Prime Number Mechanism Research: Skip-Multiplication Theory

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Abstract

Current research on the distribution of prime numbers either narrows down to counting due to a focus on applied fields, meaning it only counts without knowing which numbers are prime, or it explores patterns but, because of a cause-from-effect order, leads to unclear results and only approximate rather than definitive distribution patterns. This paper attempts to discuss a common type of multiplication—multiplication whose factors do not contain 1, called skip-multiplication—introducing skip-numbers and skip-products, to explain the essence of prime numbers as full-skip numbers and their distribution patterns, and to achieve a fundamental grasp of prime numbers in terms of identity rather than merely quantity through the method of reverse selection from the skip-product table. Furthermore, because tables rely on enumeration, their role in application is at most auxiliary rather than a replacement for any tool; thus, this paper mainly provides a new number-theoretic understanding of prime and composite numbers in the pure mathematical theory domain. The appendix supplements some patterns of the skip-multiplication theory tables and discussions on the oddness of primes and the Riemann Hypothesis.

Keywords: prime number, composite number, skip-multiplication, skip-product, skip-number

Introduction: The Four Quadrants of Prime Number Theory

Research on prime numbers roughly has two directions/dimensions: theory and application, that is, understanding the essence of prime numbers or skipping the understanding of the essence and directly realizing computation of prime numbers for practical application. The former grasps only theory, the latter examines only computability. Of course, both can be grasped simultaneously. Thus, the two axes representing the theoretical x-dimension and the applied y-dimension can form a coordinate system, and prime number theory naturally has four quadrant-based classifications of emphasis—

- First quadrant $(+x, +y)$: knows why, and is polynomial-time computable;
- Second quadrant $(-x, +y)$: does not know why, but is polynomial-time computable;
- Third quadrant $(-x, -y)$: does not know why, and is not polynomial-time computable;
- Fourth quadrant $(+x, -y)$: knows why, but is not polynomial-time computable.

Currently, frontier prime number research is mostly effort on the y-axis, belonging to the second quadrant—applied counting, without knowing why it is so or which numbers are prime. They proceed from effect to cause, only finding patterns of existing phenomena (phenomena caused by prime numbers), not the patterns of prime numbers themselves. Moreover, the degree of completion in counting is not 100%, as in the prime number theorem[2], the Riemann equation[3], etc., there exist inaccuracies/only estimates, or unproven deficiencies. Therefore, it is not that one must reach the

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[2] Hadamard, J. (1896). Sur la distribution des zéros de la fonction ζ (s) et ses conséquences arithmétiques.

[3] Riemann, B. (1859). Über die Anzahl der Primzahlen unter einer gegebenen Größe.

extreme to belong to a certain quadrant, but rather that providing/wanting to provide the tools required by that quadrant means belonging to that quadrant. A quadrant is only a direction; for example, the second quadrant does not really need to provide an exact prime-counting tool, it only needs to embody the desire to solve the error upper bound problem in order to obtain density. So it is not that everything in the second quadrant is a direct computational tool, but rather that everything in this quadrant serves computation.

This paper attempts to discuss prime numbers from the x-axis, that is, essentially rather than appliedly, discussing a first or fourth quadrant prime number theory, and in degree differs from current second quadrant prime number theories, attempting to reach the extreme/100%—completely and exactly understanding the essence of prime numbers and their mechanism (including distribution patterns) and being provable. And in the end,

- if it fails to grasp the essence of prime numbers, then it is merely a third quadrant prime number theory or accidentally becomes a second quadrant prime number theory,

if it grasps the essence of prime numbers, then

- if it satisfies polynomial-time computability, or directly exhibits the aforementioned accident, then it is a first quadrant prime number theory;
- if it does not satisfy this, it is the fourth quadrant, such as a method that sacrifices computability/relies on enumeration but can understand prime numbers.

It is worth noting the difference from the third quadrant—even if in the end it can only become a fourth quadrant prime number theory, that is, like the third quadrant, it is not polynomial-time computable/belongs to enumeration, the enumeration of the fourth quadrant is also different from the ordinary traditional enumeration of the third quadrant—even if it lists all prime numbers, it clearly knows in advance the pattern by which the prime number sequence is listed and in what manner it is listed, it is merely that the action still requires listing. Whereas third quadrant prime number theory, i.e., traditional enumeration such as the Sieve of Eratosthenes, counts one by one, having no concept at all of "why this one was obtained now" or "how the next will be obtained." Therefore, if a fourth quadrant prime number theory exists, its enumeration is merely an unavoidable operation caused by methodological limitations, but it has already predictively grasped the structure of prime numbers—already knowing what all the later prime numbers will be and how they will emerge; the focus is on understanding prime numbers rather than computation, which also determines that fourth quadrant prime number theory cannot independently replace second quadrant prime number theory. Of course, second quadrant prime number theory, because it does not understand the mechanism of prime numbers, likewise cannot replace the theoretical role of the fourth quadrant—the first quadrant is perfect, the second and fourth quadrants are complementary, and the third quadrant is backward.

1 Prime Numbers: Skip-Multiplication and Its Skip-Numbers

When seeking the pattern of prime numbers, we should focus on why prime numbers exist/why the phenomenon of prime numbers occurs. Once its occurrence is clearly understood, its pattern will naturally also be known. The reason the pattern of prime numbers has not been found is that we are not understanding it, but merely understanding its "phenomenon"—the traditional definition of a prime number is a natural number greater than 1 that has no other factors besides 1 and itself, and this definition starts too much from the prime number itself, as if the prime number were originally there, not born because of others (that is, "a number that has no other factors besides 1 and itself" seems to need no cause). In fact, when we change perspective, we find that prime numbers are not so independent; their appearance is very dependent on multiplication:

What is a prime number? Or, what does the definition "has no other factors besides 1 and itself" itself mean? This is actually only saying that a prime number is a number skipped by some products. A number is prime only when it cannot be multiplied out between numbers greater than 1, that is, when it does not serve as the product of multiplication between numbers greater than 1. And "does not serve as the product of multiplication between numbers greater than 1" means it is skipped by all multiplications between numbers greater than 1, that is, prime numbers are actually these interval numbers between the products of all numbers greater than 1. For example, the products between numbers greater than 1, that is, numbers starting from 2, each multiplied by numbers starting from 2 (in discussions about prime numbers, non-integers are not in the topic, so it is all integers starting from 2), namely

$$\begin{gathered} 2\times 2, 2\times 3, 2\times 4\cdots \\ 3\times 2, 3\times 3, 3\times 4\cdots \\ 4\times 2, 4\times 3, 4\times 4\cdots \\ \vdots \end{gathered}$$

the products are respectively

$$\begin{gathered} 4, 6, 8\cdots \\ 6, 9, 12\cdots \\ 8, 12, 16\cdots \\ \vdots \end{gathered}$$

the skipped numbers, that is, the numbers that fail to become products, are respectively

$$\begin{gathered} 4, \{5\}, 6, \{7\}, 8\cdots \\ 6, \{7, 8\}, 9, \{10, 11\}, 12\cdots \\ 8, \{9, 10, 11\}, 12, \{13, 14, 15\}, 16\cdots \\ \vdots \end{gathered}$$

namely

$$\begin{gathered} 5, 7\cdots \\ 7, 8, 10, 11\cdots \\ 9, 10, 11, 13, 14, 15\cdots \\ \vdots \end{gathered}$$

We first call these numbers "skip-numbers s," the set of skip-numbers is S, and prime numbers must be among them. Prime and composite numbers are denoted p and c respectively, and the set of prime numbers and the set of composite numbers are denoted P and C respectively:

$$\mathrm{P} \subset \mathrm{S}.$$

Merely "belongs to," not "is." Prime numbers must belong to skip-numbers, but skip-numbers are not necessarily prime, such as 8, 9, 10, 14, 15⋯ among them are composite:

$$\mathrm{P}, \mathrm{C} \subset \mathrm{S}.$$

And, as just said, a number is prime only when it cannot be multiplied out between numbers greater than 1, so that it has no other factors besides 1 and itself. The reason these composite numbers among the skip-numbers still have factors besides 1 and itself even though they are among the skip-numbers is that they simply "were not skipped clean." Very simply, when a number is composite, it must be the product of multiplication between numbers greater than 1, and appearing as the product of such multiplication, it cannot be one of the intervals skipped by this multiplication (our definition of skip-number is the numbers skipped by these products), such as 9 skipped in the multiplication of 2 against numbers starting from 2 appears in the product of 3 against numbers starting from 2 (3×3). Therefore, skip-numbers can have prime and composite numbers, but the numbers that are skip-numbers in all multiplications of numbers greater than 1 are only prime numbers, because in any such multiplication it is an interval between products and fails to become a product, such as 7 is skipped in the multiplications of 2, 3, 4 against 4, 3, 2 respectively, while composite numbers are only skipped in these multiplications, yet are products in other multiplications. Therefore, when we limit the scope of discussion to "in all cases are skip-numbers," that is, skip-rate 100%, then the set of skip-numbers is the set of prime numbers:

$$\mathrm{P} = \mathrm{S},$$

and composite numbers are merely skip-numbers in specific cases. Or we divide skip-numbers into half-skip numbers/non-full-skip numbers and full-skip numbers, with composite numbers being non-full-skip numbers and prime numbers being full-skip numbers, denoted $\mathrm{S}_{\mathrm{Partial}}$ and $\mathrm{S}_{\mathrm{Full}}$ respectively:

$$(\mathrm{C} = \mathrm{S}_{\mathrm{P}} := \{n \in \mathbb{N} | \exists a, b \geq 2, ab = n\}) + (\mathrm{P} = \mathrm{S}_{\mathrm{F}} := \{n \in \mathbb{N} | \forall a, b \geq 2, ab \neq n\}) = \mathrm{S}.$$

And thus, we have a clear definition of prime numbers—full-skip numbers S_{F}—numbers skipped by all multiplications of numbers greater than 1/numbers that do not serve as any product of all multiplications of numbers greater than 1; it can only be the product of itself and 1.

Summary Because the concept of "skipping in multiplication" is examined, it must be a phenomenon of multiplication between numbers starting from 2 — 1 multiplied by any number is obtained by stacking itself (that is, from 1 to the product, everything is filled by 1 itself), and the multiplication of numbers less than 2 has no skip-numbers. Only the multiplication of numbers greater than 1, $n(n > 1)$, skips at least $n - 1$ numbers, such as 2 multiplied by itself gives the product 4, skipping $2 - 1 = 1$ number, 3:

$$2 \times 2 = 4 \text{ skips } 3,$$
$$2, \{3\}, 4.$$

Since the factors are all $n(n > 1)$, the product cannot still be itself (the case of multiplying by 1), nor can it be merely itself $+1$ (the case of 1 multiplied by another number greater than 1), that is, $2n(n > 1)$ must skip at least $n - 1$ numbers, so the factor and the product are separated by an interval of at least $n - 1$—skip-numbers. From this it can be known that the skip-number is neither itself (the factor) nor the product with respect to the factors of this multiplication, that is, the product of any number $n(n > 1)$ multiplied by any number $m(m > 1)$ must lie in the range after the factors and before the product, which also defines the "skip-number interval" $(n \wedge m, nm)$. When 2 begins to multiply numbers greater than itself, such as 2 times 3 (or no longer using 2 to multiply, but looking at the multiplication of numbers greater than 2), the skipped numbers increase from one to many, such as the product of 2 times 3 is 6, for 2 it skips 3, 4, 5, for 3 it skips 4, 5:

$$\begin{cases} 2 \times 3 = 6 \ \text{ skips } \ 3, 4, 5, \\ 3 \times 2 = 6 \ \text{ skips } \ 4, 5. \end{cases}$$
$$\begin{cases} 2, \{3, 4, 5\}, 6, \\ 3, \{4, 5\}, 6. \end{cases}$$

Thus

- We call this kind of multiplication containing skip-numbers skip-multiplication $\mathcal{M}_s := \{ab | a, b \in \mathbb{N}, a, b \geq 2\}$, that is, multiplication based on 2×2 (factors not containing 1), and the reason skip-multiplication produces skip-numbers is that factors are all greater than 1/starting from 2. Having defined skip-multiplication, there's no need to describe it as "multiplication of numbers greater than 1."
- Here 2×2 is the smallest skip-multiplication in skip-multiplication, the basic unit of skip-multiplication. And the smallest skip-multiplication of any number greater than 1/number starting from 2 is itself times 2, that is, $2n$ is their smallest skip-number interval, and the number of skip-numbers is still $n - 1$. Therefore it can also be seen that when multiplying larger numbers, the number of skip-numbers skipped each time does not change; it is repeatedly skipping intervals of the same quantity, or when multiplying more multipliers (such as $a \times b \times c \times \cdots$), the smallest skip-multiplication interval can absolutely contain the numbers they will skip.
- It can also be found that to find prime numbers within any natural number N, one only needs to search among the skip-numbers of natural numbers before $\frac{1}{2} + 1$, that is, $\frac{N}{2} + 1$:

$$\mathrm{P} < \mathrm{S}_{\frac{N}{2}+1},$$

because the smallest skip-multiplication of any number is itself times 2, so one only needs to look at the skip-multiplication of those numbers whose product with 2 just exceeds N, because it can at most skip N and will not skip larger numbers; N will first appear as a skip-number in the skip-numbers of $\frac{N}{2} + 1$, that is, the skip-numbers of numbers after $\frac{N}{2} + 1$ all contain invalid numbers greater than N.

- Products are divided into two categories: skip-products and non-skip-products; products belonging to this kind of multiplication are called skip-products $p_s := \mathcal{M}_s = \{n \in \mathbb{N} | \exists a, b \geq 2, n = ab\}$, and only products of multiplication whose factors contain 1 are non-skip-products.
- The definition of prime numbers as full-skip numbers thus changes from "numbers skipped by all multiplications of numbers greater than 1" to "skip-numbers skipped by all skip-multiplications" (so the earlier statement that "prime numbers are not independent but depend on multiplication" means exactly this. Prime numbers are not so much the material of multiplication as commonly thought, but rather the numbers skipped by multiplication. The material of multiplication is more 2, because multiplication by 1 is simple stacking, and only 2 and the multiplication of numbers starting from 2 have "skipping," and thus produce a characteristic distinct from addition). "Multiplication of numbers greater than 1" = "skip-multiplication," and the "distribution pattern" of prime numbers is actually the "full-skip pattern" in skip-multiplication; answering how prime numbers are distributed is essentially answering how full-skip occurs, that is, why full-skip numbers are fully skipped. Composite numbers and prime numbers are understood through this subdivision of multiplication—skip-multiplication—

and are dependent, not independent. From this, although the mechanism is still unclear, it can be known that multiplication, or specifically skip-multiplication within multiplication, must be the entire cause of the zeros oscillating in the imaginary part yet pinned at 0.5 in the real part in the Riemann Hypothesis. Thus, we have grasped the first anchor point for discussion, whether for discussing the Riemann Hypothesis or simply for seeking the distribution pattern of prime numbers.

In fact, the interpretation of full-skip numbers is precisely dragging the definition of prime numbers back from "multiplicative material" to "irreducible element." Therefore, full-skip numbers are equivalent to irreducible elements, but the concept of "full-skip number" depends on skip-multiplication, and can be analyzed from the perspective of skip-multiplication theory, possessing more discussion angles in the direction of skip-multiplication theory.

The previous definition of prime numbers gave them too much independence, causing us, when facing a concept that itself has no content, to merely chase shadows. Only when we clarify the reason for the appearance of prime numbers themselves, that is, being skipped in skip-multiplication, do we discover some possible substantive traces.

2 The Distribution Pattern of Prime Numbers/The Full-Skip Pattern of Full-Skip Numbers

2.1 "Distribution"

Since current prime-number-related theories are mostly for applied purposes (the second quadrant mentioned in the Introduction), "distribution" only concerns counting, which is reasonable—but from a non-applied, purely mathematical theoretical perspective (first and fourth quadrants), the "distribution" of prime numbers should, besides quantity, also include another dimension of information—the value of each prime number. When we speak of "prime number distribution," we are essentially speaking of which numbers among all natural numbers are prime (their specific positions in the natural number sequence), and when we know which numbers are prime, we simultaneously know the quantity. If we only count, even if we accurately calculate the number of prime numbers in any interval, we will not know which numbers are prime. If what we know is the specific values of prime numbers, that is, which numbers are prime, then in fact we also know their positions in the natural number sequence, and grasping the identity of prime numbers itself logically completes the counting (although not the kind of function/formula desired by second quadrant prime number theory). Therefore, research on prime number distribution should also include research on the identity of prime numbers, which better conforms to the requirements of the concept of "distribution," while counting only includes information about density in the distribution.

2.2 Composite Number Distribution Pattern

Taking "distribution should include value information" as a starting point, let us look at the part concerning the pattern of values — after obtaining the concepts of skip-multiplication and the skip-numbers it skips, prime numbers are defined as full-skip numbers, and the distribution pattern is defined as the full-skip pattern. Now we begin to think about the distribution pattern of full-skip numbers/the full-skip pattern:

First, skip-numbers are the intervals between skip-products, so skip-numbers are definitely not skip-products; at least prime numbers, as full-skip numbers, will not be skip-products; second, composite numbers require factors all greater than 1, so composite numbers must be products of $n \times$

$m(n, m > 1)$, and $n \times m(n, m > 1)$, because it is multiplication of numbers greater than 1, is necessarily a kind of skip-multiplication, so the product of $n \times m(n, m > 1)$ is of course a skip-product—composite numbers equal skip-products, and all composite numbers must be in the set of skip-products (so the earlier case of composite numbers as skip-numbers is accidental; now the case as skip-products is essential), and no composite number will be missed (after all, all factors required by all composite numbers are included). Thus, we can make the skip-product table/composite number table for $n \times m(n, m > 1)$ (considering page size, let us first look from 2 to 10):

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

This is the skip-product table when both n and m take $[2, 10]$; extending infinitely in both axis directions gives the complete skip-product table/skip-product set when both n and m take $[2, +\infty)$ (it is known that prime numbers are infinite and prime numbers are full-skip numbers, so theoretically full-skip numbers, i.e., intervals between skip-products, appear infinitely, so skip-products also appear infinitely). Since both n and m start from 2, the factors do not contain 1, and prime numbers absolutely do not appear in the table/all skip-products appearing in the table are absolutely composite; and, as just said, the smallest skip-multiplication interval is 1 skip-number, that is, the skip-number interval of 2×2, so examining the product table of $n \times m$ with two factors can cover all possible skip-multiplication cases. This is an important material of this paper.

And, it is known that prime numbers after 2 are all odd, and listing the set of odd numbers is trivial content in mathematics; we can at any time list any interval of odd numbers starting from 3 (can be all odd numbers, i.e., the set of odd numbers itself), by any means, whether a table or a sequence, such as within 100:

	3	5	7	9	11	13	15	17	19
21	23	25	27	29	31	33	35	37	39
41	43	45	47	49	51	53	55	57	59
61	63	65	67	69	71	73	75	77	79
81	83	85	87	89	91	93	95	97	99

Now, we have two tables, the composite number table just now and the odd number table listed now— since the composite number table contains all composite numbers, it of course covers the composite numbers among odd numbers, and we can always use the composite number table to mark out all composite numbers in the odd number table:

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160

	3	5	7	9	11	13	15	17	19
21	23	25	27	29	31	33	35	37	39
41	43	45	47	49	51	53	55	57	59
61	63	65	67	69	71	73	75	77	79
81	83	85	87	89	91	93	95	97	99

(Due to page size limitations, 51, 57, 69, 87 can be found in the composite number table at $n = 3$ with $m = 17, 19, 23, 29$, and 85, 95 at $n = 5$ with $m = 17, 19$.) Composite numbers appearing in the odd number table will definitely be in the composite number table. In this way, through reverse selection, we can always obtain all prime numbers in any interval:

	3	5	7	9	11	13	15	17	19
21	23	25	27	29	31	33	35	37	39
41	43	45	47	49	51	53	55	57	59
61	63	65	67	69	71	73	75	77	79
81	83	85	87	89	91	93	95	97	99

When we understand through the concept of skip-multiplication that composite numbers are actually skip-products, and obtain the skip-product table, with the aid of computers, we can always sieve out prime numbers in any interval, even astronomical numbers, in polynomial time — the "screening" operation is even trivial in difficulty for computer use. When the table has already listed all possible composite numbers, and we can always list all odd numbers and reverse-select, then we can predict all possible prime numbers, know all positions of prime numbers in the natural number sequence, without needing to examine the target interval in advance, and theoretically it is non-enumerative:

- In application it is still enumeration, because listing the table itself is enumeration;
- In theory, in terms of "grasping the pattern," we do not need to list the table to verify this composite number distribution pattern; rather, by explaining that this pattern itself depends on multiplication, we explain that if multiplication exists, this pattern exists—by discovering its manner of dependence on multiplication, we discover its distribution pattern structurally rather than enumeratively, so for the grasp of the composite number distribution pattern in any interval, it satisfies predictiveness rather than enumerativeness. One of the differences between the fourth quadrant and the third quadrant.

From this, we know the distribution pattern of prime numbers—the full-skip pattern of full-skip numbers: skip-multiplication proceeds with skip-products as nodes, and the numbers in the table are exactly the places where each skip of skip-multiplication lands. That is, what does not appear in the skip-product table must have been skipped. The distribution pattern of prime numbers is strongly correlated with the bypass pattern of skip-multiplication; prime numbers are distributed in the form of non-skip-products (distributed by bypassing skip-products). And because listing depends on enumeration, this method relying on listing, although able to expound the distribution pattern of primes, largely sacrifices computability, leaving only understanding value in the pure theoretical domain.

Supplement 1: The Even-Odd Pattern of the Skip-Product Table If one merely uses the composite number table to mark the odd number table, a very obvious problem will be found—there exist even composite numbers in the composite number table, with many invalid interference items (always close to half). For computers, screening out even composite numbers/screening out odd composite numbers is not difficult, but for us it is somewhat laborious. However, there is a method of handling this—the odd-even distribution in the composite number table has an obvious pattern—the odd-even of rows and columns is exactly opposite to the odd-even of the cell contents; odd rows/columns are all even, and even rows/columns are all odd:

Because both n, m start from 2, that is, the first row/column is $n, m = 2$, and an even number multiplied by any natural number is even, so even numbers are all allocated to odd rows/columns—the first, third, fifth⋯ rows/columns are all even (conversely, even rows/columns are all odd):

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Thus we obtain the skip-product table when n and m are odd, and odd times odd must give odd, so we will definitely obtain the odd skip-product table/odd composite number table:

	3	**5**	**7**	**9**
3	9	15	21	27
5	15	25	35	45
7	21	35	49	63
9	27	45	63	81

$$9, 15, 21, 25, 27, 35, 45, 49, 63, 81 \cdots \in C.$$

Thus invalid interference items are eliminated, and now the skip-products that appear are directly the composite numbers that will appear among odd numbers without needing screening, so this table largely approaches the odd number table itself.

Supplement 2: The Skip-Product Table as a Computational Tool Although the skip-product table relies on enumeration because it is a table, it is itself a square, and $n \times m$ can yield all cell counts—in a sense, if the number of cells equals the number of composite numbers, then it is a very concise counting form, that is, it seems it could in fact belong to the second quadrant. But the reason it cannot be directly calculated this way is also obvious:

3 Limitations in Counting

3.1 The Repetition Pattern of the Skip-Product Table

Using the skip-product table only through the above can only let us theoretically predict prime numbers in any interval (equivalent to grasping the pattern), but only know which prime numbers there will be, without intuitively directly knowing how many prime numbers there will be/the quantity of prime numbers ("not intuitive" means that listing the table certainly includes quantity information, but without counting and without using a computer, one will still encounter the following problems)—although the composite number table is $n \times m$, the number of composite numbers is not simply nm (that is, equal to the number of cells), because there exist repeated composite numbers in the composite number table:

"First type" of repetition: commutative law repetition Since the skip-product table is essentially a product table, and the factors n and m both start from the same number (2)—the commutative law of multiplication will certainly take effect. For example, the skip-product of the first row second column and the first column second row is both 6, because $n_1 = 2$ encountering $m_2 = 3$ is equivalent to $m_1 = 2$ encountering $n_2 = 3$, both being $2 \times 3 = 6$. This causes no repetition only when the row and column numbers are equal, such as the first row encountering the first column, the second row encountering the second column, that is, when n and m have the same value, i.e., n^2 or m^2, because the commutative law of multiplication cannot take effect, and there is no difference before and after exchange. That is, when n and m have different values, half of the total skip-products are repeated, and each $(nm)_1 (n \neq m)$ has a corresponding $(nm)_2$ due to the commutative law.

	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Since each time the n-axis increases by $+1$, the m-axis has a corresponding equal $m+1$ (and vice versa), the case of n^2 occurs strictly according to the pattern—extending along the diagonal, and the two sides of this line are completely identical. Then we can always simply use this diagonal as a boundary to determine the parts to retain/remove:

	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3		9	12	15	18	21	24	27	30
4			16	20	24	28	32	36	40
5				25	30	35	40	45	50
6					36	42	48	54	60
7						49	56	63	70
8							64	72	80
9								81	90
10									100

or

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	4								
3	6	9							
4	8	12	16						
5	10	15	20	25					
6	12	18	24	30	36				
7	14	21	28	35	42	49			
8	16	24	32	40	48	56	64		
9	18	27	36	45	54	63	72	81	
10	20	30	40	50	60	70	80	90	100

Thus, we obtain the upper triangular table or lower triangular table—

$$\frac{nm - n\vee m}{2} + n\vee m. \tag{1}$$

Because the number of cells in the square total table before removing commutative law repetition is nm, and since each $(nm)_1$ has a corresponding $(nm)_2$, the two sides of the n^2/m^2 diagonal are equal in number, and because the number of cells in the n^2/m^2 diagonal is $n\vee m$ (that is, the row/column numbers), the triangular table after removing "commutative law repetition" is: the square total table nm minus the n^2/m^2 diagonal, then half of the two halves, then adding back the n^2/m^2 diagonal—this is the composite number table without commutative law repetition. Therefore, when we are given a target interval and asked for the number of composite numbers, since composite numbers in the target interval will not repeat, we need to remove all $(nm)_2$, that is, the number of all n^2 plus $\frac{1}{2}$ of the number of all other nm. With Q as quantity and c as complement:

$$Q_{n^2} + \frac{Q_{(n^2)^c}}{2}.$$

Or taking the complement in quantity:

$$Q_{n^2} + \frac{{Q_{n^2}}^c}{2}.$$

Thus, the repetitions caused by the commutative law among non-n^2 exist in a pattern, and these repeated quantities can always be determined and predicted non-enumeratively, so they can always be removed. This is speaking of the structure of the skip-product table; if according to Supplement 2.2, following the actually superior prime-counting method of removing even composite numbers, it is also the same. Here the skip-product table containing even numbers is used only to express that this pattern is a pattern of cell counts in the skip-product table; it makes no difference what numbers are placed in the cells.

However, when our topic involves N, that is, discussing composite numbers below N, the triangular table is still not enough—for example, if I want to find composite numbers below 10, that is, $N = 10$, 10 itself is not a square number (not on the n^2/m^2 diagonal):

	2	**3**	**4**	**5**
2	4			10
3		9		
4			16	
5				25

So there is no anchor point to serve as the maximum value at the lower right corner of the table; we do not know which square number to choose to restore a square table, and cannot draw a square table, let alone a triangular table based on it. Therefore, if we want to find composite numbers below any N, and N is not a square number—

1. First see whether the requirement "below N" includes N; if it includes, enter step 2; if it does not include, enter step 3.

2. See whether N itself is composite. If it is, find the square number n^2/m^2 closest to N (that is, with the smallest difference) and greater than N (that is, the first square number after N) to enter step 4; if it is not, then see whether there is a composite number between the square number closest to N and less than N (that is, the first square number before N) and N. If there is, still find the square number n^2/m^2 closest to N and greater than N, then enter step 4; if there is not, directly use the square number closest to N and less than N to enter step 4.

3. See whether there is a composite number between the square number closest to N and less than N and N. If there is, find the square number closest to N and greater than N and enter step 4; if there is not, directly use the square number closest to N and less than N to enter step 4.

4. Find the square root x of this square number, and use x as n and m to draw the square total table nm.

5. Obtain the total number of cells in the square total table $nm = x^2$ and substitute into (1):

$$\frac{x^2 - x}{2} + x$$

to obtain the sum of the number of composite numbers and the number of errors, and construct the triangular table.

Next, even if we know how to draw the triangular table when finding composite numbers below N and N is not a square number, this triangular table does not necessarily contain all composite numbers below N (as can already be glimpsed in the table in 2.2). For example, when finding composite numbers below 9, that is, $N = 9$, the triangular table containing 9 ($n, m = 3$) does not contain 8 (need multiply number 4):

	2	**3**	**4**
2	4	6	8
3		9	12

Such parts outside the triangular table need to be discovered in larger values, such as $N = 16$:

	2	**3**	**4**	**5**	**6**	**7**
2	4	6	8	10	12	14
3		9	12	15	18	21
4			16	20	24	28

$N = 25$:

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
2	4	6	8	10	12	14	16	18	20	22	24
3		9	12	15	18	21	24	27	30	33	36
4			16	20	24	28	32	36	40	44	48
5				25	30	35	40	45	50	55	60

$N = 36$:

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
3		9	12	15	18	21	24	27	30	33	36	39	42	45	48	51
4			16	20	24	28	32	36	40	44	48	52	56	60	64	68
5				25	30	35	40	45	50	55	60	65	70	75	80	85
6					36	42	48	54	60	66	72	78	84	90	96	102

It can be found that, because the interval by which N exceeds n_i/m_i is certainly smaller than the interval by which N exceeds n_{i+1}/m_{i+1}, such as $N-2$ is certainly larger than $N-3$, the smaller n/m extends farther, more fragmentedly, and larger (the last product); the larger n/m is, the farther the product closest to N but less than N. The smaller the multiplier, the smaller the skip-number interval, and the larger the product closest to N but less than N, so the missed parts outside the triangular table also form a triangle, but not as regular as our original triangular table. Taking the upper triangular table as an example, each n_i extends to m being $\frac{N}{n_i} - 1$, such as when $N = 36$, the row $n = 2$ extends to m being $\frac{36}{2} - 1 = 17$ (34), and the row $n = 3$ extends to m being $\frac{36}{3} - 1 = 11$ (33)$\cdots$ because the corresponding value at $m = \frac{N}{n}$ has already appeared once as the maximum anchor point (if it is the lower triangular table, each m_i extends to n being $\frac{N}{m_i} - 1$; if $\frac{N}{n_i}$ or $\frac{N}{m_i}$ is not an integer, the decimal part can be removed, that is, rounded down, and there is no need to subtract 1, because after removing the decimal part it is less than the original decimal form, and will not multiply out the anchor point). We can actively avoid some repetitions during the table-drawing process. Then the number of n_i is the side length $\sqrt{nm}$ of the total table, and because the last row n_i only has the maximum anchor point and does not extend, it is the total number of rows $\sqrt{nm}$ minus 1, so it extends $\sqrt{nm}-1$ times to $\frac{N}{n_i}-1$ or $\lfloor \frac{N}{n_i} \rfloor$, but no matter which one it is, the right half-region is

$$\left(\sqrt{nm}-1\right)\left[\left(\frac{nm}{n_i}-2\right)-\left(\frac{n_i\sqrt{nm}}{n_i}-1\right)\right],$$

where $\frac{nm}{n_i}-2$ is the number of all products that the factor n_i of any row will pass through on the way to multiplying all the way to N, so it also includes the part of the left half-region. Therefore, we subtract from $\frac{nm}{n_i}-2$ the part of the left half-region in that row, that is, subtract $\frac{n_i\sqrt{nm}}{n_i}-1$. Simplified:

$$\frac{nm\sqrt{nm}}{n_i}-nm-\frac{nm}{n_i}+1.$$

Therefore, composite numbers below N should be distributed in our regular triangular table and this irregular triangular table — an irregular inverted triangle extending upward with N or the square number anchor point closest to N as the maximum value:

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
3		9	12	15	18	21	24	27	30	33	36	39	42	45	48	51
4			16	20	24	28	32	36	40	44	48	52	56	60	64	68
5				25	30	35	40	45	50	55	60	65	70	75	80	85
6					36	42	48	54	60	66	72	78	84	90	96	102

$$\frac{nm-n}{2}+n+\frac{nm\sqrt{nm}}{n_i}-nm-\frac{nm}{n_i}+1,$$

simplified:

$$\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i},$$

and the original regular triangular table will miss many composite numbers. We can now regard the original triangular table $\frac{nm-n}{2}+n$ as the left half-region, and the missed part $\frac{nm\sqrt{nm}}{n_i}-nm-\frac{nm}{n_i}+1$ as the right half-region — the characteristic of the left half-region is that it can cover part of the composite numbers and provide the maximum square number anchor point for drawing the table and the right half-region, while the characteristic of the right half-region is that it can supplement the missed composite numbers to cover all composite numbers, but relies on the maximum square number of the left half-region as an anchor point, and with this maximum square number as the boundary, will approach this value without extending beyond it—although the right half-region is not covered by the left half-region we first discussed, it is also strictly dependent on left half-region, so it is not an uncontrolled independent missed region, but a clear extension of left half-region. If we originally took square total table, then it is not an inverted triangle, but an irregular polygon of a square + a triangle:

$$nm + \frac{nm\sqrt{nm}}{n_i} - nm - \frac{nm}{n_i} + 1 = \frac{nm\sqrt{nm} - \mathrm{nm} + n_i}{n_i},$$

but the characteristics of the left and right regions remain unchanged.

Returning to this section—the commutative law indeed does not affect counting, and the triangular table (left half-region $\frac{nm-n}{2} + n$) handles this very well. The key point is the second type of repetition:

"Second type" of repetition: multi-factor repetition It seems that besides commutative law repetition, there are other repetitions, that is, even after removing commutative law repetition, there are still repetitions in the remaining triangular table region—a composite number may have multiple different factorizations, and each factor pair in the skip-product table generates it once, such as 12 is both 2×6 and 3×4; 2, 3, 4, 6 are all factors of 12. This cannot be solved by resolving commutative law repetition:

	2	**3**	**4**	**5**	**6**
2	4	6	8	10	12
3		9	12	15	18

In fact, the essence of commutative law repetition is also a variant of this kind of repetition—multi-factor-pair repetition, that is, not only complexity in quantity, but their order can further strengthen the complexity. The reason a product nm repeats due to the commutative law is essentially that the factors in any factor pair can be swapped between axes to produce a new factor pair—although the factor pair (n, m) and the factor pair (m, n) contain the same factors, the axes on which the factors lie are different. Therefore, "first type" and "second type" need quotation marks; in reality the skip-product table only has repetition caused by multiple factors/factor pairs.

But the reason we still first independently discuss "commutative law repetition" is that, although it is caused by multi-factor repetition, it is the only part that can be solved first—the pure multi-factor repetition now mentioned logically cannot be independently solved—if we can find the distribution pattern of the factors of any product, we essentially also know the distribution pattern of prime numbers. If we can count the factor pairs of any product/accurately calculate how many factor pairs each product has, we essentially also certainly know how many products have only "itself and 1" as their factor pair, so there is no order such as first solving multi-factor repetition and then obtaining the composite/prime distribution pattern. Solving the multi-factor repetition problem is equivalent to solving the prime distribution (counting) problem. Therefore, this is a repetition we cannot handle, but we can first solve "commutative law repetition" to narrow the range as much as possible.

But being unable to completely solve multi-factor repetition does not affect that we already know the distribution of prime numbers—we are discussing the theoretical distribution pattern (x-axis/fourth quadrant), not the applied one, that is, focusing on understanding the pattern, not on operation. Theoretically, if the content of 2.1 is consensus, "listing all prime numbers" itself should include counting. So multi-factor repetition only prevents us from simply obtaining the quantity through

$\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i}$, that is, the number of cells does not equal the number of composite numbers, but $\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i}$ was not the theoretically preferred counting method in the first place.

3.2 Table Counting and Its Advantages and Disadvantages

Even if the multi-factor repetition problem is solved, leaving an accurate odd composite number table, this is still not an applied exact formula for density—as said in 2.2, the tabular method itself belongs to enumeration. As long as it is in this tabular counting form, it is generally not considered capable of finding prime density.

But, precisely because finding density is usually an applied-level problem, entering the applied level generally means one may use a computer. As said in 2.2, in most cases, sieving out all composite numbers below any large number is only a polynomial-time or even trivial operation for a computer. Therefore, when the skip-product table is paired with a computer, even if the multi-factor repetition problem is not solved (batch filtering of repetitions is still a trivial operation for computers), to a large extent it is still a proven exact method, with certain advantages compared to, for example, unproven, inexact/still-estimating methods. So the skip-product table does have deficiencies in counting, but when practical counting generally uses computers, such differences are largely erased, leaving only advantages, even if not suitable for all situations.

Appendix

A The Ratio Pattern of "Commutative Law Repetition" in the Skip-Product Table

In the skip-product table, the ratio of the repeated number $(nm)_2$ caused by "commutative law repetition" to the non-repeated number ${(nm)_2}^{c}=(nm)_1+n^2$ also has a pattern—drawing a square region starting from $n=2$ (that is, n and m have the same value, and the number of rows and columns is consistent), such as

4	6
6	9

$n,m=2$

4	6	8
6	9	12
8	12	16

$n,m=3$

4	6	8	10
6	9	12	15
8	12	16	20
10	15	20	25

$n,m=4$

(Note: at this time n,m refer to the nth row and mth column, not the corresponding factors. For example, because skip-multiplication starts from 2, the first row/column $n,m=1$ corresponds to the skip-multiplication multiplier 2, and the second row/column corresponds to 3—row/column numbers $n,m\neq$ multipliers; looking at multipliers starting from 2, row number $n=$ multiplier $-\ 1$.) As said before—because n and m start from the same number, $n+x$ always has a corresponding position $m+x$, so n^2/m^2 has a pattern of appearing along the diagonal, causing each row/column to have an n^2/m^2, and when truncated to the xth row/column, that is, when n and m take the xth value, there are x n^2/m^2; the number of n^2/m^2 is x.

4	6
6	9

$n, m = 2$

4	6	8
6	9	12
8	12	16

$n, m = 3$

4	6	8	10
6	9	12	15
8	12	16	20
10	15	20	25

$n, m = 4$

At this time, excluding n^2/m^2, the quantities of the remaining two kinds of nm, namely $(nm)_1$ and its repetition $(nm)_2$, are consistent:

$$Q_{(nm)_1} = Q_{(nm)_2},$$

so either quantity plus the number of n^2/m^2, that is, x, is naturally greater than the other by x. Therefore, there is always the ratio:

$$\begin{cases} Q_{(nm)_1} + x : Q_{(nm)_2} \\ Q_{(nm)_2} + x : Q_{(nm)_1}. \end{cases}$$

And what we want to remove is $(nm)_2$, so the final remaining quantity considered is $Q_{(nm)_1} + x$ (although it is the same as $Q_{(nm)_2} + x$, it is logically distinguished), taking the first ratio. Then it can be found that the ratio of the non-repeated skip-product $Q_{(nm)_1} + \left(x = Q_{n^2}\right)$ to the repeated skip-product $Q_{(nm)_2}$ has a pattern:

4	6
6	9

$n, m = 2$
$3 : 1$

4	6	8
6	9	12
8	12	16

$n, m = 3$
$6 : 3$

4	6	8	10
6	9	12	15
8	12	16	20
10	15	20	25

$n, m = 4$
$10 : 6$

By analogy:

$$\begin{gathered} (n, m = 5) \Rightarrow (15 : 10), \\ (n, m = 6) \Rightarrow (21 : 15), \\ (n, m = 7) \Rightarrow (28 : 21), \\ \vdots \end{gathered}$$

Thus, the ratio pattern of non-repetition and repetition:
$Q_{(nm)_1}$ pattern: when n and m are x, the quantity $Q_{(nm)_1}$ of $(nm)_1$ is always $Q_{(nm)_1} + x$ when n and m are $x - 1$, such as when the third row/column, that is, $n, m = x = 3$, $Q_{(nm)_1} = 6$ is $\left(Q_{(nm)_1} = 3\right) + (x = 3) = 6$ when the second row/column, that is, $n, m = x = 3 - 1 = 2$;
$Q_{(nm)_2}$ pattern: when n and m are x, $Q_{(nm)_2}$ is always $Q_{(nm)_1}$ when n and m are $x - 1$, such as when the third row/column, that is, $n, m = x = 3$, $Q_{(nm)_2} = 3$ is $Q_{(nm)_1} = 3$ when the second row/column, that is, $n, m = x = 2$.
That is

$$\left(Q_{(nm)_1}\right)_{n,\,m=\mathrm{x}-1}+x=\left(Q_{(nm)_1}\right)_{n,\,m=x}=\left(Q_{(nm)_2}\right)_{n,\,m=x+1}.$$

From this, it seems we have a second method to determine the number of repetitions caused by the commutative law in order to remove repetitions, but because this is based on the discovery that "the two sides of the n^2/m^2 diagonal are consistent," it essentially still belongs to the first method mentioned earlier.

B An Independent Proof of the Oddness of Prime Numbers: The Repetition Pattern of Skip-Numbers

a Proof of the Oddness of Prime Numbers

We all know that prime numbers greater than 2 are odd/have oddness, but even if we do not know this information in advance, we can independently discover it by studying skip-multiplication:

This time we list the skip-number table rather than the skip-product table—listing all numbers that have skip-numbers (all numbers capable of performing skip-multiplication) and their smallest skip-number intervals, that is, all numbers starting from 2 and their smallest skip-number intervals, namely all numbers n starting from 2 and from the next number $n+1$ to the number before $2n$, $2n-1$:

2	3									**4**
3	4	5								**6**
4	5	6	7							**8**
5	6	7	8	9						**10**
6	7	8	9	10	11					**12**
7	8	9	10	11	12	13				**14**
8	9	10	11	12	13	14	15			**16**
9	10	11	12	13	14	15	16	17		**18**
10	11	12	13	14	15	16	17	18	19	**20**

We can find that, not only does the skip-product table have repetition, the skip-number table also has repetition—the kth skip-number of n is always the $k-1$th skip-number of $n+1$, such as the 2nd skip-number 5 of 3 is the 1st skip-number 5 of 4, and the 3rd skip-number 7 of 4 is the 2nd skip-number 7 of 5. This actually causes each number to be distributed along a diagonal in the lower-left direction (even including the row heads of non-skip-numbers), so unlike the repetition of skip-products, the repetition of skip-numbers is obviously more numerous, increasing with the number of rows. In fact it is like the same row repeating with the pattern of decreasing by 1 number each time (the first skip-number of $n-1$) and increasing by 2 numbers ($2n-1$ and $2n-2$). It is like the same row continuously moving to the left, and the range of visible values constantly changing.

But from this, in fact, if we determine the property of a certain number in a certain row, the properties of all its occurrences in this table are determined. For example, although 11 appears 5 times, as long as 11 is prime, these 5 occurrences of 11 are of course all prime. Using this property that "the same number in different rows has equal properties," we can find:

The second-to-last skip-number $2n_i - 2$ is always $2n_{i-1}$, such as the second-to-last number 4 of 3 is 2 times 2, and the second-to-last number 6 of 4 is 3 times 2.

2	3									**4**
3	4	5								**6**
4	5	6	7							**8**
5	6	7	8	9						**10**
6	7	8	9	10	11					**12**
7	8	9	10	11	12	13				**14**
8	9	10	11	12	13	14	15			**16**
9	10	11	12	13	14	15	16	17		**18**
10	11	12	13	14	15	16	17	18	19	**20**

In fact, according to the skip-number repetition pattern just mentioned—that the kth skip-number of n is always the $k-1$th skip-number of $n+1$—which gives rise to the diagonal structure, the upper right corner of each $2n_i - 2$ is exactly $2n_{i-1}$ (and continuing to extend to the lower left, it is the same number):

2	3	**4**							
3	4	5	**6**						
4	5	6	7	**8**					
5	6	7	8	9	**10**				
6	7	8	9	10	11	**12**			
7	8	9	10	11	12	13	**14**		
8	9	10	11	12	13	14	15	**16**	
9	10	11	12	13	14	15	16	17	**18**
10	11	12	13	14	15	16	17	18	19

Also because of the skip-number repetition pattern, the case of more skip-numbers from the end is merely repetition of the last first and second skip-numbers $2n-1$ and $2n-2$; in fact, the only new information appearing each time is $2n-1$ and $2n-2$, so all remaining skip-numbers can be divided into two halves—repetitions of $2n-1$ and repetitions of $2n-2$, that is, all skip-numbers are actually $2n-1$ and its repetitions and $2n-2$ and its repetitions, so all $2n_i - 2$ and their repetitions are $2n_{i-1}$, that is, half of the skip-numbers are $2n_{i-1}$.

Next,

1. Because it is the smallest skip-number interval table, and the right endpoint of the smallest skip-number interval is $2n-1$;

2. Any number times 2 is even, that is, $2n$ is even;

3. Any even number minus 1 is odd, that is, $2n-1$ is odd—

then all the last skip-numbers $2n-1$ of all rows in this table are odd:

2	3									**4**
3	4	5								**6**
4	5	6	7							**8**
5	6	7	8	9						**10**
6	7	8	9	10	11					**12**
7	8	9	10	11	12	13				**14**
8	9	10	11	12	13	14	15			**16**
9	10	11	12	13	14	15	16	17		**18**
10	11	12	13	14	15	16	17	18	19	**20**

And because any odd number minus 1 is even, the second-to-last skip-number $2n-2$ of each row is of course even again (because it involves the "second," whether the second from the beginning or the second from the end, only numbers starting from 3 have a second skip-number):

2	3									**4**
3	4	5								**6**
4	5	6	7							**8**
5	6	7	8	9						**10**
6	7	8	9	10	11					**12**
7	8	9	10	11	12	13				**14**
8	9	10	11	12	13	14	15			**16**
9	10	11	12	13	14	15	16	17		**18**
10	11	12	13	14	15	16	17	18	19	**20**

This is to say, $2n_i-2$ and its repetitions are all even—half of the skip-numbers are even, and they all equal $2n_{i-1}$:

2	3									**4**
3	4	5								**6**
4	5	6	7							**8**
5	6	7	8	9						**10**
6	7	8	9	10	11					**12**
7	8	9	10	11	12	13				**14**
8	9	10	11	12	13	14	15			**16**
9	10	11	12	13	14	15	16	17		**18**
10	11	12	13	14	15	16	17	18	19	**20**

And if a prime number has a factor greater than 2, that factor must be itself and the other factor is 1—so when a skip-number always has a number n_{i-1} greater than 2 that can be obtained by multiplying by 2, it is necessarily composite—the second-to-last number $2n_i - 2$ and all its repetitions are always composite. Then if this half of the skip-numbers is also the even part of the skip-numbers, prime numbers greater than 2 are forever among the odd numbers, that is, among $2n - 1$ and its repetitions. In other words, this half of the skip-numbers all fail to become full-skip numbers because they are $2n_{i-1}$, and are merely non-full-skip numbers.

In fact, this is just rediscovering from the perspective of the skip-number table a very basic logical inference: because skip-multiplication starts from 2, and if n is greater than 2 and even, it can necessarily be written as

$$n = 2 \times k,$$

then if any even number greater than 2 necessarily has factor 2, of course it cannot be prime.

b The Odd-Even Ratio Pattern of Skip-Number Repetition

We look at the skip-number table starting from 2, with the right endpoint of the interval unrestricted, such as

3	
4	5

$n = 2$
$[2, 3]$

3		
4	5	
5	6	7

$n = 3$
$[2, 4]$

3			
4	5		
5	6	7	
6	7	8	9

$n = 4$
$[2, 5]$

(same as above, here n refers to the nth row, not the right endpoint of the interval/the multiplier corresponding to the skip-multiplication.) We can find—because the last skip-number of each number is always odd, and as n (row number)/the right endpoint of the interval increases, one number is added row by row, resulting in an outermost column in any interval that contains the largest quantity of numbers and is entirely odd:

3	
4	**5**

$n = 2$
$[2, 3]$

3		
4	**5**	
5	6	**7**

$n = 3$
$[2, 4]$

3			
4	**5**		
5	6	**7**	
6	7	8	**9**

$n = 4$
$[2, 5]$

—in the skip-number table of any interval, odd numbers are always more numerous. Also with Q as quantity, odd numbers as O, even numbers as E, then:

$$(Q_{\mathrm{O}})_{n_i} > (Q_{\mathrm{E}})_{n_i}.$$

And as the skip-number table/n/the right endpoint increases, the odd-even ratio has the following pattern:

3	
4	**5**

$n = 2$
$[2, 3]$
$2 : 1$

3		
4	**5**	
5	6	**7**

$n = 3$
$[2, 4]$
$4 : 2$

3			
4	**5**		
5	6	**7**	
6	**7**	8	**9**

$n = 4$
$[2, 5]$
$6 : 4$

3				
4	**5**			
5	6	**7**		
6	**7**	8	**9**	
7	8	**9**	10	**11**

$n = 5$
$[2, 6]$
$9 : 6$

3					
4	**5**				
5	6	**7**			
6	**7**	8	**9**		
7	8	**9**	10	**11**	
8	**9**	10	**11**	12	**13**

$n = 6$
$[2, 7]$
$12 : 9$

The odd-even quantity difference, starting from $n = 3$/right endpoint 4, always increases by 1 every other row, such as when $n = 3$/right endpoint 4 the odd-even quantity difference is $4 - 2 = 2$, and this difference will be maintained once in the next row, that is, $n = 4$/right endpoint 5 ($6 - 4 = 2$), then increase by 1 in the row after that, that is, $n = 5$/right endpoint 6 ($9 - 6 = 3, 3 = 2 + 1$). By analogy, when $n = 6$/right endpoint 7, $n = 7$/right endpoint 8, $n = 8$/right endpoint 9, the odd-even ratios are respectively $12 : 9$, $16 : 12$, $20 : 16$, with differences respectively 3, 4, 4, and together with the previous differences arranged as

$$1, 2, 2, 3, 3, 4, 4.$$

At the same time it can be found that, starting from $n = 2$/right endpoint 3, the number of odd numbers at n_i is always the number of even numbers at n_{i+1}, such as the number of odd numbers at $n = 2$/right endpoint 3 is 2, while the number of even numbers at $n = 3$/right endpoint 4 is also 2; the number of odd numbers at $n = 3$ /right endpoint 4 is 4, and the number of even numbers at $n = 4$/right endpoint 5 is also 4⋯:

$$(Q_{\mathrm{O}})_{n_i} = (Q_{\mathrm{E}})_{n_{i+1}}.$$

It can also be found that, starting from $n = 2$/right endpoint 3, when n is even/right endpoint is odd (that is, the "next row" between the current row and the row after next mentioned above), n is always twice the odd-even quantity difference, such as when $n = 2$/right endpoint 3 the odd-even quantity difference is $2 \div 2 = 1$, and when $n = 4$/right endpoint 5 the odd-even quantity difference is $4 \div 2 = 2$⋯:

$$n_{\mathrm{E}} = 2Q_{\mathrm{O}} - 2Q_{\mathrm{E}}.$$

Thus, the number of odd skip-numbers in any interval is not only certainly greater than the number of even skip-numbers, but the manner of being greater also has a pattern.

C On the Riemann Hypothesis

Randomness Theory

The current view of the Riemann Hypothesis by people may be rather plain and simplified:

First, even among mathematicians, most believe that RH assumes there exists a definite upper bound for the error in prime counting, so the distribution of prime numbers has an exact pattern—however, logically speaking, claiming that there certainly exists a phenomenon of "an upper bound for the error" already claims that the error first exists, before there can be an "upper bound for the error," that is, it holds that error certainly exists in any prime counting (non-enumerative algorithms; sieves of course can have no error), so this claim itself implies the nonexistence of an exact pattern, because exactness should mean no error. Thus, "the error has an upper bound" is merely discovering a pattern, just as claiming "prime numbers are all in the natural numbers" is also discovering a pattern, but likewise neither is an exact pattern among patterns, only a weak conclusion (of course, among weak conclusions there are also stronger and weaker ones). So RH in fact assumes that the distribution of prime numbers cannot be exactly calculated by a non-enumerative algorithm, but this "cannot" itself has an exact structure determined by the zeros—the distribution of prime numbers has a certain determinism but is irreducible, that is, it can be said to be a pattern to a certain degree (a pattern accompanied by a small degree of randomness), or it can be said to be random to a certain degree (randomness accompanied by a known pattern). But when we say an object has a pattern, if it does not have a completely exact pattern, but only a pattern to a certain degree, then from the requirement of "only when an exact pattern is grasped can it be considered that a pattern exists," this itself does not meet the requirement and leans more toward randomness, so in this part, saying "RH holds that the distribution of prime numbers is random" is more consistent with the facts; it is only asking "what is the upper bound of this randomness" (the precise narrowing of the zero range only gives the maximum variation range of the error; within this range there is still no pattern to follow. Even if a fixed value is given, this value is only an upper bound, that is, only an oscillation range, without the oscillation pattern within that range) — although the cost is deviating from the terminology habits of the mathematical community;

Second, if one holds that RH assumes a randomness with an upper bound (that is, the view just stated), so RH holds that the distribution of prime numbers is random—this is also imprecise. As said above, "saying 'RH holds that the distribution of prime numbers is random' is more consistent with the facts" is only "in this part," that is, when we have only thought up to this point should we draw this conclusion, but RH does not stop thinking here — RH assumes the upper bound is $O(\sqrt{N}\ln N)$, requiring only that the specific value be less than it, not equal to it. This means the error can also be 0, and when the error is 0, it means an exact calculation algorithm has been found—an exact pattern exists, that is, it is not random. So, claiming that error exists indeed indicates that there is an error term, but the upper bound including 0 also indicates that the error term can equal 0—then the error term can be used to indicate that there is no error. Thus, RH itself does not determine randomness, but only more broadly assumes that the error is below a given upper bound;

Finally, still, "the error can be 0" is only not thinking further. If one continues—it is known that Littlewood proved that the error term changes sign infinitely many times[4], and $\pi(N)$ is a step function rather than a smooth function, and moreover after Mellin transform the pole structures on both sides are unequal… most of these are elementary trivial observational facts and do not need proof. So, only the Riemann upper bound tells us "the error can be 0," but other mathematics at the same time also provides a lower bound—in known mathematics, we already know the error cannot be 0. Therefore, the truly correct posture is that we indeed hold that "the error is not 0," that is, we indeed take randomness theory as the starting point, but not because of RH, but because of more basic theorems. So directly concluding from RH allowing the error to be 0 that the distribution of prime numbers can have an exact pattern is overinterpretation beyond RH itself — the distribution of prime numbers should still be considered random, and we should still be randomness theorists on this issue, but the reason is not RH.

So, we first clarify our basic viewpoint—the distribution of prime numbers is random, and clarify where this viewpoint comes from—mathematics other than RH, and only then can we, on the basis of holding that the distribution of prime numbers is random, begin to prove some small patterns within this randomness, such as RH (because proving RH can only show that this randomness is not so random, and cannot directly eliminate the randomness at its foundation; that would require a stronger exact distribution pattern than an oscillation range upper bound, so objectively speaking it is only a small pattern—relative to an exact pattern, any pattern is small/weak. Of course, weak does not mean easy to prove, and strong does not mean hard to prove; for example, the skip-product table can reveal the structure of prime and composite numbers, and is exact in numerical identity, but is not hard to prove—on the contrary, it is very trivial, a table-listing operation; while RH has troubled humanity to this day).

Supplement "Random" refers to the conclusions we can draw, not an assertion about whether prime numbers are in fact random in the world. Whether prime numbers are in fact random we do not know; their random manifestations can always be explained as "we just have not found the pattern yet."

[4] Littlewood, J. E. (1914). Sur la distribution des nombres premiers.

a Extension

The RH error term $\pi(N)-\mathrm{Li}(N)$ has an optimal absolute upper bound $O\left(\sqrt{N}\ln N\right)$, and this error term has a unique mapping in skip-multiplication theory (specifically the skip-product table):

First, in Riemann's equation there is only one error term, $\pi(N)-\mathrm{Li}(N)$, that is, it contains all errors in any prime counting—there is no other term in the equation used to represent "other errors"; theoretically all possible errors are in $\pi(N)-\mathrm{Li}(N)$. Thus, $\pi(N)-\mathrm{Li}(N)$ is a closed system, and the final errors of all prime-counting methods will naturally be uniformly packed and regarded as this one part $\pi(N)-\mathrm{Li}(N)$. This means that it is not that some prime-counting method must actively correspond to the content of RH, but that RH itself claims that any prime-counting method theoretically follows its requirements—the extension is more required by RH itself, not work that prime-counting methods need to specially do additionally. RH naturally provides support for the rough extension; this is determined by its own nature (assertion).

Second, if the prime-counting method used has errors in multiple directions, that is, the error is caused by multiple factors, then to find the optimal absolute upper bound of $\pi(N)-\mathrm{Li}(N)$ one needs to separately find the optimal upper bound of each error factor, because $\pi(N)-\mathrm{Li}(N)$ must absorb error sources from all directions (approximate integration, choice of main term, local fluctuations in prime distribution, etc.). For a single-source-error prime-counting method with only one error factor, its error term does not need to be "packed" and is directly equivalent to $\pi(N)-\mathrm{Li}(N)$, and the optimal upper bound of this error term is directly $O\left(\sqrt{N}\ln N\right)$.

Then, skip-multiplication theory has only one factor causing counting error — multi-factor repetition—so the skip-product table is exactly such a single-error-factor prime-counting method—as said in Supplement 2 of Chapter 2, $n\times m$ itself is a computation, so the triangular table $\frac{nm-n\vee m}{2}+n\vee m$ is of course also a computation—and because Riemann's equation allows one error term, the reason we originally thought $n\times m$ or $\frac{nm-n\vee m}{2}+n\vee m$ could not be directly used, namely "multi-factor repetition," happens at this time to become the reason the formula can be used — it conforms to the requirement of the error term in Riemann's equation. That is, if $\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i}$ is taken as $\mathrm{Li}(N)$, then that error term is exactly "multi-factor repetition"—using $\pi(N)$ minus $\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i}$. Or conversely, it is precisely because error exists that it becomes an algorithm rather than a sieve, because sieves generally can achieve zero-error enumeration. Thus, the skip-product table, besides the fourth quadrant, can now also involve the second quadrant and enter the topic of RH (RH is a conjecture about error, and error refers to the error in counting prime numbers, so it only applies to counting methods such as the first and second quadrants. The third and fourth quadrants do not count, so naturally there is no counting error). The error caused by multi-factor repetition is directly $\pi(N)-\mathrm{Li}(N)$, and the optimal repetition upper bound of multi-factor repetition should be $O\left(\sqrt{N}\ln N\right)$ (it must be the "optimal" upper bound and not merely an ordinary upper bound. If it were merely an ordinary upper bound, it would not be the upper bound Riemann wanted, but rather an upper bound already proposed long ago by Gauss, Legendre, and others). Thus, Riemann's hypothesis is now transformed—there is no longer any need to deal with the abstract "zeros and $\mathrm{Li}(N)$"; one only needs to prove that "the total amount of multi-factor repetition" has an optimal upper bound, that is, the error caused by multi-factor repetition and its optimal absolute upper bound are equivalent to Riemann's error and its optimal absolute upper bound. With E denoting error, we have

$$\begin{cases} E_{\text{multi-factor repetition}} = \pi(N) - \mathrm{Li}(N) \\ O_{E_{\text{multi-factor repetition}}} = O\left(\sqrt{N}\ln N\right). \end{cases}$$

And, the cause of multi-factor repetition is that the factors are "many"; skip-products/composite numbers have "many" factors/factor pairs (the second part of 3.1). Then the absence of an optimal upper bound for the error caused by multi-factor repetition manifests as "the manyness of factors/factor pairs" having no optimal upper bound, that is, the factors/factor pairs of skip-products/composite numbers increasing non-smoothly, suddenly, and in large quantities. Thus, $O\left(\sqrt{N}\ln N\right)$ not existing/the zeros not being on the real part 0.5 manifests as factor explosion/factor-pair explosion. Therefore, Riemann's tools can be extended into skip-multiplication theory; proving the RH is to prove that the factors/factor pairs of skip-products will not explode—having a limited "manyness" (and a rather good limitation).

b Non-Trivial Upper Bounds

Current Situation

At present the best tool in our hands is the odd inverted triangular table (having removed even numbers and "commutative law repetition"). Let us see how many repetitions exist in the result of $\frac{n-nm}{2}+1+\frac{nm\sqrt{nm}-nm}{n_i}$—when we input an N, if the number of repetitions is greater than $\sqrt{N}\ln N$ then this triangular table does not satisfy the RH requirement; otherwise it satisfies it.

We can simply enumerate and find,

- When $N = 100$, $\sqrt{N}\ln N \approx 46$, while the number of repetitions is only 5:

	3	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**	**31**	**33**
3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	99
5		25	35	45	55	65	75	85	95							
7			49	63	77	91										
9				81	99											
11					(121)											

(marked up to 121 because according to 3.1, if there is a composite number between the square number closest to N and less than N and N, then find the square number closest to N and greater than N.)

$5 < 46$, so when $N = 100$ it is below the upper bound required by Riemann;

- When $N = 200$, $\sqrt{N}\ln N \approx 74$, while the number of repetitions is only 19 (when N is too large the extension of the row $n = 3$ becomes very long and the table becomes very long, so multiplication expressions are used; among them the products of the multiplication by 3, that is, the row $n = 3$, all appear for the first time, so repetitions are examined starting from the row $n = 5$):

$5 \times 9 = 45, 7 \times 9 = 63, 5 \times 15 = 75, 9 \times 9 = 81, 9 \times 11 = 99, 5 \times 21 = 105, 7 \times 15 = 105, 9 \times 13 = 117, 5 \times 27 = 135, 9 \times 15 = 135, 7 \times 21 = 147, 9 \times 17 = 153, 5 \times 33 = 165, 11 \times 15 = 165, 9 \times 19 = 171, 7 \times 27 = 189, 9 \times 21 = 189, 5 \times 39 = 195, 13 \times 15 = 195.$
(And the first 5 are exactly the repetitions when $N = 100$ above.) $19 < 74$, so when $N = 200$ it still satisfies the Riemann requirement.

But problems begin to appear—Riemann allows the absolute error to increase, but does not allow the relative error to increase, that is, the relative error must gradually decrease. We can see that from $100 : 5$ to $200 : 19$, the ratio has increased—from 0.05 to 0.095. If it continues to increase, it will eventually exceed the upper bound required by Riemann at some future input value of N;

- When $N = 300$, $\sqrt{N}\ln N \approx 98$, and the number of repetitions is 36:

$5 \times 9 = 45, 7 \times 9 = 63, 5 \times 15 = 75, 9 \times 9 = 81, 9 \times 11 = 99, 5 \times 21 = 105, 7 \times 15 = 105, 9 \times 13 = 117, 5 \times 27 = 135, 9 \times 15 = 135, 7 \times 21 = 147, 9 \times 17 = 153, 5 \times 33 = 165, 11 \times 15 = 165, 9 \times 19 = 171, 7 \times 27 = 189, 9 \times 21 = 189, 5 \times 39 = 195, 13 \times 15 = 195, 9 \times 23 = 207, 5 \times 45 = 225, 9 \times 25 = 225, 15 \times 15 = 225, 7 \times 33 = 231, 11 \times 21 = 231, 9 \times 27 = 243, 5 \times 51 = 255, 15 \times 17 = 255, 9 \times 29 = 261, 7 \times 39 = 273, 13 \times 21 = 273, 9 \times 31 = 279, 5 \times 57 = 285, 15 \times 19 = 285, 9 \times 33 = 297, 11 \times 27 = 297.$
$36 < 98$, satisfying the Riemann requirement, but as said above, from $200 : 19$ to $300 : 36$, it is 0.095 to 0.12—the ratio is still increasing. Moreover, a new problem has appeared: purely from the perspective of growth rate, when N is 100, 200, 300 respectively, the growth rate of $\sqrt{N}\ln N$ is slower than the growth rate of repetitions. $(74 - 46 = 28) > (98 - 74 = 24)$, $\sqrt{N}\ln N$ decreased by 4. By analogy, when N is 400, 500, 600, $\sqrt{N}\ln N$ is respectively 119, 138, 156, and the gaps change from 24 to 21, 19, 18⋯ while multi-factor repetition increases with N as 5, 19, 36, 54, 76⋯ with gaps from 14 to 17, 18, 22⋯ increasing, so it will eventually catch up with and surpass $\sqrt{N}\ln N$ at some future input value of N.

Thus, combining the relative error also increasing just mentioned, these two points determine that although the odd triangular table counting method initially, when the input value of N is not large, still satisfies the Riemann requirement, the gap will grow larger and larger, so this is only a trivial upper bound. Since most repetitions are repetitions relative to small row numbers, such as rows $n = 3$、$n = 5$, it seems possible to avoid repetitions by removing some small row numbers as the input N becomes larger/taking only part of the rows each time—making what were originally repetitions turn into first appearances. But in this way, although for the left half-region the reduction of skip-products in small row numbers is not much for the remaining rows (even if it is just the next row, the reduction is merely it minus 1, and when the remaining rows are stacked on top, there will still be many left), for the inverted triangle that has added the right half-region, the small row numbers include many skip-products. If the right half-region is not considered, then the correct true value of the remaining composite numbers in the right half-region itself becomes an error—two types of errors appear: the number of correct composite numbers that were not counted and the number of multi-factor repetitions. Therefore, we still need to discuss a more non-trivial upper bound. There are now two:

α Upper Bound for the Growth Rate of the Left Half-Region

The first—after clarifying N, the growth rate of the left half-region (approximately half) of the skip-product table composed of the left and right half-regions has a non-trivial upper bound:

First, according to the first part of 3.1, whether it is the square total table or the triangular table, the left half-region must be drawn with square numbers as anchor points. Then this composite-counting method has, from the moment of its birth, limited the growth mode of counting to the pattern of growth

by square numbers—the maximum anchor point cannot be arbitrarily taken in the natural number sequence, but is taken in the square number sequence, such as among 4, 9, 16, 25⋯ That is, when we draw the table, no matter how many composite numbers below what number we want (that is, no matter what N is input), the continuous change of the table must proceed by increasing/decreasing one row/column or one pair of rows/columns each time:
for example, the left half-region when we first find composite numbers below 9:

	2	**3**
2	4	6
3		9

now if we want to find a larger range, that is, input a larger N, then it is finding composite numbers below the next square number 16 after 9:

	2	**3**	**4**
2	4	6	8
3		9	12
4			16

if we want to find an N not as large as 9, as long as we want to change the range, it is always to composite numbers below the previous square number 4 before 9:

	2
2	4

This is because non-square numbers simply cannot be used to draw the table—the table is drawn according to square number anchor points. That is, even if the input N is not a square number, the anchor point taken when drawing the table must be the square number n^2/m^2 closest to N; otherwise this counting method cannot occur—it proceeds by drawing tables, and drawing tables requires square numbers. This causes me, whether wanting to change to finding the "previous range" or the "next range," to have to find the previous square number or the next square number, rather than any nearby natural number—a table cannot be drawn from non-square numbers. This makes the change in the number of cells between each consecutive change of N (whether increase or decrease) necessarily smooth—strictly according to a certain quantity, and the value of this quantity is

$$\sqrt{N}-1,$$

or

$$\begin{cases}\sqrt{n^2}-1\\ \sqrt{m^2}-1,\end{cases}$$

because each time only one column is added—the column where the new maximum square number anchor point n^2/m^2 is located, and the value of this column must be the square root of this square number—according to 3.1, because at first it is drawn as a square, and the area of a square is the direct multiplication of the side lengths, the number of rows and columns must both be the square root of the anchor point:

	2	3	4
2	4	6	8
3		9	12
4			16

Next, because skip-multiplication does not contain 1, counting from 2, any square root x is of course the $x-1$th number to appear:

	2	3	4
2	4	6	8
3		9	12
4			16

The actual number of cells added/reduced each time must be the square root of the square number anchor point minus 1. If according to Supplement 2.2, even numbers are removed (and in fact they should be), the choice of anchor point changes from choosing in the square number sequence to choosing in the square odd number sequence, then the change value unit will change from $\sqrt{N}-1/\sqrt{n^2}-1/\sqrt{m^2}-1$ to

$$\begin{cases}\sqrt{N}-1-x \\ \sqrt{n^2}-1-x \\ \sqrt{m^2}-1-x,\end{cases} \tag{2}$$

that is, looking at the number x of even numbers appearing in the rows/columns. Besides subtracting 1, one also needs to subtract all even rows/columns. For example, from input 9 to input 16, 0 cells are added—whether input 9 or input 16, it is

	3
3	9

because

$$\sqrt{9}-1-1=1,$$
$$\sqrt{16}-1-2=1.$$

The new increase in the number of cells is both 1. A pattern can also be found—without removing even numbers, as the square numbers continuously increase, every other larger composite number, the number of even numbers contained increases by $+1$, and the number of newly added cells also increases by 1 every other. For example, from input 4 to the next square number 9, the number of even numbers does not change, both having only the one even number 2:

	2
2	4

	2	3
2	4	6
3		9

And from input 9 to the next square number 16, the number of even numbers changes from 1 (only 2) to 2 (2 and 4):

	2	3
2	4	6
3		9

	2	3	4
2	4	6	8
3		9	12
4			16

By analogy, when inputting the next square number after 16, namely 25, the number of even numbers does not change; when inputting the next square number 36, the number of even numbers $+1\cdots$

$$\begin{gathered}\sqrt{25}-1-2=2,\\ \sqrt{36}-1-3=2,\\ \vdots\end{gathered}$$

Correspondingly, the newly added number of cells is staggered: from input 9 to input 16 it does not change, and from input 16 to input 25 it increases by 1—as the square numbers increase, when the number of even numbers changes the number of newly added cells does not change, and when the number of newly added cells changes the number of even numbers does not change (this also makes x exactly become $\frac{\sqrt{N}-1}{2}/\frac{\sqrt{n^2}-1}{2}/\frac{\sqrt{m^2}-1}{2}$ each time the number of even numbers changes). This is the unit value of the change in the number of cells of the triangular table—although we can input N values far apart and seemingly obtain very different ranges, but within the internal continuous change between N and N of this system, the rate of change is controllable and has no sudden explosion/collapse—the continuous change process is smooth, following (2) or the skip-product table without removing even numbers. And the triangular table has only columns and no rows (if the lower triangular table is used, it is the reverse), so it increases by only one (2); if it is the square total table, it is two (2)s minus 1, because there is overlap at n^2/m^2, and the number of newly added cells is not directly two side lengths, but two side lengths minus one cell:

$$\begin{cases}2\left(\sqrt{N}-1-x\right)-1=2\sqrt{N}-2x-3\\ 2\left(\sqrt{n^2}-1-x\right)-1=2\sqrt{n^2}-2x-3\\ 2\left(\sqrt{m^2}-1-x\right)-1=2\sqrt{m^2}-2x-3.\end{cases}$$

Thus, we obtain the strict unit value of change of the left half-region—(2) or two (2)s minus 1. In the continuous change of the left half-region, the change in the number of cells between consecutive times is locked, fixedly increasing/decreasing only by the above values.

Then, a very simple logic—multi-factor repetition occurs inside the table, with cells as its carrier; the continuous change of the number of cells/the growth rate between consecutive times has an upper bound, and repetitions occurring depending on the cells naturally cannot exceed this upper bound in the continuous smooth change of N values. By limiting the continuous change upper bound of the carrier of repetition, we can obtain the continuous change upper bound of repetition itself—between each adjacent input of N, the error cannot suddenly increase/suddenly decrease, because the change in the number of cells is only so much. And whether it is the square total table or the triangular table, this upper bound is smaller than the $O(\sqrt{N}\ln N)$ required by Riemann.

When x changes, it must be $\frac{\sqrt{N}-1}{2}/\frac{\sqrt{n^2}-1}{2}/\frac{\sqrt{m^2}-1}{2}$, and when x does not change, it is the previous $\frac{\sqrt{N}-1}{2}/\frac{\sqrt{n^2}-1}{2}/\frac{\sqrt{m^2}-1}{2}$ (because the input value has not increased, it is even smaller, no need to discuss), then $2(\sqrt{N}-1-x)-1$ can actually be written as

$$2\left(\sqrt{N}-1-\frac{\sqrt{N}-1}{2}\right)-1=\sqrt{N}-2,$$

and the triangular table's $\sqrt{N}-1-\frac{\sqrt{N}-1}{2}$ simplifies to

$$\frac{\sqrt{N}-1}{2},$$

$\frac{\sqrt{N}-1}{2}$ equals $\sqrt{N}\ln N$ when $N=1$, but the discussion about prime numbers does not need to go as far as 1—the skip-product table itself excludes 1, so it is not less than or equal to but directly less than:

$$\begin{cases}\sqrt{N}-2<\sqrt{N}\ln N\\ \dfrac{\sqrt{N}-1}{2}<\sqrt{N}\ln N.\end{cases}$$

Thus, we can non-trivially limit the error range of the left half-region between consecutive times. The counting framework itself is smooth; the sudden change of error cannot come from the framework itself; if it is to occur, it can only come from inside the framework.

However, this is only saying that the change of error between consecutive times remains smooth, while the total repetition value of each independent time is not necessarily less than or equal to the $O(\sqrt{N}\ln N)$ required by Riemann. So we certainly need to discuss other content — the second non-trivial upper bound:

$\boldsymbol{\beta}$ Discrete Error Upper Bound

When $\pi(N)-\mathrm{Li}(N)$ is extended into the skip-product table as multi-factor repetition, since multi-factor repetition is discrete, it is something happening to multiple composite numbers, that is, there are multiple composite numbers that will repeat, then $\pi(N)-\mathrm{Li}(N)$ itself can also be endowed with discreteness; the total repetition obtained by summing the respective repetition values of the composite numbers is the single-time total error $\pi(N)-\mathrm{Li}(N)$ each time N is input, so $\pi(N)-\mathrm{Li}(N)$ can actually be regarded as discretized, as the sum of multiple repetitions $[\pi(N)-\mathrm{Li}(N)]_i$.

And, a very basic number-theoretic fact—in any non-trivial factor pair (n,m) of any product $nm=n\times m$, there must be one factor $\le\sqrt{nm}$ and the other $\ge\sqrt{nm}$. This comes from the most basic arithmetic: if two numbers greater than $\sqrt{nm}$ are multiplied, the result is necessarily greater than nm. This is even more the most fundamental, unshakable structural feature of the skip-product table—

the above content manifests in the skip-product table as: the smaller factor $n \vee m$ of any skip-product nm must be chosen from the finite set $\{2, 3, \cdots, \sqrt{nm}\}$ (skip-multiplication starts from 2, so the left endpoint is 2):

	2	**3**	$\cdots$	$\sqrt{nm}$
2	4	6	$\cdots$	$2\sqrt{nm}$
3	6	9	$\cdots$	$3\sqrt{nm}$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\sqrt{nm}$	$2\sqrt{nm}$	$3\sqrt{nm}$	$\cdots$	nm

And because each skip-product has corresponding rows and columns, it means that when a repeated skip-product appears, new rows and columns must also appear—and the number-theoretic fact just mentioned has already locked down the number of rows and columns below any large N. The rows and columns of factors that must accompany any skip-product nm are finite, so the number of times any skip-product nm is repeated in the skip-product table is of course also finite—naturally not exceeding $\sqrt{nm}$. From this, even if we do not know how multi-factor repetition is specifically distributed in position, we have grasped the most fundamental hard restriction on it in counting (that is, we still have not found an exact multi-factor repetition pattern like we did for commutative law repetition, but the magnitude of repetition can be completely determined—and Riemann only requires counting, not position). Define the repetition number $Q_r(nm)$ of a skip-product—for a skip-product nm, let

$$Q_r(nm) = \#\{(n, m) | n \geq 2, m \geq 2\}$$

called the repetition number of nm. Next, the lemma—"in any non-trivial factor pair of any product, there must be one factor $\leq \sqrt{nm}$ and the other $\geq \sqrt{nm}$," that is, for any nm, its repetition number satisfies

$$Q_r(nm) \leq \sqrt{nm} - 1, \tag{3}$$

that is, if $n > m$, then necessarily $m \leq \sqrt{nm}$; if $m > n$, then necessarily $n \leq \sqrt{nm}$. Since the values of $n \vee m$ can only reach $\sqrt{nm}$, there are at most $\sqrt{nm} - 1$ such $n \vee m$ (because skip-multiplication can only start from 2, so the natural number 1 is naturally missing), and each corresponds to a unique $n \vee m = \frac{\sqrt{nm}}{m \vee n}$. The lemma is proved; $O\left(\sqrt{nm} - 1\right)$ is the strict, universal numerical boundary. Thus, the upper limit on the number of rows and columns directly locks down the total amount of each part $[\pi(N) - \mathrm{Li}(N)]_i$ after $\pi(N) - \mathrm{Li}(N)$ is discretized.

Of course, if commutative law repetition is retained, that is, if one looks at square table rather than the triangular table, it still holds, except that one more question needs to be discussed—which factor is smaller and which is larger—and the result is consistent, because "smaller factor" is always finite.

Returning to the table, it is represented as an inverse proportional curve distribution—even if we have not grasped the exact pattern of multi-factor repetition, it is not that we have grasped no pattern at all—the assertion that "grasping the exact pattern of multi-factor repetition equals finding the exact pattern of prime distribution" only limits us from seeking an exact pattern, but the pattern of discrete error upper bound $O\left(\sqrt{nm} - 1\right)$—the inverse proportional curve distribution pattern—is not exact:

	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36

$$n = \frac{nm}{m}.$$

If it is the square total table, it is an inverse proportional hyperbola, which makes it more obvious. We can find that repetitions must be distributed along such curves. But although this is intuitive, it is not non-trivial, because it is not exact. We still need to study the error upper bound of $[\pi(N) - \mathrm{Li}(N)]_i$.

However, if it is only the upper bound of discrete error, then even if it is non-trivial for itself, it has no non-trivial contribution to limiting the range of total error. For example, if the number of repetitions Q is already close to or even greater than or equal to $\sqrt{N}\ln N$ (as said in Current Situation), then even if highly composite numbers are scarce, the composite numbers whose repetition count truly reaches $\sqrt{nm} - 1$ are few/fewer and fewer, the total error will certainly be greater than $\sqrt{N}\ln N$. So, although the skip-product table structurally naturally prohibits factor explosion/factor-pair explosion, it only prohibits the error between each input of N (α) and each part of the error at each input of N (β) from exploding, and does not directly show that the single-time total error at each input of N will not explode (the accumulation among repetitions is not the same-frequency resonance of zero-point waves in RH. Our extension is a rough extension—only through the simple logic "because there is no other error term, it must be this" making the two equivalent, without proving that discrete repetitions correspond one-to-one to zeros, so we are only separately handling the accumulation among repetitions, unrelated to zero-point resonance). Define $Q_{\mathcal{R}}(N)$ as total number of repetitions of all nm below N:

$$Q_{\mathcal{R}}(N) = \sum\nolimits_{nm<N} Q_r(nm),$$

due to (3),

$$Q_{\mathcal{R}}(N) = \sum\nolimits_{nm<N} Q_r(nm) \leq \sum_{k=2}^{N} \left(\sqrt{nm} - 1\right),$$

In this way, the range upper bound will not be $\sqrt{nm} - 1$ but $\sum_{k=2}^{N} \left(\sqrt{nm} - 1\right)$. This is the disadvantage of the skip-product table compared to other prime-counting methods, the cost of seeing the structure clearly—although because of the lemma just stated, new factor pairs must be accompanied by new rows and columns, and rows and columns are always finite, and combined with our existing screening such as handling the even part and "commutative law repetition," it is impossible for all skip-products with repetitions to really repeat $\sqrt{nm} - 1$ times (that is, it is impossible for all skip-products with repetitions to happen to be highly composite numbers), that is, the integral approximation will not really reach $\frac{2}{3}N^{3/2}$, but in any case, it is still greater than Riemann's requirement. So, this is not the result we want, but only tells us that $\pi(N) - \mathrm{Li}(N)$ in skip-multiplication theory can be transformed from $E_{\text{multi-factor repetition}}$ to $E_{Q_{\mathcal{R}}(N)}$—$\pi(N) - \mathrm{Li}(N)$ is narrowed from "the error caused by multi-factor repetition $E_{\text{multi-factor repetition}}$" to "the 'cumulative' error caused by multi-factor repetition $E_{Q_{\mathcal{R}}(N)}$," translating the problem into a clearer skip-multiplication theory problem, and also corresponding to the divisor function problem in number theory, pulling it into this problem domain. Define $E_{Q_{\mathcal{R}}(N)}$:

- The ideal skip-product table we want is one with no repetition at all, where the number of cells = the number of skip-products/the number of composite numbers, and the number of cells/the number of skip-products/the number of composite numbers of such a table is the ideal value, denoted Q_α;
- Because repetition exists, the number of cells is not equal to the number of skip-products (there are repeated skip-products), and the distorted value we obtain is denoted Q_β; Q_β deviates from Q_α, and the deviation itself is $E_{Q_{\mathcal{R}}(N)}$:

$$E_{Q_{\mathcal{R}}(N)} = Q_\alpha - Q_\beta,$$

and we have now only found broader and narrower (more general and more detailed) upper bounds—the error upper bound between each time and the error upper bound of each part within each time, exactly missing the error upper bound of each time located in the middle—the $E_{Q_{\mathcal{R}}(N)}$ upper bound that Riemann wanted. But, although the exact order $O\left(\sqrt{N}\ln N\right)$ has not yet been fully reached, it has already explained structurally why the error cannot exhibit exaggerated runaway.

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