{3}\beta_k^2}}$	$\gamma_{\text{eff}} = \cosh\zeta = \cosh\sqrt{\sum_{k=1}^{3}\zeta_k^2}$
Existence condition	$\sum_{k=1}^{3}\beta_k^2 < 1$ required	Always defined
Permutation symmetry	Exact	Exact
Pure boost?	Yes	Yes
One-dimensional limit	Agrees with Form B	Agrees with Form A
Relation to sequential product	Generally different	Generally different

Table 1: Comparison of the velocity-sum form (Form A) and the rapidity-sum form (Form B) for three mutually orthogonal velocities.

The two forms coincide only in the one-dimensional limit, where at most one of the velocities is nonzero. In general, Form A uses a velocity-vector sum, while Form B uses a rapidity-vector sum. These are different definitions of a symmetric pure boost.

6 Relation to sequential boost composition

The sequential product

$$\Lambda = B(\mathbf{v}_3)B(\mathbf{v}_2)B(\mathbf{v}_1) \tag{34}$$

is generally order-dependent and is not equal to either Form A or Form B. In general it can be decomposed as

$$\Lambda = R_{\mathrm{W}}\, B(\mathbf{w}), \tag{35}$$

where R_{W} is a product of Wigner rotations and $B(\mathbf{w})$ is an effective pure boost. For three mutually orthogonal velocities, the Wigner rotation structure is generally more complicated than in the two-velocity case.

Therefore, the symmetric forms should be interpreted as follows:

- Form A and Form B define a *single* pure boost whose orthogonal components are specified symmetrically.
- They are not obtained by symmetrizing the order-dependent sequential product $B(\mathbf{v}_3)B(\mathbf{v}_2)B(\mathbf{v}_1)$.
- If the physical situation is a sequence of three boosts in a specified order, one must use the sequential product and include the corresponding Wigner rotations.

7 Degenerate cases

For Form A, if

$$\boldsymbol{\beta} = 0, \tag{36}$$

the expression

$$\frac{\gamma - 1}{\beta^2} \tag{37}$$

is not defined directly. In this case the transformation is the identity:

$$ct' = ct, \qquad \mathbf{r}' = \mathbf{r}. \tag{38}$$

The limit $\beta \to 0$ reproduces the identity continuously.

For Form B, if

$$\zeta = 0, \tag{39}$$

the unit vector

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\zeta}}{\zeta} \tag{40}$$

is not defined. Again the transformation is the identity:

$$ct' = ct, \qquad \mathbf{r}' = \mathbf{r}. \tag{41}$$

For $\zeta > 0$, Eq. (29) is well defined.

8 Conclusion

We have clarified how to construct symmetric vector Lorentz transformations for mutually orthogonal velocities. The sequential composition of boosts is order-dependent and generally contains Wigner rotations; it should not be confused with a symmetric pure boost. Two natural symmetric pure-boost definitions were presented. Form A is based on the velocity-vector sum $\boldsymbol{\beta} = \sum_k \mathbf{v}_k/c$

and requires $\sum_k \beta_k^2 < 1$. Form B is based on the rapidity-vector sum $\boldsymbol{\zeta} = \sum_k \boldsymbol{\zeta}_k$ and is always defined, with effective Lorentz factor

$$\gamma_{\text{eff}} = \cosh \sqrt{\sum_{k=1}^{3} \zeta_k^2}. \tag{42}$$

Both forms are pure boosts and are exactly symmetric under permutations of the three orthogonal velocities, but they are generally inequivalent. The results are intended as a pedagogical clarification of the distinction between symmetric pure-boost definitions and order-dependent sequential boost composition.

References

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