Quantum energy levels and Riemann Zeros

Payam Danesh [1], **Raoul Bianchetti** [2]

[1] *Department of biosystems engineering, University of Tehran*

[2]*Information Physics Institute, Gosport, Hampshire, United Kingdom*

Abstract

In this paper, we develop an observation into a precise operator-theoretic program. For a zero written as $\rho = \sigma + it$, the imaginary part of the energy is $t(1 - 2\sigma)$. Hence, for every non-real zero in the critical strip, $E_\rho$is real precisely on the critical line. The result shows the Riemann hypothesis is the assertion that all quadratic zeta energies are real. This paper then formulates the Hamiltonian structure required to turn this equivalence into a proof. The proposed operator is not a finite spectral fit, instead, it is a modular-arithmetic Hamiltonian built on logarithmic space, prime translations, an arithmetic boundary domain, a prime-power trace formula and determinant closure with the completed zeta function. The main theorem is conditional since if such a self-adjoint operator has determinant $\xi(s)$ in the spectral coordinate $s(1 - s)$, then all non-trivial zeros lie on critical line.

Keywords: Riemann Hypothesis; quantum energy levels; Hilbert–Pólya program; self-adjoint Hamiltonian; spectral determinant.

1. Introduction

Riemann Hypothesis (RH) is one of the open problems whose statement is short but whose consequences reach deep into analysis, arithmetic and mathematical physics. Its central point can be stated without technical decoration: the prime numbers fluctuate around their average distribution according to complex frequencies, and the location of those frequencies is controlled by the zeros of the zeta function. Riemann's original memoir introduced this zero-prime connection [1] and the classical analytic theory is described in detail by Montgomery in his book [2].

For $\operatorname{Re} s > 1$, the zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \qquad (1)$$

After analytic continuation, its non-trivial zeros lie in the critical strip and control the oscillatory terms in explicit prime-counting formulas. RH states

$$\zeta(\rho) = 0, 0 < \operatorname{Re} \rho < 1 \Longrightarrow \operatorname{Re} \rho = \frac{1}{2}. \qquad (2)$$

The importance of (2) is not only geometric. It gives the strongest possible zero-theoretic control over the error terms in the prime number theorem. In physical language, the statement resembles a spectral reality condition: a family of complex quantities is forced onto a symmetry line, in the same way that the spectrum of a self-adjoint Hamiltonian is forced onto the real axis.

The spectral interpretation became especially compelling after Goldston's pair-correlation work and Odlyzko's large-scale computations showed that high zeta zeros exhibit the spacing statistics associated with eigenvalues of random Hermitian matrices [3], [4]. This evidence does not prove (2) but it gives a strong structural clue: the zeros behave like spectral data.

Several mathematical-physics frameworks have attempted to turn that clue into an operator mechanism. For example, Berry and Keating connected the leading zero-counting law with a semiclassical xp-type Hamiltonian, while Connes interpreted the zeros as missing spectral lines in an absorption spectrum [5], [6]. These two pictures differ in a decisive way. In one, the zeros resemble energy levels. In the other, they resemble gaps or resonances in a continuum. A proof of the Riemann Hypothesis through spectral theory requires the stronger interpretation: the zeros must correspond to spectral values of a self-adjoint operator, and the relevant states must belong to the Hilbert space. Further quantum models have explored boundary wavefunctions, scattering structures, and continuum states [7]. Arithmetic quantum-statistical constructions have also shown that zeta-type structures can appear in equilibrium and phase-transition formalisms [8], [9]. A recent Hamiltonian construction is especially close to the present formulation because it uses the quadratic levels $\rho(1-\rho)$, while also noting that the resulting states are not normalizable and therefore do not yet give a Hilbert–Pólya proof [10]. Dynamical quantum models have likewise encoded zeta zeros as special events in time evolution [11].

Given the status of the previous works, this paper isolates the part of the spectral program that is already exact and separates it from the part that remains open. The exact part is the quadratic energy map: a zero has real quadratic energy precisely when it lies on the critical line. The open part is the construction of a self-adjoint modular-arithmetic Hamiltonian whose determinant is the completed zeta function in the same quadratic coordinate. This paper does not claim an unconditional proof of the RH. It gives a complete conditional theorem and a precise operator program.

2. The quadratic zeta-energy map

The technical core of the paper is the map $s \mapsto s(1-s)$. The usual ordinate t is natural after one has already written a zero as $1/2 + it$. The quadratic coordinate is stronger because it can be assigned before the critical line is assumed. It records whether a zero in the critical strip is compatible with a real quantum spectrum.

The quadratic zeta energy associated with a zero ρ is

$$E_\rho = \rho(1-\rho). \qquad (3)$$

The completed zeta function is

$$\xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad (4)$$

And it satisfies the functional equation

$$\xi(s) = \xi(1-s). \qquad (5)$$

The same reflection symmetry is built into the quadratic coordinate

$$s(1-s) = (1-s)s. \qquad (6)$$

This shared symmetry is the first reason to regard $E_\rho$as the natural spectral variable. The second reason is more decisive which converts the critical-line condition into the realness of an energy.

Definition 1 (Quadratic zeta energy). For a non-trivial zero ρ of $\zeta(s)$, the associated quadratic zeta energy is the complex number $E_\rho$defined in (3).

We write a zero in the critical strip as

$$\rho = \sigma + it. \qquad (7)$$

Substituting (7) into (3) gives

$$E_\rho = \sigma(1-\sigma) + t^2 + i\,t(1-2\sigma), \qquad (8)$$

so that

$$\operatorname{Im} E_\rho = t(1-2\sigma). \qquad (9)$$

Proposition 1 (Spectral-reality equivalence). Let ρ be a non-real zero of $\zeta(s)$in the critical strip. Then

$$E_\rho \in \mathbb{R} \Leftrightarrow \operatorname{Re}\rho = \frac{1}{2}.$$

Proof. For a non-real zero, $t \neq 0$. By (9), $\operatorname{Im} E_\rho = 0$ if and only if $1 - 2\sigma = 0$, which is exactly $\sigma = 1/2$. Conversely, substituting $\sigma = 1/2$in (8) gives a real value.

If a zero is on the critical line, it can be written $\rho_n = 1/2 + it_n$ and its quadratic energy becomes

$$E_n = \frac{1}{4} + t_n^2. \qquad (10)$$

Equation (10) is compatible with the standard spectral intuition. The difference is the logical order. The value $t_n$is introduced after the critical line is known, whereas $E_\rho$in (3) is defined for an arbitrary zero in the strip.

3. Spectral-reality theorem

The previous section reduces the zero-location problem to a real-spectrum problem. A self-adjoint Hamiltonian is the natural object at this point because its spectrum is real. The following formulation states the determinant condition needed to make the connection exact and proves the conditional implication.

We can define the quadratic zeta spectrum by

$$\mathcal{E}_\zeta = \{\rho(1-\rho) : \xi(\rho) = 0,\ 0 < \operatorname{Re}\rho < 1\}. \qquad (11)$$

Since the zeta function has no zeros on the real interval $0 < s < 1$, the zeros relevant to (11) are non-real. Proposition 1 therefore gives the equivalent spectral form of RH:

$$\mathrm{RH} \Leftrightarrow \mathcal{E}_\zeta \subset \mathbb{R}. \qquad (12)$$

The proposed Hilbert-space object is a self-adjoint Hamiltonian $\widehat{H}_{RH}$whose spectral determinant satisfies

$$\det_\zeta \left(\widehat{H}_{RH} - s(1-s)\right) = C\,\xi(s), C \in \mathbb{C} \setminus \{0\}. \qquad (13)$$

Here $\det_\zeta$ denotes a zeta-regularized or renormalized determinant appropriate to the final operator. The content of (13) is exact: zeros of $\xi(s)$ become spectral zeros of $\widehat{H}_{RH}$ in the quadratic variable.

Theorem 1 (Spectral-reality implication). If there exist a Hilbert space $\mathcal{H}_{RH}$, a dense domain $\mathcal{D}_{arith}$, and a self-adjoint operator

$$\widehat{H}_{RH}: \mathcal{D}_{ar} \quad \subset \mathcal{H}_{RH} \to \mathcal{H}_{RH}$$

satisfying (13), then the Riemann Hypothesis is true.

Proof. Let us ρ be a non-trivial zero. Since $\xi(\rho) = 0$, (13) implies that $\rho(1-\rho)$ is a spectral value of $\widehat{H}_{RH}$. Self-adjointness gives $\mathrm{Spec}(\widehat{H}_{RH}) \subset \mathbb{R}$. Hence E_ρ is real. Proposition 1 then gives $\mathrm{Re}\,\rho = 1/2$.

The theorem is not a numerical statement. It does not assert that a finite Hermitian matrix can reproduce a finite list of zeros. It requires one operator whose determinant and trace structure generate the completed zeta function. Matching several levels would be evidence at most; determinant closure is the mathematical target.

4. Modular-arithmetic Hamiltonian

The operator must be built from arithmetic structure rather than from the zeros inserted by hand. The natural arena is logarithmic space, where multiplication by a prime becomes translation. In that coordinate, prime powers can appear as translation lengths, and the explicit formula can be read as a trace formula with arithmetic orbit data.

The proposed decomposition is

$$\widehat{H}_{RH} = \widehat{H}_0 + \widehat{V}_{\mathbb{P}}. \qquad (14)$$

The first term is the free geometric part. The second term carries the prime data. The zeros should appear only after the spectral analysis of the full operator.

We introduce the logarithmic coordinate

$$u = \log x. \qquad (15)$$

Then multiplication $x \mapsto px$ becomes translation $u \mapsto u + \log p$. The arithmetic Hilbert space is taken in the form

$$\mathcal{H}_{RH} = L^2(\mathcal{X}, d\mu), \qquad (16)$$

where $\mathcal{X}$is a logarithmic or modular configuration space carrying the prime translations, and $d\mu$ is the corresponding invariant measure.

By setting

$$\hat{P} = -i\frac{d}{du}. \qquad (17)$$

the free Hamiltonian is chosen as

$$\hat{H}_0 = \frac{1}{4} + \hat{P}^{\,2}. \qquad (18)$$

This is the operator version of the critical-line energy in (10): a momentum eigenstate with parameter $t$has the energy $1/4 + t^2$.

For each prime p, define the translation operator

$$T_p = \exp\left[-i(\log p)\hat{P}\right]. \qquad (19)$$

The k-fold power of T_p corresponds to the prime power p^k. A regularized prime interaction is then

$$\hat{V}_{\mathbb{P}}^{(\varepsilon)} = -\sum_{p\in\mathbb{P}}\sum_{k\geq 1}\frac{\log p}{p^{k(1/2+\varepsilon)}}\left(T_p^k + T_p^{-k}\right). \qquad (20)$$

For absolute convergence of the coefficients in (20), the exponent must be large enough; with the displayed regulator, $\varepsilon > 1/2$ is sufficient. The range $0 < \varepsilon \leq 1/2$, and especially the critical limit $\varepsilon \downarrow 0$, belongs to the renormalized trace problem rather than to the bounded-perturbation regime.

For fixed admissible ε, we set

$$\hat{H}_{RH}^{(\varepsilon)} = \hat{H}_0 + \hat{V}_{\mathbb{P}}^{(\varepsilon)}. \qquad (21)$$

A standard route is to prove that the perturbation is relatively bounded with respect to $\hat{H}_0$and then apply the Kato–Rellich theorem [12]. The final operator must be self-adjoint on a precisely defined domain, and the general theory of self-adjoint Hamiltonians makes the domain issue unavoidable [13]. This point leads directly to the quantitative part of the construction. Once the arithmetic domain has selected the admissible spectrum, the resulting eigenvalue count must be compared with the known average distribution of the non-trivial zeros. The smooth Riemann–von Mangoldt counting term used for comparison is

$$\bar{N}(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}. \qquad (22)$$

This expression gives the leading average number of zeta-zero ordinates in the range $0 < t \leq T$. In the proposed Hamiltonian picture, it plays the role of the smooth background density: the free part of the operator must reproduce this main growth, while the arithmetic prime interaction must account for the remaining oscillatory deviation. Thus the domain problem and the counting formula are not separate issues. The domain decides the spectrum; the Riemann–von Mangoldt term tells us whether that spectrum has the correct large-scale shape.

In this case**, Figure 1** makes this comparison explicit and shows the first positive non-trivial zeros of the Riemann zeta function after transformation by the quadratic spectral map $E_n = \frac{1}{4} + t_n^2$, where each zero is written as $\rho_n = \frac{1}{2} + it_n$. The solid curve represents the resulting energy sequence and shows the rapid growth of the spectral levels as the ordinates $t_n$increase. The dashed curve gives the residual $n - \bar{N}(t_n)$, where $\bar{N}(T)$is the smooth Riemann–von Mangoldt counting approximation. This residual is essential because it isolates the arithmetic fluctuation that is not captured by the leading asymptotic density. In the proposed Hamiltonian picture, the smooth curve corresponds to the free spectral scale, while the residual encodes the prime-driven correction that the arithmetic interaction must reproduce.

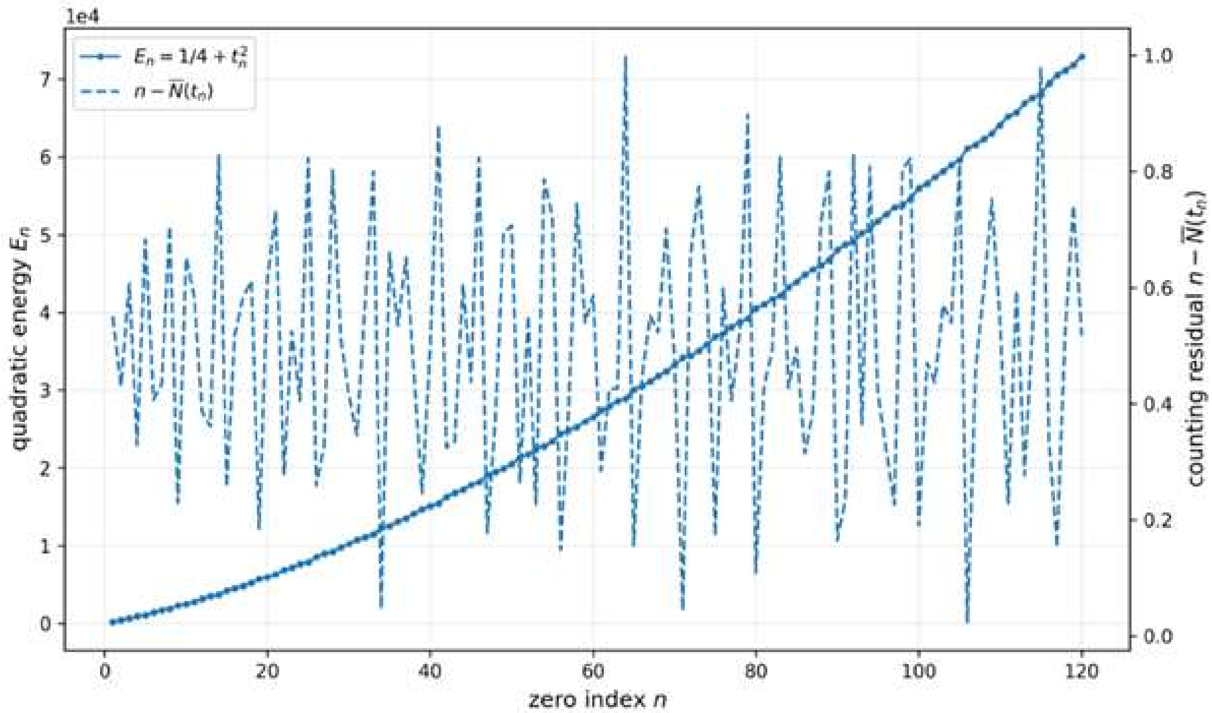

Figure 1. First Riemann zeros mapped to quadratic quantum energies.

The trace formula requires the prime powers to appear with lengths and amplitudes

$$L_{p,k} = k\log p, A_{p,k} = \frac{\log p}{p^{k/2}}. \qquad (23)$$

Each plotted point in **figure 2** represents one prime-power orbit (p,k), with logarithmic length $L_{p,k} = k\log p$on the horizontal axis and trace amplitude $A_{p,k} = (\log p)/p^{k/2}$on the vertical axis. For example, the first primitive prime orbits occur at $L_{2,1} = 0.6931$, $L_{3,1} = 1.0986$, $L_{5,1} = 1.6094$, and $L_{7,1} = 1.9459$, with corresponding amplitudes $A_{2,1} = 0.4901$, $A_{3,1} = 0.6343$, $A_{5,1} = 0.7198$, and $A_{7,1} = 0.7350$. Repetitions of the same prime move linearly to the right while their amplitudes decay by powers of $p^{-1/2}$; for instance, the repeated orbit of $p = 2$gives $A_{2,2} = 0.2451$at $L_{2,2} = 1.3863$and $A_{2,3} = 0.1225$at $L_{2,3} = 2.0794$. This figure therefore shows, in concrete numerical form, how primes and their repetitions enter the Hamiltonian as logarithmic translations with rapidly weakening weights.

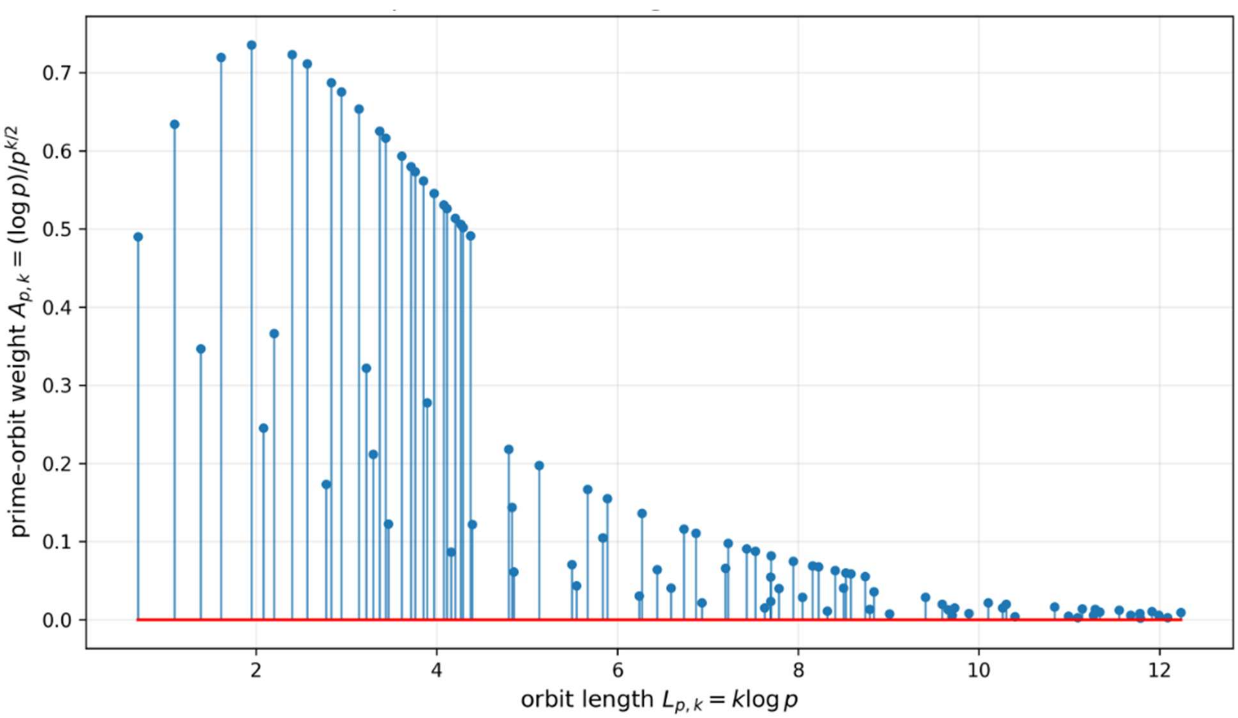

Figure 2. Prime-power data entering the arithmetic trace term.

The completed zeta function on the critical line after changing variables from the height t to the quadratic spectral energy $E = \frac{1}{4} + t^2$ is plotted in **figure 3**. The vertical axis shows $\log_{10} | \xi(\frac{1}{2} + it) |$, so the sharp downward troughs identify the energies where the completed zeta function vanishes. The first marked spectral levels occur at $E_1 = 200.040455$, $E_2 = 442.176151$, $E_3 = 625.792997$, $E_4 = 925.923087$, and $E_5 = 1084.968282$, obtained directly from the first critical-line ordinates $t_1 = 14.1347251417$, $t_2 = 21.0220396388$, $t_3 = 25.0108575801$, $t_4 = 30.4248761259$, and $t_5 = 32.9350615877$.

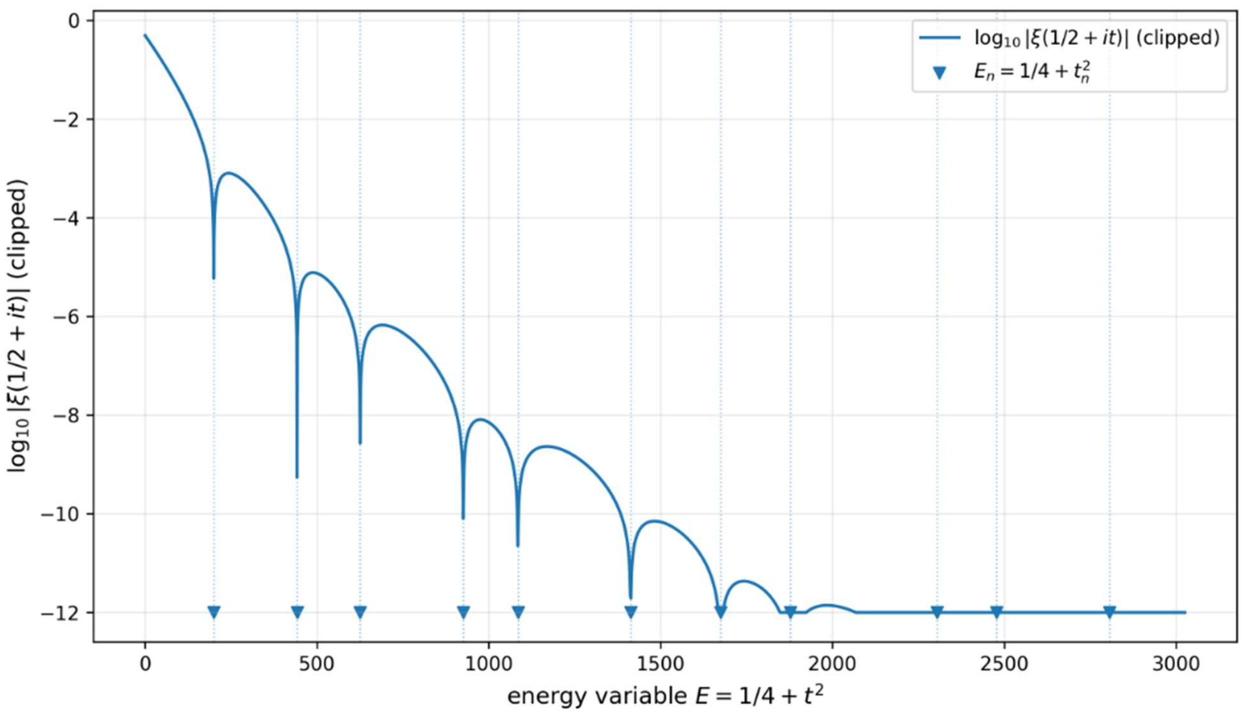

Figure 3. Zeros of the completed zeta function as spectral levels.

The alignment between the marked energies and the troughs of $\log_{10}|\xi|$ is the numerical signature of the determinant condition: in the proposed operator formulation, these are not merely zeros on a complex graph, but the target locations of quantum energy levels. Thus the plot shows exactly where the completed zeta determinant must vanish in the real energy coordinate.

5. Prime trace formula and determinant closure

A spectral model is mathematically convincing only if it reproduces the prime-zero duality. The smooth density gives the leading scale, but the arithmetic information lies in the oscillatory prime-power terms. The correct target is therefore a trace formula, not only a list of eigenvalues.

For a suitable even test function g, the normalized trace-formula target has the form

$$\sum_{\rho} g(t_\rho) = M(g) - \sum_{p\in\mathbb{P}} \sum_{k\geq 1} A_{p,k}\left[\hat{g}(L_{p,k}) + \hat{g}(-L_{p,k})\right], \qquad (24)$$

where $M(g)$ denotes the smooth contribution, including the gamma-factor and pole terms under the chosen normalization. The structure is analogous to geometric trace formulas, where spectral data are dual to periodic-orbit data [14]. In the present arithmetic setting, the primitive orbits are primes and the repetitions are prime powers.

The energy coordinate is

$$\lambda = s(1-s). \qquad (25)$$

Logarithmic differentiation of (13), with the determinant written as $\det(\hat{H}_{RH} - \lambda)$, gives the resolvent trace condition

$$\mathrm{Tr}\,\left(\hat{H}_{RH}-\lambda\right)^{-1} = \frac{1}{2s-1}\,\frac{\xi'(s)}{\xi(s)}. \qquad (26)$$

The sign in (26) follows from $d[s(1-s)]/ds = 1-2s$ and $d(\hat{H}_{RH} - \lambda)/ds = 2s-1$. This condition is a concrete analytic test for determinant closure. If the final self-adjoint realization satisfies (26), then (13) follows up to the nonzero constant C. Regularized spectral determinants of this kind are standard in spectral analysis and zeta-function theory [15]. The limiting Hamiltonian cannot be obtained by simply setting $\varepsilon = 0$in (20). The proper object must be defined through

$$\hat{H}_{RH} = \lim_{\varepsilon\to 0^+} \mathcal{R}_\varepsilon\left(\hat{H}_{RH}^{(\varepsilon)}\right), \qquad (27)$$

Here $\mathcal{R}_\varepsilon$ subtracts the divergent smooth component while preserving the finite arithmetic oscillation. The relevant convergence may be strong-resolvent or norm-resolvent convergence, depending on the final construction. The essential requirement is that renormalization preserve the prime trace term in (24).

6. Conclusion

The most delicate part of the program is the distinction between formal spectral values and genuine quantum energy levels. In a self-adjoint theory, an energy level must correspond to a state in the Hilbert space. A resonance or scattering pole may carry important information, but it is not enough for proof route in Theorem 1.

The required eigenvalue equation is

$$\hat{H}_{RH}\psi_\rho = E_\rho\psi_\rho, \psi_\rho \in \mathcal{H}_{RH}, \parallel \psi_\rho \parallel < \infty. \qquad (28)$$

For a differential or pseudodifferential Hamiltonian, the domain determines the boundary conditions and therefore the spectrum. The proposed arithmetic domain has the form

$$\mathcal{D}_{ar} \quad = \{\psi \in \mathcal{D}_{\max}: \mathcal{B}_{\xi}[\psi] = 0\}, \qquad (29)$$

where $\mathcal{B}_{\xi}$is a boundary functional determined by the completed-zeta symmetry and the prime-translation structure. The physical Hamiltonian is the restriction

$$\hat{H}_{RH} = \hat{H}_{formal} \mid_{\mathcal{D}_{arith}}. \qquad (30)$$

The condition in (29) must not be a disguised insertion of the zeros. It must arise from the arithmetic geometry of the operator. The desired boundary condition enforces the quantization rule

$$\xi(s) = 0 \Leftrightarrow s(1-s) \in \mathrm{Spec}_p(\hat{H}_{RH}). \qquad (31)$$

This statement says that zeros of the completed zeta function are precisely point-spectrum energies of the confined arithmetic system. The proposed physical mechanism is standing-wave selection in logarithmic arithmetic space. Prime translations create phases indexed by $k\log p$. The completed zeta function organizes the interference of those phases. The arithmetic boundary condition selects the standing waves whose determinant vanishes. Normalizability turns the zero into a genuine quantum level, and self-adjointness forces the associated energy to be real.

Conjecture 1 (Modular-arithmetic spectral realization): There exist a Hilbert space $\mathcal{H}_{RH}$, a dense arithmetic domain $\mathcal{D}_{arit}$, and a self-adjoint Hamiltonian $\hat{H}_{RH}$such that the determinant identity (13) holds. This conjecture is the operator-theoretic core of the paper. It is not a numerical approximation and not a diagonal matrix with the zeros built in. It is a statement about a self-adjoint Hamiltonian whose determinant is the completed zeta function in the quadratic spectral variable.

If Conjecture 1 is established, Theorem 1 gives the RH immediately. This proof is short because the construction carries the burden: a zero of ξ becomes a spectral value, self-adjointness makes that value real, and Proposition 1 places the zero on the critical line. The remaining technical obligations are now clear. First, the logarithmic Hilbert space and the prime-translation representation in (16) and (19) give the representation-theoretic foundation, but the boundary functional in (29) must still be derived from arithmetic structure rather than imposed from the zero set. Second, the regularized self-adjointness mechanism is standard in the bounded regime. In particular, the coefficient condition

$$\sum_{p\in\mathbb{P}}\sum_{k\geq 1}\frac{\log p}{p^{k(1/2+\varepsilon)}}<\infty \qquad (32)$$

makes $\hat{V}_{\mathbb{P}}^{(\varepsilon)}$ a bounded symmetric operator because each T_p^k is unitary. With the regulator used in (20), (32) is guaranteed for $\varepsilon > 1/2$. The final theory must then prove that the arithmetic boundary restriction and the renormalized limit in (27) preserve self-adjointness.

In this work, we complete the algebraic spectral-reality step and give an explicit modular-arithmetic Hamiltonian architecture. We also identify the regularized self-adjoint mechanism, the exact trace-formula target, the point-spectrum requirement and the determinant identity that must be verified. The main point can be stated in one sentence; the RH is equivalent, through the quadratic coordinate $E_\rho = \rho(1-\rho)$, to the realness of the zeta energy spectrum. The novelty is that this energy is defined before the critical line is assumed. If one self-adjoint modular-arithmetic Hamiltonian realizes the real-energy map, the prime trace data, the arithmetic point spectrum, and determinant closure, then spectral reality gives the RH.

References

[1] B. Riemann, "On the Number of Prime Numbers less than a Given Quantity. (Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.)".

[2] H. Montgomery, A. Nikeghbali, and M. T. Rassias, "Exploring the Riemann Zeta Function: 190 years from Riemann's Birth," *Exploring the Riemann Zeta Function: 190 years from Riemann's Birth*, pp. 1–298, Sep. 2017, doi: 10.1007/978-3-319-59969-4/SAVE-RESEARCH.

[3] A. M. Odlyzko, "On the Distribution of Spacings Between Zeros of the Zeta Function," vol. 48, pp. 273–308.

[4] D. A. Goldston, S. M. Gonek, A. E. Özlük, and C. Snyder, "On the Pair Correlation of Zeros of the Riemann Zeta-Function," *Proceedings of the London Mathematical Society*, vol. 80, no. 1, pp. 31–49, Jan. 2000, doi: 10.1112/S0024611500012211.

[5] M. V. Berry and J. P. Keating, "The Riemann Zeros and Eigenvalue Asymptotics," *https://doi.org/10.1137/S0036144598347497*, vol. 41, no. 2, pp. 236–266, Aug. 2006, doi: 10.1137/S0036144598347497.

[6] A. Cormes, “Trace formula in noncommutative geometry and the zeros of the Riemann zeta function,” *Selecta Mathematica, New Series*, vol. 5, no. 1, pp. 29–106, 1999, doi: 10.1007/S000290050042/METRICS.

[7] G. Sierra, “A quantum mechanical model of the Riemann zeros,” *New J. Phys.*, vol. 10, Dec. 2007, doi: 10.1088/1367-2630/10/3/033016.

[8] J. B. Bost and A. Connes, “Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory,” *Selecta Mathematica 1995 1:3*, vol. 1, no. 3, pp. 411–457, Dec. 1995, doi: 10.1007/BF01589495.

[9] M. Planat, P. Solé, and S. Omar, “Riemann hypothesis and Quantum Mechanics,” *J. Phys. A Math. Theor.*, vol. 44, no. 14, Feb. 2011, doi: 10.1088/1751-8113/44/14/145203.

[10] X. Suo, “Hamiltonian with Energy Levels Corresponding to Riemann Zeros,” *Phys. Rev. A (Coll. Park).*, vol. 112, no. 6, Dec. 2025, doi: 10.1103/n9ch-xhw6.

[11] S. Wei *et al.*, “The Riemann Hypothesis Emerges in Dynamical Quantum Phase Transitions,” Nov. 2025, Accessed: Jun. 07, 2026. [Online]. Available: https://arxiv.org/pdf/2511.11199

[12] T. Kato, “Perturbation Theory for Linear Operators,” vol. 132, 1995, doi: 10.1007/978-3-642-66282-9.

[13] P. F. Zweifel, “A review of: ‘Methods of modern mathematical physics, Volume II. Fourier analysis, self-adjointness. by Michael Reed and Barry Simon. Academic Press, New York (1975).,’” *Transp. Theory Stat. Phys.*, vol. 5, no. 2–3, pp. 205–206, Jan. 1976, doi: 10.1080/00411457608230829.

[14] A. Selberg, “Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces With Applications to Dirichlet Series,” *The Journal of the Indian Mathematical Society*, pp. 47–87, Sep. 1956, doi: 10.18311/JIMS/1956/16985.

[15] A. Voros, “Spectral functions, special functions and the Selberg zeta function,” *Commun. Math. Phys.*, vol. 110, no. 3, pp. 439–465, Sep. 1987, doi: 10.1007/BF01212422/METRICS.

2 September 2026

Here rests in God

Georg Friedrich Bernhard Riemann

Professor at Göttingen

Born in Breselenz on September 17, 1826

Died in Selasca on July 20, 1866

For those who love God, all things must work together for good