Homogeneous Mixture Flow Field and Separated Flow Fields Dynamic Equations

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Abstract

A multicomponent, multiphase fluid can be modelled as either homogeneous mixture flow field or separated flow fields. The key concept is that fluid flows will be considered as continuum fields through suitable volumetric "average" procedures, so that the differential calculus tools can be used. For homogeneous mixture flow field, the mixture is treated as a single system (single-phase flow). Starting from the mixture momentum density 1-form per unit volume, taking the exterior derivative of this 1-form yields a field strength tensor in a differential 2-form. The interior product (contraction) of this 2-form with the tangent velocity yields the dynamic equation of the homogeneous mixture flow, $\vec{E} + \vec{\Omega} \times \vec{V} = 0$, (or expressed as a differential 1-form of eq. (20)). It is equivalent to the Euler-Lagrange equation, but the gradient of kinetic energy doesn't equal zero for continuum fields, in contrast to the classical Euler-Lagrange equation for a discrete isolated particle. For separated flow fields, different components travel at their own velocities; there exists a velocity difference between components, and each component is treated as an individual system to consider the interactions between different components. The homogeneous mixture dynamic equation turns into two equations for separated flow fields. The two equations become tightly entangled. It shows there are very complicated interaction forces between the two flows.

1. Introduction

Multicomponent, multiphase fluid flows occur commonly in both nature and technology. In everyday life, we commonly perceive matter as particles filling in space. For example, clouds are droplets of liquid moving in a gas; particles are moved by interacting with air [1]. Through proper “average” procedures, the multicomponent, multiphase fluid flows can be treated as interacting continua or homogeneous mixture. Over the past few decades, there have been many extensive, rigorous mathematical “average” operations appearing in many excellent literatures, such as [2,3,4]. Here we take the concept that multiphase fluid flows will be considered as continuum interacting materials. Continuum fields are “idealizations” to be treated using the differential calculus tools. More precisely, in this paper, we will adopt the exterior differential 1-form and 2-form to describe the dynamics of the continuum fields and the interactions between different continua. We will see that the exterior differential forms (and the wedge products) are an elegant and powerful mathematical tool to describe the continuum field dynamics.

For simplification, in this paper, we assume the space is filled by two components, two-phase flow fields, but the idea can be simply extended to multicomponent (many particles), multiphase fluid flow fields.

Section 2 introduces the mixture property formulations. It defines the homogeneous mixture properties per unit volume and gives the definition of the center-of-mass velocity of the mixture.

Section 3 treats the homogeneous mixture as a single continuum field; thereby, the interactions between two components are regarded as interior forces inside the system and are not taken into consideration automatically. A Euler-Lagrange equation can be recovered from these exterior differential forms for the continuum fields. But the fundamental difference is that the gradient of kinetic energy cannot be zero in continuum field dynamic equations.

Section 4 deals with the separated flow, supposing two components have differential flow velocities; each component is treated as an individual system. The homogeneous mixture equation turns into two equations, which become entangled with each other, revealing complicated interactions between the two phases.

2. Mixture Property Formulations

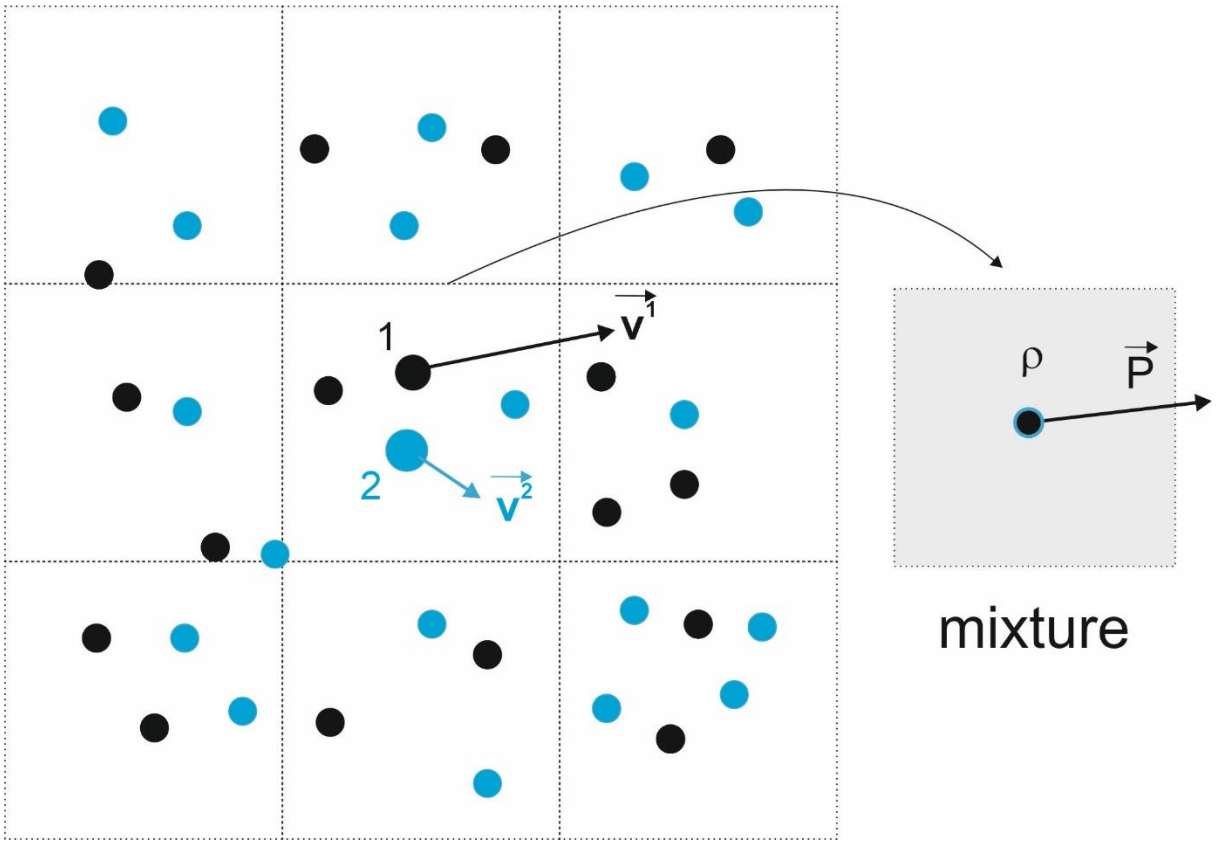


Fig. 1. Two-component (or two-particle) flow modelled as a homogeneous continuum field; the mixture volumetric momentum density is the sum of individual components weighted by the smear densities.

Giving a two-component two-phase flow field in the space (designated as particle 1 and particle 2), as shown in Fig. 1. The tangent velocities are denoted as $\vec{v}^1$ and $\vec{v}^2$, respectively.

The smear mass densities per unit volume (written with the bar over the symbol) are defined by

$$\overline{\rho}_i = \phi_i \rho_i \,, \qquad (i = 1, 2) \tag{1}$$

where ρ_i are the thermodynamic densities and ϕ_i are the volume fractions, and are defined by the requirement that

$$\phi_1 + \phi_2 = 1 \tag{2}$$

The total volumetric mass density of the mixture, thus, is

$$\rho = \overline{\rho}_1 + \overline{\rho}_2 = \phi_1 \rho_1 + \phi_2 \rho_2 \tag{3}$$

The mixture volumetric momentum density is the sum of individual components weighted by the smear densities (hereafter, the mixture properties are denoted with capital letters)

$$\vec{P} = \rho \vec{V} = \vec{P}_1 + \vec{P}_2 = \overline{\rho}_1 \vec{v}^1 + \overline{\rho}_2 \vec{v}^2 \tag{4}$$

By definition, the mixture velocity (center-of-mass velocity) equals

$$\vec{V} = \frac{\vec{P}}{\rho} = \frac{\phi_1 \rho_1 \vec{v}_1 + \phi_2 \rho_2 \vec{v}_2}{\phi_1 \rho_1 + \phi_2 \rho_2} \tag{5}$$

Similarly, the mixture volumetric potential energy density is then defined by

$$U = \phi_1 U_1 + \phi_2 U_2 = \overline{U}_1 + \overline{U}_2 \tag{6}$$

3. Homogeneous Mixture Flow Field

In this section, we shall be concerned with the dynamics of the homogeneous mixture flow in which mixture volumetric momentum density, the mixture velocity (center-of-mass), the mixture volumetric potential energy density, etc., are relevant variables, see Fig.1.

For a general curvilinear coordinate system or a general manifold (or curved manifold), the differential 1-form A is defined as:

$$A = P_i dq^i - (U/C)(Cdt) = P_i dq^i - Udt \tag{7}$$

Here, A is the action density 1-form per unit volume. C represents the wave propagation speed in the field. P_i is the canonical momentum density conjugated to the general coordinate q^i, is defined as

$$P_i = \rho g_{ij} \dot{q}^j \tag{8}$$

where, g_{ij} is Riemannian metric tensor, it assigns to each point of the manifold and is a smooth, symmetric, positive-definite (0,2)-tensor field on a manifold M.

For simplicity, in this paper we apply the standard Cartesian coordinate system, where the components g_{ij} of the Riemannian metric tensor are given by the Kronecker delta:

$$g_{ij} = \delta_{ij} \tag{9}$$

Thus, the action 1-form density per unit volume can be expressed explicitly in the Cartesian coordinate system as

$$A = P_x dx + P_y dy + P_z dz - U dt. \tag{10}$$

Accordingly, the tangent velocity along the physical trajectory is

$$\vec{V} = V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^z \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \tag{11}$$

The exterior derivative of the differential 1-form density yields a differential 2-form density. Taking the exterior derivative of eq. (10) yields:

$$F = dA = dP_x \wedge dx + dP_y \wedge dy + dP_z \wedge dz - dU \wedge dt. \tag{12}$$

The differentials of P_i and U are expressed as

$$\begin{cases} dP_i = \dfrac{\partial P_i}{\partial x^\mu} dx^\mu \\ -dU = -\dfrac{\partial U}{\partial x^\mu} dx^\mu \end{cases}, \quad for\ \mu = (t, x, y, z) \tag{13}$$

If each term is written out explicitly, e.g.:

$$\begin{cases} dP_x = \dfrac{\partial P_x}{\partial t}dt + \dfrac{\partial P_x}{\partial x}dx + \dfrac{\partial P_x}{\partial y}dy + \dfrac{\partial P_x}{\partial z}dz \\ -dU = -\dfrac{\partial U}{\partial t}dt - \dfrac{\partial U}{\partial x}dx - \dfrac{\partial U}{\partial y}dy - \dfrac{\partial U}{\partial z}dz \end{cases}. \tag{14}$$

Using the antisymmetric property of the wedge product of

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \qquad dx^\mu \wedge dx^\mu = 0, \tag{15}$$

and by collecting the same basis terms of $dx^\mu \wedge dx^\nu$, with some algebra manipulation, finally, we get the differential 2-form density:

$$\begin{aligned} F = &\left(\frac{\partial P_x}{\partial t} + \frac{\partial U}{\partial x}\right) dt \wedge dx + \left(\frac{\partial P_y}{\partial t} + \frac{\partial U}{\partial y}\right) dt \wedge dy + \left(\frac{\partial P_z}{\partial t} + \frac{\partial U}{\partial z}\right) dt \wedge dz \\ &+ \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right) dz \wedge dx. \end{aligned} \tag{16}$$

The differential 2-form density can be rewritten more compactly:

$$\begin{aligned} F = &E_{01} dt \wedge dx + E_{02} dt \wedge dy + E_{03} dt \wedge dz \\ &+ \Omega_z dx \wedge dy + \Omega_x dy \wedge dz + \Omega_y dz \wedge dx \end{aligned} \tag{17}$$

Here

$$\begin{cases} E_{0i} = \left(\dfrac{\partial P_i}{\partial t} + \dfrac{\partial U}{\partial x}\right) dt \wedge dx^i \\ \Omega_k = \left(\dfrac{\partial P_j}{\partial x^i} - \dfrac{\partial P_i}{\partial x^j}\right) dx^i \wedge dx^j \end{cases}, \qquad for\ \ i,j,k = (x,y,z) \tag{18}$$

The differential 2-form represents the field strength tensor, which is an antisymmetric (0,2)-tensor field.

The interior product (contraction) of a differential 2-form with the tangent velocity vector (eq. (11)) yields a differential 1-form, defined by inserting the tangent velocity vector $\vec{V}$ into the first slot of the 2-form of F. It is expressed as:

$$\iota_{\vec{V}}(F) = F\left(\vec{V}, -\right) = 0 \tag{19}$$

It can be expressed in index notation as:

$$\begin{cases} \left(E_{0i} + \varepsilon_{ijk}\Omega_j V^k\right)dx^i = 0 \\ \varepsilon_{ijk}\Omega_j V^k = \left(\vec{\Omega} \times \vec{V}\right)_i \end{cases} \tag{20}$$

Here we employ the Einstein summation convention, where repeated indices j and k are implicitly summed over. ε_{ijk} is the Levi-Civita Symbol in 3D.

It can also be expressed more compactly in vector form in 3D space (actually it lives in cotangent space, thus they are covectors):

$$\vec{E} + \vec{\Omega} \times \vec{V} = 0 \tag{21}$$

where

$$\vec{\Omega} = \nabla \times \vec{P} \tag{22}$$

The mathematical full derivations of the eq. (20) are extensive, for detailed algebra calculations please refer to reference [5].

We will see that eq. (21) is equivalent to the Euler-Lagrange equations. If we write eq. (21) explicitly, it reads

$$\frac{\partial \vec{P}}{\partial t} + \nabla \mathrm{U} + \left(\nabla \times \vec{P}\right) \times \vec{V} = 0. \tag{23}$$

The mixture kinetic energy is defined as

$$\mathrm{T} = \frac{1}{2} P_i V^i = \frac{1}{2} \rho \vec{V}^2 \tag{24}$$

We have the following vector calculus identity

$$\left(\nabla \times \vec{P}\right) \times \vec{V} = \left(\vec{V} \cdot \nabla\right)\left(\vec{P}\right) - \nabla \mathrm{T} \tag{25}$$

Substituting eq (25) into the eq. (23) yields:

$$\frac{\partial \vec{P}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{P} = \nabla(T - \mathrm{U}) \tag{26}$$

The LHS of eq. (26) is the total derivative of the mixture momentum, and the RHS is the gradient of the Lagrangian density per unit volume:

$$\frac{d\vec{P}}{dt} = \nabla \mathcal{L} \tag{27}$$

Accordingly, the momentum density per unit volume conjugated to the tangent velocity $\vec{V}$ is defined as

$$\vec{P} = \frac{\partial T}{\partial \vec{V}} = \frac{\partial \mathcal{L}}{\partial \vec{V}} \tag{28}$$

Finally, we get the field Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{V}}\right) - \nabla \mathcal{L} = 0 \tag{29}$$

It must be noted that in continuum fields, the gradient of the kinetic energy doesn't equal zero,

$$\nabla T \neq 0 \tag{30}$$

Since the partial derivatives of the kinetic energy with respect to the coordinates should be included to consider the momentum exchanges between neighboring particles in continuum fields, in contrast to isolated discrete particle motions, where the gradient of kinetic energy is assumed to be zero.

If we assume the gradient of kinetic energy is zero, eq. (26) becomes the unsteady Euler equation for fluid flow:

$$\frac{\partial \vec{P}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{P} = -\nabla U \tag{31}$$

This is just the classical Euler-Lagrange equation for a discrete isolated particle without considering the interactions between neighboring particles.

If the gradient of kinetic energy is modelled as a symmetric viscous stress tensor using the Stokes hypothesis and ignoring the rotational effect (actually, the gradient of kinetic energy, namely the velocity gradient tensor, contains

stretch/dilatation and shear deformations – captured by the symmetric part, and vorticity/spinning tensor – captured by the antisymmetric part, which represents a rigid-body rotation, causing turbulent flow), eq. (26) becomes the Navier-Stokes equations:

$$\frac{\partial \vec{P}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{P} = -\nabla U + \nabla \cdot \tau \tag{32}$$

For details, please refer to reference [5].

If the compressibility of the fluids caused by the fluid flow cannot be neglected,

$$\vec{P} = (\gamma \rho_0)\vec{V}\ ;\ \ U = \alpha \rho_0 C^2 \tag{33}$$

where, ρ_0 is the mixture density per unit volume when the fluid is at rest relative to an inertial observer ($\vec{V} = 0$), and γ is the volume compression factor caused by the relative velocity to the observer (or called the length contraction effect, namely the Lorentz factor, in the theory of relativity),

$$\gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}};\quad \vec{\beta} = \frac{\vec{V}}{C} \tag{34}$$

α is the reciprocal of the volume compression factor, $\alpha = 1/\gamma$. For detailed derivations please refer to [6].

4. Separated Fluid Flow Fields

It is clear, in general, that two different particles (components) travel in space at different velocities with $\vec{v}^1$ and $\vec{v}^2$, and indeed such relative motion will produce an interaction between two different components (particles). In this case, we shall be concerned with the separated flow model, in contrast to the homogeneous model; see Fig. (2).

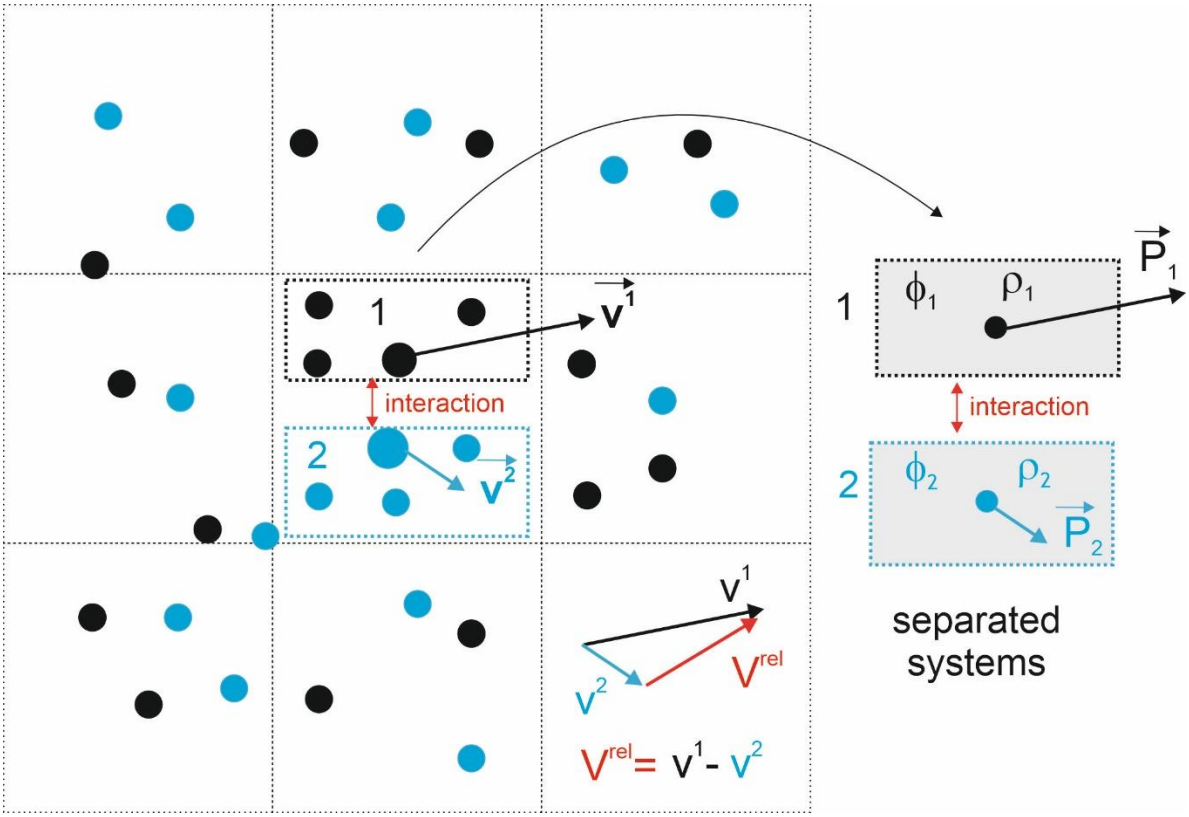


Fig. 2. Separated flow model, each component has its own flow velocity and momentum density.

In a separated multicomponent multiphase flow model, each component is physically distinct from every other component, at every spatial point, each component has its own flow velocity.

Define the velocity difference (slip velocity):

$$\vec{v}^1 - \vec{v}^2 = \vec{v}^{rel} \tag{35}$$

With the slip velocity definition, the mixture momentum density, eq. (4) can be expressed as

$$\begin{cases} \rho\vec{V} = \rho\vec{v}^1 - \overline{\rho}_2\vec{v}^{rel} & \text{(36a)} \\ \rho\vec{V} = \rho\vec{v}^2 + \overline{\rho}_1\vec{v}^{rel} & \text{(36b)} \end{cases}$$

The center-of-mass velocity can be rewritten as

$$\begin{cases} \vec{V} = \vec{v}^1 - \lambda_2\vec{v}^{rel} & \text{(37a)} \\ \vec{V} = \vec{v}^2 + \lambda_1\vec{v}^{rel} & \text{(37b)} \end{cases}$$

where

$$\begin{cases} \lambda_1 = \overline{\rho}_1/\rho \\ \lambda_2 = \overline{\rho}_2/\rho \end{cases} \tag{38}$$

By definition:

$$\lambda_1 + \lambda_2 = 1 \tag{39}$$

In mathematics, differential forms of the same degree constitute a vector space. Consequently, two 1-forms (or two 2-forms) can be added together pointwise, and the resulting sum is likewise a 1-form (or 2-form). Accordingly, the eq. (20) can be split into 2 parts:

$$\left[\left(E_{0i}^{1}+\varepsilon_{ijk}\Omega_{j}^{1}V^{k}\right)+\left(E_{0i}^{2}+\varepsilon_{ijk}\Omega_{j}^{2}V^{k}\right)\right]dx^{i}=0 \tag{40}$$

The superscripts 1 and 2 represent the components 1 and 2.

It can be written more compactly in vector form in 3D space, but it should be noted that it represents the coefficients of the 1-form; mathematically, it lives in the cotangent space, and thus is a covector:

$$\left(\vec{E}_1+\vec{E}_2\right)+\left(\vec{\Omega}_1+\vec{\Omega}_2\right)\times\vec{V}=0 \tag{41}$$

According to the equations (18) and (20), we have the definition

$$\begin{cases}\vec{E}_1=\dfrac{\partial\vec{P}_1}{\partial t}+\nabla\overline{\mathrm{U}}_1\\ \vec{E}_2=\dfrac{\partial\vec{P}_2}{\partial t}+\nabla\overline{\mathrm{U}}_2\\ \vec{E}=\vec{E}_1+\vec{E}_2\end{cases} \tag{42}$$

and the following equations:

$$\begin{cases}\vec{\Omega}_1=\nabla\times\vec{P}_1\\ \vec{\Omega}_2=\nabla\times\vec{P}_2\\ \vec{\Omega}=\vec{\Omega}_1+\vec{\Omega}_2\end{cases} \tag{43}$$

Substituting eq. (37a) and (37b) into eq. (41), with some algebraic manipulation, we get

$$\begin{cases} \vec{E}_1 + \vec{\Omega}_1 \times \vec{v}^1 = -\vec{E}_2 + \vec{v}^1 \times \vec{\Omega}_2 - \lambda_2 \vec{v}^{rel} \times \vec{\Omega} & \text{(44a)} \\ E_2 + \vec{\Omega}_2 \times \vec{v}_2 = -E_1 + \vec{v}_2 \times \vec{\Omega}_1 + \lambda_1 \vec{v}_{rel} \times \vec{\Omega} & \text{(44b)} \end{cases}$$

For each component, we have the following vector calculus identity

$$\begin{cases} \vec{\Omega}_1 \times \vec{v}^1 = (\vec{v}^1 \cdot \nabla)\vec{P}_1 - \nabla \overline{T}_1 & \text{(45a)} \\ \vec{\Omega}_2 \times \vec{v}^2 = (\vec{v}^2 \cdot \nabla)\vec{P}_2 - \nabla \overline{T}_2 & \text{(45b)} \end{cases}$$

Substituting eq. (45) into eq. (44) yields

$$\begin{cases} \vec{E}_1 + (\vec{v}^1 \cdot \nabla)\vec{P}_1 - \nabla \overline{T}_1 = -\vec{E}_2 + \vec{v}^1 \times \vec{\Omega}_2 - \lambda_2 \vec{v}^{rel} \times \vec{\Omega} & \text{(46a)} \\ \vec{E}_2 + (\vec{v}^2 \cdot \nabla)\vec{P}_2 - \nabla \overline{T}_2 = -\vec{E}_1 + \vec{v}^2 \times \vec{\Omega}_1 + \lambda_1 \vec{v}^{rel} \times \vec{\Omega} & \text{(46b)} \end{cases}$$

Here $\overline{T}$ represents the smear kinetic energy per unit volume, e.g. for component 1:

$$\overline{T}_1 = \frac{1}{2}\vec{P}_1 \vec{v}^1 = \frac{1}{2}\overline{\rho}_1 (\vec{v}^1)^2 \quad \text{(47)}$$

the smear Lagrangian density per unit volume for component 1, thus, is defined as:

$$\overline{\mathcal{L}}_1 = \overline{T}_1 - \overline{U}_1 \quad \text{(48)}$$

and the smear momentum density for component 1 is:

$$\vec{P}_1 = \frac{\partial \overline{T}_1}{\partial \vec{v}^1} = \frac{\partial \overline{\mathcal{L}}_1}{\partial \vec{v}^1} \tag{49}$$

If the equations are written as the Euler-Lagrange equation formation, for component 1, it reads

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathcal{L}}_1}{\partial \vec{v}^1}\right) - \nabla \overline{\mathcal{L}}_1 = -\vec{E}_2 + \vec{v}^1 \times \vec{\Omega}_2 - \lambda_2 \vec{v}^{rel} \times \vec{\Omega} \tag{50}$$

Similarly, we can get the equation for component 2 in the Euler-Lagrange formation:

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathcal{L}}_2}{\partial \vec{v}^2}\right) - \nabla \overline{\mathcal{L}}_2 = -\vec{E}_1 + \vec{v}^2 \times \vec{\Omega}_1 + \lambda_1 \vec{v}^{rel} \times \vec{\Omega} \tag{51}$$

The RHS of eq. (50) and (51) represent the interaction forces between components 1 and 2.

Conclusions

The multicomponent multiphase flow can be treated as homogeneous flow field or separated flow fields. In the homogeneous flow model, the mixture will be regarded as a single flow system, neglecting the interactions between components (particles). Starting from the differential 1-form of the mixture momentum density per unit volume (mathematically, it lives in cotangent vector space), taking the exterior derivative of this 1-form gives the field strength differential 2-form. The interior product (contraction) of a differential 2-form with the tangent velocity vector yields a differential 1-form, which is

the dynamic equation for the homogeneous flow. It is equivalent to the Euler-Lagrange equations for continuum fields, but the gradient of kinetic energy should not be equal to zero.

Considering the slip velocities between two components (two particles), the mixture velocity (center-of-mass velocity) can be expressed by this velocity difference and individual phase velocities. The homogeneous dynamic equation splits into two equations. The two equations are tightly coupled to each other, showing very complicated interactions between the two phases.

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