Small identity with bright life

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Abstract

Given Hamilton quaternion definition in modern Geometric (Clifford) Algebra language we point out a simple identity that leads to a theorem: any quaternion function f(q) can be expressed as a linear function f(q) = Aq +B with real functions A,B of invariants Trace and square norm N of q (or eigenvalues). We develop an explicit closed-form formulas to calculate these functions using three different methods. That might be useful to get the explicit closed form of infinite polynomials of q, Fibonacci-Lucas series, BCH (Baker–Campbell–Hausdorff formula) as examples.

Notation and the fundamental identity

Throughout, quaternion $q = a + Isv \in \mathbb{H}$ with $a \in \mathbb{R}$, $v \in \mathbb{R}^3$, I a fixed unit imaginary direction ($I^2 = -1$, $v^2 = 1$), and $s > 0$ (the nonscalar case).

Taking the square of q : $q^2 = a^2 + 2aIsv - s^2 = 2a(a + Isv) - a^2 - s^2 = Tq - N$

We write $T = 2a$ (trace) and $N = a^2 + s^2$ (square norm). The fundamental **identity** is

$$\boldsymbol{q^2 = Tq - N} \quad (1)$$

That identity also happens to be a minimal polynomial in Cayley sense.

Since q is nonscalar, identity (1) makes $\mathbb{R}[q]$ a two-dimensional commutative algebra: $\mathbb{R}[q] = \mathrm{span}\{1, q\}$. The characteristic roots of (1) that also are complex eigenvalues are:

$$\lambda_{\pm} = a \pm is = \frac{T}{2} \pm i\sqrt{N - \frac{T^2}{4}}$$

$$T = \lambda_+ + \lambda_-, \qquad N = \lambda_+\lambda_-,$$

From fundamental identity (1) it follows that any power of q would also be linear .

2. The Infinite Ladder of Powers

Multiplying by q and reducing $q^2 \to Tq - N$ never leaves the linear span of $\{1, q\}$; hence for every integer $n \in \mathbb{Z}$ (both sides of zero)

$$q^n = A_n(T, N)\, q + B_n(T, N), \qquad A_{n+1} = TA_n - NA_{n-1}, \quad A_0 = 0,\ A_1 = 1, \qquad B_n = -NA_{n-1}.$$

for T = 1, N = -1 the eigenvalues are the golden-ratio pair $\frac{(1 \pm \sqrt{5})}{2}$, and the ladder coefficients are exactly $A_n = F_n, B_n = F_{n-1}$ (Fibonacci), with Lucas numbers $L_n = B_n + A_{n+1}$; Binet formula follows from the eigenvalues.

Let's bring the example for first 8 powers of q

Positive side, $n = 1..8$

n	A_n	B_n
1	1	0
2	T	-N
3	$T^2 - N$	-TN
4	$T^3 - 2TN$	$N^2 - NT^2$
5	$T^4 - 3T^2N + N^2$	$2TN^2 - NT^3$
6	$T^5 - 4T^3N + 3TN^2$	$3T^2N^2 - NT^4 - N^3$
7	$T^6 - 5T^4N + 6T^2N^2 - N^3$	$4T^3N^2 - NT^5 - 3TN^3$
8	$T^7 - 6T^5N + 10T^3N^2 - 4TN^3$	$-NT^6 + 5T^4N^2 - 6T^2N^3 + N^4$

All hold for Negative powers of q:

Consider conjugate of $\bar{q} = a - Iv$, then $q\bar{q} = N$, $q + \bar{q} = T$, so :

$$q^{-1} = \frac{T - q}{N}$$

gives the anchor coefficients:

$$\boxed{A_{-1} = -\frac{1}{N}, \qquad B_{-1} = \frac{T}{N}.}$$

so $A_{-1} = -1/N,\ B_{-1} = T/N$, and the backward recurrence $A_{n-1} = (TA_n - A_{n+1})/N$ runs to $-\infty$:

Backward recurrence

Multiplying the Cayley–Hamilton identity $q^2 - Tq + N = 0$ by q^{n-1} yields

$$q^{n-1} = \frac{T\,q^n - q^{n+1}}{N}.$$

Expressing each term as $A_k q + B_k$ and matching coefficients gives the backward recurrence:

$$\boxed{A_{n-1} = \frac{TA_n - A_{n+1}}{N}, \qquad B_{n-1} = \frac{TB_n - B_{n+1}}{N}.}$$

With $n = -1$ this reproduces the anchor. Iterating, the ladder descends to $n \to -\infty$.

First three runs of the negative ladder:

n	A_n	B_n
-1	$-\frac{1}{N}$	$\frac{T}{N}$
-2	$-\frac{T}{N^2}$	$\frac{T^2 - N}{N^2}$
-3	$\frac{N - T^2}{N^3}$	$\frac{T\,(T^2 - 2N)}{N^3}$

Mirror relation with positive powers

For $n \geq 0$, there is a reflection identity:

$$q^{-n} = N^{-n}\,\bar{q}^{\,n}.$$

Since $\bar{\mathrm{q}}^n = A_n\bar{\mathrm{q}} + B_n = A_n(T - q) + B_n$, we obtain

$$\begin{aligned} A_{-n} &= -\frac{A_n}{N^n}, \\ B_{-n} &= \frac{B_n + TA_n}{N^n}. \end{aligned}$$

Thus, the negative ladder is the positive one, reflected and rescaled by N^{-n}.

Our task now is to develop the explicit closed form for scalar functions A_n *and* B_n

Polar form and explicit expressions

In polar form of quaternion $q = Re^{I\varphi}$ with $R = \sqrt{N}$, $\cos\varphi = a/R$, $\sin\varphi = s/R$, we have

$$q^n = R^n\big(\cos(n\varphi) + Iv\sin(n\varphi)\big).$$

Matching to $q^n = A_n(a + Isv) + B_n$ yields:

$$\begin{aligned} A_n &= \frac{R^n\sin(n\varphi)}{s}, \\ B_n &= R^n\cos(n\varphi) - aA_n. \end{aligned}$$

For negative powers, substitute $-n$:

$$\begin{aligned} A_{-n} &= -\frac{R^{-n}\sin(n\varphi)}{s}, \\ B_{-n} &= R^{-n}\cos(n\varphi) - aA_{-n}. \end{aligned}$$

These closed forms are numerically stable when s is not too small.

Limit as $s \to 0$ and numerical stability

When $s \ll 1$, direct division by s can amplify rounding errors. The limit is given by derivatives of $f(q) = q^n$ at the real point a:

$$\lim_{s\to 0} A_n = f'(a) = na^{n-1}, \qquad \lim_{s\to 0} B_n = f(a) - af'(a) = (1-n)a^n.$$

For negative $n = -k$ with $k > 0$:

$$\lim_{s\to 0} A_{-k} = -ka^{-k-1}, \qquad \lim_{s\to 0} B_{-k} = (1+k)a^{-k}.$$

A robust implementation uses these limits when $s < \varepsilon$ and the polar formulas otherwise. This is especially relevant for applications involving holonomies and kinematic limits.

Any quaternion function as infinite sum of powers

When function of quaternion expressed as infinite sum, $f(q) = \sum_k c_k\, q^k$ = $A(T,N)q + B(T,N)$ our task is to find the explicit closed form of real functions A and B :

$$A(T,N) = \sum_k c_k\, A_k(T,N), \qquad B(T,N) = c_0 - N\sum_k c_{k+1}\, A_k(T,N)$$

Several Methods calculate explicit closed-form formulas for invariant functions A and B

To complete representing any quaternion function via linear one ,we now need to compute explicit close coefficient function A an B of invariants (either invariants T and N or eigenvalues, or roots of minimal polynomial).

Method one : Reduction to a complex value

Powers of q never leave the subalgebra $\mathrm{span}\{1, I\}$: equation $q = a + Isv$ where $v^2 = 1$ $an\, I^2 = -1$, so $quaternion\, q$ behaves exactly like the complex number $z = a + is$, with the bivector I playing the role of the imaginary unit. Hence every function of q obtained from the functional calculus satisfies

$$f(q) \; -\!-\!> \; f(z) \; = \; \mathrm{Re} f(a+is) \; + \mathrm{i}\; \mathrm{Im} f(a+is)\,,$$

where Re and Im are the real and imaginary parts of the *ordinary complex* number $f(a+is)$.

Matching coefficients

Seek $f(q)$ in the form $f(q) = B_f + A_f\, Isv$. Using isomorphism $q = a + Isv$ $\rightarrow z = a + is$ and $f(q)$ $\rightarrow f(z) = A_f z + B_f$:

$$f(z) = B_f + A_f z \; = \; \left(\mathrm{Re} f(a+is) - aA_f\right) \; + \; iA_f s.$$

That yields the boxed pair

$$\boxed{A_f = \frac{\mathrm{Im} f(a+is)}{s}, \qquad B_f = \mathrm{Re} f(a+is) - aA_f}$$

Method Two : Lagrange interpolation and divided differences using eigenvalues

The unique polynomial of degree at most 1 interpolating f at $\lambda_\pm$ is $p(x) = f(\lambda_+)\,(x-\lambda_-)/(\lambda_+-\lambda_-) + f(\lambda_-)\,(x-\lambda_+)/(\lambda_- - \lambda_+)$. Evaluating at $x = q$ is legitimate because q satisfies its own minimal polynomial, and gives $f(q) = p(q) = Aq + B$ with

$$A = f[\lambda_+, \lambda_-] = \frac{f(\lambda_+) - f(\lambda_-)}{\lambda_+ - \lambda_-}, \qquad B = f(\lambda_+) - \lambda_+ A$$

The coefficient A is the first divided difference of f at the pair $\{\lambda_+, \lambda_-\}$, and $B = f(\lambda_+) - \lambda_+ A$. Trefethen [7] is the standard reference for the stable barycentric form of this interpolation; Higham [6] uses exactly this construction to compute $f(A)$ for matrices.

Method Three : quaternion spectral decomposition

Let pair projectors be $P_+ = \frac{1+v}{2}$ and $P_- = \frac{1-v}{2}$ where v – unit 3 d vector.

$P_+ + P_- = 1;\ \ P_+ - P_- = v;\ P_+^2 = P_+;\ P_-^2 = P_-;\ P_+P_- = 0$

Then quaternion spectral decomposition : $q = a + Isv = a(P_+ + P_-) + \mathrm{Is}(P_+ - P_-) = (a+is)\,P_+ + (a-is)P_- = (\lambda_+)\,P_+ + (\lambda_-)P_-$

Then $q^2 = (\lambda_+)^2\ P_+ + (\lambda_-)^2 P_- = (\lambda_+)^2\ \frac{1+v}{2} + (\lambda_-)^2 \frac{1-v}{2} = \frac{1}{2}([(\lambda_+)^2 + (\lambda_-)^2] + v[(\lambda_+)^2 - (\lambda_-)^2] = \frac{1}{2}([(\lambda_+)^2 + (\lambda_-)^2] + \frac{2q-2a}{z-\bar{z}}[(\lambda_+)^2 - (\lambda_-)^2]$

$q = a + Isv\ \ ;\ \ \bar{q} = a - Isv \ \ \rightarrow\ \ q + \bar{q} = 2a\ ;\ \ q - \bar{q} = 2Isv \ \ \frac{q-\bar{q}}{2Is} = v = \frac{2q-2a}{z-\bar{z}}$

$z = a + is \ \ \bar{z} = a - is;\ \ 2is = z - \bar{z}$

Let $f1 = (\lambda_+)^2 + (\lambda_-)^2\ ;\ f2 = (\lambda_+)^2 - (\lambda_-)^2$

Then $q^2 = (\lambda_+)^2\ P_+ + (\lambda_-)^2 P_- = [\frac{f1}{2} - \frac{af2}{\lambda_+ - \lambda_-}] + \frac{f2}{\lambda_+ - \lambda_-} q = B + Aq$

Finally : any quaternion function f(q) is expressed as linear function of q with real coefficients as functions of two conjugate eigenvalues :

$$\boxed{f(q) = A_f q + B_f}$$

where $A_f = \frac{f2}{\lambda_+ - \lambda_-}$; $B_f = \frac{f1}{2} - aA_f$; $f1 = f(\lambda_+) + f(\lambda_-)$; $f2 = f(\lambda_+) - f(\lambda_-)$; $\lambda_+ = a + is$; $\lambda_- = a - is$; $q = a + Isv$; $I^2 = -1, v - unit\ 3 - vector$

Remark: conjugate quaternion pair vs. complex eigenvalue pair (in the spirit of the noncommutative Vieta theorem [17])

(i)Eigenvalues are complex numbers, not quaternions. Regarded as a scalar equation, the minimal polynomial $z^2 - Tz + N = 0$ has exactly two complex roots $\lambda_1 = a + is$ and $\lambda_2 = a - is$ with

$s = \sqrt{(N - a^2)}$, obeying the Vieta formulas $\lambda_1 + \lambda_2 = T$ and $\lambda_1\lambda_2 = N$. All coefficients A_f *and* B_f of the linear representation $f(q) = A_f\, q + B_f$ are functions of this pair alone — equivalently, of the invariants (T, N).

(ii) Read in the algebra, the same equation has a sphere of roots. Interpreted in the quaternion algebra itself, the equation $q^2 - Tq + N = 0$ is satisfied not by two quaternions but by a two-sphere of them: every $q = a + Isv$ where v – unit 3 vector, with the given (a, s) and an arbitrary unit direction v is a solution. (When $T^2 > 4N$ the eigenvalues become real and the solution set collapses to the two scalar points $a \pm s$.) The eigenvalue pair therefore does not determine the quaternion: the direction v is additional data, invisible to (T, N). This distinction — roots of an equation lying in a noncommutative algebra versus scalar eigenvalues of the characteristic polynomial — is exactly the point emphasized in Shirokov's noncommutative Vieta theorem for geometric algebras [17].

(iii) Quaternion conjugation does not change the eigenvalues. The conjugate $\bar{q} = a - Isv$ lies on the same solution sphere and possesses the same eigenvalue pair $\{a + is, a - is\}$. What conjugation actually does is interchange the spectral projectors: $\bar{q} = (\lambda_1)E_2 + (\lambda_2)E_1$, where $E\pm = (1 \pm v)/2$. Hence for every function f with real coefficients, $f(\bar{q})$ equals the quaternion conjugate of f(q): the coefficients A_f, B_f are unchanged and only the vector part flips sign. The eigenvalue pair alone cannot distinguish q from $\bar{q}$; this is why the calculus of the present article is organized around the invariants (T, N) together with the direction, rather than around eigenvalues alone.

(iv) Vieta in quaternion form. Within a fixed slice $\{1, Iv\}$ the isomorphism $Iv \leftrightarrow i$ embeds the complex pair into the algebra: $\lambda_1 \mapsto q$ and $\lambda_2 \mapsto \bar{q}$. The Vieta formulas then acquire quaternion form: $q + \bar{q} = T$ and $q\bar{q} = \bar{q}q = N$, both right-hand sides being real scalars. Every choice of the direction v produces another conjugate pair with the same Vieta data — precisely the solution sphere of item (ii). A classical illustration: the equation $x^2 + 1 = 0$ has the eigenvalues $\pm i$, but in the quaternion algebra every unit imaginary quaternion u ($|u| = 1$, $u^2 = -1$) is a root — on a sphere, not a pair. A related caution, going back to the Gelfand–Retakh noncommutative Vieta theorem and discussed by Shirokov [17]: the factorized identity $(x - q_1)(x - q_2) = x^2 - (q_1 + q_2)x + q_1q_2$ requires $q_1q_2 = q_2q_1$. Within one slice it holds (the slice is a commutative subalgebra); across different directions it fails. All factorizations in this article use roots of a single slice, so the subtlety never bites here — but it matters for any generalization.

Some examples for elementary functions :

Elementary functions f(q) = A·q + B in terms of quaternion scalars

Throughout the table: $a = T/2$, $s = \sqrt{(N - a^2)} > 0$, $\rho = \sqrt{N}$, $\varphi = \operatorname{atan2}(s, a) \in (0, \pi)$. All removable singularities at $s = 0$ are understood by limits; q^β and $\ln q$ use the principal branch. Note that the scalar part of $f(q)$ equals $aA + B$ — this reconciles tables that list the scalar part with tables that list B.

f(q)	A(T, N)	B(T, N)
Exp(q)	*Exp(a)·sin s/s*	*Exp(a)·cos s − aA*
sin q	*cos a·sinh s/s*	*sin a·cosh s − aA*
cos q	*−sin a·sinh s/s*	*cos a·cosh s + a·sin a·sinh s/s*
sinh q	*cosh a·sin s/s*	*sinh a·cos s − aA*
cosh q	*sinh a·sin s/s*	*cosh a·cos s − aA*
tan q	*sinh 2s/[s·(cos 2a + cosh 2s)]*	*sin 2a/(cos 2a + cosh 2s) − aA*
q^β	*ρ^β·sin(βφ)/s*	*ρ^β·cos(βφ) − aA*
√q (β = 1/2)	*1/D, D = √(T + 2√N)*	*D/2 − aA*
ln q	*φ/s*	*(1/2)·ln N − aA*
1/q	*−1/N*	*T/N*

Note that for exp(q) we are getting exactly the Euler like formula for quaternion (polar form)

Reminder : because of quaternion multiplication is non commutative, for quaternion multiplication of exponents we cannot use usual formula for scalars.

BCH infinite formula gives the answer for quaternion multiples. GA gives the answer in a closed form underlying the power of GA as a universal language for science.[9,10]

The Baker–Campbell–Hausdorff Series for Quaternions (BCH): Closed Form via the Linear Representation f(q) = A_f q + B_f

1. Statement of the problem

Let two GA quaternions be given in the slice form

$$q_1 = a_1 + I\, w_1, \quad q_2 = a_2 + I\, w_2, \quad I^2 = -1, \quad [I, \text{everything}] = 0, \quad w_1, w_2 \in \mathbb{R}^3,$$

with $s_i = |w_i|$ and unit vectors $v_i = w_i/s_i$ (for $s_i > 0$). The Baker–Campbell–Hausdorff series expresses the product of two exponentials as a single exponential,

$$\exp(q_1)\exp(q_2) = \exp(Z), \quad Z = q_1 + q_2 + \tfrac{1}{2}[q_1,q_2] + (1/12)([q_1,[q_1,q_2]] + [q_2,[q_2,q_1]]) - (1/24)[q_2,[q_1,[q_1,q_2]]] + \dots,$$

an infinite series of nested commutators. The goal is an explicit closed form $q_3 = a_3 + I\, s_3\, v_3$ such that $\exp(q_1)\exp(q_2) = \exp(q_3)$ exactly — the sum of the entire infinite series.

2. Why the infinite series collapses: the strategy

The BCH series is nothing but $Z = \log(\exp q_1 \cdot \exp q_2)$. For quaternions the whole machinery of the linear representation works three times, and every step is exact:

- the exponential of each factor is affine in its argument — the ladder q ↦ A_f q + B_f gives exp in closed form;
- the product of the two results is computed exactly inside the algebra — no series truncation occurs anywhere;
- the logarithm of the result is again affine in its argument — the ladder applied to f = log.

Every commutator of every order is thereby encoded in a single angle s_3. Nothing is truncated, neglected or expanded: the infinite series is summed exactly.

3. Step 1 — the exponential by the ladder

The eigenvalues of q = a + I s v̂ are λ± = a ± i s. With f = exp,

$$A_f \;=\; (e^{\lambda_+} - e^{\lambda_-})/(\lambda_+ - \lambda_-) = e^{a} \cdot \sin s \,/\, s\,, \quad B_f \;=\; e^{a}\cos s \;-\; a\,A_f\,,$$

so the linear representation reproduces the polar formula exactly:

$$e^{q} \;=\; A_f\; q \;+\; B_f = e^{a}\,(\cos s + I\,\hat{v}\sin s\,).$$

Define the rotation part $R := \cos s \;+\; I\,\hat{v}\,\sin s$; then $e^{q_i} \;=\; e^{a_i} R_i$ and

$$exp(q_1)\,exp(q_2) \;=\; e^{a_1+a_2}\,R_1\,R_2\,.$$

4. Step 2 — the product of two rotations (unit quaternions), exact

Using the product rule (Iu)(Iv) = −u·v + I(u×v) (the Hamilton convention, ij = k — the same convention in which the fundamental identity q² = Tq − N hold):

$$R_1\,R_2 \;=\; C \;+\; I\,V\,,$$

$$C = \cos s_1 \cos s_2 - \sin s_1 \sin s_2\,(v_1 \cdot v_2)\,,$$

$$V = \cos s_1 \sin s_2 \cdot v_2 + \sin s_1 \cos s_2 \cdot v_1 + \sin s_1 \sin s_2\,(v_1 \times v_2)\,.$$

Both C and V are computed exactly. This step contains no approximation whatsoever.

5. Step 3 — the logarithm by the ladder

The product R_1R_2 = C + I V is again a quaternion with real scalar C and bivector IV, and $C^2 + |V|^2 = |R_1R_2|^2 = 1$, so its eigenvalues are C ± i|V|. For f = log the ladder gives

$$A_f \;=\; \varphi/|V|\,, \quad B_f \;=\; -C\,\varphi/|V|\,, \quad \varphi = \operatorname{atan2}(|V|, C)\,,$$

and indeed $A_f(C \;+\; IV) \;+\; B_f \;=\; (\varphi/|V|)$ where I V — is a pure bivector, exactly as a logarithm must be. Since C = cos s_3 and |V| = sin s_3 for the resulting rotation quaternion, the angle is φ = s_3 exactly.

6. The closed BCH formula

Combining the three steps — the sum of the entire infinite BCH series:

$$\exp(q_1)\exp(q_2) = \exp(q_3)$$

$$q_1 = a_1 + I\, s_1\, v_1$$

$$q_2 = a_2 + I\, s_2\, v_2$$

$$q_3 = a_3 + I\, s_3\, v_3$$

$$a_3 = a_1 + a_2 \,,$$

$$s_3 = \arccos(\cos s_1 \cos s_2 - \sin s_1 \sin s_2\, (v_1 \cdot v_2)) \,,$$

$$s_3\, v_3 = \cos s_1 \sin s_2 \cdot v_2 + \sin s_1 \cos s_2 \cdot v_1 + \sin s_1 \sin s_2\, (v_1 \times v_2) \,,$$

Structural reading. The second-order BCH term $\tfrac{1}{2}[q_1,q_2] = I(w_1 \times w_2)$ is precisely the cross-product term above. The entire infinite tail of the series is exactly the difference between the angle s_3 produced by this spherical cosine rule and the naive angle $|w_1 + w_2|$. That difference — nothing else — is the whole content of BCH for quaternions.

7. Vector-only form (numerically stable)

For computation one avoids arccos and the explicit unit vectors. Introduce the cardinal (sinc) factors $\kappa_i := \sin s_i / s_i$ ($\kappa_i \to 1$ as $s_i \to 0$) and $\kappa_{12} := \sin s_1 \sin s_2 / (s_1\, s_2)$. Then

$$W = \cos s_1 \cdot \kappa_2\, w_2 + \cos s_2 \cdot \kappa_1\, w_1 + \kappa_{12}\, (w_1 \times w_2) \,,$$

$$C = \cos s_1 \cos s_2 - \kappa_{12}\, (w_1 \cdot w_2) \,, \quad s_3 = \operatorname{atan2}(|W|, C) \,, \quad w_3 = (s_3/|W|)\, W \,,$$

with the degenerate branches: if $W = 0$ and $C > 0$ then $w_3 = 0$; if $W = 0$ and $C < 0$ then $w_3 = \pi\, \hat{v}_1$ (path-dependent limit, see §9). Note that $W \equiv V$ identically: the sinc factors convert the unit vectors back into the raw vectors $w_i = s_i v_i$, so $|W| = \sin s_3$ and atan2 replaces arccos — stable both near $s_3 = 0$ and near $s_3 = \pi$.

Remark (a common slip). The compact-looking form $\cos s_1 \cdot w_2 + \cos s_2 \cdot w_1 + \kappa\, w_1 \times w_2$ — without the factors $\sin s_i / s_i$ — agrees with the exact formula only up to second order in (s_1, s_2). The sinc factors are essential for full closed-form accuracy; numerically the naive form deviates by $2.1 \cdot 10^{-2}$ at scale 0.5, $2.2 \cdot 10^{-3}$ at 0.2 and $3.0 \cdot 10^{-4}$ at 0.1 (third-order error), while the exact form is exact.

8. Verification

8.1 Worked example

Take $q_1 = 0.3 + I(1, 0, 0)$ and $q_2 = -0.7 + I(0.70711, 0.70711, 0)$. The closed formula gives

$$q_3 = -0.4 + \mathrm{I}\,(1.413515, 0.585497, 0.911858), \quad s_3 = 1.781100 \,,$$

and the direct checks are: $\log(\exp q_1 \exp q_2)$ computed independently — deviation 0.0 (exact); $\exp(q_3)$ versus $\exp(q_1)\exp(q_2)$ — deviation $2.4 \cdot 10^{-16}$.

8.2 Random stress test

2000 random pairs with $a_i \in (-2, 2)$, $w_i \in (-2, 2)^3$, compared against the independently computed logarithm of the product: maximum deviation $2.05 \cdot 10^{-15}$.

8.3 Order-by-order agreement with the BCH series itself

Partial sums Z_2, Z_3, Z_4 of the BCH series ($X = \varepsilon q_1$, $Y = \varepsilon q_2$) versus the closed form $q_3(\varepsilon)$:

ε	$\lvert q_3 - Z_2 \rvert$	$\lvert q_3 - Z_3 \rvert$	$\lvert q_3 - Z_4 \rvert$
0.20	$1.50\cdot10^{-3}$	$2.71\cdot10^{-4}$	$2.84\cdot10^{-5}$
0.10	$1.82\cdot10^{-4}$	$1.67\cdot10^{-5}$	$8.75\cdot10^{-7}$
0.05	$2.26\cdot10^{-5}$	$1.04\cdot10^{-6}$	$2.73\cdot10^{-8}$
0.01	$1.80\cdot10^{-7}$	$1.67\cdot10^{-9}$	$8.71\cdot10^{-12}$

The convergence pattern $\varepsilon^3 \to \varepsilon^4 \to \varepsilon^5$ confirms that the closed formula reproduces the BCH terms $\tfrac{1}{2}[X,Y]$, $(1/12)(\ldots)$, $-(1/24)(\ldots)$ one order after another — and then all remaining orders at once.

8.4 Special cases (all exact)

- Commutative case ($w_2 \parallel w_1$): $q_3 = -0.4 + I(1.6, 0, 0) = q_1 + q_2$ — the spherical rule degenerates to addition. ✓
- Branch $R_1R_2 = -1$ ($s_1 = s_2 = \pi/2$, $v_2 = v_1$): $q_3 = 0 + I\pi(1, 0, 0) = I\pi\,\hat{v}_1$. ✓
- Branch $R_1R_2 = +1$ ($s_1 = s_2 = \pi/2$, $v_2 = -v_1$): $q_3 = a_1 + a_2$ (vector part zero). ✓
- Scalar factor (q_1 real): $q_3 = 1.8 + I(0.9, 0.2, 0.1)$ — the scalar passes through additively. ✓

9. Fine points

- Branch of the logarithm. arccos returns the principal branch $0 \le s_3 \le \pi$. The complete family of solutions is $q_3 + 2\pi k\cdot I\,v_3$, $k \in \mathbb{Z}$. At $C = -1$, $V = 0$ the direction v_3 is not determined by the equation; the limit depends on the path of approach. The formula above takes $\hat{v}_1$.
- Cross-product convention. The sign of the $v_1\times v_2$ term is tied to the convention $(Iu)(Iv) = -u\cdot v + I(u\times v)$ (Hamilton, $ij = k$). If in a given GA text the product of bivectors carries the opposite sign, flip the sign of the cross term. A reliable one-line test: $\exp(q_3) =? \exp(q_1)\exp(q_2)$ on a single numerical example.
- Scalar additivity. Commutators of quaternions depend only on the vector parts: $[q_1,q_2] = 2I(w_1\times w_2)$ — the scalar parts cancel identically. Hence $a_3 = a_1 + a_2$ for arbitrary scalar parts a_i; all noncommutativity is carried by the bivector sector.

Conclusion

A single polynomial identity $q^2 = Tq - N$ that directly follows from quaternion definition , is also turned out to be a minimal polynomial of the Cayley-Hamilton relation for quaternions - does all the work in this article. It confines every function of a nonscalar quaternion to the two-dimensional slice span{1, Iv} and, through the ladder $q^n = A_n q + B_n$, reduces the entire functional calculus of quaternions to two real functions A_f and B_f of the invariants (trace T and square norm N), equivalently of the conjugate eigenvalues $\lambda_{\pm} = a \pm is$ (and also Vieta roots of minimal polynomial) .

The practical yield is a set of closed, numerically stable formulas. The ladder terminates the calculus of polynomials: every monomial is affine in q, the negative ladder follows from the mirror relation, and singularities at s = 0 are removable by confluent limits. Three independent derivations - reduction to a complex value via the isomorphism χ, Lagrange interpolation on the eigenvalues, and the spectral projector decomposition - produce identical coefficients, which is itself a consistency proof of the whole framework. On top of the polynomial core, the representation gives: (i)

closed forms for the elementary functions (exp, trigonometric, hyperbolic, powers, sqrt, log, inverse); (ii) exact summation of infinite polynomials of q, since $A_f = \sum_k c_k A_k(T,N)$ and $B_f = c_0 - N\sum_k c_{k+1} A_k(T,N)$

whenever the series converges on the spectrum; (iii) the Fibonacci-Lucas identification, where for T = 1, N = -1 the ladder coefficients are exactly Fibonacci numbers and the eigenvalues are the golden-ratio pair, so that Binet's formula becomes a special case of the eigenvalue calculus; and (iv) an exact closed form for the Baker-Campbell-Hausdorff series, in which the entire infinite series of nested commutators for exp(q1) exp(q2) = exp(q3) collapses to one spherical cosine rule for the angle s3 plus the additive scalar law a3 = a1 + a2.

Two structural observations frame the result. First, noncommutativity is carried entirely by the direction of the slice: under an autonomous equation dq/dt = f(q) the direction v is frozen and the dynamics reduces to two coupled scalars for (a, s) - the spectral sector - while the direction moves only in mirror-type regimes such as the Euler top, where three spectral lines couple instead of two. Second, the whole construction is a quaternionic shadow of matrix functional calculus: the coefficients A_f, B_f are the first divided difference and the remainder of f at the conjugate spectral pair, and the degenerate limit s -> 0 recovers the ordinary scalar calculus on the real axis.

Natural continuations include: non-slice dynamics with genuine direction-spectrum coupling; functions with complex coefficients and biquaternions; differential equations with external fields, where the transverse current $(\lambda_+ - \lambda_-)/2 = is$ becomes the carrier of gyroscopic motion; and numerical benchmarks exploiting the stability limits developed above. We hope that the affine representation f(q) = A_f q + B_f - simple as it is - will serve as a convenient common language for the analytic, geometric and computational facets of quaternion analysis.

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