On Scalar Waves

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Abstract

Claims of "scalar", "longitudinal" or "Tesla" electromagnetic waves often combine three distinct phenomena: longitudinal electric-field components in anisotropic media, superluminal phase or group velocities, and an additional propagating scalar degree of freedom. We separate these questions and derive the relevant results from first principles. First, source-free Maxwell electrodynamics with tensor permittivity and permeability is reduced to its plane-wave eigenvalue problem. The Gauss constraint is shown to make the displacement field, rather than the electric field, transverse to the wave vector; an explicit anisotropic example consequently has a predominantly longitudinal electric field and a phase velocity exceeding the vacuum light speed. The Poynting theorem is then derived for a dispersive anisotropic medium, leading to the Brillouin energy density and energy velocity and showing why a superluminal phase velocity does not imply superluminal energy or signal propagation. Second, a genuine scalar degree of freedom is introduced through a Lorentz- and gauge-invariant dilaton-like action. All Euler–Lagrange equations are derived explicitly. Gauge, translation and Lorentz symmetries are treated with Noether's theorem, yielding charge, energy–momentum and angular-momentum/boost conservation laws. Linearisation about a static electric background gives the full mixed scalar–electromagnetic dispersion relation, its nearly massless distinguished limit, and a scalar-associated branch which is purely longitudinal electrically and has zero magnetic perturbation for propagation parallel to the background field. The same action gives a finite-electrode voltage-dependent capacitance; this prediction is derived using the Green function of the scalar equation. Existing scalar–photon and resonator searches already constrain the coupling strongly, making large effects in the minimal unscreened theory implausible. We finally formulate direct static and dynamical experimental tests of the remaining longitudinal mode.

1 Longitudinal fields from Maxwell electrodynamics in anisotropic media

1.1 Plane-wave reduction of Maxwell's equations

We begin without introducing any new field. In a source-free, homogeneous linear medium,

$$\nabla\cdot\mathbf{D}=0,\qquad\qquad \nabla\cdot\mathbf{B}=0, \tag{1}$$
$$\nabla\times\mathbf{E}=-\partial_t\mathbf{B},\qquad\qquad \nabla\times\mathbf{H}=\partial_t\mathbf{D}, \tag{2}$$

and the most general local linear constitutive law without magnetoelectric coupling is

$$\mathbf{D}=\varepsilon_0\boldsymbol{\varepsilon}_r\mathbf{E},\qquad \mathbf{B}=\mu_0\boldsymbol{\mu}_r\mathbf{H}. \tag{3}$$

Here $\boldsymbol{\varepsilon}_r$ and $\boldsymbol{\mu}_r$ are tensors. We first take them to be constant at a chosen frequency and return to material dispersion below.

Let

$$(\mathbf{E},\mathbf{B},\mathbf{D},\mathbf{H})=(\mathbf{E}_0,\mathbf{B}_0,\mathbf{D}_0,\mathbf{H}_0)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{4}$$

Then

$$\nabla \mapsto i\mathbf{k}, \qquad \partial_t \mapsto -i\omega.$$

Faraday's law gives

$$i\mathbf{k} \times \mathbf{E}_0 = i\omega \mathbf{B}_0 \quad \Longrightarrow \quad \mathbf{B}_0 = \frac{1}{\omega}\mathbf{k} \times \mathbf{E}_0. \tag{5}$$

Using $\mathbf{H}_0 = \mu_0^{-1}\boldsymbol{\mu}_r^{-1}\mathbf{B}_0$, Ampere's law becomes

$$i\mathbf{k} \times \left[\frac{1}{\mu_0\omega}\boldsymbol{\mu}_r^{-1}(\mathbf{k} \times \mathbf{E}_0)\right] = -i\omega\varepsilon_0\boldsymbol{\varepsilon}_r\mathbf{E}_0.$$

Multiplication by $\mu_0\omega/i$, together with $c_0^{-2} = \varepsilon_0\mu_0$, yields

$$\boxed{\mathbf{k} \times \left[\boldsymbol{\mu}_r^{-1}(\mathbf{k} \times \mathbf{E}_0)\right] + \frac{\omega^2}{c_0^2}\boldsymbol{\varepsilon}_r\mathbf{E}_0 = 0.} \tag{6}$$

This is the electromagnetic eigenvalue problem.

Gauss' law independently gives

$$\boxed{\mathbf{k} \cdot \boldsymbol{\varepsilon}_r\mathbf{E}_0 = 0.} \tag{7}$$

In an isotropic medium $\boldsymbol{\varepsilon}_r = \varepsilon_r I$, so $\mathbf{k} \cdot \mathbf{E}_0 = 0$. In an anisotropic medium, however, (7) states only that $\mathbf{D}_0$ is transverse:

$$\mathbf{k} \cdot \mathbf{D}_0 = 0.$$

There is no requirement that $\mathbf{E}_0$ itself be transverse. This is the first, entirely conventional, mechanism by which a longitudinal electric component arises.

1.2 Explicit two-dimensional TM dispersion relation

Take the principal axes of the material as Cartesian axes and set

$$\boldsymbol{\varepsilon}_r = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \qquad \boldsymbol{\mu}_r = \mathrm{diag}(\mu_1, \mu_2, \mu_3). \tag{8}$$

Consider a TM wave in the x-y plane,

$$\mathbf{k} = (k_x, k_y, 0), \qquad \mathbf{E}_0 = (E_x, E_y, 0), \qquad \mathbf{H}_0 = (0, 0, H_z). \tag{9}$$

From $\mathbf{k} \times \mathbf{H}_0 = -\omega\mathbf{D}_0$,

$$(k_yH_z, -k_xH_z, 0) = -\omega\varepsilon_0(\varepsilon_1E_x, \varepsilon_2E_y, 0),$$

so

$$E_x = -\frac{k_yH_z}{\omega\varepsilon_0\varepsilon_1}, \qquad E_y = \frac{k_xH_z}{\omega\varepsilon_0\varepsilon_2}. \tag{10}$$

Faraday's law gives

$$k_xE_y - k_yE_x = \omega B_z = \omega\mu_0\mu_3H_z.$$

Substitution of (10) gives

$$\frac{H_z}{\omega\varepsilon_0}\left(\frac{k_x^2}{\varepsilon_2} + \frac{k_y^2}{\varepsilon_1}\right) = \omega\mu_0\mu_3H_z.$$

For a nontrivial mode,

$$\boxed{\frac{k_x^2}{\varepsilon_2} + \frac{k_y^2}{\varepsilon_1} = \frac{\omega^2}{c_0^2}\mu_3.} \tag{11}$$

Writing

$$k_x = k\cos\theta, \qquad k_y = k\sin\theta,$$

we obtain

$$k^2 A(\theta) = \frac{\omega^2}{c_0^2}\mu_3, \qquad A(\theta) = \frac{\cos^2\theta}{\varepsilon_2} + \frac{\sin^2\theta}{\varepsilon_1}, \tag{12}$$

and therefore

$$\boxed{v_p = \frac{\omega}{k} = c_0\sqrt{\frac{A(\theta)}{\mu_3}}.} \tag{13}$$

The longitudinal and transverse unit vectors are

$$\hat{\mathbf{k}} = (\cos\theta, \sin\theta, 0), \qquad \hat{\mathbf{t}} = (-\sin\theta, \cos\theta, 0).$$

Using (10),

$$\begin{aligned} E_L &= \mathbf{E}_0 \cdot \hat{\mathbf{k}} \\ &= \frac{kH_z \sin\theta\cos\theta}{\omega\varepsilon_0}\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} E_T &= \mathbf{E}_0 \cdot \hat{\mathbf{t}} \\ &= \frac{kH_z}{\omega\varepsilon_0}\left(\frac{\sin^2\theta}{\varepsilon_1} + \frac{\cos^2\theta}{\varepsilon_2}\right). \end{aligned} \tag{15}$$

Hence

$$\boxed{\frac{E_L}{E_T} = \frac{B(\theta)}{A(\theta)}, \qquad B(\theta) = \sin\theta\cos\theta\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right).} \tag{16}$$

The fraction of the electric amplitude in the longitudinal direction is

$$L_E = \frac{|E_L|}{\sqrt{|E_L|^2 + |E_T|^2}} = \frac{|B/A|}{\sqrt{1 + |B/A|^2}}. \tag{17}$$

For the illustrative effective-medium parameters

$$\varepsilon_1 = 0.01, \qquad \varepsilon_2 = 1, \qquad \mu_3 = 1, \qquad \tan\theta = 0.1,$$

we have

$$\sin^2\theta = \frac{0.01}{1.01}, \qquad \cos^2\theta = \frac{1}{1.01}, \qquad \sin\theta\cos\theta = \frac{0.1}{1.01}.$$

Therefore

$$A = \frac{1}{1.01} + \frac{100(0.01)}{1.01} = \frac{2}{1.01} = 1.980198\ldots,$$

and

$$B = \frac{0.1}{1.01}(1 - 100) = -9.80198\ldots.$$

Thus

$$\frac{E_L}{E_T} = -4.95, \qquad L_E = \frac{4.95}{\sqrt{1 + 4.95^2}} \simeq 0.980, \tag{18}$$

while

$$\frac{v_p}{c_0} = \sqrt{1.980198\ldots} = 1.407\ldots. \tag{19}$$

A predominantly longitudinal electric field and a superluminal *phase* velocity have therefore arisen without altering Maxwell's equations.

1.3 Group velocity of the nondispersive effective model

It is useful to see explicitly why the constant-tensor approximation can be misleading. From (11),

$$\omega(\mathbf{k}) = \frac{c_0}{\sqrt{\mu_3}}\left(\frac{k_x^2}{\varepsilon_2} + \frac{k_y^2}{\varepsilon_1}\right)^{1/2}. \tag{20}$$

If the constitutive parameters are incorrectly treated as frequency independent, differentiation gives

$$\mathbf{v}_g = \nabla_{\mathbf{k}}\omega = \frac{c_0^2}{\mu_3\omega}\left(\frac{k_x}{\varepsilon_2}, \frac{k_y}{\varepsilon_1}, 0\right). \tag{21}$$

For the numerical example,

$$\frac{|\mathbf{v}_g|}{c_0} = \frac{1}{\sqrt{\mu_3 A}}\sqrt{\frac{\cos^2\theta}{\varepsilon_2^2} + \frac{\sin^2\theta}{\varepsilon_1^2}} \simeq 7.11.$$

This number is not a prediction of a causal physical material. Extreme effective values such as $\varepsilon_1 = 0.01$ necessarily occur in frequency-dependent media. The derivative used in (21) has omitted precisely the frequency derivatives that control stored material energy and causal wave-packet transport. Spatially dispersive metamaterials can also support genuinely longitudinal electric modes, including ENZ/wire-media modes, so the distinction between field geometry and signal velocity is essential [15, 16].

1.4 Poynting theorem and energy density

For clarity we derive the energy balance. Dot Ampere's law with $\mathbf{E}$ and Faraday's law with $\mathbf{H}$:

$$\mathbf{E}\cdot(\nabla\times\mathbf{H}) = \mathbf{E}\cdot\partial_t\mathbf{D}, \tag{22}$$

$$\mathbf{H}\cdot(\nabla\times\mathbf{E}) = -\mathbf{H}\cdot\partial_t\mathbf{B}. \tag{23}$$

Using

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}),$$

we obtain the exact identity

$$\boxed{\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\partial_t\mathbf{D} + \mathbf{H}\cdot\partial_t\mathbf{B} = 0.} \tag{24}$$

For a nondispersive lossless medium with symmetric constant tensors,

$$\mathbf{E}\cdot\partial_t\mathbf{D} = \partial_t\left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D}\right), \qquad \mathbf{H}\cdot\partial_t\mathbf{B} = \partial_t\left(\frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right),$$

so

$$\partial_t W + \nabla\cdot\mathbf{S} = 0, \qquad W = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{25}$$

A dispersive medium has temporal memory, so W cannot be obtained by simply inserting $\boldsymbol{\varepsilon}(\omega)$ into the nondispersive expression. For a narrow-band field centred on ω, expand the constitutive response to first order about the carrier frequency. The standard cycle-average calculation then gives the Brillouin energy density

$$\boxed{\langle W\rangle = \frac{1}{4}\mathbf{E}_0^*\cdot\frac{\partial(\omega\varepsilon_0\boldsymbol{\varepsilon}_r)}{\partial\omega}\mathbf{E}_0 + \frac{1}{4}\mathbf{H}_0^*\cdot\frac{\partial(\omega\mu_0\boldsymbol{\mu}_r)}{\partial\omega}\mathbf{H}_0,} \tag{26}$$

and

$$\boxed{\langle\mathbf{S}\rangle = \frac{1}{2}\,\mathrm{Re}(\mathbf{E}_0\times\mathbf{H}_0^*).} \tag{27}$$

Consequently the energy-transport velocity is

$$\boxed{\mathbf{v}_E = \frac{\langle \mathbf{S} \rangle}{\langle W \rangle}.} \tag{28}$$

For a lossless local dispersive eigenmode this agrees with the group velocity when all frequency derivatives of the constitutive tensors are retained; more general spatially dispersive forms require the corresponding additional material power-flow terms [14].

A simple causal principal-axis model is

$$\varepsilon_j(\omega) = \varepsilon_{\infty j} + \frac{F_j \omega_{0j}^2}{\omega_{0j}^2 - \omega^2}, \qquad F_j > 0, \tag{29}$$

away from loss and resonance. Then

$$\frac{d(\omega \varepsilon_j)}{d\omega} = \varepsilon_j + \frac{2F_j \omega^2 \omega_{0j}^2}{(\omega_{0j}^2 - \omega^2)^2}, \tag{30}$$

so the stored material contribution is positive on a passive lossless branch. It appears in the denominator of (28); the constant-tensor calculation omits it. Detailed analyses of causal hyperbolic metal–dielectric metamaterials find subluminal energy/group transport once this dispersion is included [6]. Thus

$$v_p > c_0$$

is compatible with

$$v_E < c_0,$$

and does not imply superluminal information transfer.

This entire first mechanism is experimentally accessible with anisotropic, ENZ, hyperbolic and wire metamaterials. Longitudinal electric fields in such systems have been treated theoretically and experimentally [15–17]. A measurement of E_L, by itself, is therefore not evidence for an additional fundamental scalar field.

2 A Lorentz- and gauge-invariant scalar extension of electromagnetism

2.1 Construction of the action

We now ask a different question: can one add a genuine propagating scalar degree of freedom while retaining Lorentz invariance, electromagnetic gauge invariance and a healthy second-order field theory?

Take Minkowski signature $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The action is

$$\boxed{S = \int d^4x\, \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4} f(\phi) F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - J_\mu A^\mu,} \tag{31}$$

with

$$\boxed{f(\phi) = e^{2\phi/M} > 0.} \tag{32}$$

Every index is contracted, so the action is Lorentz scalar. Under

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda$$

we have

$$F_{\mu\nu} \mapsto F_{\mu\nu} + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda = F_{\mu\nu}.$$

The field terms are therefore exactly gauge invariant. The source term changes by

$$\delta S_J = -\int d^4x \, J^\mu \partial_\mu \Lambda = -\int d^4x \, \partial_\mu (J^\mu \Lambda) + \int d^4x \, \Lambda \, \partial_\mu J^\mu.$$

With vanishing boundary variation, gauge invariance of the sourced action requires

$$\partial_\mu J^\mu = 0. \tag{33}$$

The positivity $f > 0$ is important: it preserves the sign of the Maxwell kinetic term. The scalar kinetic term is canonical. The theory is second order because $\mathcal{L}$ contains only first derivatives of A_μ and ϕ. Finally,

$$M \to \infty \quad \Longrightarrow \quad f(\phi) \to 1,$$

so Maxwell theory plus a decoupled Klein–Gordon scalar is recovered.

2.2 Variation with respect to the electromagnetic potential

Vary A_ν while holding ϕ fixed. Since

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

the electromagnetic variation is

$$\begin{aligned} \delta S_A &= -\frac{1}{4} \int d^4x \, f \, \delta(F_{\mu\nu} F^{\mu\nu}) - \int d^4x \, J^\nu \delta A_\nu \\ &= -\frac{1}{2} \int d^4x \, f F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) - \int d^4x \, J^\nu \delta A_\nu. \end{aligned} \tag{34}$$

Because $F^{\mu\nu} = -F^{\nu\mu}$, the two derivative terms are equal, giving

$$\delta S_A = -\int d^4x \, f F^{\mu\nu} \partial_\mu \delta A_\nu - \int d^4x \, J^\nu \delta A_\nu.$$

Integrating by parts,

$$\delta S_A = \int d^4x \, [\partial_\mu (f F^{\mu\nu}) - J^\nu] \, \delta A_\nu - \int_{\partial\Omega} f F^{\mu\nu} \delta A_\nu \, d\Sigma_\mu.$$

For admissible variations vanishing on the boundary, stationarity for arbitrary δA_ν gives

$$\boxed{\partial_\mu [f(\phi) F^{\mu\nu}] = J^\nu.} \tag{35}$$

The homogeneous Maxwell equations,

$$\partial_{[\lambda} F_{\mu\nu]} = 0, \tag{36}$$

follow identically from the definition $F = dA$.

2.3 Variation with respect to the scalar

Now vary ϕ:

$$\delta S_\phi = \int d^4x \left[-\frac{1}{4} f'(\phi) F_{\mu\nu} F^{\mu\nu} \delta\phi + \partial_\mu \phi \, \partial^\mu \delta\phi - m^2 \phi \, \delta\phi \right]. \tag{37}$$

Integrating the kinetic term by parts,

$$\int d^4x \, \partial_\mu \phi \, \partial^\mu \delta\phi = - \int d^4x \, (\Box\phi) \delta\phi + \int_{\partial\Omega} \partial^\mu \phi \, \delta\phi \, d\Sigma_\mu.$$

Hence

$$\delta S_\phi = - \int d^4x \left[\Box\phi + m^2 \phi + \frac{1}{4} f'(\phi) F_{\mu\nu} F^{\mu\nu} \right] \delta\phi,$$

and therefore

$$\boxed{\Box\phi + m^2 \phi = -\frac{1}{4} f'(\phi) F_{\mu\nu} F^{\mu\nu}.} \tag{38}$$

With the $+---$ convention,

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$$

in natural units, so a purely electric background is a scalar source of the opposite sign to a purely magnetic background.

2.4 Noether symmetries

2.4.1 Gauge symmetry and charge conservation

Taking ∂_ν of (35),

$$\partial_\nu J^\nu = \partial_\nu \partial_\mu (f F^{\mu\nu}).$$

Since partial derivatives commute,

$$\partial_\nu \partial_\mu (f F^{\mu\nu}) = \frac{1}{2} \partial_\mu \partial_\nu \left[f (F^{\mu\nu} + F^{\nu\mu}) \right] = 0.$$

Thus

$$\boxed{\partial_\mu J^\mu = 0.} \tag{39}$$

This is the local Noether identity associated with $U(1)$ gauge invariance. Integrating over a fixed spatial volume and using

$$J^\mu = (\rho, \mathbf{J}),$$

gives

$$\frac{d}{dt} \int_V \rho \, d^3x = - \int_{\partial V} \mathbf{J} \cdot \hat{\mathbf{n}} \, dA.$$

For an isolated system the flux vanishes at infinity and the total electric charge is conserved.

2.4.2 Translations and energy–momentum

For a constant infinitesimal translation

$$x^\mu \mapsto x'^\mu = x^\mu + a^\mu,$$

the active field variations at the same coordinate point are

$$\delta\phi = -a^\nu \partial_\nu \phi, \qquad \delta A_\alpha = -a^\nu \partial_\nu A_\alpha.$$

Because the source-free Lagrangian has no explicit x^μ-dependence,

$$\delta\mathcal{L} = -a^\nu\partial_\nu\mathcal{L} = -\partial_\mu(a^\mu\mathcal{L}).$$

Noether's theorem therefore gives four conserved translation currents. The canonical tensor can be improved by the standard gauge-invariant Belinfante/Hilbert procedure. Varying the matter-free action with respect to the metric gives

$$\begin{aligned} T^{\mu\nu} = f(\phi)\left[-F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \\ + \partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left[\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}m^2\phi^2\right]. \end{aligned} \tag{40}$$

We can verify its conservation directly. The divergence of the electromagnetic part is

$$\partial_\mu T^{\mu\nu}_{\mathrm{EM}} = -F^{\nu}{}_{\alpha}\partial_\mu(fF^{\mu\alpha}) + \frac{1}{4}f'(\phi)F_{\alpha\beta}F^{\alpha\beta}\partial^\nu\phi,$$

where the Bianchi identity has been used. In the source-free case the first term vanishes by (35). The scalar part obeys

$$\partial_\mu T^{\mu\nu}_{\phi} = (\Box\phi + m^2\phi)\partial^\nu\phi = -\frac{1}{4}f'F^2\partial^\nu\phi.$$

With consistent index/sign convention the exchange terms cancel between the two sectors, yielding

$$\boxed{\partial_\mu T^{\mu\nu} = 0.} \tag{41}$$

Thus

$$P^\nu = \int_{\mathbb{R}^3} T^{0\nu}\,d^3x$$

is conserved when boundary fluxes vanish. P^0 is total energy and P^i are the three momentum components.

The energy density is especially transparent:

$$T^{00} = \frac{f(\phi)}{2}(E^2 + B^2) + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2. \tag{42}$$

Since $f = e^{2\phi/M} > 0$, every term is non-negative. This is the basic energetic reason for preferring (31) to higher-derivative attempts to make $\partial_\mu A^\mu$ physical.

2.4.3 Lorentz transformations

For an infinitesimal Lorentz transformation,

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

The scalar transforms as a scalar,

$$\delta\phi = -\delta x^\alpha\partial_\alpha\phi,$$

while A^μ transforms as a four-vector. Since the action is Lorentz invariant and $T^{\mu\nu}$ is symmetric, the conserved Lorentz current is

$$\boxed{J^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}, \qquad \partial_\lambda J^{\lambda\mu\nu} = 0.} \tag{43}$$

Indeed,

$$\partial_\lambda J^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} + x^\mu\partial_\lambda T^{\lambda\nu} - x^\nu\partial_\lambda T^{\lambda\mu} = 0.$$

The three spatial pairs $(\mu,\nu)=(i,j)$ give angular momentum; the three pairs $(0,i)$ give boost charges.

There is no independent scalar shift symmetry. Under $\phi \mapsto \phi + C$,

$$\frac{1}{2}m^2\phi^2 \mapsto \frac{1}{2}m^2(\phi+C)^2$$

when $m \neq 0$, and even if $m=0$,

$$f(\phi+C) = e^{2C/M} f(\phi) \neq f(\phi).$$

Hence no Noether current follows from a constant scalar shift in this model.

2.5 Linearisation about a static electric background

Set $J^\mu = 0$ locally and consider

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{e}, \qquad \mathbf{B} = \mathbf{b}, \qquad \phi = \phi_0 + \varphi,$$

where

$$\mathbf{E}_0 = E_0 \mathbf{e}_z, \qquad \mathbf{B}_0 = 0, \qquad \phi_0 = \text{constant}.$$

The background scalar equation requires

$$m^2\phi_0 = \frac{1}{2} f_0' E_0^2, \tag{44}$$

where $f_0 = f(\phi_0)$. Locally rescale the electromagnetic normalization so that $f_0 = 1$, and define

$$\alpha = \frac{f_0'}{f_0} = \frac{2}{M}. \tag{45}$$

To first order,

$$f(\phi) = 1 + \alpha\varphi + O(\varphi^2).$$

The modified Gauss law is

$$\nabla \cdot (f\mathbf{E}) = 0.$$

Keeping first-order terms,

$$\boxed{\nabla \cdot \mathbf{e} + \alpha \mathbf{E}_0 \cdot \nabla\varphi = 0.} \tag{46}$$

The modified Ampere law is

$$\nabla \times (f\mathbf{B}) - \partial_t (f\mathbf{E}) = 0,$$

hence

$$\boxed{\nabla \times \mathbf{b} - \partial_t \mathbf{e} - \alpha \mathbf{E}_0\, \partial_t \varphi = 0.} \tag{47}$$

The homogeneous equations are

$$\nabla \cdot \mathbf{b} = 0, \qquad \nabla \times \mathbf{e} + \partial_t \mathbf{b} = 0. \tag{48}$$

For the scalar equation, use

$$F^2 = 2(B^2 - E^2) = -2E_0^2 - 4\mathbf{E}_0 \cdot \mathbf{e} + O(2).$$

Also

$$f'(\phi) = f_0' + f_0''\varphi + O(2).$$

Subtracting the background equation (44), the linear scalar equation becomes

$$\Box\varphi + m_{\text{eff}}^2 \varphi - f_0' \mathbf{E}_0 \cdot \mathbf{e} = 0, \tag{49}$$

where

$$m_{\text{eff}}^2 = m^2 - \frac{1}{2} f_0'' E_0^2. \tag{50}$$

For $f = e^{2\phi/M}$,

$$f_0' = \frac{2}{M}, \qquad f_0'' = \frac{4}{M^2},$$

so

$$\boxed{m_{\text{eff}}^2 = m^2 - \frac{2E_0^2}{M^2}.} \tag{51}$$

2.6 Plane-wave eigenvectors and dispersion relation

Let every perturbation be proportional to

$$e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}.$$

Then (46) gives

$$\mathbf{k}\cdot\mathbf{e} = -\alpha(\mathbf{E}_0\cdot\mathbf{k})\varphi. \tag{52}$$

If θ is the angle between $\mathbf{k}$ and $\mathbf{E}_0$, the longitudinal electric amplitude is therefore

$$\boxed{e_L = -\alpha E_0 \cos\theta\,\varphi = -\frac{2E_0}{M}\cos\theta\,\varphi.} \tag{53}$$

Faraday's law gives

$$\boxed{\mathbf{b} = \frac{\mathbf{k}\times\mathbf{e}}{\omega}.} \tag{54}$$

Choose $\hat{\mathbf{t}}$ in the plane of $\mathbf{k}$ and $\mathbf{E}_0$, perpendicular to $\hat{\mathbf{k}}$, so

$$\mathbf{E}_0 = E_0(\cos\theta\,\hat{\mathbf{k}} + \sin\theta\,\hat{\mathbf{t}}).$$

Fourier transforming (47) and eliminating $\mathbf{b}$ using (54) gives

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{e}) + \omega^2\mathbf{e} + \alpha\omega^2\mathbf{E}_0\varphi = 0.$$

Since

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{e}) = \mathbf{k}(\mathbf{k}\cdot\mathbf{e}) - k^2\mathbf{e},$$

the transverse-in-plane component is

$$(\omega^2 - k^2)e_T + \alpha\omega^2 E_0 \sin\theta\,\varphi = 0. \tag{55}$$

Writing

$$\Delta = \omega^2 - k^2, \tag{56}$$

we have

$$\boxed{e_T = -\alpha\frac{\omega^2}{\Delta}E_0\sin\theta\,\varphi.} \tag{57}$$

The polarization perpendicular to the $(\mathbf{k},\mathbf{E}_0)$-plane has no scalar coupling and obeys simply

$$\boxed{\omega^2 = k^2.} \tag{58}$$

Now Fourier transform (49). Since

$$\Box\varphi = (-\omega^2 + k^2)\varphi = -\Delta\varphi,$$

we obtain

$$(m_{\text{eff}}^2 - \Delta)\varphi - \alpha E_0(\cos\theta\, e_L + \sin\theta\, e_T) = 0.$$

Insert (53) and (57):

$$m_{\text{eff}}^2 - \Delta + \alpha^2 E_0^2 \cos^2\theta + \alpha^2 E_0^2 \frac{\omega^2}{\Delta}\sin^2\theta = 0.$$

Multiply by Δ, use $\omega^2 = k^2 + \Delta$, and collect terms:

$$\Delta^2 - \left(m_{\text{eff}}^2 + \alpha^2 E_0^2\right)\Delta - \alpha^2 k^2 E_0^2 \sin^2\theta = 0. \tag{59}$$

For the exponential coupling,

$$m_{\text{eff}}^2 + \alpha^2 E_0^2 = m^2 - \frac{2E_0^2}{M^2} + \frac{4E_0^2}{M^2} = m^2 + \frac{2E_0^2}{M^2}.$$

Therefore

$$\boxed{\Delta_\pm = \frac{1}{2}\left[m^2 + \frac{2E_0^2}{M^2} \pm \sqrt{\left(m^2 + \frac{2E_0^2}{M^2}\right)^2 + \frac{16k^2E_0^2}{M^2}\sin^2\theta}\,\right].} \tag{60}$$

Since $\omega_\pm^2 = k^2 + \Delta_\pm$, this is the full mixed dispersion relation.

2.7 Weak coupling and the distinguished nearly massless limit

For fixed $m \neq 0$ and large M, expand the square root:

$$\sqrt{(m^2+\epsilon)^2 + \delta} = m^2 + \epsilon + \frac{\delta}{2m^2} + O(M^{-4}),$$

where

$$\epsilon = \frac{2E_0^2}{M^2}, \qquad \delta = \frac{16k^2E_0^2}{M^2}\sin^2\theta.$$

Thus

$$\Delta_+ = m^2 + \frac{2E_0^2}{M^2} + \frac{4k^2E_0^2}{m^2M^2}\sin^2\theta + O(M^{-4}), \tag{61}$$

$$\Delta_- = -\frac{4k^2E_0^2}{m^2M^2}\sin^2\theta + O(M^{-4}). \tag{62}$$

The photon-like branch is therefore shifted only at $O(M^{-2})$.

This expansion fails when $m \to 0$. The two terms under the square root become comparable when

$$m^4 \sim \frac{16k^2E_0^2}{M^2}\sin^2\theta,$$

or

$$\boxed{m^2 \sim \frac{4kE_0}{M}|\sin\theta|.} \tag{63}$$

Define the $O(1)$ distinguished parameter

$$\zeta = \frac{m^2 M}{4kE_0|\sin\theta|}. \tag{64}$$

At leading $O(M^{-1})$, the smaller $2E_0^2/M^2$ term may be omitted inside the square bracket, giving

$$\boxed{\Delta_\pm \sim \frac{2kE_0}{M}|\sin\theta|\left(\zeta \pm \sqrt{1+\zeta^2}\right).} \tag{65}$$

At exact near-degeneracy, $\zeta \to 0$,

$$\Delta_\pm \simeq \pm \frac{2kE_0}{M} |\sin\theta|.$$

Since

$$\omega_\pm = \sqrt{k^2 + \Delta_\pm} = k + \frac{\Delta_\pm}{2k} + O(\Delta_\pm^2),$$

we obtain

$$\boxed{\omega_\pm \simeq k \pm \frac{E_0}{M} |\sin\theta|.} \tag{66}$$

Degenerate mixing has promoted the observable splitting from $O(M^{-2})$ to $O(M^{-1})$.

2.8 The parallel longitudinal branch

The limit $\theta \to 0$ must be taken directly because the oblique distinguished scaling contains $\sin\theta$. Set

$$\mathbf{k} \parallel \mathbf{E}_0.$$

Then $e_T = 0$ and (53) gives

$$e_L = -\frac{2E_0}{M}\varphi.$$

Because $\mathbf{e} \parallel \mathbf{k}$,

$$\mathbf{b} = \frac{\mathbf{k} \times \mathbf{e}}{\omega} = 0.$$

Equation (59) factorises into the ordinary photon solution and

$$\Delta = m_{\text{eff}}^2 + \alpha^2 E_0^2 = m^2 + \frac{2E_0^2}{M^2}.$$

Hence

$$\boxed{\omega_L^2 = k^2 + m^2 + \frac{2E_0^2}{M^2}, \qquad \mathbf{e} = e_L \hat{\mathbf{k}}, \qquad \mathbf{b} = 0.} \tag{67}$$

This is a genuine propagating scalar-associated eigenmode. It is not an additional polarization of Maxwell theory: its electromagnetic longitudinal component is tied to the independent scalar amplitude φ.

Its group velocity is

$$v_g = \frac{d\omega_L}{dk} = \frac{k}{\sqrt{k^2 + m^2 + 2E_0^2/M^2}} < 1 \tag{68}$$

in natural units, so this mode is also subluminal.

2.9 SI form and directly measurable wave predictions

Use the SI Lagrangian

$$\mathcal{L} = \frac{\varepsilon_0}{2} e^{2\phi/M_{\text{SI}}}(E^2 - c^2B^2) + \frac{1}{2}\left(\frac{\phi_t^2}{c^2} - |\nabla\phi|^2 - \mu^2\phi^2\right), \tag{69}$$

where

$$\mu = \frac{m_\phi c}{\hbar}, \qquad [M_{\text{SI}}] = \sqrt{\text{J/m}}. \tag{70}$$

The natural inverse-length coupling generated by a DC electric field is

$$\boxed{\kappa_E = \frac{\sqrt{\varepsilon_0} E_0}{M_{\text{SI}}}.} \tag{71}$$

Writing $q = \omega/c$ and $\Delta = q^2 - k^2$, the preceding calculation becomes

$$\boxed{\Delta_\pm = \frac{1}{2}\left[\mu^2 + 2\kappa_E^2 \pm \sqrt{(\mu^2 + 2\kappa_E^2)^2 + 16\kappa_E^2 k^2 \sin^2\theta}\right].} \tag{72}$$

For the parallel branch,

$$\boxed{\omega_L^2 = c^2k^2 + c^2\mu^2 + 2c^2\kappa_E^2.} \tag{73}$$

In the nearly massless oblique regime at fixed frequency,

$$k_\pm \simeq q \mp \kappa_E|\sin\theta|,$$

so

$$\boxed{\Delta k = k_- - k_+ \simeq 2\kappa_E|\sin\theta|, \qquad \Delta\Phi \simeq 2\kappa_E L|\sin\theta|.} \tag{74}$$

These are direct interferometric predictions.

2.10 Static finite-electrode prediction

The same coupling must also affect electrostatics. Set $\mathbf{B} = 0$, $\mathbf{E} = -\nabla U$, and take the massless limit first. The static field equations following from (69) are

$$\nabla \cdot \left(e^{2\phi/M_{\mathrm{SI}}}\nabla U\right) = 0, \tag{75}$$

and

$$-\nabla^2\phi = \frac{\varepsilon_0}{M_{\mathrm{SI}}} e^{2\phi/M_{\mathrm{SI}}}|\nabla U|^2. \tag{76}$$

For weak coupling expand

$$\phi = \frac{\Phi}{M_{\mathrm{SI}}} + O(M_{\mathrm{SI}}^{-3}), \qquad U = U_0 + \frac{U_2}{M_{\mathrm{SI}}^2} + O(M_{\mathrm{SI}}^{-4}). \tag{77}$$

At leading order,

$$\nabla^2 U_0 = 0, \qquad \boxed{-\nabla^2\Phi = \varepsilon_0|\nabla U_0|^2.} \tag{78}$$

The free-space Green function satisfies

$$-\nabla^2 \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} = \delta(\mathbf{x} - \mathbf{x}'),$$

hence

$$\boxed{\Phi(\mathbf{x}) = \frac{\varepsilon_0}{4\pi}\int_{\mathbb{R}^3} \frac{E_0^2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'.} \tag{79}$$

Unlike the infinite-plate problem with an artificially imposed scalar boundary value, a finite capacitor allows $\phi \to 0$ naturally at spatial infinity.

The electrostatic energy at fixed geometry is

$$\mathcal{U} = \frac{\varepsilon_0}{2}\int e^{2\phi/M_{\mathrm{SI}}} E^2\, d^3x.$$

At fixed electrode voltage, the $O(M^{-2})$ correction can be evaluated on the leading Maxwell field; the first variation with respect to U vanishes because U_0 already solves the Maxwell boundary-value problem. Thus

$$\delta\mathcal{U} = \frac{\varepsilon_0}{M_{\mathrm{SI}}^2}\int \Phi E_0^2\, d^3x.$$

Since $\mathcal{U} = \frac{1}{2}CV^2$,

$$\frac{1}{2}\Delta C\, V^2 = \frac{\varepsilon_0}{M_{\rm SI}^2} \int \Phi E_0^2\, d^3x, \tag{80}$$

or

$$\boxed{\frac{\Delta C}{C_0} = \frac{2\varepsilon_0}{M_{\rm SI}^2 C_0 V^2} \int \Phi E_0^2\, d^3x.} \tag{81}$$

Substituting (79),

$$\boxed{\frac{\Delta C}{C_0} = \frac{\varepsilon_0^2}{2\pi M_{\rm SI}^2 C_0 V^2} \iint \frac{E_0^2(\mathbf{x}) E_0^2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x\, d^3x'.} \tag{82}$$

The correction is manifestly positive.

For two large circular plates of radius R, separation $d \ll R$, use the leading uniform gap field

$$E_0 \simeq \frac{V}{d}, \qquad C_0 \simeq \frac{\varepsilon_0 \pi R^2}{d}.$$

Neglecting $O(d/R)$ corrections in the kernel reduces the double volume integral to

$$E_0^4 d^2 J_R,$$

where

$$J_R = \iint_{|\boldsymbol{\rho}|,|\boldsymbol{\rho}'|<R} \frac{d^2\rho\, d^2\rho'}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|}.$$

Introduce the separation $s = |\boldsymbol{\rho} - \boldsymbol{\rho}'|$. The overlap area of two radius-R disks displaced by s is

$$A_{\rm ov}(s) = 2R^2 \cos^{-1}\frac{s}{2R} - \frac{s}{2}\sqrt{4R^2 - s^2}.$$

Therefore

$$J_R = 2\pi \int_0^{2R} A_{\rm ov}(s)\, ds = \frac{16\pi R^3}{3}. \tag{83}$$

Substituting into (82),

$$\begin{aligned} \frac{\Delta C}{C_0} &\simeq \frac{\varepsilon_0^2}{2\pi M_{\rm SI}^2 (\varepsilon_0 \pi R^2/d) V^2} \frac{V^4}{d^4} d^2 \frac{16\pi R^3}{3} \\ &= \boxed{\frac{8}{3\pi}\frac{R}{d}\frac{\varepsilon_0 V^2}{M_{\rm SI}^2}.} \end{aligned} \tag{84}$$

Writing

$$C(V) = C_0(1 + aV^2), \qquad a = \frac{8}{3\pi}\frac{R}{d}\frac{\varepsilon_0}{M_{\rm SI}^2},$$

the charge is

$$Q(V) = C(V)V = C_0(V + aV^3).$$

Hence the differential capacitance is

$$C_{\rm diff} = \frac{dQ}{dV} = C_0(1 + 3aV^2),$$

so

$$\boxed{\frac{\Delta C_{\rm diff}}{C_0} = \frac{8}{\pi}\frac{R}{d}\frac{\varepsilon_0 V_{\rm DC}^2}{M_{\rm SI}^2}.} \tag{85}$$

For finite scalar mass, (78) becomes

$$(-\nabla^2 + \mu^2)\Phi = \varepsilon_0 E_0^2.$$

The Green function is

$$G_\mu(r) = \frac{e^{-\mu r}}{4\pi r},$$

so every $1/r$ in (79)–(82) is replaced by $e^{-\mu r}/r$. Varying R and d therefore probes the scalar Compton range μ^{-1}.

2.11 Connection with existing experiments

Expanding the interaction at small ϕ/M,

$$-\frac{1}{4}e^{2\phi/M}F^2 = -\frac{1}{4}F^2 - \frac{\phi}{2M}F^2 + O(M^{-2}). \tag{86}$$

Thus the model belongs to the familiar dilaton-like scalar–photon class. Searches using resonant cavities and capacitors have explicitly analysed such couplings [10, 11]. Consequently the coupling $1/M$ cannot be chosen arbitrarily large merely to make the longitudinal mode easy to observe.

The static result gives a direct way to use voltage-coefficient data. If a capacitor with known geometry is measured at two voltages V_1, V_2, and no fractional change larger than δ is seen, (84) implies, for the large-disk massless approximation,

$$\boxed{M_{\mathrm{SI}} > \left[\frac{8\varepsilon_0}{3\pi}\frac{R}{d}\frac{V_2^2 - V_1^2}{\delta}\right]^{1/2}.} \tag{87}$$

For a real precision capacitor, (82), rather than the disk approximation, should be evaluated using the actual Maxwell field. Mechanical deformation, dielectric electrostriction, temperature and ordinary voltage coefficients are nuisance effects and must be fitted simultaneously.

Dedicated scalar–photon searches are generally much more constraining because they target the same interaction directly. The important conclusion is therefore not that existing capacitance measurements show a scalar signal; they do not. Rather, existing null experiments force the minimal unscreened theory into a weak-coupling regime.

2.12 A direct longitudinal-wave experiment

The parallel solution (67) suggests a more specific test than a generic scalar search. Consider an RF emitter in a controlled DC electric field $\mathbf{E}_0$, followed by an electromagnetic shield and a resonant receiver whose electric mode is axial. Ordinary propagating Maxwell radiation is suppressed by the shield. The scalar-associated mode is sought through its longitudinal electric component.

The theory predicts several correlated signatures:

$$e_L = -\frac{2E_0}{M}\varphi, \qquad B = 0 \quad (\mathbf{k} \parallel \mathbf{E}_0), \tag{88}$$

and, for oblique near-degenerate propagation,

$$\Delta k \simeq 2\frac{\sqrt{\varepsilon_0}E_0}{M_{\mathrm{SI}}}|\sin\theta|. \tag{89}$$

Thus a coherent amplitude generated by mixing is odd under

$$E_0 \mapsto -E_0,$$

whereas its power is even:

$$A_{\text{sig}} \propto E_0, \qquad P_{\text{sig}} \propto E_0^2. \tag{90}$$

A phase-sensitive heterodyne measurement can therefore test the sign reversal, while a power measurement can test the quadratic field scaling. Rotating the apparatus tests the angular dependence in (89). Measuring both electric and magnetic pickup tests the defining longitudinal signature $B/E_L \to 0$.

Shielded resonator technology already exists in searches for non-Maxwell fields, including longitudinal massive-vector modes [12, 13]. Those experiments do not test the same field theory, but they demonstrate the relevant metrology. The proposed test differs by making the controlled DC electric background an essential part of the signal model.

2.13 Conclusions

The calculations lead to two sharply separated conclusions. First, Maxwell theory itself permits longitudinal electric components in anisotropic media because $\mathbf{D}$, not necessarily $\mathbf{E}$, is constrained to be transverse. The explicit TM calculation produced a 98% longitudinal electric amplitude and $v_p \simeq 1.407c_0$ for an illustrative anisotropic effective tensor. The apparent superluminal group speed obtained by freezing that tensor is not a causal prediction: the Poynting theorem and Brillouin energy density show where material dispersion enters the energy velocity, and causal metamaterial models retain subluminal energy transport.

Second, a genuine scalar field can be coupled to electromagnetism without sacrificing Lorentz invariance, $U(1)$ gauge invariance or positivity of the quadratic energy. The variational calculation gives the modified Maxwell and scalar equations; Noether's theorem gives local charge conservation, energy–momentum conservation and Lorentz angular-momentum/boost currents. Linearisation in a DC electric background yields the exact mixed dispersion relation (60). Its nearly massless limit is singular in ordinary perturbation theory and requires the distinguished scaling (63), under which the mode splitting becomes $O(M^{-1})$. Parallel propagation supports the especially clear eigenmode

$$E_L \neq 0, \qquad B = 0, \qquad \omega_L^2 = c^2k^2 + c^2\mu^2 + 2c^2\kappa_E^2.$$

The same theory predicts the positive voltage-dependent capacitance (84). Existing scalar–photon experiments constrain the interaction strongly, so the minimal theory does not support large, readily observable "Tesla-wave" effects. What remains is a conventional falsifiable field-theory question: whether the specific weak longitudinal electric branch survives a dedicated electric-background experiment. The simultaneous tests of field scaling, phase reversal, angular dependence, dispersion and B/E_L provide a substantially more discriminating experiment than a search for unexplained RF transmission alone.

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