Study On Primes Of Form $p \times q \pm r$

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Abstract

Prime numbers can be represented in many ways. This work focuses on representing them using other primes: p, q, and r as: $p \times q \pm r$. The aim of this research is to explore interesting properties of such prime representation, including the smallest product of p, q, and r that can represent a given prime number and 4 related hypotheses that were formulated when exploring this topic.

1 Introduction

Chen's famous theorem states that every large enough even integer N can be expressed as $N = p + q$ or $N = p + q_1 \times q_2$ (p, q, q_1 and q_2 are primes) [1]. A100952 [2] provided a conjecture that every integer >60 may be partitioned into a prime p and a semiprime $q \times r$, where the prime p is bounded by $log(min(q, r))$. This work focuses on case when N is prime and has a form: $N = p \times q \pm r$ where all three numbers: p, q and r are also primes. Experiments are led with letter and spirit of finding interesting properties of such representation.

For instance, representations in the mentioned form $p \times q \pm r$ for primes 2 and 11 are as follows: $2 = 2 \times 2 - 2$, $11 = 2 \times 3 + 5 = 2 \times 7 - 3$.

Lemma 1. *Every prime P can be written as: $P = p \times q \pm r$, where p, q and r are primes.*

Proof. Lemma does not exlude that p, q and r are distinct primes. In that spirit, for every prime P it is possible to write: $P = 2 \times P - P$, and both 2 and P are primes ($2 \times P - P$ is of form $p \times q \pm r$). □

Let's denote $PQR(x)$ as a logical formula for a product of p, q and r for prime x: $PQR(x) = |p \times q \times r|$, where either $x = p \times q + r$ or $x = p \times q - r$. PQR is then a set: $\{x \in PQR : PQR(x)\}$, and $PQR_{min}(p)$ is the smallest member of PQR for prime p, $PQR_{max}(p)$ - the greatest one. As a consequence of Lemma 1, we know that both $PQR_{min}(p)$ and $PQR_{max}(p)$ exist for every prime.

Lemma 2. $8 \leq PQR(p) \leq 2 \times p^2$

Proof. Lower limit comes from the fact that if $p = q = r = 2$ (the smallest possible prime), then the smallest member of PQR set is $|2 \times 2 \times 2| = 8$. Upper limit is a conclusion from Lemma 1: maximal member of PQR set is $|2 \times P \times P| = 2 \times P^2$. □

Of course, lower limit of Lemma 2 is fullfiled for one prime only ($2 = 2 \times 2 - 2$ and other possible option: $2 \times 2 + 2$ is not a prime). Also upper limit generally seems to be too secure for all the primes. The smallest ever PQR is for prime 2 : $PQR_{min}(2) = PQR_{max}(2) = 8$.

Lemma 3. *If P is prime and $P = p \times q \pm r$ and p, q are primes >2, then we have only three candidates for $\pm r$: either -2 or $+2$ or $-(p \times q - 2)$.*

Proof. If p, q are primes >2 , then both are odd. If two odd numbers are multiplied, then result is also odd ($X = p \times q$) and X>4. We also have one even prime (2). Let's analyze four possible cases now for r. 1) If from X we substract an odd number, we will have an even number as a result. If result is prime, it needs to be 2, thus we are subtracting $r = X - 2 = p \times q - 2$. 2) If to X we add an odd number, we will have an even number as a result. But result will be >2, thus cannot be prime. 3) If from X we substract an even number, we will have odd number as a result. The only possibility for substracted r here is 2. 4) If to X we add an even number, we will have an odd number as a result. Again, the only possibility for added r is 2. □

Lemma 3 is important to optimize algorithm to search for PQR - we have only 3 cases to check if p and q are primes >2.

Lemma 4. *Only 13 has two non-trivial permutations of p, q and r in PQR_{min}. All other primes have one.*

Proof. A non-trivial permutation in PQR_{min} is the one that has a different value of r. Please note that p and q are mutliplied and can be swapped (due to commutative property of multiplication) but this is a trivial permutation, excluded here. Without loss of generality we have two cases to analyse, A and B.

A) $p \times q + r = p \times r - q$ which is giving us: $p \times (r - q) = r + q$, then: $p = \frac{r+q}{r-q}$, then: $p = 1 + 2 \times \frac{q}{r-q}$. q is prime by definition, thus can be divided by either 1 or q only. Let's analyse these two subcases, $'$ and $''$. A') If denominator $= 1$, then $r - q = 1$, then $r = q + 1$. The only primes that are fulfilling A' are: $r = 3$ and $q = 2$ (all other primes are separated at least by 2). A") If denominator $= q$, then $r - q = q$, then $r = 2 \times q$, which cannot be true (r is prime by definition).

B) $p \times q + r = p \times r + q$ which is giving us: $p \times (q - r) = q - r$, then ($q \neq r$, which is true because otherwise we would have the same permutation): $p = \frac{q-r}{q-r} = 1$ which is false (1 is not a prime).

To sum up, we just have one possible case with: $r = 3$ and $q = 2$ which is giving us: $p \times 2 + 3 = p \times 3 - 2$, resulting in: $p = 5$. At the end we have: $13 = 3 \times 5 - 2 = 2 \times 5 + 3$. □

2 Algorithm

The algorithm A to search for $PQR_{min}(x)$ is based on Lemma 1 and Lemma 3.

For every combination of primes p, q, r ($p \leq q$):

1. Intially D is an empty dictonary holding prime as a key and PQR_{min} as a value ($key \rightarrow value$).

2. if p>2 and q>2:

   if $N_1 = p \times q - 2$ is prime:

   check if $|p \times q - 2|$ is the lowest ever value for key N_1 in D; If yes, record $(N_1 \rightarrow -2 \times p \times q)$ in D.

   check if $|p \times q - 2|$ is the lowest ever value for key 2 in D; If yes, record $(2 \rightarrow -2 \times p \times q)$ in D.

   if $N_2 = p \times q + 2$ is prime, check if $|p \times q \times 2|$ is the lowest ever value for key N_2 in D; If yes, record $(N_2 \rightarrow 2 \times p \times q)$ in D.

3. else:

   if $N_3 = p \times q - r$ is prime, check if $|p \times q \times r|$ is the lowest ever value for key N_3 in D; If yes, record $(N_3 \rightarrow -p \times q \times r)$ in D.

   if $N_4 = p \times q + r$ is prime, check if $|p \times q \times r|$ is the lowest ever value for key N_4 in D; If yes, record $(N_4 \rightarrow p \times q \times r)$ in D.

4. Once planed set of primes is analysed, D holds the results.

Algorithm A was coded in [3] and all experiments were carried out with the use of this framework.

3 Initial observations

Initial results are presented in Table 1 and Figures 1-3 - they are also suggestions for the four hypotheses below:

- **Hypothesis 1** (suggested by Figure 1): $PQR_{min}(x)$ delta to x increase is linear for big x.
- **Hypothesis 2** (suggested by Figure 2): $\frac{x}{PQR_{min}(x)}$ may stabilize for big x.
- **Hypothesis 3** (suggested by Figure 3): Actual $PQR_{min}(x)$ is far from possible extremes for bigger x.
- **Hypothesis 4** (suggested by Table 1): $+r$ is more frequent than $-r$ in PQR_{min}.

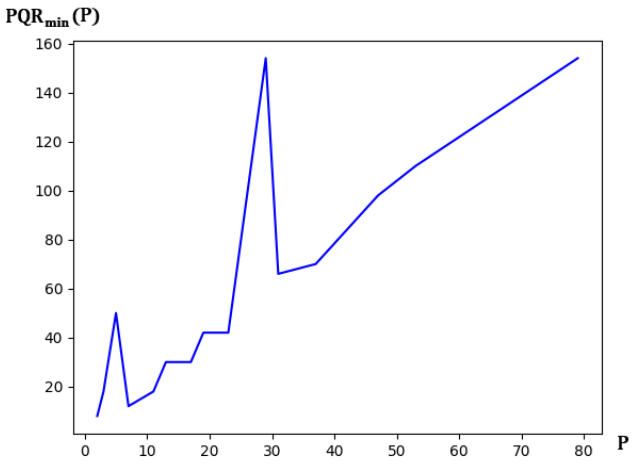

Figure 1: PQR_{min} for the first 5 iterations of A.

Table 1: Primes with their PQR_{min} for the first 5 iterations of A.

Prime P	p	q	r	PQR_{min}
2	2	2	-2	8
3	2	3	-3	18
5	2	5	-5	50
7	2	2	+3	12
11	3	3	+2	18
13	2	5	+3	30
17	3	5	+2	30
19	3	7	-2	42
23	3	7	+2	42
29	2	11	+7	154
31	3	11	-2	66
37	5	7	+2	70
47	7	7	-2	98
53	5	11	-2	110
79	7	11	+2	154

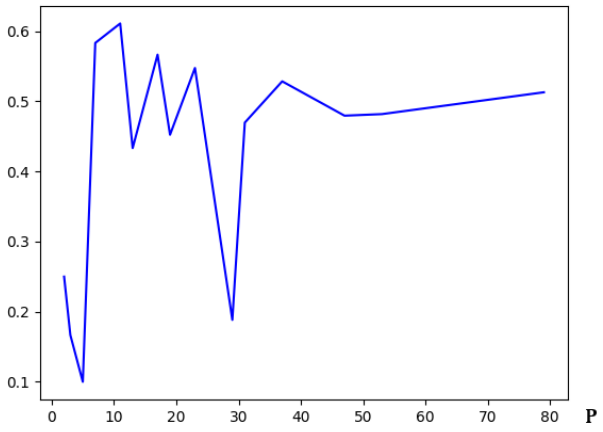

Figure 2: $\frac{x}{PQR_{min}(x)}$ for the first 5 iterations of A.

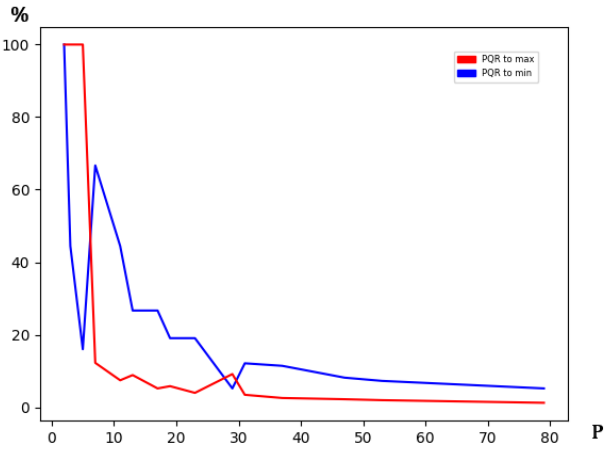

Figure 3: PQR_{min} to extremes from Lemma 2 for the first 5 iterations of A.

4 Results of experiments

Executed experiments focused to checks around hypotheses 1-4 and ended up with the following conclusions:

- **Hypothesis 1** may be true (see: Figure 4), also sup-

ported by the Hypothesis 2 (see: Figure 5). Slope coefficient of a straight line is ≈ 0.5.

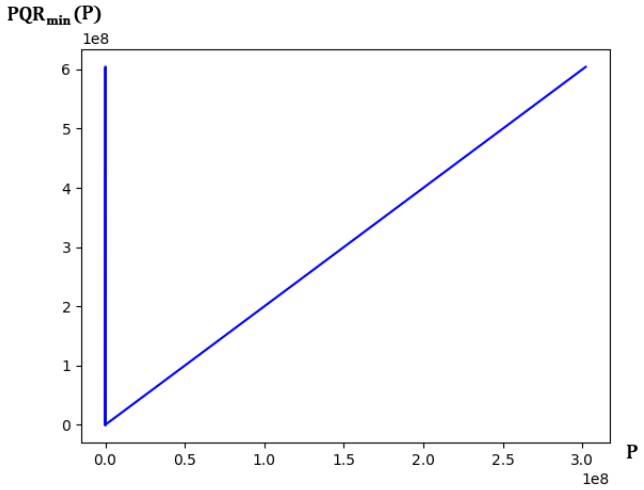

Figure 4: PQR_{min} for the first 2000 iterations of A.

- **Hypothesis 2** may be true (see: Figure 5), with a very prompt stabilization after initial fluctuations.

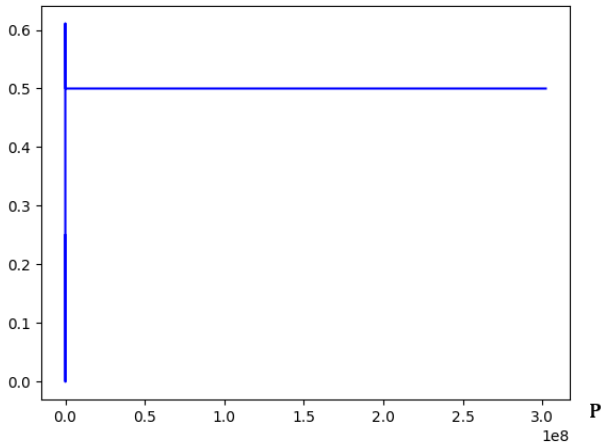

Figure 5: $\frac{x}{PQR_{min}(x)}$ for the first 2000 iterations of A.

- **Hypothesis 3** seems to be true either. Figure 6 suggests that PQR_{min} is far from the possible extremes and most probably minimal and maximum values Lemma 2 could be significantly improved.

- **Hypothesis 4** may be true although difference between positive and negative r is lower and lower with more primes checked and maybe eventually both have the same probability (see: Figure 7). What is also interesting is that for the first iterations $-r$ is more frequent but then $+r$ comes to voice (see: Figures 8 and 9).

5 Conclusions

Executed experiments allowed to investigate all 4 formulated hypotheses, in particular allowed to study

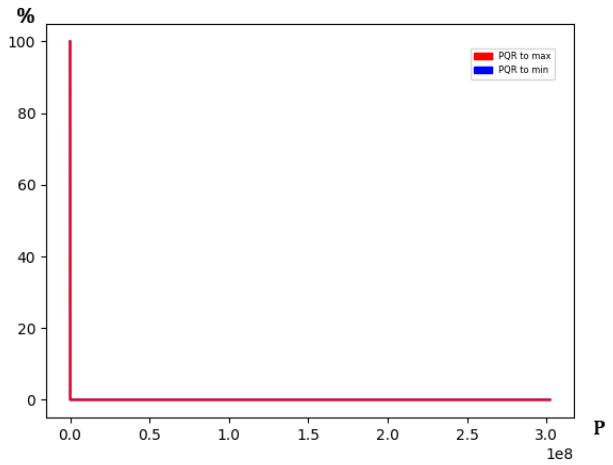

Figure 6: PQR_{min} to extremes from Lemma 2 for the first 2000 iterations of A.

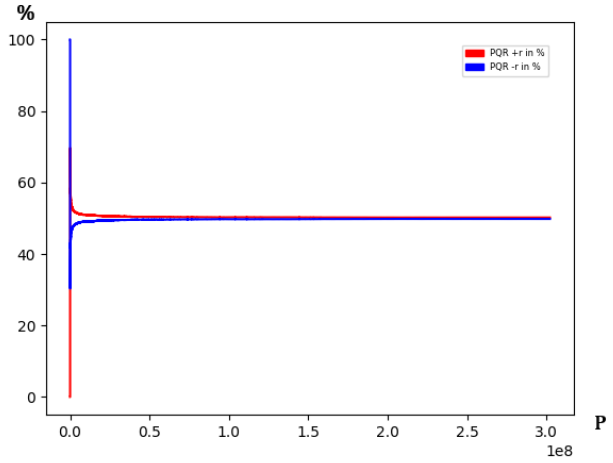

Figure 7: Percentage of either positive or negative r in PQR_{min} for the first 2000 iterations of A.

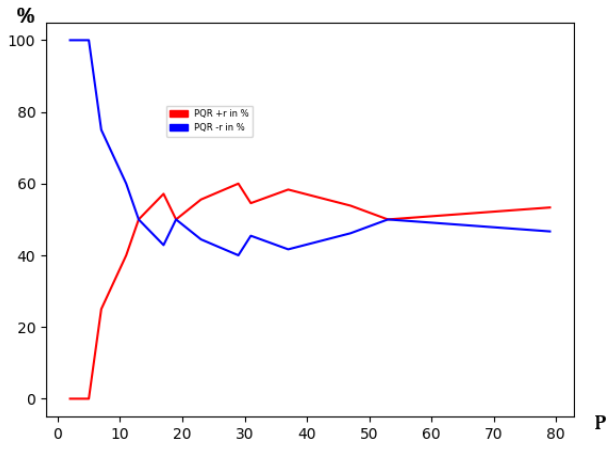

Figure 8: Percentage of either positive or negative r in PQR_{min} for the first 5 iterations of A.

PQR_{min} in primes of form $p \times q \pm r$. Additional figures (Figure 10 and 11) allowed to observe that sign of PQR_{min} (noted as $sPQR_{min}$) is distributed rather evenly, with notable positive extremum for smaller primes (it is also confirmed in Figure 6) which may be foundation of further studies.

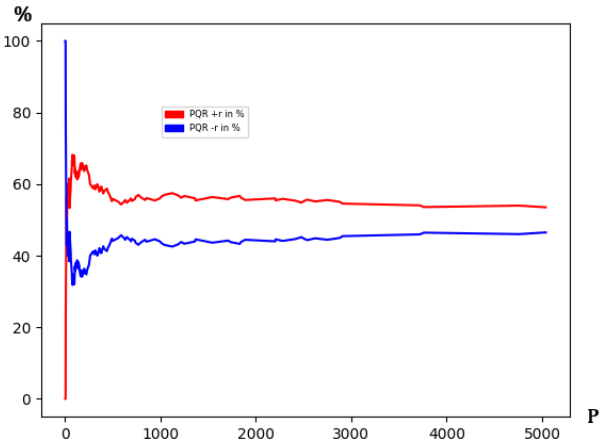

Figure 9: Percentage of either positive or negative r in PQR_{min} for the first 20 iterations of A.

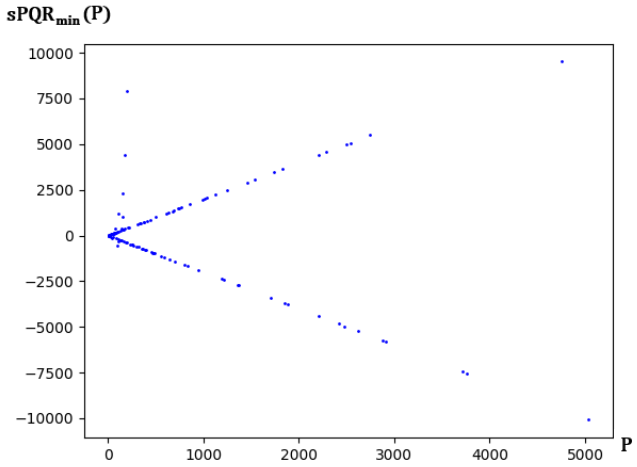

Figure 10: $sPQR_{min}$ for the first 20 iterations of A, takes into account sign of r in PQR_{min}.

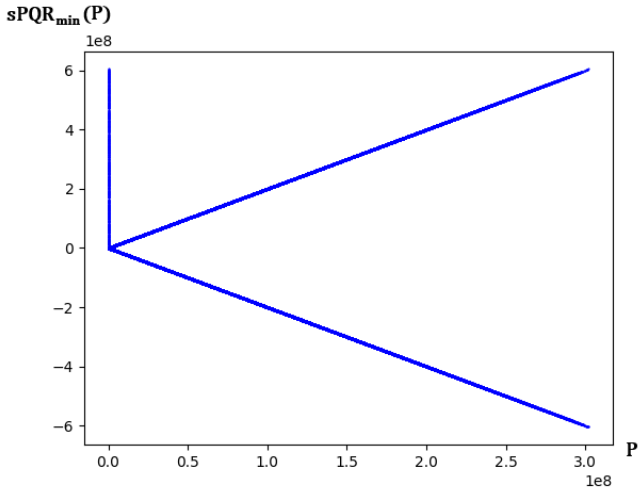

Figure 11: $sPQR_{min}$ for the first 2000 iterations of A, takes into account sign of r in PQR_{min}.

References

[1] J.-R. Chen, *On the Representation of a Large Even Number as the Sum of a Prime and the Product of at Most Two Primes, II*, Scientia Sinica 16, 1973, pp 157–176.

[2] OEIS A100952, *OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A100952. Numbers that cannot be written as p*q+r with three distinct primes p, q and r.*

[3] *Library for various operations on primes.* https://github.com/mbarylsk/primes