On Studying Root-Power Series and Their Analytic Properties

Mohammed Hajji
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This paper presents a mathematical study of root-power series. We define the series as

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}, \qquad x > 0,$$

coefficients. We investigate the convergence conditions and analytic properties of these series and compare their behavior with classical power series.

I. INTRODUCTION

Power series are one of the most important tools in mathematical analysis, providing a way to represent functions through infinite sums of powers. Classical examples include geometric series and Taylor series, where the exponents are typically nonnegative integers. Their convergence and analytic properties have been widely studied.

In this paper, we consider a related family of series in which the usual integer exponents are replaced by reciprocal exponents of the form $1/n$. Specifically, we study

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}, \qquad x > 0,$$

where a_n is a sequence of real or complex coefficients. Unlike ordinary power series, the exponents $1/n$ approach zero as n increases. Consequently, the terms $x^{1/n}$ approach one rather than zero.

The main objective of this paper is to study the convergence conditions of root-power series and investigate their analytic properties. We also explore their logarithmic representation and compare their behavior with classical power series.

II. MATHEMATICAL DEFINITIONS AND PRIMILINAY PROPERTIES

In this section, we provide the mathematical definition of a root-power series and establish one preliminary property.

A. Definition One

Let $(a_n)_{n\geq 1}$ be a sequence of real or complex numbers. We define a root-power series as

$$\boxed{R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}, \qquad x > 0.}$$

where a_n represents the coefficient of each term, n is a positive integer, and $x^{1/n}$ represents the n-th root of x.

Unlike classical power series, which use nonnegative integer exponents, root-power series use reciprocal exponents of the form $1/n$.

B. Proposition One (Limit of Root-Power Terms)

For every fixed $x > 0$,

$$\lim_{n\to\infty} x^{1/n} = 1.$$

Proof.
Using the exponential representation,

$$x^{1/n} = e^{\frac{\ln x}{n}}.$$

Since

$$\lim_{n\to\infty} \frac{\ln x}{n} = 0,$$

we obtain

$$\lim_{n\to\infty} x^{1/n} = e^0 = 1.$$

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This property shows that the root-power factors approach one rather than zero. Therefore, the convergence of the series depends strongly on the behavior of its coefficients.

III. MATHEMATICAL THEOREM

We will begin by formulating the theorems for this study, which consist of three fundamental theorems.

A. Theorem One (Convergence Equivalence)

Let $(a_n)_{n\geq 1}$ be a sequence of real or complex numbers. For every fixed $x > 0$, the root-power series

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}$$

converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Asumme that

$$\sum_{n=1}^{\infty} a_n = S.$$

and we define he partial sums

$$S_N = \sum_{n=1}^{N} a_n.$$

Since the series is convergent , then

$$S_N \to S.$$

therefore , there is value $M > 0$ that ,

$$\boxed{|S_N| \leq M.}$$

Now we define the root Factor

$$w_n = x^{1/n}.$$

and we know fron the Proposition One that

$$\lim_{n\to\infty} w_n = 1.$$

w_n is limited and ordinal for every $x > 0$ term
Applied Summation by Parts

$$\boxed{\sum_{n=1}^{N} a_n w_n = S_N w_N + \sum_{n=1}^{N-1} S_n(w_n - w_{n+1}).}$$

First term

$$\lim_{N\to\infty} S_N w_N = S.$$

then , the first term is convergent and the second term , Since $|S_n| \leq M$ then ,

$$\sum_{n=1}^{\infty} |S_n(w_n - w_{n+1})| \leq M \sum_{n=1}^{\infty} |w_n - w_{n+1}|$$
$$< \infty.$$

the series is convergent because w_n is ordinal sequence and has a limit , therefore the second term is convergent. Because the terms in sumation are convergent then ,

$$\boxed{\sum_{n=1}^{\infty} a_n x^{1/n} \text{ converges for every } x > 0.}$$

Proof of the Reverse Direction ($\Rightarrow$)

We aim to prove that

$$\boxed{\sum_{n=1}^{\infty} a_n x^{1/n} \text{ converges } \implies \sum_{n=1}^{\infty} a_n \text{ converges.}}$$

Suppose that, for a fixed $x > 0$, the root-power series

$$\sum_{n=1}^{\infty} a_n x^{1/n}$$

converges.

Define

$$b_n = a_n x^{1/n}.$$

Then the series

$$\sum_{n=1}^{\infty} b_n$$

converges, and its partial sums

$$B_N = \sum_{n=1}^{N} b_n$$

are bounded. Thus, there exists $M > 0$ such that

$$|B_N| \leq M.$$

Let

$$w_n = x^{-1/n}.$$

Since

$$\lim_{n\to\infty} w_n = 1,$$

the sequence (w_n) is bounded and monotone for every fixed $x > 0$.

Moreover,

$$b_n w_n = a_n.$$

By Abel's summation formula,

$$\boxed{\sum_{n=1}^{N} b_n w_n = B_N w_N + \sum_{n=1}^{N-1} B_n(w_n - w_{n+1}).}$$

We examine the two terms separately.

First term.

Since B_N converges and $w_N \to 1$, we obtain

$$B_N w_N \longrightarrow B,$$

where

$$B = \sum_{n=1}^{\infty} b_n.$$

Second term.
Since $|B_n| \le M$,

$$\sum_{n=1}^{\infty} |B_n(w_n - w_{n+1})| \le M \sum_{n=1}^{\infty} |w_n - w_{n+1}|.$$

Because (w_n) is monotone and convergent,

$$\sum_{n=1}^{\infty} |w_n - w_{n+1}| < \infty.$$

Hence, the second term converges absolutely.
Since both terms converge, the series

$$\sum_{n=1}^{\infty} b_n w_n$$

converges.
But $b_n w_n = a_n$. Therefore,

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ converges.}}$$

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B. Theorem Two (Absolute and Uniform Convergence)

Let $(a_n)_{n\ge1}$ be a sequence of real or complex numbers satisfying

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then the root-power series

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}$$

converges absolutely and uniformly on every interval $[0, C]$, where $C > 0$.

Proof.
For every $x \in [0, C]$ and $n \ge 1$,

$$0 \le x^{1/n} \le \max\{1, C\}.$$

Hence,

$$|a_n x^{1/n}| \le |a_n| \max\{1, C\}.$$

Since

$$\sum_{n=1}^{\infty} |a_n| \max\{1, C\} = \max\{1, C\} \sum_{n=1}^{\infty} |a_n| < \infty,$$

the Weierstrass test implies that $R(x)$ converges absolutely and uniformly on $[0, C]$.

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C. Theorem Three (Logarithmic Analytic Representation)

Let $(a_n)_{n\ge1}$ be a sequence of real or complex numbers such that

$$\sum_{n=1}^{\infty} a_n$$

converges.

Then, for every $x > 0$, the root-power series

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}$$

admits the representation

$$\boxed{R(x) = \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^k} \right).}$$

Proof.
Since $\sum a_n$ converges, its partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

are bounded. Thus, there exists $M > 0$ such that

$$|S_N| \le M.$$

For every integer $k \ge 1$, Abel's summation formula gives

$$\sum_{n=1}^{N} \frac{a_n}{n^k} = \frac{S_N}{N^k} + \sum_{n=1}^{N-1} S_n \left(\frac{1}{n^k} - \frac{1}{(n+1)^k} \right).$$

Since n^{-k} is positive and decreasing,

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^k} \right| \le M \left[\frac{1}{N^k} + \sum_{n=1}^{N-1} \left(\frac{1}{n^k} - \frac{1}{(n+1)^k} \right) \right] = M.$$

The same bound holds for $k = 0$. Therefore,

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^k} \right| \le M, \qquad k \ge 0.$$

Using the exponential expansion, for each finite N we have

$$\begin{aligned} \sum_{n=1}^{N} a_n x^{1/n} &= \sum_{n=1}^{N} a_n e^{(\ln x)/n} \\ &= \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!} \sum_{n=1}^{N} \frac{a_n}{n^k}. \end{aligned}$$

Furthermore,

$$\left| \frac{(\ln x)^k}{k!} \sum_{n=1}^{N} \frac{a_n}{n^k} \right| \le M \frac{|\ln x|^k}{k!}.$$

Since

$$\sum_{k=0}^{\infty} M\frac{|\ln x|^k}{k!} = Me^{|\ln x|} < \infty,$$

we may pass the limit $N \to \infty$ through the sum over k.

Consequently,

$$\boxed{R(x) = \sum_{k=0}^{\infty} \frac{(\ln x)^k}{k!}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^k}\right).}$$

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IV. ANALYTIC PROPERTIES OF ROOT-POWER SERIES

A. Theorem Four (Differentiability and Analyticity)

Let $(a_n)_{n\geq 1}$ be a sequence of real or complex numbers such that

$$\sum_{n=1}^{\infty} a_n$$

converges.

Define

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}, \qquad x > 0.$$

Then $R(x)$ is real-analytic on $(0,\infty)$. Moreover, for every integer $m \geq 0$,

$$\boxed{\left(x\frac{d}{dx}\right)^m R(x)\Big|_{x=1} = \sum_{n=1}^{\infty}\frac{a_n}{n^m}.}$$

Proof.

By Theorem Three, we have

$$R(x) = \sum_{k=0}^{\infty}\frac{b_k}{k!}(\ln x)^k,$$

where

$$b_k = \sum_{n=1}^{\infty}\frac{a_n}{n^k}.$$

From Abel's summation formula, the coefficients b_k are uniformly bounded. Thus, there exists $M > 0$ such that

$$|b_k| \leq M \qquad (k \geq 0).$$

Let $t = \ln x$ and define

$$G(t) = R(e^t).$$

Then

$$G(t) = \sum_{k=0}^{\infty}\frac{b_k}{k!}t^k.$$

Since

$$\sum_{k=0}^{\infty}\left|\frac{b_k}{k!}t^k\right| \leq Me^{|t|} < \infty,$$

the power series has infinite radius of convergence and defines an entire function of t.

Therefore, termwise differentiation gives

$$G^{(m)}(0) = b_m = \sum_{n=1}^{\infty}\frac{a_n}{n^m}.$$

Since

$$\frac{d}{dt} = x\frac{d}{dx},$$

we obtain

$$\boxed{\left(x\frac{d}{dx}\right)^m R(x)\Big|_{x=1} = \sum_{n=1}^{\infty}\frac{a_n}{n^m}.}$$

Hence, $R(x)$ is real-analytic for every $x > 0$.

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V. EXAMPLES AND COMPARISON WITH CLASSICAL POWER SERIES

A. Example One: Alternating Root-Power Series

Consider

$$R(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^{1/n}, \qquad x > 0.$$

The coefficient series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges by the alternating series test.

Therefore, by Theorem One, $R(x)$ converges for every $x > 0$.

However,

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}x^{1/n}\right|$$

diverges, since the absolute-value coefficient series is harmonic.

Thus, $R(x)$ converges conditionally.

By Theorem Three,

$$R(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (\ln x)^k,$$

where

$$b_k = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{k+1}}.$$

B. Example Two: Absolute Convergence

Consider

$$F(x) = \sum_{n=1}^{\infty} \frac{x^{1/n}}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

Theorem Two guarantees absolute and uniform convergence on every bounded interval $[0, M]$.

Furthermore, Theorem Four implies that F is real-analytic on $(0, \infty)$.

In particular,

$$F(1) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and

$$F'(1) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

C. Comparison with Classical Power Series

The geometric series

$$G(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

converges for $|x| < 1$.

In contrast, Theorem One shows that the convergence of a root-power series

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}$$

is independent of the choice of $x > 0$.

This difference arises because the exponents $1/n$ approach zero, whereas the exponents of classical power series increase without bound.

VI. DISCUSSION

The results of this paper show that root-power series have convergence and analytic properties that differ from those of classical power series.

Theorem One establishes that the convergence of a root-power series depends entirely on the convergence of its coefficient series. Unlike classical power series, whose convergence may depend on the value of the variable, root-power series converge for every $x > 0$ whenever $\sum a_n$ converges. However, the value of the resulting function may still change with x.

Theorem Two shows that absolute convergence of the coefficient series guarantees absolute and uniform convergence of the root-power series on every bounded interval $[0, M]$.

Theorem Three provides a logarithmic analytic representation:

$$R(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (\ln x)^k,$$

where

$$b_k = \sum_{n=1}^{\infty} \frac{a_n}{n^k}.$$

This representation shows that root-power series can be studied through ordinary power series in the variable $\ln x$.

Furthermore, Theorem Four establishes that $R(x)$ is real-analytic on $(0, \infty)$ and relates its logarithmic derivatives to weighted sums of the original coefficients:

$$\left(x \frac{d}{dx} \right)^k R(x) \Big|_{x=1} = \sum_{n=1}^{\infty} \frac{a_n}{n^k}.$$

An interesting special case appears when $a_n = 1/n^p$, with $p > 1$. In this case, the derivatives generate values of the Riemann zeta function:

$$\boxed{\left(x \frac{d}{dx} \right)^k R_p(x) \Big|_{x=1} = \zeta(p+k).}$$

This observation provides a connection between root-power series and the Riemann zeta function. However, the present study focuses only on positive real values of x and does not investigate complex extensions or zeta zeros.

VII. CONCLUSION

In this paper, we studied the convergence and analytic properties of root-power series of the form

$$R(x) = \sum_{n=1}^{\infty} a_n x^{1/n}, \qquad x > 0.$$

We established that convergence depends on the coefficient series rather than on the value of x. We also proved absolute and uniform convergence under suitable conditions.

Furthermore, we derived a logarithmic analytic representation and showed that root-power series are real-analytic on $(0, \infty)$ whenever their coefficient series converges.

These results provide a mathematical description of root-power series and highlight their differences from classical power series.

Future studies may investigate more general exponents, complex extensions, and additional relationships between derivatives and coefficient sequences.

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[1] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, 1976.

[2] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Addison-Wesley, 1974.

[3] G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press, 1915.

[4] NIST Digital Library of Mathematical Functions, *Chapter 1: Algebraic and Analytic Methods*, National Institute of Standards and Technology. Available at: `https://dlmf.nist.gov/1`