# A note on “Power series associated with reciprocals of generalized central binomial coefficients”, Integers 26 (2026), #A2

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## Abstract

In this note, we analyze a binomial series established by R. Zurita in 2026

Keywords: binomial series, hypergeometric functions

# Introduction

In reference [9, p.19, Eq.(d)], appears the following formula:

$$S=\sum_{k=3}^{\infty}\frac{(-1)^k}{(k-2)\binom{4k}{2k}}=-\frac{219}{4900}-\frac{\sqrt{34}}{595}\left(\left(2\sqrt{\sqrt{17}+1}-7\sqrt{\sqrt{17}-1}\right)\sin^{-1}\left(\frac{\sqrt{5-2\sqrt{2}}-\sqrt{5+2\sqrt{2}}}{4}\right)-\right.$$
$$\left.\left(2\sqrt{\sqrt{17}-1}+7\sqrt{\sqrt{17}+1}\right)\ln\left(\frac{\sqrt{2\left(\sqrt{17}-3\right)}+\sqrt{5-2\sqrt{2}}+\sqrt{5+2\sqrt{2}}}{4}\right)\right)$$

The original Zurita formula contained a minor misprint which has been corrected here.
This series also appears in [1,5,7,8].
Notation: The generalized hypergeometric function is defined as

$${}_pF_q\left(\{a_1,\ \ldots,\ a_p\},\{b_1,\ \ldots,\ b_q\},z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{z^k}{k!}\quad,\ |z|<1$$

See [6, chapter 16].
The Gauss hypergeometric function: ${}_2F_1\left(\{a,\ b\},\{c\},z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}\quad,\ |z|<1$
See [6, chapter 15].

In this note, we provide several alternative representations for the sum S.

# Some Remarks on Zurita’s Formula

**Entry 1.**

$$S=\sum_{k=3}^{\infty}\frac{(-1)^k}{(k-2)\binom{4k}{2k}}=-\frac{1}{924}\,{}_4F_3\left(\left\{1,\ 1,\ \frac{7}{2},\ 4\right\},\left\{2,\ \frac{13}{4},\ \frac{15}{4}\right\},-\frac{1}{16}\right)$$

$$S =$$

$$\sum_{k=3}^{\infty} \frac{(-1)^k}{(k-2)\binom{4k}{2k}} = -\frac{219}{4900} - \frac{\sqrt{34}}{595}\left(\sqrt{53\sqrt{17}-157}\,\tan^{-1}\left(\sqrt{\sqrt{17}-4}\right) - \sqrt{53\sqrt{17}+157}\,\ln\left(\frac{\sqrt{1+\sqrt{17}+\sqrt{2\left(1+\sqrt{17}\right)}}}{2}\right)\right)$$

$$S = \sum_{k=3}^{\infty} \frac{(-1)^k}{(k-2)\binom{4k}{2k}} = -\frac{219}{4900} + \left(\frac{9}{70}+\frac{i}{14}\right)\sqrt{\frac{1}{34}+\frac{2i}{17}}\; {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},-\frac{i}{4}\right) + \left(\frac{9}{70}-\frac{i}{14}\right)\sqrt{\frac{1}{34}-\frac{2i}{17}}\; {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},\frac{i}{4}\right)$$

$$S = \sum_{k=3}^{\infty} \frac{(-1)^k}{(k-2)\binom{4k}{2k}} = -\frac{219}{4900} + \frac{1}{\sqrt{17}}\left(\frac{9}{140}+\frac{i}{28}\right)\left(\sqrt{\sqrt{17}+1} + i\sqrt{\sqrt{17}-1}\right) {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},-\frac{i}{4}\right) +$$

$$\frac{1}{\sqrt{17}}\left(\frac{9}{140}-\frac{i}{28}\right)\left(\sqrt{\sqrt{17}+1} - i\sqrt{\sqrt{17}-1}\right) {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},\frac{i}{4}\right)$$

$$-S = \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{(k-2)\binom{4k}{2k}} =$$

$$\frac{1}{924}\; {}_3F_2\left(\left\{1,\frac{7}{2},4\right\},\left\{\frac{13}{4},\frac{15}{4}\right\},-\frac{1}{16}\right) + \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{(k-2)(k-1)\binom{4k+4}{2k+2}}\; {}_3F_2\left(\left\{1,k+\frac{3}{2},k+2\right\},\left\{k+\frac{5}{4},k+\frac{7}{4}\right\},-\frac{1}{16}\right)$$

$$-S = \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{(k-2)\binom{4k}{2k}} =$$

$$\frac{1}{924}\; {}_3F_2\left(\left\{1,\frac{7}{2},4\right\},\left\{\frac{13}{4},\frac{15}{4}\right\},\frac{1}{16}\right) - \sum_{k=3}^{\infty} \frac{(-1)^{k-1}(2k-3)}{(k-2)(k-1)\binom{4k+4}{2k+2}}\; {}_3F_2\left(\left\{1,k+\frac{3}{2},k+2\right\},\left\{k+\frac{5}{4},k+\frac{7}{4}\right\},\frac{1}{16}\right)$$

**Entry 2.**

$$\sin^{-1}\left(\frac{\sqrt{5+2\sqrt{2}} - \sqrt{5-2\sqrt{2}}}{4}\right) = \tan^{-1}\left(\sqrt{\sqrt{17}-4}\right)$$

$$\sin^{-1}\left(\frac{\sqrt{5+2\sqrt{2}} - \sqrt{5-2\sqrt{2}}}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2\left(5-\sqrt{17}\right)}}{4}\right)$$

$$\sin^{-1}\left(\frac{\sqrt{5+2\sqrt{2}} - \sqrt{5-2\sqrt{2}}}{4}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3+\sqrt{17}}{2}}\right)$$

$$\ln\left(\frac{\sqrt{2\left(\sqrt{17}-3\right)} + \sqrt{5-2\sqrt{2}} + \sqrt{5+2\sqrt{2}}}{4}\right) =$$

$$\frac{i}{4}\left(\sqrt{5+2\sqrt{2}} - \sqrt{5-2\sqrt{2}}\right) {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},\frac{5-\sqrt{17}}{8}\right) + \left(\frac{1-i}{2\sqrt{2}}\right) {}_2F_1\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},\frac{i}{4}\right)$$

$$\ln\left(\frac{\sqrt{2\left(\sqrt{17}-3\right)} + \sqrt{5-2\sqrt{2}} + \sqrt{5+2\sqrt{2}}}{4}\right) = \frac{1}{2}\ln\left(1+\frac{\sqrt{17}-3}{8}\right) + \ln\left(1+\sqrt{\sqrt{17}-4}\right)$$

$$\ln\left(\frac{\sqrt{2\left(\sqrt{17}-3\right)} + \sqrt{5-2\sqrt{2}} + \sqrt{5+2\sqrt{2}}}{4}\right) =$$

$$\left(\frac{2\sqrt{17}-7}{19}\right) {}_2F_1\left(\left\{\frac{1}{2},1\right\},\left\{\frac{3}{2}\right\},\left(\frac{2\sqrt{17}-7}{19}\right)^2\right)+\left(\frac{2\sqrt{\sqrt{17}-4}}{2+\sqrt{\sqrt{17}-4}}\right) {}_2F_1\left(\left\{\frac{1}{2},1\right\},\left\{\frac{3}{2}\right\},\frac{\sqrt{17}-4}{\left(2+\sqrt{\sqrt{17}-4}\right)^2}\right)$$

**Entry 3.**

$${}_4F_3\left(\left\{1,1,\frac{7}{2},4\right\},\left\{2,\frac{13}{4},\frac{15}{4}\right\},-\frac{1}{16}\right)=$$

$$\frac{7227}{175}+A\,\pi+B\tan^{-1}\left(\frac{\sqrt{(5+2\sqrt{5})(\sqrt{17}-4)}-1}{\sqrt{5+2\sqrt{5}}+\sqrt{\sqrt{17}-4}}\right)-C\ln\left(\frac{\sqrt{2(\sqrt{17}-3)}+\sqrt{5-2\sqrt{2}}+\sqrt{5+2\sqrt{2}}}{4}\right)$$

where

$$A=\frac{66}{25}\sqrt{\frac{2}{17}}\left(7\sqrt{\sqrt{17}-1}-2\sqrt{\sqrt{17}+1}\right)$$

$$B=\frac{132}{5}\sqrt{\frac{2}{17}}\left(7\sqrt{\sqrt{17}-1}-2\sqrt{\sqrt{17}+1}\right)$$

$$C=\frac{132}{5}\sqrt{\frac{2}{17}}\left(7\sqrt{\sqrt{17}+1}+2\sqrt{\sqrt{17}-1}\right)$$

# Other Formulas

**Entry 4.**

$${}_5F_4\left(\left\{1,1,\frac{7}{3},\frac{8}{3},3\right\},\left\{2,\frac{13}{6},\frac{5}{2},\frac{17}{6}\right\},\frac{1}{64}\right)=$$

$$\frac{8547}{50}-A\,\pi+B\tan^{-1}\left(\frac{2\sqrt{3}-3-\sqrt{2\sqrt{21}-9}}{\sqrt{3}+(2-\sqrt{3})\sqrt{2\sqrt{21}-9}}\right)+C\ln\left(\frac{\sqrt{3}+\sqrt{7}+\sqrt{2\sqrt{21}-3}}{4}\right)$$

where

$$A=\frac{539}{5\sqrt{3}}+\frac{11}{5}\sqrt{\frac{7(\sqrt{21}-3)}{6}}\;,\;B=\frac{22}{5}\sqrt{42(\sqrt{21}-3)}\;,\;C=\frac{22}{5}\sqrt{42(\sqrt{21}+3)}$$

**Entry 5.**

$${}_4F_3\left(\left\{1,1,\frac{4}{3},\frac{5}{3}\right\},\left\{\frac{7}{6},\frac{3}{2},\frac{11}{6}\right\},-\frac{1}{512}\right)=$$

$$A\,\pi+B\tan^{-1}\left(\frac{7\sqrt{5-2\sqrt{5}}-\sqrt{9+2\sqrt{57}}}{7+\sqrt{(5-2\sqrt{5})(9+2\sqrt{57})}}\right)+C\ln(2)+D\ln\left(13+\sqrt{57}+\sqrt{26\sqrt{57}-30}\right)$$

where

$$A=-\frac{32}{57}\sqrt{114\sqrt{57}+342}$$

$$B=\frac{160}{57}\sqrt{114\sqrt{57}+342}$$

$$C=-160\left(-\frac{1}{3}+\frac{2}{57}\sqrt{114\sqrt{57}-342}\right)$$

$$D=\frac{80}{57}\sqrt{114\sqrt{57}-342}$$

**Entry 6.**

$$ {}_3F_2\left(\left\{1,\,1,\,\frac{3}{2}\right\},\,\left\{\frac{4}{3},\,\frac{5}{3}\right\},\,\frac{1}{5}\right) = \frac{5\sqrt{15}}{36}\,\pi - \frac{10\sqrt{15}}{9}\,\tan^{-1}\left(\frac{1-\left(1+\sqrt{2}\right)\sqrt{3}\,\left(\sqrt{5}-2\right)}{1+\sqrt{2}-2\sqrt{3}+\sqrt{15}}\right) - \frac{5}{9}\,\ln\left(\frac{5}{2}\right) $$

**Entry 7.**

$$ \sum_{k=2}^{\infty}\frac{(4/15)^k}{(k-1)\binom{2k}{k}} = \frac{2}{15} - \frac{13\pi}{90\sqrt{14}} + \frac{13}{15}\sqrt{\frac{2}{7}}\,\tan^{-1}\left(\frac{28-14\sqrt{3}-\sqrt{14}}{14+2\sqrt{14}-\sqrt{42}}\right) $$

$$ {}_2F_1\left(\{2,\,2\},\,\left\{\frac{7}{2}\right\},\,\frac{1}{15}\right) = -\frac{10\,125}{4} + \frac{48\,375\,\pi}{16\sqrt{14}} - \frac{145\,125}{4\sqrt{14}}\,\tan^{-1}\left(\frac{28-14\sqrt{3}-\sqrt{14}}{14+2\sqrt{14}-\sqrt{42}}\right) $$

**Entry 8.**

$$ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\binom{3k}{k}} = \ln(2) - \frac{\ln(2-2a)}{3-2a} - \frac{a}{(1+a)^2+a}\,\ln\left(8+4a+4a^2\right) - \frac{\pi\sqrt{4+3a^2}}{2\left(13+6a+4a^2\right)} + \frac{2\sqrt{4+3a^2}}{\left(13+6a+4a^2\right)}\,\tan^{-1}\left(\frac{2+a-\sqrt{4+3a^2}}{2+a+\sqrt{4+3a^2}}\right) $$

where

$$ a = \left(\frac{2}{3\left(\sqrt{93}-9\right)}\right)^{1/3} - \frac{1}{3^{2/3}}\left(\frac{\sqrt{93}-9}{2}\right)^{1/3} $$

$$ a^3 + a - 1 = 0 $$

Remark:

$$ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\binom{3k}{k}} = \frac{1}{3}\,{}_3F_2\left(\left\{1,\,1,\,\frac{3}{2}\right\},\,\left\{\frac{4}{3},\,\frac{5}{3}\right\},\,-\frac{4}{27}\right) $$

**Entry 9.**

$$ \sum_{k=3}^{\infty}\frac{1}{(k-2)\,k\binom{2k}{k}} = \frac{17}{36} - \frac{\pi}{4\sqrt{3}} $$

$$ \sum_{k=3}^{\infty}\frac{k-1}{(k-2)\,k\binom{2k}{k}} = \frac{\pi}{12\sqrt{3}} - \frac{1}{9} $$

$$ \sum_{k=3}^{\infty}\frac{17k-13}{(k-2)\,k\binom{2k}{k}} = \frac{5\pi}{12\sqrt{3}} $$

$$ \sum_{k=3}^{\infty}\frac{3k-4}{(k-2)^2\binom{2k}{k}} = \frac{11}{108} + \frac{\pi^2}{54} $$

# Final Remarks

**Entry 10.**

$$ S = \sum_{k=3}^{\infty}\frac{(-1)^k}{(k-2)\binom{4k}{2k}} = $$

$$ \frac{1}{24}\left(\frac{21\,493}{156\,800} - \frac{41\pi}{1120\sqrt{2}} - \frac{2\ln(2)}{35} - \frac{253\ln(2)}{8960\sqrt{2}} + \frac{253\ln\left(2-\sqrt{2}\right)}{4480\sqrt{2}}\right) + \frac{1}{512}\sum_{k=1}^{\infty}\left(\frac{1}{\binom{4k}{2k}} - \frac{1}{\binom{4k+4}{2k+2}}\right)\left(\frac{(-1)^{k-1}\,(135+80k)}{(5+4k)\,(7+4k)}\right) + $$

$$\frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{1}{\binom{4k}{2k}}-\frac{1}{\binom{4k+4}{2k+2}}\right)(-1)^n\left(\frac{2}{35}\Phi(-1,\,1,\,1+k)+\frac{83}{2560}\Phi\left(-1,\,1,\,\frac{9}{4}+k\right)+\frac{15}{3584}\Phi\left(-1,\,1,\,\frac{11}{4}+k\right)\right)$$

where

$$\Phi(z,\,s,\,v)=\sum_{k=0}^{\infty}\frac{z^k}{(v+k)^s}\ ,\ |z|<1$$

$$\Phi(z,\,1,\,v)=\frac{1}{v}\,{}_2F_1\,(\{1,\,v\},\,\{1+v\},\,z)$$

$$\Phi(-1,\,1,\,v)=\frac{1}{v}\,{}_2F_1\,(\{1,\,v\},\,\{1+v\},\,-1)=\frac{1}{2\,v}\,{}_2F_1\left(\{1,\,1\},\,\{1+v\},\,\frac{1}{2}\right)=\frac{1}{v\,2^v}\,{}_2F_1\left(\{v,\,v\},\,\{1+v\},\,\frac{1}{2}\right)$$

$$\Phi(-1,\,1,\,1+k)=(-1)^k\ln(2)+(-1)^{k-1}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{m}\ ,\ k=1,\,2,\,3,\,\ldots$$

$$\Phi\left(-1,\,1,\,\frac{9}{4}+k\right)=(-1)^k\left(-\frac{16}{5}+\frac{\pi}{\sqrt{2}}+\sqrt{2}\,\ln\left(\sqrt{2}+1\right)\right)+(-1)^{k-1}\,4\sum_{m=1}^{k}\frac{(-1)^{m-1}}{4\,m+5}\ ,\ k=1,\,2,\,3,\,\ldots$$

$$\Phi\left(-1,\,1,\,\frac{11}{4}+k\right)=(-1)^k\left(-\frac{16}{21}+\frac{\pi}{\sqrt{2}}+\sqrt{2}\,\ln\left(\sqrt{2}-1\right)\right)+(-1)^{k-1}\,4\sum_{m=1}^{k}\frac{(-1)^{m-1}}{4\,m+7}\ ,\ k=1,\,2,\,3,\,\ldots$$

$\Phi(z,\,s,\,v)$ is the Lerch's Transcendent. See [6, chapter 25, p.612, Eq.(25.14.1)].

**Entry 11.**

$$S=\sum_{k=3}^{\infty}\frac{(-1)^k}{(k-2)\binom{4k}{2k}}=\frac{4\ln(2)}{35}+\frac{2\sqrt{5}}{25}\ln(2)\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{\pi\sqrt{3}}{27}\ln(2)-\sum_{n=3}^{\infty}\sum_{k=n+1}^{\infty}\frac{(-1)^n}{(n-2)\binom{4k}{2k}}-\sum_{n=3}^{\infty}\sum_{k=n+1}^{\infty}\frac{(-1)^k}{(k-2)\binom{4n}{2n}}$$

$$S=\sum_{k=3}^{\infty}\frac{(-1)^k}{(k-2)\binom{4k}{2k}}=\frac{4\ln(2)}{35}+\frac{2\sqrt{5}}{25}\ln(2)\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{\pi\sqrt{3}}{27}\ln(2)-\sum_{n=4}^{\infty}\frac{(-1)^n}{n-2}\sum_{k=3}^{n-1}\frac{1}{\binom{4k}{2k}}-\sum_{n=4}^{\infty}\frac{1}{\binom{4n}{2n}}\sum_{k=3}^{n-1}\frac{(-1)^k}{k-2}$$

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