Real–Phase Gauge Field Theory Transformations in U(1), SU(2), and SU(4)

N. B. Cook

September 1, 2026

1 Abstract

Reference 1 shows that when multipath interference is treated correctly, the imaginary term, i sin S, is unnecessary; the real term cos S suffices, therefore Euler's exp(iS) = i sin S + cos S reduces to exp(iS) $\to$ cos S. Can this same principle be applied to complex phase amplitudes in gauge theory phase transformations? Gauge invariance only requires a continuous group action in real space, not the complex phase structure suggested by textbooks. The complex exponential hides the physics. The conversion of gauge theory to real space phase changes makes explicit the physical mechanism of gauge theory.

2 U(1): Full Derivation

We show explicitly, by step–by–step calculus, how replacing the usual unitary gauge transformation by its real projection preserves gauge invariance of the Dirac equation and covariant derivative.

Start from the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{1}$$

Standard local U(1) transformation:

$$\psi(x) \to \psi'(x) = e^{i\chi(x)}\psi(x). \tag{2}$$

We replace the complex phase by its real part:

$$\psi(x) \to \psi'(x) = \cos\chi(x)\,\psi(x). \tag{3}$$

Compute the transformed derivative:

$$\begin{aligned}
\partial_\mu \psi' &= \partial_\mu\big(\cos\chi\,\psi\big) && (4)\\
&= (\partial_\mu \cos\chi)\,\psi + \cos\chi\,\partial_\mu\psi && (5)\\
&= -\sin\chi\,(\partial_\mu\chi)\,\psi + \cos\chi\,\partial_\mu\psi. && (6)
\end{aligned}$$

Insert into the Dirac operator:

$$(i\gamma^\mu\partial_\mu - m)\psi' = i\gamma^\mu\partial_\mu\psi' - m\psi' \tag{7}$$

$$= i\gamma^\mu\big[-\sin\chi\,(\partial_\mu\chi)\,\psi + \cos\chi\,\partial_\mu\psi\big] - m\cos\chi\,\psi. \tag{8}$$

Use the original equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$, i.e.

$$i\gamma^\mu\partial_\mu\psi = m\psi, \tag{9}$$

to rewrite the $\cos\chi$ term:

$$(i\gamma^\mu\partial_\mu - m)\psi' = i\gamma^\mu\big[-\sin\chi\,(\partial_\mu\chi)\,\psi\big] + \cos\chi\,(i\gamma^\mu\partial_\mu\psi - m\psi) \tag{10}$$

$$= -i\gamma^\mu\sin\chi\,(\partial_\mu\chi)\,\psi. \tag{11}$$

Thus the transformed Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi' = -i\gamma^\mu\sin\chi\,(\partial_\mu\chi)\,\psi. \tag{12}$$

The extra term proportional to $\partial_\mu\chi$ must be cancelled by the gauge field via the covariant derivative.

Define the covariant derivative

$$D_\mu = \partial_\mu + igA_\mu, \tag{13}$$

and require homogeneous transformation:

$$D'_\mu\psi' = \cos\chi\,D_\mu\psi. \tag{14}$$

Compute $D'_\mu\psi'$ explicitly:

$$D'_\mu\psi' = \partial_\mu(\cos\chi\,\psi) + igA'_\mu\,\cos\chi\,\psi \tag{15}$$

$$= -\sin\chi\,(\partial_\mu\chi)\,\psi + \cos\chi\,\partial_\mu\psi + igA'_\mu\,\cos\chi\,\psi. \tag{16}$$

Right–hand side:

$$\cos\chi\,D_\mu\psi = \cos\chi\,(\partial_\mu\psi + igA_\mu\psi) \tag{17}$$

$$= \cos\chi\,\partial_\mu\psi + igA_\mu\,\cos\chi\,\psi. \tag{18}$$

Equate both:

$$-\sin\chi\,(\partial_\mu\chi)\,\psi + \cos\chi\,\partial_\mu\psi + igA'_\mu\,\cos\chi\,\psi = \cos\chi\,\partial_\mu\psi + igA_\mu\,\cos\chi\,\psi. \tag{19}$$

Cancel $\cos\chi\,\partial_\mu\psi$:

$$-\sin\chi\,(\partial_\mu\chi)\,\psi + igA'_\mu\,\cos\chi\,\psi = igA_\mu\,\cos\chi\,\psi. \tag{20}$$

Divide by ψ:

$$-\sin\chi\,\partial_\mu\chi + igA'_\mu\,\cos\chi = igA_\mu\,\cos\chi. \tag{21}$$

Solve for A'_μ:

$$igA'_\mu \cos\chi = igA_\mu \cos\chi + \sin\chi\,\partial_\mu\chi, \tag{22}$$

$$A'_\mu = A_\mu + \frac{\sin\chi}{g\cos\chi}\,\partial_\mu\chi = A_\mu + \frac{1}{g}\tan\chi\,\partial_\mu\chi. \tag{23}$$

Thus the Dirac equation with D_μ is invariant under the real transformation $\psi \to \cos\chi\,\psi$ provided the gauge field transforms as

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{g}\tan\chi\,\partial_\mu\chi. \tag{24}$$

3 SU(2): Matrix Calculus Derivation

Standard SU(2) transformation:

$$\psi \to \psi' = U(x)\psi, \qquad U(x) = \exp\big(i\chi^a(x)\tau^a\big), \tag{25}$$

with Pauli matrices τ^a.

We replace $U(x)$ by its real projection:

$$\psi' = R(x)\psi, \qquad R(x) = \Re\,U(x), \tag{26}$$

where R is a real 2×2 matrix.

Compute the derivative:

$$\partial_\mu\psi' = \partial_\mu(R\psi) \tag{27}$$

$$= (\partial_\mu R)\psi + R\,\partial_\mu\psi. \tag{28}$$

Define the covariant derivative

$$D_\mu = \partial_\mu + igW^a_\mu\tau^a, \tag{29}$$

and demand homogeneous transformation:

$$D'_\mu\psi' = R\,D_\mu\psi. \tag{30}$$

Compute $D'_\mu\psi'$:

$$D'_\mu\psi' = \partial_\mu(R\psi) + igW'^a_\mu\tau^a R\psi \tag{31}$$

$$= (\partial_\mu R)\psi + R\partial_\mu\psi + igW'^a_\mu\tau^a R\psi. \tag{32}$$

Right–hand side:

$$R\,D_\mu\psi = R(\partial_\mu\psi + igW^a_\mu\tau^a\psi) \tag{33}$$

$$= R\partial_\mu\psi + igRW^a_\mu\tau^a\psi. \tag{34}$$

Equate:

$$(\partial_\mu R)\psi + R\partial_\mu\psi + igW'^{\,a}_\mu\tau^a R\psi = R\partial_\mu\psi + igRW^a_\mu\tau^a\psi. \tag{35}$$

Cancel $R\partial_\mu\psi$:

$$(\partial_\mu R)\psi + igW'^{\,a}_\mu\tau^a R\psi = igRW^a_\mu\tau^a\psi. \tag{36}$$

Factor out ψ:

$$\partial_\mu R + igW'^{\,a}_\mu\tau^a R = igRW^a_\mu\tau^a. \tag{37}$$

Solve for W'_μ:

$$igW'^{\,a}_\mu\tau^a R = igRW^a_\mu\tau^a - \partial_\mu R, \tag{38}$$

$$W'^{\,a}_\mu\tau^a = RW^a_\mu\tau^a R^{-1} - \frac{i}{g}(\partial_\mu R)R^{-1}. \tag{39}$$

This is the usual SU(2) gauge transformation law, with U replaced by its real projection R.

4 SU(4): Matrix Calculus Derivation

Standard SU(4) transformation:

$$\psi \to \psi' = U(x)\psi, \qquad U(x) = \exp\left(i\chi^A(x)T^A\right), \tag{40}$$

with generators T^A.

Replace $U(x)$ by its real projection:

$$\psi' = R(x)\psi, \qquad R(x) = \Re\, U(x), \tag{41}$$

where R is a real 4×4 matrix.

Compute the derivative:

$$\partial_\mu\psi' = \partial_\mu(R\psi) \tag{42}$$

$$= (\partial_\mu R)\psi + R\,\partial_\mu\psi. \tag{43}$$

Define the covariant derivative

$$D_\mu = \partial_\mu + ig_4 A^A_\mu T^A, \tag{44}$$

and demand

$$D'_\mu\psi' = R\,D_\mu\psi. \tag{45}$$

Compute $D'_\mu\psi'$:

$$D'_\mu\psi' = \partial_\mu(R\psi) + ig_4 A'^{\,A}_\mu T^A R\psi \tag{46}$$

$$= (\partial_\mu R)\psi + R\partial_\mu\psi + ig_4 A'^{\,A}_\mu T^A R\psi. \tag{47}$$

Right–hand side:

$$R\,D_\mu\psi = R(\partial_\mu\psi + ig_4A_\mu^AT^A\psi) \tag{48}$$

$$= R\partial_\mu\psi + ig_4RA_\mu^AT^A\psi. \tag{49}$$

Equate:

$$(\partial_\mu R)\psi + R\partial_\mu\psi + ig_4{A'_\mu}^AT^AR\psi = R\partial_\mu\psi + ig_4RA_\mu^AT^A\psi. \tag{50}$$

Cancel $R\partial_\mu\psi$:

$$(\partial_\mu R)\psi + ig_4{A'_\mu}^AT^AR\psi = ig_4RA_\mu^AT^A\psi. \tag{51}$$

Factor out ψ:

$$\partial_\mu R + ig_4{A'_\mu}^AT^AR = ig_4RA_\mu^AT^A. \tag{52}$$

Solve for A'_μ:

$$ig_4{A'_\mu}^AT^AR = ig_4RA_\mu^AT^A - \partial_\mu R, \tag{53}$$

$${A'_\mu}^AT^A = RA_\mu^AT^AR^{-1} - \frac{i}{g_4}(\partial_\mu R)R^{-1}. \tag{54}$$

Thus the SU(4) gauge field transforms exactly as in the standard non–Abelian case, with the real projection R replacing the unitary matrix U.

5 Summary

In U(1), SU(2), and SU(4), the detailed calculus shows that replacing the unitary gauge transformation by its real projection preserves gauge invariance of the Dirac equation and covariant derivative, provided the gauge field transforms with the usual compensating derivative term, with U replaced by $R = \Re U$.

6 Reference

Understanding Euclidean Quantum Mechanics and the Path Integral of Quantum Force Fields, https://vixra.org/pdf/2607.0096v1.pdf,