De Broglie's Phase Waves as Missing Link Between Single-Electron Wave Functions and Electronic Orbits

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Abstract

According to a recently proposed vibrational interpretation of the Schrödinger equation for hydrogen-like atoms, stationary single-electron wave functions describe vibrations that are driven by phase waves associated with a bound electron moving on precessing orbits similar to orbits of the old quantum theory by Bohr and Sommerfeld. Based on this interpretation, the objective of this work is to provide insights into the structure of these wave functions by constructing similar wave functions from the corresponding electronic orbits and their associated phase waves without explicitly solving the three-dimensional Schrödinger equation. The proposed construction extends solutions of the two-dimensional Schrödinger equation into three dimensions and takes precession of the orbital angular momentum into account by a weighted average over all rotations about the quantization axis. The constructed wave functions are structurally similar to the standard wave functions of hydrogen-like atoms in spherical coordinates; thus, the proposed construction provides a new way of understanding the structure of these wave functions.

1 Introduction

Shortly after the publication of the Schrödinger equation, de Broglie presented his pilot wave theory [BV09, chapter 2]. This theory includes a guiding equation, which can be used to extract electronic trajectories from single-electron wave functions. Bohmian mechanics [Boh52] includes a similar feature, while Planck proposed "modified wave mechanics" (MWM), which includes a criterion to identify wave functions that correspond to classical trajectories of electrons [Pla40, Rob17]. Schrödinger and other founders of quantum mechanics, however, were skeptical of the concept of electronic orbits in the atomic domain [BV09, page 401], and, in fact, theories that could relate wave functions to electronic trajectories had very little effect on the early development of quantum mechanics.

There were, however, other (often pedagogically motivated) efforts to relate single-electron wave functions of hydrogen-like atoms to electronic orbits [PW35, Buc08]. Table 1 summarizes the correspondences as described by Bucher [Buc08], which are employed in the present work. In the past, discussions of the relation between electronic orbits and corresponding wave functions usually have been limited to comparisons of invariants (e.g., angular momentum) and average values (e.g., expectation value of the distance between electron and nucleus). Robert explained why it is difficult to relate wave functions to electronic orbits: "There is a 'statistical character' in these solutions, from which average values and fluctuations of dynamical operators are obtained using the very rules of statistical mechanics" [Rob17]. Thus, they "might be not accurate enough to describe particle motion because,

Table 1: Hydrogen-like wave functions characterized by standard quantum numbers (n, l, m) and corresponding electronic orbits according to Bucher [Buc08].

Wave functions	Corresponding electronic orbits
$n > l = 0 \wedge m = 0$	"Coulomb oscillators" a.k.a. "pendulum paths"
$n > l > 0 \wedge m = 0$	elliptical orbits in planes parallel to z-axis
$n > l > 0 \wedge m \neq 0$	elliptical orbits in planes tilted by τ relative to x-y-plane with $\tau = \arccos(m/\sqrt{l(l+1)})$

for instance, they do not separate the directions of motion, and the trajectories that they construct correspond to a statistical (with uniform probability) averaging of [Planck's] (MWM) waves corresponding to each motion" [Rob17]. In spite of these difficulties, Littman et al. searched for patterns in the nodal structure of hydrogen wave functions by comparing them with de Broglie phase waves on specific electronic orbits. They concluded: "Wavefunctions and charge densities for atomic hydrogen display patterns that are understandable in terms of the planetary models of the old quantum theories. The seemingly complex myriad of nodes and anti-nodes of hydrogen are recognizable in comparison with de Broglie-wave-analyzed Bohr–Sommerfeld orbits" [LGZA23].

The hypothesis that single-electron wave functions are strongly related to phase waves on electronic orbits motivated the present work, which constructs similar wave functions based on electronic orbits and their phase waves without solving a three-dimensional Schrödinger equation. The construction method, which is presented in Section 4, was inspired in part by the role of Larmor precession in a vibrational interpretation of the Schrödinger equation [Kra26], which is reviewed in Section 2. Another ingredient of the construction method are solutions of the two-dimensional Schrödinger equation, which are introduced in Section 3. Section 5 discusses insights that are provided by the present work, and Section 6 concludes this paper.

2 Vibrational Interpretation of the Schrödinger Equation for a Spinless Electron in Hydrogen-Like Atoms

The mentioned vibrational interpretation of the Schrödinger equation [Kra26] interprets de Broglie's internal clock of an electron and the associated phase wave as electromagnetic phenomena. In this interpretation, the internal clock of an atom's electron is presumed to drive a stationary electromagnetic vibration of the atom's electromagnetic field, which is in a fixed phase relation with the electron's phase wave. It is further assumed that this vibration is stationary not only for a single, closed Kepler orbit of the electron but for all electronic orbits that are generated by the presumably inevitable precession of the electronic orbit. The time-independent Schrödinger equation has been shown to take all these orbits into account by stating a stationary wave equation in a volume of space that includes all points that are covered by these orbits. Thus, precession of orbits appears to be a baked-in feature of the Schrödinger equation—at least in this interpretation. Therefore, a stationary wave function solving the time-independent Schrödinger equation does not correspond to a single, closed Kepler orbit of the electron but to a (relatively slowly) precessing orbit, which may be conceptualized as a family of closed Kepler orbits that are generated by precession.

In a two-dimensional hydrogen-like atom, relativistic apsidal precession (see Section 3) is inevitable for elliptical orbits. In the three-dimensional case (see Section 4), even weak magnetic fields (e.g., due to the magnetic moment of the nucleus) result in Larmor precession of the electronic orbit. In general, Larmor precession causes the electron to leave a single orbital plane, such that the precessing orbit sweeps a volume of points as described above.

3 From 2D Orbits to 2D Wave Functions

For a two-dimensional hydrogen-like atom, the rotational symmetric potential of a spinless electron is

$$V(\tilde{r}) = \frac{Ze^2}{4\pi\varepsilon_0\tilde{r}} \tag{1}$$

with the atomic number Z, the elementary charge e, the vacuum permittivity ε_0, and the distance $\tilde{r}$ between the electron and the nucleus. The time-independent Schrödinger equation for a two-dimensional stationary wave function $\tilde{\Psi}$ of energy $\tilde{E}$ is then given by

$$-\frac{\hbar^2}{2\mu}\nabla^2\tilde{\Psi} + V(\tilde{r})\tilde{\Psi} = \tilde{E}\tilde{\Psi} \tag{2}$$

with the reduced mass of the electron $\mu = (m_\mathrm{e}m_\mathrm{N})/(m_\mathrm{e} + m_\mathrm{N})$ where m_e and m_N are the masses of the electron and the nucleus. Note that this work uses tilde overscripts for symbols that are used only in the context of the two-dimensional Schrödinger equation.

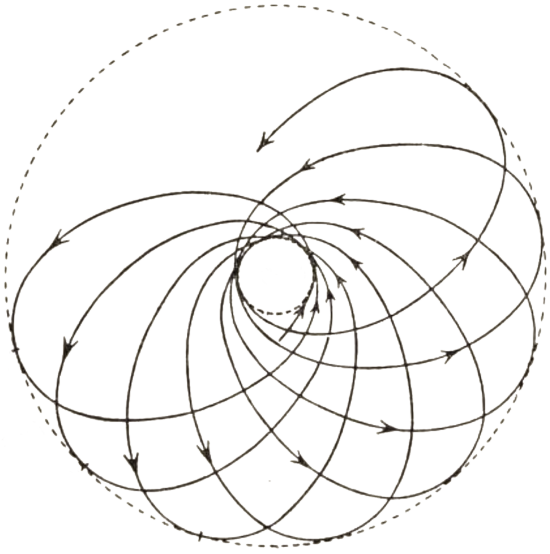

Figure 1: Rosetta orbit caused by relativistic apsidal precession. Illustration adapted from figure 110 in Sommerfeld's book "Atomic structure and spectral lines" [Som23, page 467].

The classical orbits of an electron in the potential $V(\tilde{r})$ stay in one plane. Here, this plane is parameterized by polar coordinates $\tilde{r}$ and $\tilde{\phi}$ with $\tilde{r} = 0$ marking the position of the nucleus. As illustrated in Figure 1, relativistic apsidal precession of elliptical orbits forms rosetta orbits [Som23]. The mentioned vibrational interpretation [Kra26] of the two-dimensional Schrödinger equation suggests that every point of a two-dimensional stationary wave function is in a fixed phase relation with phase waves along all parts of a precessing orbit that pass through the point. Figure 1 shows that a precessing orbit passes through most of its points in two ways: either in outward direction or in inward direction. There is no reason why a stationary wave function should single out one of these directions; thus, it is plausible to expect that the direction of the phase velocity of a stationary wave function is the average of these two directions, i.e., the direction parameterized by the $\tilde{\phi}$ coordinate. This is in fact the case (except for $\tilde{m} = 0$) as the two-dimensional stationary wave functions $\tilde{\Psi}_{\tilde{n},\tilde{m}}(\tilde{r},\tilde{\phi},t)$ for integer quantum numbers $\tilde{n} \geq 0$ and $\tilde{m}$ with $|\tilde{m}| \leq \tilde{n}$ [ZZ67, HK79, PP02] show:

$$\tilde{\Psi}_{\tilde{n},\tilde{m}}(\tilde{r},\tilde{\phi},t) = \tilde{C}_{\tilde{n},\tilde{m}}\tilde{R}_{\tilde{n},\tilde{m}}(\tilde{r})\exp(\mathrm{i}\tilde{m}\tilde{\phi})\exp(-\mathrm{i}\tilde{\omega}_{\tilde{n},\tilde{m}}t) = \tilde{C}_{\tilde{n},\tilde{m}}\tilde{R}_{\tilde{n},\tilde{m}}(\tilde{r})\exp(\mathrm{i}(\tilde{m}\tilde{\phi} - \tilde{\omega}_{\tilde{n},\tilde{m}}t)) \tag{3}$$

with normalization constants $\tilde{C}_{\tilde{n},\tilde{m}}$, angular frequencies $\tilde{\omega}_{\tilde{n},\tilde{m}}$ and radial functions

$$\tilde{R}_{\tilde{n},\tilde{m}}(\tilde{r}) = \exp\left(\frac{-Z\tilde{r}}{(\tilde{n}+\frac{1}{2})a_\mu}\right)\left(\frac{2Z\tilde{r}}{(\tilde{n}+\frac{1}{2})a_\mu}\right)^{|\tilde{m}|} L^{2|\tilde{m}|}_{\tilde{n}-|\tilde{m}|}\left(\frac{2Z\tilde{r}}{(\tilde{n}+\frac{1}{2})a_\mu}\right), \tag{4}$$

where $L^{2|\tilde{m}|}_{\tilde{n}-|\tilde{m}|}$ are generalized Laguerre polynomials, and a_μ is similar to the Bohr radius but with the electron's reduced mass μ instead of the electron's mass m_e:

$$a_\mu = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}. \tag{5}$$

While the wave functions $\tilde{\Psi}_{\tilde{n},\tilde{m}}$ for $\tilde{m} \neq 0$ correspond to elliptical orbits (as noted by Hassoun [Has81]), the apsidal precession illustrated in Figure 1 leads to a circular symmetry of these wave functions. This supports the interpretation that apsidal precession is a baked-in feature of the two-dimensional Schrödinger equation for hydrogen-like atoms.

4 From 2D Wave Functions to 3D Wave Functions

The three-dimensional hydrogen-like atom requires a spherically symmetric potential

$$V(r) = \frac{Ze^2}{4\pi\varepsilon_0 r} \tag{6}$$

Table 2: Two-dimensional wave functions $\tilde{\Psi}_{\tilde{n},\tilde{m}}$ for $\tilde{n} \leq 2$ and the corresponding three-dimensional wave functions $\Psi_{n,l,m}$.

2D wave functions	corresponding 3D wave functions
$\tilde{n} = 0,\ \tilde{m} = 0$	$n = 1,\ l = 0,\ m = 0$
$\tilde{n} = 1,\ \tilde{m} = 0$	$n = 2,\ l = 0,\ m = 0$
$\tilde{n} = 1,\ \tilde{m} = \pm 1$	$n = 2,\ l = 1,\ m = \pm 1$
$\tilde{n} = 2,\ \tilde{m} = 0$	$n = 3,\ l = 0,\ m = 0$
$\tilde{n} = 2,\ \tilde{m} = \pm 1$	$n = 3,\ l = 1,\ m = \pm 1$
$\tilde{n} = 2,\ \tilde{m} = \pm 2$	$n = 3,\ l = 2,\ m = \pm 2$

where r denotes the distance between the electron and the nucleus in three dimensions. In analogy to the two-dimensional case of the previous section, the time-independent Schrödinger equation for a three-dimensional stationary wave function Ψ of energy E is given by

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi + V(r)\Psi = E\Psi. \tag{7}$$

The solutions $\Psi_{n,l,m}$ for quantum numbers n, l, and m in spherical coordinates r, ϕ, and θ may be written as:

$$\Psi_{n,l,m}(r,\phi,\theta,t) = C_{n,l,m} R_{n,l}(r) P_l^m(\cos\theta) \exp(\mathrm{i}m\phi) \exp(-\mathrm{i}\omega_{n,l}t) \tag{8}$$

with normalization constants $C_{n,l,m}$, angular frequencies $\omega_{n,l}$, associated Legendre polynomials $P_l^m(\cos\theta)$ (including the Condon-Shortley phase $(-1)^m$), and the radial functions

$$R_{n,l}(r) = \exp\left(\frac{-Zr}{na_\mu}\right)\left(\frac{2Zr}{na_\mu}\right)^l L_{n-1-l}^{2l+1}\left(\frac{2Zr}{na_\mu}\right). \tag{9}$$

Zaslow and Zandler [ZZ67] have related these wave functions for the hydrogen problem ($Z = 1$) to the two-dimensional wave functions discussed in the previous section. Based on their results, Table 2 lists quantum numbers $\tilde{n}$ and $\tilde{m}$ of two-dimensional wave functions for $\tilde{n} \leq 2$ and the quantum numbers n, l, and m of the corresponding three-dimensional wave functions. As noted by Zaslow and Zandler, this is not a one-to-one mapping since three-dimensional wave functions with $|m| < l$ do not correspond to any two-dimensional wave function. In terms of Table 1, wave functions with $|m| < l$ correspond to electronic orbits that are neither pendulum paths nor "minimally tilted" orbits, i.e., orbits with minimum tilt angle τ for a specific value of l.

From Table 2, the following correspondences may be inferred:

$$\tilde{n} \longleftrightarrow n - 1, \qquad |\tilde{m}| \longleftrightarrow l, \qquad \tilde{m} \longleftrightarrow m. \tag{10}$$

With these correspondences, the two-dimensional wave functions $\tilde{\Psi}_{\tilde{n},\tilde{m}}$ are structurally similar to the three-dimensional wave functions $\Psi_{n,l,m}$. In particular, $\exp(\mathrm{i}\tilde{m}\tilde{\phi})$ matches $\exp(\mathrm{i}m\phi)$, and "[e]xcept for the shift in the electron cloud towards the nucleus, radial wavefunctions [$R_{n,l}$ and $\tilde{R}_{\tilde{n}=n-1,\tilde{m}=l}$] are otherwise very similar, since both are described by associated Laguerre polynomials" [ZZ67].

Unfortunately, Zaslow and Zandler's discussion does not address the associated Legendre polynomials $P_l^m(\cos\theta)$ in $\Psi_{n,l,m}$, and it does not relate the three-dimensional wave functions $\Psi_{n,l,m}$ with $|m| < l$ to any two-dimensional wave functions. The following sections try to fill these gaps by presenting a method to construct wave functions $\psi_{n,l,m}$, which are similar to the three-dimensional wave functions $\Psi_{n,l,m}$, based on the two-dimensional wave functions $\tilde{\Psi}_{\tilde{n}=n-1,\tilde{m}=l}$.

4.1 Case $l = 0 \wedge m = 0$

If $l = 0$ and $m = 0$, $P_l^m(\cos\theta)\exp(\mathrm{i}m\phi) = 1$; thus, $\Psi_{n,0,0}$ does not depend on θ and ϕ, i.e., it is spherically symmetric. In terms of the electronic pendulum paths mentioned in Table 1, this spherical symmetry could be interpreted as the result of an ergodic deflection of the electron each time it passes the nucleus [Kra25].

As noted above, the radial function $R_{n,0}$ is very similar to $\tilde{R}_{n-1,0}$, thus, $\tilde{\Psi}_{n-1,0}$ is similar to $\Psi_{n,0,0}$. Therefore, $\psi_{n,0,0} = c_{n,0,0}\tilde{\Psi}_{n-1,0}$ (with a normalization constant $c_{n,0,0}$) is used as the result of the proposed construction in this special case.

4.2 Case $l > 0 \wedge m = 0$

According to Table 1, three-dimensional wave functions for $l > 0$ and $m = 0$ correspond to an elliptical electronic orbit in a plane parallel to the z-axis, for example, the x-z-plane. The proposed construction method begins by assuming a two-dimensional wave function on this orbital plane, which covers the apsidal precession inside the orbital plane as discussed in Section 3. If the orbital plane is placed in the x-z-plane, the role of the two-dimensional coordinate $\tilde{\phi}$ is taken over by the spherical coordinate θ on one half of the orbital plane and $-\theta$ on the other half.

Note that this two-dimensional wave function does not cover the precession of the orbital plane about the z-axis (for example, due to Larmor precession). This precession is taken into account by a weighted average of all rotations (parameterized by $\phi_0 \in [0, 2\pi]$) of the two-dimensional wave function about the z-axis, where the "weight" at a point (r, ϕ, θ) is chosen as $|\cos(\phi - \phi_0)|$ such that points on the rotated orbital plane at ϕ_0 receive weight 1 and points on the orthogonal plane receive weight 0. Another interpretation of $|\cos(\phi - \phi_0)|$ is that it provides a smooth extension of the two-dimensional wave function on the plane $\phi = \phi_0 \vee (\phi + \pi) \bmod 2\pi = \phi_0$ to three dimensions. Presumably, there are alternatives to the weight factor $|\cos(\phi - \phi_0)|$ (featuring the same periodicity and symmetry), but those alternatives were not further researched.

The wave function $\psi_{n,l,m}$ is constructed as the mentioned weighted average over all rotations for ϕ_0 from 0 to 2π. It is convenient to split the integration over ϕ_0 into one integration from $\phi - \frac{\pi}{2}$ to $\phi + \frac{\pi}{2}$ and one integration from $\phi + \frac{\pi}{2}$ to $\phi + \frac{3\pi}{2}$. This allows us to handle the two halves of the orbital plane separately, which helps with the mentioned use of θ and $-\theta$ in the role of $\tilde{\phi}$:

$$
\begin{aligned}
\psi_{n,l,0}(r,\phi,\theta,t) &= c_{n,l,0}\int_{\phi-\pi/2}^{\phi+\pi/2} \mathrm{d}\phi_0\, |\cos(\phi-\phi_0)|\, \tilde{\Psi}_{n-1,l}(r,\theta,t) + \\
&\quad c_{n,l,0}\int_{\phi+\pi/2}^{\phi+3\pi/2} \mathrm{d}\phi_0\, |\cos(\phi-\phi_0)|\, \tilde{\Psi}_{n-1,l}(r,-\theta,t) && (11)\\
&= c_{n,l,0}\int_{\phi-\pi/2}^{\phi+\pi/2} \mathrm{d}\phi_0 \cos(\phi-\phi_0)\tilde{\Psi}_{n-1,l}(r,\theta,t) - \\
&\quad c_{n,l,0}\int_{\phi+\pi/2}^{\phi+3\pi/2} \mathrm{d}\phi_0 \cos(\phi-\phi_0)\tilde{\Psi}_{n-1,l}(r,-\theta,t) && (12)\\
&= c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,\theta,t)\,[-\sin(\phi-\phi_0)]_{\phi_0=\phi-\pi/2}^{\phi_0=\phi+\pi/2} - \\
&\quad c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,-\theta,t)\,[-\sin(\phi-\phi_0)]_{\phi_0=\phi+\pi/2}^{\phi_0=\phi+3\pi/2} && (13)\\
&= c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,\theta,t)\,(-\sin(-\pi/2)+\sin(\pi/2)) - \\
&\quad c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,-\theta,t)\,(-\sin(-3\pi/2)+\sin(-\pi/2)) && (14)\\
&= 2c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,\theta,t) + 2c_{n,l,0}\tilde{\Psi}_{n-1,l}(r,-\theta,t) && (15)\\
&= 2c_{n,l,0}\tilde{C}_{n-1,l}\tilde{R}_{n-1,l}(r)\exp(-\mathrm{i}\tilde{\omega}_{n-1,l}t)\,(\exp(\mathrm{i}l\theta)+\exp(-\mathrm{i}l\theta)) && (16)\\
&= 4c_{n,l,0}\tilde{C}_{n-1,l}\tilde{R}_{n-1,l}(r)\exp(-\mathrm{i}\tilde{\omega}_{n-1,l}t)\cos(l\theta) && (17)
\end{aligned}
$$

The factor $\cos(l\theta)$ is a Chebyshev polynomial of the first kind, which can be expanded in terms of Legendre polynomials. In leading order of Legendre polynomials, $\cos(l\theta)$ is proportional to $P_l(\cos\theta) = P_l^0(\cos\theta)$. Thus, with a new normalization constant $c'_{n,l,0}$, $\psi_{n,l,0}$ becomes approximately:

$$\psi_{n,l,0}(r,\phi,\theta,t) \approx c'_{n,l,0}\tilde{R}_{n-1,l}(r)\exp(-\mathrm{i}\tilde{\omega}_{n-1,l}t)P_l^0(\cos\theta), \tag{18}$$

which is similar to

$$\Psi_{n,l,0}(r,\phi,\theta,t) = C_{n,l,0}R_{n,l}(r)\exp(-\mathrm{i}\omega_{n,l}t)P_l^0(\cos\theta). \tag{19}$$

The common factor $P_l^0(\cos\theta)$ in $\psi_{n,l,0}$ and $\Psi_{n,l,0}$ may be understood as a consequence of the averaging of rotated two-dimensional wave functions, which takes precession of the orbital angular

momentum about the z-axis into account. Therefore, precession of the orbital angular momentum appears to be a baked-in feature of the three-dimensional Schrödinger equation for hydrogen-like atoms (not unlike the role of apsidal precession for the two-dimensional Schrödinger equation mentioned at the end of Section 3).

4.3 Case $l > 0 \wedge m \neq 0$

For $m = 0$, the corresponding electronic orbits are in orbital planes that are parallel to the z-axis. For increasing $|m|$, however, the orbital planes tilt away from the z-axis toward the x-y-plane. In the construction described in the previous section, this tilting can be taken into account by reducing the number of oscillations of the wave function with varying θ while at the same time increasing the number of oscillations with varying ϕ. The former change can be implemented by replacing $\tilde{\Psi}_{n-1,l}$ with $\tilde{\Psi}_{n-1,l-|m|}$, while the latter change can be implemented by an additional factor $\exp(\mathrm{i}m\phi)$.

There is yet another effect of the tilting of orbital planes: with increasing $|m|$, orbits stay increasingly close to the x-y-plane, i.e., the θ values associated with points of orbits stay increasingly close to $\pi/2$. Thus, there is an increasingly strong attenuation of the wave function for $\theta \neq \pi/2$. The proposed construction implements this effect by an additional factor $(\sin\theta)^{|m|}$ where it is assumed that $\sin\theta \geq 0$ because $0 \leq \theta \leq \pi$.

With these changes, the calculation in Eqs. (11)–(17) becomes:

$$
\begin{aligned}
\psi_{n,l,m}(r,\phi,\theta,t) &= c_{n,l,m} \int_{\phi-\pi/2}^{\phi+\pi/2} \mathrm{d}\phi_0 \, |\cos(\phi-\phi_0)| \, \tilde{\Psi}_{n-1,l-|m|}(r,\theta,t) \exp(\mathrm{i}m\phi)(\sin\theta)^{|m|} + \\
&\quad c_{n,l,m} \int_{\phi+\pi/2}^{\phi+3\pi/2} \mathrm{d}\phi_0 \, |\cos(\phi-\phi_0)| \, \tilde{\Psi}_{n-1,l-|m|}(r,-\theta,t) \exp(\mathrm{i}m\phi)(\sin\theta)^{|m|} \quad (20) \\
&= 2c_{n,l,m} \Big(\tilde{\Psi}_{n-1,l-|m|}(r,\theta,t) + \tilde{\Psi}_{n-1,l-|m|}(r,-\theta,t) \Big) \exp(\mathrm{i}m\phi)(\sin\theta)^{|m|} \quad (21) \\
&= 2c_{n,l,m} \tilde{C}_{n-1,l-|m|} \tilde{R}_{n-1,l-|m|}(r) \exp(-\mathrm{i}\tilde{\omega}_{n-1,l-|m|}t) \times \\
&\quad \Big(\exp(\mathrm{i}(l-|m|)\theta) + \exp(-\mathrm{i}(l-|m|)\theta) \Big) \exp(\mathrm{i}m\phi)(\sin\theta)^{|m|} \quad (22) \\
&= 4c_{n,l,m} \tilde{C}_{n-1,l-|m|} \tilde{R}_{n-1,l-|m|}(r) \exp(-\mathrm{i}\tilde{\omega}_{n-1,l-|m|}t) \times \\
&\quad \exp(\mathrm{i}m\phi)(\sin\theta)^{|m|} \cos((l-|m|)\theta) \quad (23)
\end{aligned}
$$

The expression $(\sin\theta)^{|m|}\cos((l-|m|)\theta)$ is similar to $k_{l,m}P_l^m(\cos\theta)$ with appropriately chosen constants $k_{l,m}$ as explained next. First, note that the associated Legendre polynomials $P_l^m(\cos\theta)$ with $m < 0$ can be defined as [Wei26]:

$$
P_l^m(\cos\theta) = (-1)^{|m|} \frac{(l-|m|)!}{(l+|m|)!} P_l^{|m|}(\cos\theta), \quad (24)
$$

thus, for $m < 0$, $P_l^m(\cos\theta)$ is proportional to $P_l^{|m|}(\cos\theta)$. On the other hand, for $m \geq 0$, $P_l^m(\cos\theta)$ is trivially proportional to $P_l^{|m|}(\cos\theta)$. Therefore, $P_l^m(\cos\theta)$ is proportional to $P_l^{|m|}(\cos\theta)$ for all values of m with constants of proportionality depending on l and m.

For $P_l^{|m|}(\cos\theta)$ (or equivalently for $P_l^m(\cos\theta)$ with $m \geq 0$) we have [Wei26]:

$$
P_l^{|m|}(\cos\theta) = (-1)^{|m|} (\sin\theta)^{|m|} \frac{\mathrm{d}^{|m|}}{\mathrm{d}(\cos\theta)^{|m|}} P_l(\cos\theta). \quad (25)
$$

Since the leading order of $\cos\theta$ in $P_l(\cos\theta)$ is $(\cos\theta)^l$, this means that $P_l^{|m|}(\cos\theta)$ is proportional to $(\sin\theta)^{|m|}(\cos\theta)^{l-|m|}$ in leading order of $\cos\theta$. The same is true for $(\sin\theta)^{|m|}\cos((l-|m|)\theta)$ (since $\cos((l-|m|)\theta)$ is a Chebyshev polynomial of the first kind, which is in leading order proportional to $P_{l-|m|}(\cos\theta)$).

Thus, with appropriately chosen constants $k_{l,m}$, we arrive at (in leading order of $\cos\theta$):

$$
(\sin\theta)^{|m|}\cos((l-|m|)\theta) \approx k_{l,m} P_l^m(\cos\theta). \quad (26)
$$

Therefore, the result of the proposed construction (with new normalization constants $c'_{n,l,m}$)

$$\psi_{n,l,m}(r,\phi,\theta,t) = c'_{n,l,m}\tilde{R}_{n-1,l-|m|}(r)\exp(-\mathrm{i}\tilde{\omega}_{n-1,l-|m|}t)\exp(\mathrm{i}m\phi)(\sin\theta)^{|m|}\cos((l-|m|)\theta) \tag{27}$$

is similar to

$$\Psi_{n,l,m}(r,\phi,\theta,t) = C_{n,l,m}R_{n,l}(r)\exp(-\mathrm{i}\omega_{n,l}t)\exp(\mathrm{i}m\phi)P_l^m(\cos\theta). \tag{28}$$

This completes the description of the proposed construction of three-dimensional wave functions $\psi_{n,l,m}$ that are similar to the actual three-dimensional wave functions $\Psi_{n,l,m}$ for hydrogen-like atoms. The wave functions $\psi_{n,l,m}$ and the process of their construction illuminate several features of the structure of the actual wave functions $\Psi_{n,l,m}$ as discussed in the next section.

5 Discussion

The mentioned vibrational interpretation of the Schrödinger equation for hydrogen-like atoms (see Section 2) describes stationary wave functions as vibrations driven by an electron in a volume of space close to the nucleus, which includes electrons on precessing orbits. The construction of similar wave functions in this work supports this interpretation and provides additional details: apsidal precession appears to be a baked-in feature of the solutions of the two-dimensional Schrödinger equation (see Section 3), and precession of the orbital angular momentum appears to be a baked-in feature of the solutions of the three-dimensional Schrödinger equation (see Section 4).

However, the non-relativistic Schrödinger equation without magnetic fields discussed in this work does not include any reference to the physical processes causing precession of electronic orbits; e.g., relativistic apsidal precession or Larmor precession. Nonetheless, the Schrödinger equation subtly requires its stationary solutions to "accommodate" precessing electronic orbits. It is almost as if Schrödinger had designed the Schrödinger equation for precessing electronic orbits but never wanted to admit to this feature, because he very much doubted the existence of electronic orbits in atoms [Sch28, page 9][BV09, page 401].

In any case, this work shows that the standard wave functions $\Psi_{n,l,m}$ for hydrogen-like atoms may be understood to some degree in terms of de Broglie's phase waves on precessing electronic orbits. Thus, it provides an explanation for the nodal patterns in wave functions described by Littman et al. [LGZA23]. It also shows that several features of these wave functions may be related to specific features of the corresponding electronic orbits, e.g., the interpretation of the term $(\sin\theta)^m$ in associated Legendre polynomials as an expression of the tilt angle of electronic orbits (see Section 4.3). Moreover, it shows that precession of the orbital angular momentum about the z-axis is a baked-in feature of three-dimensional wave functions $\Psi_{n,l,m}$ in spherical coordinates and links this precession (via the integration over ϕ_0; see Section 4.2) to the "statistical (with uniform probability) averaging" mentioned by Robert [Rob17].

Furthermore, this averaging explains why these wave functions fail Planck's criterion for wave functions that correspond to classical trajectories [Pla40, Rob17], and why trajectories defined by de Broglie's guiding equation [BV09, chapter 2] are circular instead of elliptic [BV09, page 401].

6 Conclusion and Future Work

While a recently proposed vibrational interpretation [Kra26] explained the Schrödinger equation for hydrogen-like atoms as a description of vibrations driven by de Broglie's phase waves on precessing electronic orbits, this work explores this interpretation in terms of stationary solutions of the Schrödinger equation by constructing similar wave functions based on the corresponding electronic orbits and their associated phase waves without explicitly solving the Schrödinger equation. The structure of the constructed wave functions as well as the method of constructing them provide several insights that are difficult to extract directly from the Schrödinger equation or its actual solutions.

Future work includes extending the presented approach to electrons with non-relativistic and relativistic spin.

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A Revisions

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