

A Bridge Between Mechanics and Thermodynamics: An Exploratory Study of Entropy in Damped Oscillators

Joshua T. Dizon

October 10, 2026

Abstract

We show that the decay of the energy envelope in a lightly damped harmonic oscillator can be explicitly connected to the increase of entropy in the surroundings. The result is an exploratory bridge between mechanics and the Second Law of Thermodynamics. The framework is extended to the electrical analog, the series RLC circuit, where the same expressions hold with $2\zeta\omega_0 = R/L$.

1 Introduction

A damped harmonic oscillator is an oscillator that is subjected to a restoring force and a velocity-dependent damping force, causing its mechanical energy to dissipate over time [1].

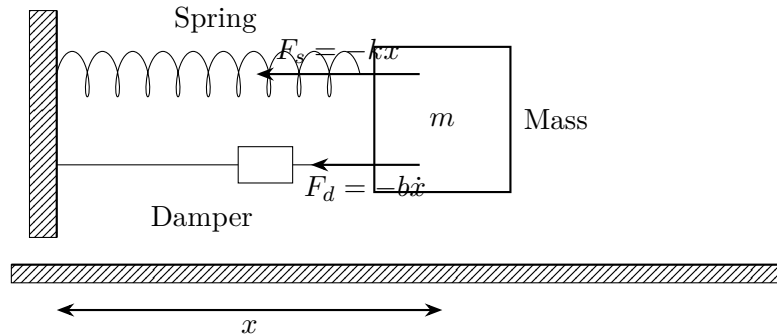
This note explores the connection between the mechanical energy loss of a damped oscillator and the entropy increase of its surroundings [2, 3].

Related work on frictionless piston oscillations traces the damping to irreversibility within the gas itself [4, 5, 6]. Here we start instead from ordinary friction-type damping and follow the dissipated heat into the surroundings.

2 Damped Oscillator Models

2.1 Spring–Mass–Damper System

Consider a mass attached to a spring and a damper.



The equation of motion for this system is [1]:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}(t) \quad (1)$$

Dividing through by m , we obtain the standard form:

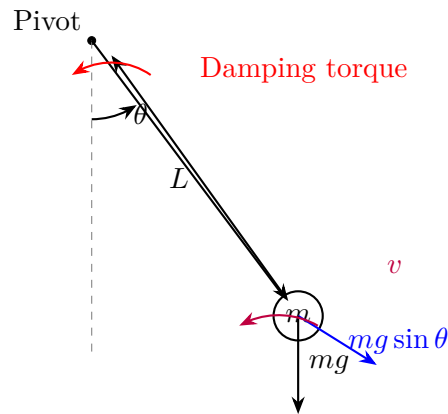
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{F_{\text{ext}}(t)}{m} \quad (2)$$

where:

- $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped natural frequency,
- $\zeta = \frac{b}{2\sqrt{km}}$ is the damping ratio,
- $F_{\text{ext}}(t)$ is any external forcing.

In the absence of external forcing ($F_{\text{ext}} = 0$), the system is a freely damped oscillator.

2.2 Damped Pendulum



For small angular displacements, the equation of motion is [1]:

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + \omega_0^2\theta = 0 \quad (3)$$

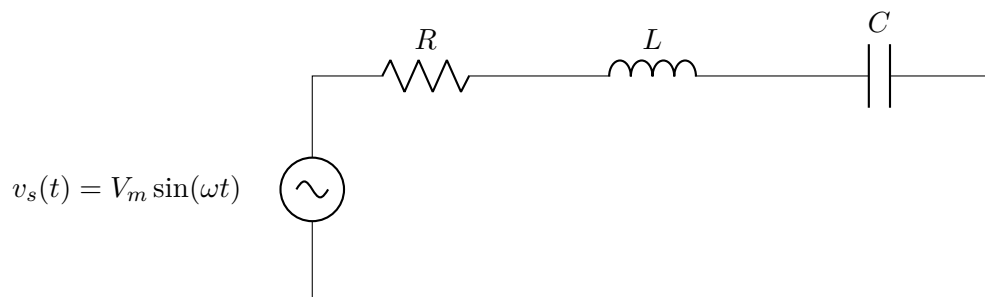
where:

$$\omega_0 = \sqrt{\frac{g}{L}}$$

and ζ is the damping ratio.

2.3 RLC Circuit: The Electrical Analog

Consider a series RLC circuit.



The equation governing the charge $q(t)$ on the capacitor is [2]:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V_s(t) \quad (4)$$

where $V_s(t)$ is the source voltage.

2.3.1 Damping Regimes of the RLC Circuit

In the absence of an external source ($V_s(t) = 0$), the circuit behaves as a freely damped oscillator. The undamped natural frequency and damping ratio are:

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}} \quad (5)$$

The characteristic equation is:

$$Ls^2 + Rs + \frac{1}{C} = 0 \quad (6)$$

with roots:

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad (7)$$

The three damping regimes are:

- **Underdamped** ($R < 2\sqrt{L/C}$): Oscillatory decay.
- **Critically damped** ($R = 2\sqrt{L/C}$): Fastest non-oscillatory return.
- **Overdamped** ($R > 2\sqrt{L/C}$): Slow non-oscillatory return.

2.3.2 Mechanical–Electrical Analogy

Mechanical	Electrical
Mass m	Inductance L
Damping b	Resistance R
Spring constant k	Inverse capacitance $1/C$
Displacement x	Charge q
Velocity $\dot{x}$	Current i
Kinetic energy $\frac{1}{2}m\dot{x}^2$	Magnetic energy $\frac{1}{2}Li^2$
Potential energy $\frac{1}{2}kx^2$	Electric energy $\frac{1}{2}q^2/C$

The RLC circuit is therefore the electrical analog of the mechanical damped oscillator. Section 7 carries the energy and entropy results over explicitly.

3 The Amplitude Envelope

The amplitude envelope is the smooth curve that traces the peak displacement of a damped oscillator over time [1].

For an underdamped oscillator, the amplitude decays exponentially:

$$A(t) = A_0 e^{-\zeta\omega_0 t} \quad (8)$$

where:

- A_0 is the initial amplitude,
- ζ is the damping ratio,
- ω_0 is the undamped natural frequency,
- t is time.

4 The Energy Envelope

The energy envelope is the footprint of the energy decay of a damped oscillator.

Because damping exists, the oscillator's mechanical energy is continuously converted into other forms such as heat due to friction, air drag, or internal material losses [4].

The total mechanical energy of a damped harmonic oscillator can be written as:

$$E(t) = \frac{1}{2}k_{\text{eff}}[A(t)]^2 \quad (9)$$

where k_{eff} is the effective stiffness of the oscillator.

Since the amplitude decays according to:

$$A(t) = A_0 e^{-\zeta\omega_0 t}$$

we substitute:

$$E(t) = \frac{1}{2}k_{\text{eff}}A_0^2 e^{-2\zeta\omega_0 t} \quad (10)$$

Letting:

$$E_0 = \frac{1}{2}k_{\text{eff}}A_0^2$$

we obtain the energy envelope:

$$\boxed{E(t) = E_0 e^{-2\zeta\omega_0 t}} \quad (11)$$

4.1 Scope of the Envelope

Two points make the envelope precise.

First, the envelope is valid for light damping, $\zeta \ll 1$. The instantaneous energy of the spring-mass-damper obeys the exact balance $dE/dt = -b\dot{x}^2$, which oscillates within each cycle. Over one cycle $\langle \dot{x}^2 \rangle = E/m$, so the cycle-averaged decay is

$$\frac{dE}{dt} = -\frac{b}{m}E = -2\zeta\omega_0 E, \quad (12)$$

which integrates to the envelope above.

Second, the effective stiffness k_{eff} is the coefficient that converts the amplitude into energy. It equals k for the spring-mass-damper, mgL for the small-angle pendulum (with A the angular amplitude), and $1/C$ for the RLC circuit (with A the charge amplitude).

5 Lost Mechanical Energy as Heat

The energy lost by the oscillator by time t is the difference between its initial energy and its current energy:

$$E_{\text{loss}}(t) = E_0 - E(t) \quad (13)$$

Substituting the energy envelope:

$$E_{\text{loss}}(t) = E_0 - E_0 e^{-2\zeta\omega_0 t} \quad (14)$$

Factoring E_0 :

$$\boxed{E_{\text{loss}}(t) = E_0 \left(1 - e^{-2\zeta\omega_0 t}\right)} \quad (15)$$

This lost mechanical energy must go somewhere. By the First Law of Thermodynamics, it is not destroyed. Instead, it is converted into heat and dissipated into the surroundings [2].

Therefore:

$$Q(t) = E_{\text{loss}}(t) \quad (16)$$

where $Q(t)$ is the total heat dissipated into the surroundings by time t .

6 Entropy Increase of the Surroundings

By the Second Law of Thermodynamics, when heat Q is transferred into surroundings acting as an isothermal thermal reservoir at constant temperature T , the entropy of the surroundings increases [3].

For an infinitesimal heat transfer:

$$dS_{\text{surr}} = \frac{\delta Q}{T} \quad (17)$$

Since T is constant, integrating gives:

$$\Delta S_{\text{surr}}(t) = \frac{Q(t)}{T} \quad (18)$$

Substituting the expression for the lost mechanical energy:

$$\boxed{\Delta S_{\text{surr}}(t) = \frac{E_0}{T} \left(1 - e^{-2\zeta\omega_0 t}\right)} \quad (19)$$

As $t \rightarrow \infty$, all mechanical energy is dissipated, yielding the asymptotic total entropy increase:

$$\Delta S_{\text{surr}}(\infty) = \frac{E_0}{T} \quad (20)$$

6.1 The Second Law Is Satisfied

The oscillator is a mechanical system. Assuming that the oscillator and damper return to the reservoir temperature T once the motion has ceased, so that their thermodynamic state is unchanged, their entropy does not change. The total entropy change is then

$$\Delta S_{\text{tot}} = \Delta S_{\text{osc}} + \Delta S_{\text{surr}} = 0 + \frac{Q}{T} \geq 0, \quad (21)$$

with equality only when there is no damping ($Q = 0$). The time-reversed process, in which a reservoir at temperature T spontaneously sets an initially quiescent oscillator into motion, would require $Q < 0$ and hence $\Delta S_{\text{tot}} = -|Q|/T < 0$, which the Second Law forbids. The Second Law is therefore what gives the damped oscillator its arrow of time.

6.2 Environmental Temperature Dependence and the Entropy Footprint

The total asymptotic entropy footprint $\Delta S_{\text{surr}}(\infty) = E_0/T$ reveals an intriguing temperature dependence.

Consider an identical oscillator with initial energy E_0 dissipating its energy in two different isothermal surroundings: a hot environment at temperature T_h and a colder environment at temperature T_c , where $T_c < T_h$.

The total entropy generated in each respective environment as $t \rightarrow \infty$ is:

$$\Delta S_h = \frac{E_0}{T_h}, \quad \Delta S_c = \frac{E_0}{T_c} \quad (22)$$

Since $T_c < T_h$, it follows directly that:

$$\Delta S_c > \Delta S_h \quad (23)$$

A compact way to see this is through the reservoir's own entropy as a function of its energy. For heat absorbed with no work done on the reservoir, $dS = dE/T$, so

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \quad (24)$$

The temperature is the inverse slope of the reservoir's entropy curve. At low temperature the curve is steep, so the same energy increment E_0 produces a larger entropy increase.

From a statistical-mechanical perspective, the entropy change quantifies the change in the thermodynamic multiplicity associated with the reservoir's macroscopic state. For an isothermal reservoir receiving a fixed amount of heat E_0 , the entropy increase is E_0/T . Consequently, the same energy transfer produces a larger entropy change at lower reservoir temperature. Under an appropriate microstate-counting description, this corresponds to a larger ratio of final to initial statistical multiplicities. This result should not be interpreted as implying that a colder reservoir necessarily possesses fewer accessible microstates in absolute terms, or that its phase-space volume literally expands by a specified amount [7].

Thus, while the oscillator loses the same quantity of mechanical energy E_0 in both cases, the entropy change of the surroundings depends inversely on their temperature.

6.3 Limits of the Result

Equation (19) assumes a reservoir so large that its temperature stays constant while it absorbs E_0 . This cannot hold as $T \rightarrow 0$: the divergence of E_0/T signals that a real reservoir would warm appreciably, and that classical damping would give way to quantum behavior. The result applies when E_0 is small compared with the thermal energy content of the reservoir.

The oscillator also never comes to rest exactly. At any finite temperature, equipartition gives a thermal fluctuation energy of order $k_B T$ for a one-dimensional oscillator. This is negligible for macroscopic oscillators, where $E_0 \gg k_B T$, but it sets the floor of validity for microscopic ones.

7 Rate of Entropy Production

The rate at which entropy is generated in the surroundings is obtained by differentiating the entropy change with respect to time:

$$\frac{dS_{\text{surr}}}{dt} = \frac{1}{T} \frac{dQ}{dt} \quad (25)$$

Since $Q(t) = E_0 - E(t)$,

$$\frac{dQ}{dt} = -\frac{dE}{dt} \quad (26)$$

Therefore:

$$\frac{dS_{\text{surr}}}{dt} = -\frac{1}{T} \frac{dE}{dt} = \frac{2\zeta\omega_0 E_0}{T} e^{-2\zeta\omega_0 t} \quad (27)$$

This shows that as the energy envelope decays, entropy is produced continuously. The faster the energy decays, the faster entropy increases.

8 The RLC Circuit: Energy, Heat, and Entropy

For the free series RLC circuit ($v_s = 0$), the stored energy is the sum of the magnetic and electric energies,

$$E = \frac{1}{2}Li^2 + \frac{q^2}{2C}. \quad (28)$$

The analog of kinetic plus potential energy. Differentiating and using $i = \dot{q}$ together with $L\ddot{q} + R\dot{q} + q/C = 0$,

$$\frac{dE}{dt} = i \left(L \frac{di}{dt} + \frac{q}{C} \right) = -i^2 R. \quad (29)$$

This exact balance says that the energy is lost as Joule heat in R , so $\dot{Q} = i^2 R$, the analog of $b\dot{x}^2$.

For light damping, $\langle i^2 \rangle = E/L$ over a cycle, so the cycle-averaged decay is $dE/dt = -(R/L)E$. Since $2\zeta\omega_0 = R/L$, the envelope is identical in form to the mechanical case:

$$E(t) = E_0 e^{-Rt/L}. \quad (30)$$

If the circuit is released from rest with charge q_0 on the capacitor, then $E_0 = q_0^2/2C$. The entropy results follow directly:

$$\Delta S_{\text{surr}}(t) = \frac{E_0}{T} (1 - e^{-Rt/L}), \quad \Delta S_{\text{surr}}(\infty) = \frac{E_0}{T}, \quad (31)$$

$$\frac{dS_{\text{surr}}}{dt} = \frac{RE_0}{LT} e^{-Rt/L}. \quad (32)$$

Here T is the temperature of the surroundings with which the resistor is in thermal contact. As a measurement, the envelope of the ring-down on an oscilloscope can be fitted to $e^{-Rt/2L}$ to extract R/L , and the entropy generated then follows from E_0 and T .

9 Discussion

The result we have derived connects three fundamental concepts:

1. **Mechanics:** The damped harmonic oscillator loses energy exponentially [1].
2. **Thermodynamics:** That lost energy becomes heat dissipated into the environment [2].
3. **Second Law:** That heat increases the entropy of the surroundings, with a total footprint determined by E_0/T [3].

The amplitude envelope is the footprint of decaying order.

The energy envelope is the footprint of declining useful mechanical energy.

The entropy increase is the footprint of the Second Law being satisfied.

All three tell the same story: organized mechanical energy dissipates into disordered thermal energy, and entropy increases—leaving a deeper thermodynamic mark in colder surroundings.

10 Conclusion

We have shown that the energy envelope of a lightly damped harmonic oscillator can be explicitly connected to the entropy increase of its surroundings.

The result is simple and general:

$$\Delta S_{\text{surr}}(t) = \frac{E_0}{T} \left(1 - e^{-2\zeta\omega_0 t}\right)$$

This exploratory bridge shows that a damped oscillator is not merely a mechanics problem. It is a thermodynamic process unfolding in time, demonstrating how mechanical order degrades into thermal disorder [4]. The same result holds for the series RLC circuit, with $2\zeta\omega_0 = R/L$.

The result rests on three assumptions: light damping, a constant-temperature reservoir, and $E_0 \gg k_B T$. Relaxing the last of these leads naturally to thermal fluctuations, where the same friction that dissipates energy also drives random motion (fluctuation-dissipation and, in circuits, Johnson-Nyquist noise). That is a natural direction for extending this exploration.

References

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