Proof of Firoozbakht's conjecture

Kouji Takaki

November 04_{th}, 2026

Abstract

We showed that the following inequality of Firoozbakht's conjecture holds when $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds.

$$\log(p_{n+1}) - \log(p_n) < \log(p_n)/n$$

Moreover, in the other case, the following inequality holds for $n \geqq 1$ because the derivative function of $\log(x)$ decreases monotonically for $x > 0$.

$$\log(n+1) - \log(n) > p_n/(n+1) \times (\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n)$$

We showed that the inequality of this conjecture is obtained by this inequality and another inequality when $p_{n+1} - p_n \geqq \log(p_n)$ holds. From the above, we proved that Firoozbakht's conjecture is true.

Contents

Introduction	1
Proof	1
Acknowledgement	3
References	4

1. Introduction

In number theory, Firoozbakht's conjecture (or the Firoozbakht conjecture) is a conjecture about the distribution of prime numbers. It is named after the Iranian mathematician Farideh Firoozbakht from the University of Isfahan who stated it first in 1982. The conjecture states that $\sqrt[n]{p_n}$ (where p_n is the nth prime) is a strictly decreasing function of n, i.e.,

$$\sqrt[n+1]{p_{n+1}} < \sqrt[n]{p_n} \text{ for all } n \geqq 1$$

Equivalently,

$$p_{n+1} < {p_n}^{1+1/n} \text{ for all } n \geqq 1$$

(Quoted from Wikipedia)

2. Proof

When we write log in this paper, log refers to natural logarithm. Let n be a positive integer. We will prove that the following inequality holds for any n.

$$p_{n+1} < {p_n}^{1+1/n} \quad \ldots (1)$$

I When $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds

$$\log(p_{n+1})/\log(p_n) < 1 + 1/p_n$$

Since $p_n > n$ holds for $n \geqq 1$,

$$\log(p_{n+1})/\log(p_n) < 1 + 1/n$$

holds. This inequality accords with the inequality (1).

Ⅱ When $\log(p_{n+1}) - \log(p_n) \geqq \log(p_n)/p_n$ holds

ⅰ When $p_{n+1} - p_n < \log(p_n)$ holds

$$p_{n+1}/p_n < 1 + \log(p_n)/p_n$$

$$\log(p_{n+1}) - \log(p_n) < \log(1 + \log(p_n)/p_n) < \log(p_n)/p_n$$

The case of ⅰ does not exist since this inequality is contrary to the condition of Ⅱ.

ⅱ When $p_{n+1} - p_n \geqq \log(p_n)$ holds

Let x be a real number. Let $f(x) = \log(x)$. $f'(x) = 1/x$ and $f''(x) = -1/x^2$ hold. The derivative function of $f(x)$ is a monotonically decreasing function for $x > 0$ since $f''(x) < 0$ holds for $x > 0$. The following inequalities hold for all n where $n \geqq 1$ holds because $f'(x) > 0$ and $f''(x) < 0$ hold for $x > 0$.

$$\log(n+1) - \log(n) > 1/(n+1)$$

$$1/p_n > (\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n)$$

Multiplying both sides of these inequalities, the following inequality holds for $n \geqq 1$.

$$(\log(n+1) - \log(n))/p_n > (\log(p_{n+1}) - \log(p_n))/((n+1)(p_{n+1} - p_n))$$

$$\log(n+1) - \log(n) > p_n/(n+1) \times (\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) \quad \ldots (2)$$

Next, we prove that the following inequality holds for $n \geqq 5$.

$$p_n/(n+1) > (p_{n+1} - p_n)/\log(p_n) \quad \ldots (3)$$

$$\log(p_n) - (n+1)(p_{n+1}/p_n - 1) > 0$$

Let $g_1(n) = \log(p_n) - (n+1)(p_{n+1}/p_n - 1)$. According to Dusart's inequality $p_n \approx n(\log(n) + \log(\log(n)))$, where we define $g_2(n)$ as $g_2(n) = \log(n(\log(n) + \log(\log(n)))) - (n+1)((n+1)(\log(n) + \log(\log(n)))/(n(\log(n) + \log(\log(n)))) - 1)$, $g_1(n) \approx g_2(n)$ holds. Since $(\log(n+1) + \log(\log(n+1)))/(\log(n) + \log(\log(n))) < 1 + 1/n$ holds for $n \geqq 4$ by numerical calculation and this inequality can be proven through calculation, the following inequality holds for $n \geqq 4$.

$$g_2(n) > \log(n(\log(n) + \log(\log(n)))) - (n+1)(2/n + 1/n^2)$$

$$g_2(n) > \log(n(\log(n) + \log(\log(n)))) - (2 + 3/n + 1/n^2)$$

In addition, since $\log(n(\log(n) + \log(\log(n)))) > 3$ holds for $n \geqq 8$ by numerical computation, we find that $g_2(n) > 0$ holds for $n \geqq 8$, $g_2(n)$ increases monotonically and $g_2(n)$ diverges to infinity as n increases.

$g_2(2) = -13.856898662794245062650331316431 48 \ldots$

$g_2(3) = -2.385069719205142730748463780340 34\ldots$

$g_2(4) = -0.684264240936032378039315635265 96\ldots$

$g_2(5) = 0.144339320935172956627502714125 91\ldots$

$g_2(6) = 0.676202516094915469385892501170 18\ldots$

$g_2(7) = 1.063215979956958953835815230417 61\ldots$

It is confirmed that $g_2(n) > 0$ holds for $n \geqq 5$.

$g_1(1) = -0.306852819440054690582767878541 82\ldots$

$g_1(2) = -0.901387711331890308604754763077 47\ldots$

$g_1(3) = 0.009437912434100374600759333226 18\ldots$

$g_1(4) = -0.911232708087543837751790113699 67\ldots$

$g_1(5) = 1.306986181889279634971034487056 03\ldots$

$g_1(6) = 0.411103203615382889899641287719 16\ldots$

$g_1(7) = 1.892036873467980786131887559049 59\ldots$

It is confirmed that $g_1(n) > 0$ holds for $n \geqq 5$ since $g_1(n) \approx g_2(n)$ holds, $g_2(n)$ is a monotonical increasing function and $g_2(n)$ diverges to infinity. Therefore, we can consider that the inequality (3) holds for $n \geqq 5$. The following inequality holds for $n \geqq 5$ by the inequalities (2) and (3).

$$(\log(p_{n+1}) - \log(p_n))/\log(p_n) < \log(n+1) - \log(n)$$

$$\log(p_{n+1})/\log(p_n) < 1 + \log(1 + 1/n) < 1 + 1/n$$

$$\log(p_{n+1})/\log(p_n) < 1 + 1/n$$

This inequality coincides with the inequality (1). Let $h(n) = 1 + 1/n - \log(p_{n+1})/\log(p_n)$.

$h(1) = 0.035026479282072832802959592321 35\ldots$

$h(2) = 0.124271378211165776570016987785 97\ldots$

$h(3) = 0.017725594132656238139243132344 37\ldots$

$h(4) = 0.130333035575731290861398226052 04\ldots$

It is confirmed that $h(n) > 0$ holds for $1 \leqq n \leqq 4$.

From the above, it is proved that Firoozbakht's conjecture is true since the inequality (1) holds for all n where $n \geqq 1$ holds. (Q.E.D.)

3. Acknowledgement

We would like to thank my family members who sustained a research environment and people who pointed out mistakes in this paper on an anonymous bulletin board.

4. References

[1] Wikipedia https://en.wikipedia.org/wiki/Firoozbakht%27s_conjecture

[2] Wikipedia https://en.wikipedia.org/wiki/Prime_number_theorem