A Simple, Geometric Way to Find the Area of a Circle and the Value of Π

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ABSTRACT

This paper outlines a straightforward, step-by-step geometric method to find the area of a circle. Instead of relying on traditional, complex trigonometry, we initiate the framework using a simple right-angled triangle inscribed within one quarter of a circle. By recursively infilling the remaining curved gaps with smaller tiers of triangles, we completely fill the shape. Tracking the mathematical lengths of these expanding bases reveals a pattern of square roots inside square roots (nested radicals) that leads directly to the value of π.

1. Introduction

For thousands of years, mathematicians have tried to find perfect ways to calculate the area of a circle and understand the transcendental constant π. The ancient Greek mathematician Archimedes did this by placing regular polygons (like a 6-sided hexagon, then a 12-sided polygon) inside and outside a circle to trap the curved edge between increasingly tight straight lines [1]. This paper uses a different, highly organized approach. Instead of changing a whole polygon all at once around the entire circle, we focus completely on just one quadrant (a quarter) of the circle. By building right-angled triangles step-by-step to fill the curved gaps, we create an infinite sum of simple, straight-edged shapes that perfectly matches the true area of the circle.

2. Drawing the Shapes and Finding the First Areas

Imagine a circle with a radius of ***r***, centered at the origin point ***O(0,0)***. We draw two perpendicular lines from the center to the edge, ***OP*** and ***OB***, pointing straight up and straight to the side. This makes a 90° corner (***∠POB = 90°***), creating a perfect quarter circle.

2.1 The Starting Base: Tier 1

We begin by drawing a straight chord connecting point ***P*** to point ***B***. This gives us our very first inscribed right-angled triangle, ***ΔPOB***. Using the elementary formula ***Area = ½ × base × height***, the area of this first triangle is:

$$Area(\Delta POB) = \tfrac{1}{2} \cdot r \cdot r = \tfrac{1}{2}r^2$$

2.2 Splitting the First Gap: Tier 2

The chord ***PB*** leaves a single, large curved gap between the flat line and the circle's round edge. To start filling this, we draw a new line from the origin ***O*** to a point P_1 exactly halfway along the circular arc. This

radius line crosses our first chord at a 90° angle at a crossing point P_2. Using the Pythagorean theorem, we find the length of the first base and the height segment:

$$BP_2 = \tfrac{1}{2}PB = r / \sqrt{2}$$

The leftover segment space from the crossing point P_2 to the outer edge boundary point P_1 gives us the height of our next triangle:

$$P_1P_2 = OP_1 - OP_2 = r - (r / \sqrt{2}) = r(\sqrt{2} - 1) / \sqrt{2}$$

By drawing lines from P_1 to P and from P_1 to B, we form a new second-tier triangle (ΔPP_1B) sitting right on top of our first chord. Its area evaluates cleanly to:

$$Area(\Delta PP_1B) = r^2(\sqrt{2} - 1) / 2$$

2.3 Splitting the Remaining Gaps: Tier 3

Adding that second triangle leaves behind two brand-new, identical curved gaps on either side. To fill the gap on the right side, we introduce another radius line OP_3 that crosses through the chord line P_1B at a point we call P_5. By using the Pythagorean theorem step-by-step on these smaller interior triangles, we calculate the exact lengths. The new base line P_1B and the small outer height segment P_3P_5 are given by:

$$P_1B = r\sqrt{(2 - \sqrt{2})}$$

$$P_3P_5 = r(2\sqrt{2} - \sqrt{(2 + \sqrt{2})}) / 2\sqrt{2}$$

Now we can calculate the total combined area of these smaller third-tier triangles:

$$Area(Tier\ 3) = r^2(2\sqrt{2} - \sqrt{(2 + \sqrt{2})})\sqrt{(2 - \sqrt{2})} / 4\sqrt{2}$$

3. Finding the Pattern for the Entire Circle

If we continue this process forever, we can write a general rule for the geometric elements at any step(n).

3.1 The Parts of Step n

1. **Number of Triangles (t_n):** Every time we add a triangle, it splits a gap and creates two smaller ones. This means the number of triangles doubles with every new layer. For any step after the first two, the number of triangles in that layer is:

   $$t_n = 2^{n-2}$$

2. **The Base Length (b_n):** Because we keep cutting the radial angle exactly in half, the base of the new triangles follows a beautifully structured, repeating square-root pattern [2]:

   $$b_n = r\sqrt{(2 - \sqrt{(2 + \sqrt{(2 + \cdots + \sqrt{2})})})}$$

(where exactly n − 2 square roots are nested inside each other)

3.2 Adding All the Pieces Together

The total geometric area of our quarter circle is obtained by summing the areas of all these infinite layers of triangles together:

$$Area_{Quarter\ Circle} = A_1 + \sum_{n=2}^{\infty} 2^{n-2} \cdot A_n = \pi r^2 / 4$$

Since a full circle is perfectly symmetrical and made of four identical quarters, we multiply our entire triangular progression by 4 to cover all 360° of the circle:

$$Area_{Circle} = 4A_1 + \sum_{n=2}^{\infty} 2^n \cdot A_n = \pi r^2$$

4. Formal Generalization of the Series Terms

To transition this framework into a rigorous analytic format, we construct a compact closed-form representation of the recursive series. Let the radius equal unity (***r = 1***). We recursively define the nested radical term string component, R_k, containing exactly ***k*** radicals:

$$R_0 = 0, \quad R_1 = \sqrt{2}, \quad R_2 = \sqrt{(2 + \sqrt{2})}, \quad \ldots \quad R_k = \sqrt{(2 + R_{k-1})}$$

Under this convention, the general base length b_n and the geometric height segment h_n for any arbitrary structural tier $n \geq 2$ evaluate precisely to:

$$b_n = \sqrt{(2 - R_{n-2})}$$

$$h_n = 1 - \sqrt{(1 - (b_n/2)^2)} = 1 - \sqrt{(1 - \tfrac{1}{4}(2 - R_{n-2}))}$$

Substituting these component parameters directly into the infinite summation series, the final isolated closed-form structural expansion for the value of π is generalized as follows:

$$\pi = 2 + \sum_{n=2}^{\infty} 2^{n-1} \cdot \sqrt{(2 - R_{n-2})} \cdot [1 - \sqrt{(1 - \tfrac{1}{4}(2 - R_{n-2}))}]$$

5. Conclusion

This paper demonstrates that right-angled triangles built inside a quarter circle using the Pythagorean theorem recursively can define a clear, self-contained mathematical series using nested square roots. By bridging the gap between discrete straight lines and continuous curves, this method serves as an excellent, highly visual alternative to polygon boundaries [3] for learning, teaching, and studying the relationship between lines and arcs.

References

[1] T. L. Heath, *The Works of Archimedes*, Cambridge: Cambridge University Press, 1897.

[2] E. W. Weisstein, "Method of Exhaustion," from *MathWorld--A Wolfram Web Resource* [Online].

[3] P. Beckmann, *A History of Pi*, New York: St. Martin's Press, 1971.