A Two-Phase Extension of Euler's Formula in an Idempotent Representation

Vyacheslav Gorchilin

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Abstract

We consider a two-phase representation of Euler's formula in the commutative algebra $\mathcal{A} \simeq \mathbb{C} \times \mathbb{C}$. An idempotent decomposition and explicitly selected logarithm values define continuous real powers of a hyperbolic unit $\jmath$ and of $-\jmath$. Their product gives a family $J(a, b)$ with two independent phases. We derive its exponential representation, two-phase sine and cosine, the fundamental trigonometric identity, and the composition law. The phase parameter plane, the torus of values, and the linear multiplication action are distinguished. The construction uses established bicomplex analysis and presents a single sequence from selected continuous powers to two independent rotations.

1 Introduction and problem statement

Euler's formula

$$e^{i\alpha} = \cos\alpha + i\sin\alpha, \qquad \alpha \in \mathbb{R}, \tag{1}$$

describes a single phase in the complex plane. A natural question is how to combine two independent phase rotations in one expression while retaining an exponential representation and an additive phase law.

We work in a commutative four-dimensional real algebra with a complex unit and a hyperbolic unit. This is a representation of the established algebra of bicomplex numbers. Bicomplex exponentials, logarithms, and trigonometric functions have been studied previously; see, for example, [1, 2]. We use the idempotent representation of this algebra.

The purpose of this article is to construct a family $J(a, b)$ from selected continuous powers of the hyperbolic unit, formulate a two-phase Euler identity, and derive its algebraic and geometric properties. Here, extension means passing from one complex phase to two independent complex components. The known bicomplex algebra and its elementary functions provide the framework; introducing the notation J does not introduce a new algebra.

The presentation develops the mathematical part of the author's online note [3].[1] Amplitude weights, quaternionic composition, and physical interpretations are outside its scope.

2 Algebraic framework

Set

$$\mathcal{A} = \mathbb{C}[t]/(t^2 - 1), \qquad \jmath = [t], \qquad i^2 = -1. \tag{2}$$

The units i and $\jmath$ commute, $\jmath^2 = 1$, and $1, i, \jmath, i\jmath$ form a real basis. This definition excludes additional implicit relations. In particular, $\jmath \neq \pm 1$.

The unit $k = i\jmath$ satisfies $k^2 = -1$ and commutes with i. Thus $\mathcal{A}$ is isomorphic to the standard bicomplex algebra generated by two commuting square roots of -1 [2].

[1] ChatGPT was used to assist with manuscript preparation, checking calculations, and translation.

Introduce

$$\mathfrak{e} = \frac{1+\jmath}{2}, \qquad \bar{\mathfrak{e}} = \frac{1-\jmath}{2}. \tag{3}$$

$$\mathfrak{e}^2 = \mathfrak{e}, \quad \bar{\mathfrak{e}}^2 = \bar{\mathfrak{e}}, \quad \mathfrak{e}\bar{\mathfrak{e}} = 0, \quad \mathfrak{e} + \bar{\mathfrak{e}} = 1, \quad \mathfrak{e} - \bar{\mathfrak{e}} = \jmath. \tag{4}$$

The bar in $\bar{\mathfrak{e}}$ is part of its name; complex conjugation of coefficients will be indicated explicitly. Every element has a unique representation

$$Z = \mathfrak{e}z_+ + \bar{\mathfrak{e}}z_-, \qquad z_+, z_- \in \mathbb{C}. \tag{5}$$

$$(\mathfrak{e}z_+ + \bar{\mathfrak{e}}z_-)(\mathfrak{e}w_+ + \bar{\mathfrak{e}}w_-) = \mathfrak{e}z_+w_+ + \bar{\mathfrak{e}}z_-w_-. \tag{6}$$

Hence $Z \mapsto (z_+, z_-)$ is an algebra isomorphism $\mathcal{A} \simeq \mathbb{C} \times \mathbb{C}$ with componentwise operations. Equations (4) follow by direct expansion, and uniqueness in (5) follows from the complex linear independence of $1, \jmath$. An element Z is invertible if and only if both components are nonzero. The nonzero elements $\mathfrak{e}, \bar{\mathfrak{e}}$ are zero divisors.

3 Continuous powers of the hyperbolic unit

Define the exponential by its power series. Using (4) and treating the constant term separately gives

$$\exp(\mathfrak{e}u + \bar{\mathfrak{e}}v) = \mathfrak{e}e^u + \bar{\mathfrak{e}}e^v, \qquad u, v \in \mathbb{C}. \tag{7}$$

Proposition 3.1. *All logarithm values of $\jmath$ and $-\jmath$ in $\mathcal{A}$ have the form*

$$\ell_\jmath^{m,n} = 2\pi i m\,\mathfrak{e} + (2n+1)\pi i\,\bar{\mathfrak{e}}, \tag{8}$$

$$\ell_{-\jmath}^{p,q} = (2p+1)\pi i\,\mathfrak{e} + 2\pi i q\,\bar{\mathfrak{e}}, \tag{9}$$

where $m, n, p, q \in \mathbb{Z}$. For any fixed logarithm value ℓ satisfying $e^\ell = \jmath$, the function $F(a) = e^{a\ell}$ is a continuous one-parameter group and agrees with the ordinary powers of $\jmath$ at integer values of a. The analogous statement holds for $-\jmath$.

Proof. The equation $e^{\mathfrak{e}u+\bar{\mathfrak{e}}v} = \jmath = \mathfrak{e} - \bar{\mathfrak{e}}$ is equivalent to $e^u = 1$ and $e^v = -1$. Their complex solutions give (8); the same argument gives (9). Continuity follows from continuity of the exponential. For fixed ℓ, we have $e^{(a+c)\ell} = e^{a\ell}e^{c\ell}$, $F(0) = 1$, and $F(1) = \jmath$. Consequently, $F(n) = \jmath^n$ for every $n \in \mathbb{Z}$. □

Thus continuity and agreement at integers do not determine a unique continuation. We select the logarithm values

$$\ell_\jmath = i\pi\bar{\mathfrak{e}}, \qquad \ell_{-\jmath} = i\pi\mathfrak{e}, \tag{10}$$

$$\boxed{\jmath^a := e^{a\ell_\jmath} = \mathfrak{e} + \bar{\mathfrak{e}}e^{i\pi a}, \qquad (-\jmath)^b := e^{b\ell_{-\jmath}} = \mathfrak{e}e^{i\pi b} + \bar{\mathfrak{e}}.} \tag{11}$$

Throughout this article, real powers refer specifically to (10)–(11). The first family rotates only the second component, and the second family only the first. No single-valued logarithm on the entire group of invertible elements is assumed.

4 Construction of the two-phase exponential

Definition 4.1. For $a, b \in \mathbb{R}$, define the two-phase exponential family by

$$\boxed{J(a,b) := \jmath^a(-\jmath)^b = \mathfrak{e}e^{i\pi b} + \bar{\mathfrak{e}}e^{i\pi a}.} \tag{12}$$

The last equality follows from (6). Equation (7) gives two equivalent forms:

$$J(a,b) = \exp[i\pi(\mathfrak{e}b + \bar{\mathfrak{e}}a)], \tag{13}$$

$$= \exp\left[i\pi\frac{a+b}{2} + i\jmath\pi\frac{b-a}{2}\right]. \tag{14}$$

The second form involves two commuting generators i and $i\jmath$, both of square -1. On the diagonal and antidiagonal of the phase plane,

$$J(a,a) = e^{i\pi a}, \qquad J(a,-a) = e^{-i\jmath\pi a} = \cos(\pi a) - i\jmath\sin(\pi a). \tag{15}$$

In particular, the ordinary Euler formula is recovered at $a = b = \alpha/\pi$. For independent a, b, the two phases belong to distinct complex components.

5 The two-phase Euler formula

Define functions with values in the real subalgebra $\mathbb{R}[\jmath]$:

$$\mathrm{Cos}(a,b) := \mathfrak{e}\cos(\pi b) + \bar{\mathfrak{e}}\cos(\pi a), \tag{16}$$

$$\mathrm{Sin}(a,b) := \mathfrak{e}\sin(\pi b) + \bar{\mathfrak{e}}\sin(\pi a). \tag{17}$$

The notation Cos, Sin distinguishes these functions from scalar sine and cosine.

Proposition 5.1. *For all real a, b, we have*

$$\boxed{J(a,b) = \mathrm{Cos}(a,b) + i\,\mathrm{Sin}(a,b),} \tag{18}$$

$$\mathrm{Cos}(a,b)^2 + \mathrm{Sin}(a,b)^2 = 1. \tag{19}$$

Proof. Apply the ordinary Euler formula to each exponential in (12). Squaring (16) and (17) eliminates the mixed terms because $\mathfrak{e}\bar{\mathfrak{e}} = 0$. The sum is

$$\mathfrak{e}(\cos^2(\pi b) + \sin^2(\pi b)) + \bar{\mathfrak{e}}(\cos^2(\pi a) + \sin^2(\pi a)) = \mathfrak{e} + \bar{\mathfrak{e}} = 1.$$

□

Identity (19) is an equality in $\mathcal{A}$, rather than a statement about the sum of squares of four real coordinates of these functions. Furthermore,

$$\mathrm{Cos}(a,b) = \frac{J(a,b) + J(-a,-b)}{2}, \tag{20}$$

$$\mathrm{Sin}(a,b) = \frac{J(a,b) - J(-a,-b)}{2i}. \tag{21}$$

Writing $A = (a,b)$ and $B = (c,d)$, the componentwise addition formulas give

$$\mathrm{Cos}(A+B) = \mathrm{Cos}\,A\,\mathrm{Cos}\,B - \mathrm{Sin}\,A\,\mathrm{Sin}\,B, \tag{22}$$

$$\mathrm{Sin}(A+B) = \mathrm{Sin}\,A\,\mathrm{Cos}\,B + \mathrm{Cos}\,A\,\mathrm{Sin}\,B. \tag{23}$$

Here $A + B = (a+c, b+d)$, and all products are taken in $\mathcal{A}$. At $a = b$, the functions reduce to $\cos(\pi a)$ and $\sin(\pi a)$.

6 Composition and geometric interpretation

Proposition 6.1. *The family J satisfies*

$$J(a,b)J(c,d) = J(a+c, b+d), \tag{24}$$

$$J(0,0) = 1, \qquad J(a,b)^{-1} = J(-a,-b), \tag{25}$$

$$J(a+2m, b+2n) = J(a,b), \qquad m,n \in \mathbb{Z}. \tag{26}$$

If $Z^\dagger := \mathfrak{e}\overline{z_+} + \bar{\mathfrak{e}}\overline{z_-}$, then $J^\dagger = J^{-1}$ and $J^\dagger J = 1$.

Proof. Both components in (12) have modulus 1. By (6), each statement reduces to the corresponding property of the complex exponential. □

6.1 The phase plane and the torus of values

The pairs $(a,b) \in \mathbb{R}^2$ form a plane of phase parameters. The kernel of $(a,b) \mapsto J(a,b)$ is $(2\mathbb{Z})^2$. Hence its image, with the multiplication in (24), has the structure

$$\mathbb{R}^2/(2\mathbb{Z})^2 \simeq S^1 \times S^1. \tag{27}$$

This is a two-dimensional torus in the four-dimensional real space $\mathcal{A}$. In the coordinates $(\operatorname{Re} z_+, \operatorname{Im} z_+, \operatorname{Re} z_-, \operatorname{Im} z_-)$, it is specified by $|z_+| = |z_-| = 1$. The parameter plane and the surface of values are therefore distinct objects.

6.2 The multiplication action

The quantity $J(a,b)$ itself is an algebra element. A linear operator is obtained by defining

$$L_{J(a,b)} : \mathcal{A} \longrightarrow \mathcal{A}, \qquad Z \longmapsto J(a,b)Z. \tag{28}$$

$$L_{J(a,b)} = \begin{pmatrix} R(\pi b) & 0 \\ 0 & R(\pi a) \end{pmatrix}, \qquad R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{29}$$

The matrix is written in the real coordinates specified above. It describes independent rotations in two fixed orthogonal planes. These operators form a subgroup of $SO(4)$ isomorphic to $SO(2) \times SO(2)$; this family does not represent arbitrary four-dimensional rotations.

The action preserves $\|Z\|^2 = |z_+|^2 + |z_-|^2$, but $\|J(a,b)\|^2 = 2$. The algebraic equality $J^\dagger J = 1$ must be distinguished from scalar unit normalization.

For example, $J(1/2, 0) = \mathfrak{e} + i\bar{\mathfrak{e}}$ fixes the first component and rotates the second by $\pi/2$. Its square is $\jmath$. At $a = b$, both components rotate through the same angle.

7 Conclusion

The choices $\ell_\jmath = i\pi\bar{\mathfrak{e}}$ and $\ell_{-\jmath} = i\pi\mathfrak{e}$ define two continuous power families acting on different complex components. Their product yields the two-phase exponential $J(a,b)$. Euler's formula retains its form when scalar sine and cosine are replaced by their idempotent counterparts; the ordinary one-phase case is recovered on the diagonal $a = b$.

Composition corresponds to addition of phases, and periodicity gives the torus $S^1 \times S^1$. The multiplication representation produces two independent plane rotations. This construction is a particular two-phase family in the established bicomplex algebra; the stated properties follow from its idempotent decomposition.

References

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[2] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa, and A. Vajiac, *Bicomplex Numbers and their Elementary Functions*, CUBO **14**(2) (2012), 61–80. doi:10.4067/S0719-06462012000200004.

[3] V. Gorchilin, *Two-phase operator J: From Euler's formula to an extended normalized state* (in Russian), online note, 3 November 2026. `https://gorchilin.com/articles/math/operator_J_article`.