below

RAMANUJAN PI SERIES-PROOF

Leonardo ***Rubino***
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Abstract: Ramanujan Pi series – proof, and other subjects of interest in the Appendix I (“i”).

$$\boxed{\pi = 1/[\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\cdot\frac{(1103+26390n)}{396^{4n}}]}$$

1 – An overview:

Here are the first 100 decimal figures of Pi:
π=3,1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899 8628034825 3421170679

Perhaps in a 3000 years ago situation, we’d better take a rope 2R long and lay it down repeatedly on the circumference of the circle and see that 3 ropes and another piece of it were needed to cover the whole circumference. But now we start from the equation of a circle whose centre is in the origin of axes (Fig. 1): $x^2 + y^2 = R^2$

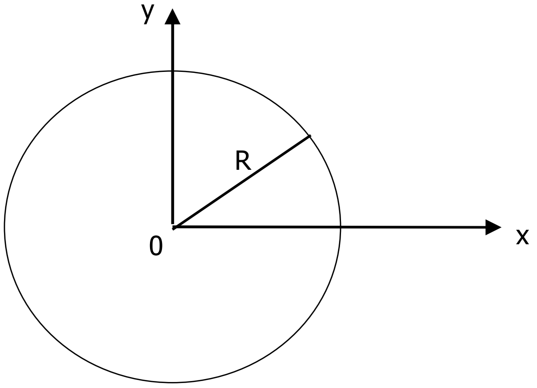


Fig. 1

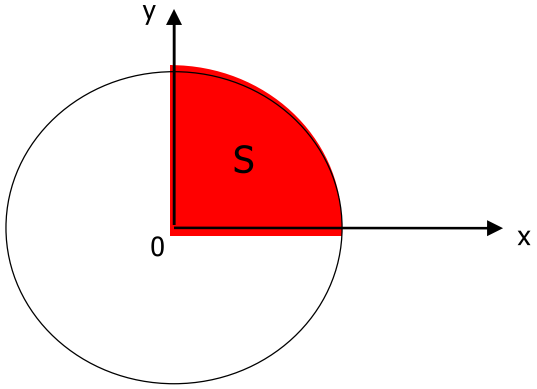


Fig. 2

If now we give the radius a value R=1, the implicit equation will become: $x^2 + y^2 = 1$. The arc of circle of the top right sector will be given by a y obtained from the previous equation: $y = +\sqrt{1-x^2}$.
By developing the last one in series of Taylor/MacLaurin, we have:

$$(f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) \cdot x^k)$$

$$\sqrt{1-x^2} = (1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \frac{21}{1024}x^{12} - \dots\dots\dots$$

The surface S under such an arc (that red one in Fig. 2, where R=1) will be:

$$S = \frac{1}{4}\pi \cdot R^2 = \frac{1}{4}\pi \cdot 1^2 = \frac{1}{4}\pi = \int_0^1 \sqrt{1-x^2}\,dx = \int_0^1 (1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \frac{21}{1024}x^{12} - \dots)dx =$$

$$= \left| x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9 - \frac{7}{2816}x^{11} - \frac{21}{13312}x^{13} - \dots \right|_0^1 = 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \frac{7}{2816} - \frac{21}{13312} - \dots$$

and so: $$\boxed{\pi = 4(1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \frac{7}{2816} - \frac{21}{13312} - \dots)}$$

Now, by using just those 7 terms in the last equation for π , we get, just as an example:
π = 4 x 0,791001164633977….. = 3,164004658535908….. , which provides just the **first decimal exact figure**; we use to say that this one is a very slowly converging series.

-<u>the Gregory-Leibniz series</u>:

let us consider the following integral: $\int_0^x \frac{1}{1+x^2} = \arctan x$ and we also consider the MacLaurin's expansion of its integrand: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, so:

$\arctan x = \int_0^x \frac{1}{1+x^2} = \int_0^x (1 - x^2 + x^4 - \dots) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ and if we fix x=1, then we will get: $\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \rightarrow \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots)$; in order to get 7 correct figures of Pi after the point, **it takes 5.000.000 terms!!**

-<u>on the convergence of the series of Ramanujan</u>:
even if we just consider the first term (n=0) of the series of Ramanujan:

$$\pi = 1/[\frac{\sqrt{8}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!}{(n!)^4} \cdot \frac{(1103 + 26390n)}{396^{4n}}] \tag{1.1}$$

we get: $\pi = 1/[\frac{\sqrt{8}}{9801} \frac{(4 \cdot 0)!}{(0!)^4} \cdot \frac{(1103 + 26390 \cdot 0)}{396^{4 \cdot 0}}] = \frac{9801}{1103\sqrt{8}} \cong 3{,}14159273\dots.$, **therefore, even with just the first term, we get six correct decimal figures!!!**
Then, if we also consider the second term, we get:

$\pi = 1/[\frac{\sqrt{8}}{9801} \sum_{n=0}^{2} \frac{(4n)!}{(n!)^4} \cdot \frac{(1103 + 26390n)}{396^{4n}}] \cong 3{,}1415926535897938779\dots..$, **so 15 correct decimal figures; so, we have just added 9 new ones!!!**
At last, with the third term, we have:

$\pi = 1/[\frac{\sqrt{8}}{9801} \sum_{n=0}^{3} \frac{(4n)!}{(n!)^4} \cdot \frac{(1103 + 26390n)}{396^{4n}}] \cong 3{,}14159265358979323846264 9065\dots.$, **so another 8 figures, with a total of 23 figures!!!** And so on.

Now, we define what the **structure** of the series of Pi of Ramanujan is going to be; in Appendix G we have presented the elliptic integrals:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{(Elliptic Integral of the first kind)}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\cdot d\theta \quad \text{(Elliptic Integral of the second kind)}$$

and in Appendix D we have also presented the Legendre's Fundamental Identity: $K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2}$ and there we also said that, by naming the parameter $k'=\sqrt{1-k^2}$, then we define K'(k)=K(k') and E'(k)=E(k') and so, when the "parameter" $k=k'=\frac{1}{\sqrt{2}}$, then, of course, K=K' and E=E' and the Legendre's Fundamental Identity, which corresponds to:

$$KE'+K'E-KK'=\frac{\pi}{2} \tag{1.2}$$

becomes, in that case: $2KE-K^2=\frac{\pi}{2}$, so:

$$\frac{E}{K}-\frac{\pi}{4K^2}=\frac{1}{2}=0{,}5\cong 0{,}3183\boxed{\approx\frac{1}{\pi}}\ . \tag{1.3}$$

Now, out of the case $k=k'=\frac{1}{\sqrt{2}}$, we evaluate anyway the quantity into the previous equation, that is the (1.3), which, out of similarity, we see like this:

$$\alpha(r)=\frac{E'(k)}{K(k)}-\frac{\pi}{[2K(k)]^2}\ . \tag{1.4}$$

Now, we are allowed to consider the case (we will see that it is useful) where the ratio between K' and K is the square root of such a number r: $\sqrt{r}=\frac{K(k')}{K(k)}$; and, among things, as well as we did also in the Appendix F, we consider (nobody can prevent us from doing that) the parameter k like a function of such a q (called "nome"), that is k=k(q) , where we decide (our choice) to analyse the cases where it has got the following form: $q=e^{-\pi\sqrt{r}}$.

(regarding the nomenclature, $\sqrt{r}$ works well, as well as $\sqrt{n}$, or also $\sqrt{N}$ …)

Well, in Appendix H we have proved that α(r) in the (1.4) really tends to 1/π when k→0 ; moreover, as such a tendency is somewhat fast, it follows that we have the reason why the series of Pi of Ramanujan is very fast. And, still in Appendix H, we saw that for k→0 we have that K'(0)=K(1)=∞ and $K(0)=\frac{\pi}{2}$, from which: $\sqrt{r}=\frac{K(k')}{K(k)}=\frac{\infty}{\pi/2}=\infty$ and so: $\lim_{k\to 0}\alpha(k)=\lim_{r\to\infty}\alpha(r)=\frac{1}{\pi}$.

Moreover, still according to Legendre, together with the expression for $\sqrt{r}$, we have that:

$$\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\left(\frac{E}{K}-1\right)\ ; \tag{1.5}$$

in fact, by starting really from the (1.4), we have that:

$\alpha(r)=\frac{E'}{K}-\frac{\pi}{(2K)^2}=\frac{4E'K-\pi}{(2K)^2}$, but according to Legendre, that is according to the (1.2), we have: $4E'K=-4K'E+4KK'+2\pi$, from which:

$$\alpha(r)=\frac{4KK'-4EK'+\pi}{(2K)^2}=\frac{4K\sqrt{r}K-4E\sqrt{r}K+\pi}{(2K)^2}=\sqrt{r}-\frac{E\sqrt{r}}{K}+\frac{\pi}{(2K)^2}=\frac{\pi}{(2K)^2}+\sqrt{r}(1-\frac{E}{K})\quad \text{qed.}$$

By the way, we write again the previous equation, that is the (1.5): $\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{1}{K}(E-K)$

and by recalling that, according to the (F.6), we have that $K(k)=\frac{\pi}{2}\theta_3^2(q)$, then: $\frac{1}{\pi\theta_3^4(q)}=\frac{\pi}{4K^2}$

and then the (1.5) becomes: $\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{1}{K}(E-K)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{4K^2}{\pi^2}\frac{\pi^2}{4K^2}\frac{1}{K}(E-K)=$

$$=\frac{1}{\pi\theta_3^4(q)}-\sqrt{r}\frac{4K}{\pi^2}\frac{1}{\theta_3^4(q)}(E-K)\ \rightarrow\ \theta_3^4(q)\alpha(r)=\frac{1}{\pi}-\sqrt{r}\frac{4K}{\pi^2}(E-K)\ \rightarrow$$

$\frac{1}{\pi}=\theta_3^4(q)\alpha(r)+\sqrt{r}\frac{4K}{\pi^2}(E-K)=\theta_3^4(q)\alpha(r)+\sqrt{r}\,4q\frac{d}{dq}\ln\theta_4(q)=\theta_3^4\alpha(r)+\sqrt{r}\,4q\frac{d}{dq}\frac{\dot{\theta}_4}{\theta_4}$, that is:

$\frac{1}{\pi}=\theta_3^4\alpha(r)+\sqrt{r}\,4q\frac{\dot{\theta}_4}{\theta_4}$, which is the (H.1) we proved there (in Appendix H), where we also prove that: $\frac{4K}{\pi^2}(E-K)=4q\frac{d}{dq}\ln\theta_4=4q\frac{\dot{\theta}_4}{\theta_4}$, which is the (H.2), an equation, among things, that we have just used. And through the (H.11) there we have also proved again the fast convergence of α(r) to 1/π (r→∞), with a tendency which is, according to the (H.13) there proved:

$$\alpha(r)-\frac{1}{\pi}\rightarrow 8(\sqrt{r}-\frac{1}{\pi})q=8(\sqrt{r}-\frac{1}{\pi})e^{-\pi\sqrt{r}}\rightarrow 0\quad !!!$$

(a very fast tendency, as well as the series of Pi of Ramanujan is, as $e^{-\pi\sqrt{r}}$ overwhelms very fast $\sqrt{r}$, when r→∞)

And by recalling (from the Appendix D) that: $\frac{dK}{dk}=\frac{E-k'^2K}{kk'^2}$, $\frac{dE}{dk}=\frac{E-K}{k}$, from which:

$E=kk'^2\frac{dK}{dk}+k'^2K$ and $E'=-kk'^2\frac{dK'}{dk}+k^2K'$, we have, at last:

$$\alpha(r)=\frac{\pi}{(2K)^2}+\sqrt{r}(1-\frac{E}{K})=\frac{1}{\pi}(\frac{\pi}{2K})^2+\sqrt{r}(1-\frac{kk'^2\frac{dK}{dk}+k'^2K}{K})=$$

$$=\frac{1}{\pi}(\frac{\pi}{2K})^2+\sqrt{r}[(1-k'^2)-kk'^2\frac{1}{K}\frac{dK}{dk}]=\frac{1}{\pi}(\frac{\pi}{2K})^2-\sqrt{r}(kk'^2\frac{1}{K}\frac{dK}{dk}-k^2)=\alpha(r),\ \text{so:}$$

$$\boxed{\frac{1}{\pi}=\sqrt{r}kk'^2[(\frac{2}{\pi})^2K\frac{dK}{dk}]+[\alpha(r)-\sqrt{r}k^2](\frac{2K}{\pi})^2\ .}\qquad(1.6)$$

Let us remind in advance that our target is that of getting to a generic form of the following kind:

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\frac{(4n)!}{(n!)^4}(G+Hn)x^n\qquad \text{(x=f(k) and G and H which are constants)}\ ,\qquad(1.7)$$

because, as a matter of facts, this is indeed the form of the series of Pi of Ramanujan we are trying to prove.

If now we consider the (C.12) in Appendix C: $\frac{2}{\pi}K(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)$ which gives the elliptic integral K as a function of the hypergeometric function, then it follows that, according to the properties of the hypergeometric functions indeed, we also have the following similar ones:

$$\frac{2}{\pi}K(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2) = \frac{1}{k'}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;\frac{-(2kk')^2}{k'^6}) = \frac{1}{k'}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;-(\frac{2k}{k'^2})^2)$$

$$\frac{2}{\pi}K(k) = \frac{1}{\sqrt{k'}}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;-(\frac{k^2}{2k'})^2)$$

$$\boxed{\frac{2}{\pi}K(k) = \frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)} \tag{1.8}$$

$$\frac{2}{\pi}K(k) = \frac{1}{\sqrt{k'^2-k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;-(\frac{2}{G^{12}+G^{-12}})^2)$$

$$\frac{2}{\pi}K(k) = \frac{1}{[1-kk'^2]^{1/4}}\cdot{}_2F_1(\frac{1}{12},\frac{5}{12};1;\frac{1}{J})$$

proof: (**a complete proof just for the (1.8), as just the (1.8) will be important for us, for the next step**): see the (C.31) in Appendix C.

Similarly, by considering the (C.11): $[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))$ then it follows that, due to the properties of the hypergeometric functions, we also have the following similar ones:

$$[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2)) = \frac{1}{k'^2}\cdot{}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{2k}{k'^2})^2) \tag{1.9}$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{k'}\cdot{}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{k^2}{2k'})^2) \tag{1.10}$$

$$\boxed{[\frac{2}{\pi}K(k)]^2 = \frac{1}{1+k^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g^{12}+g^{-12}})^2)} \tag{1.11}$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{k'^2-k^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-(\frac{2}{G^{12}+G^{-12}})^2) \tag{1.12}$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{\sqrt{k'^2-k^2}}\cdot{}_3F_2(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;\frac{1}{J}) \tag{1.13}$$

proof: (**a complete proof just for the (1.11), as just the (1.11) will be important for us, for the next steps**): see the (C.35) in Appendix C.

At this point, the key point is to notice that both the expressions found for all the $\frac{2}{\pi}K(k)$ and those of all the $[\frac{2}{\pi}K(k)]^2$ have a numerical coefficient in k (m(k)), like $\frac{1}{\sqrt{k'^2-k^2}}$ and a function of a summation given by the hypergeometric function $_iF_j$ $({}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$), that is a:

$$F(\varphi(k)) = F(\sum_{n=0}^{+\infty}a'_n\varphi^n) = \sum_{n=0}^{+\infty}a_n\varphi^n\ , \tag{1.14}$$

so, as an example:

$$[\frac{2}{\pi}K(k)]^2 = m(k)\cdot\sum_{n=0}^{+\infty} a_n\varphi^n \tag{1.15}$$

and by taking its derivative:

$$(\frac{2}{\pi})^2 2K\frac{dK}{dk} = [\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi}]$$, that is:

$$(\frac{2}{\pi})^2 K\frac{dK}{dk} = \frac{1}{2}[\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi}] \ , \tag{1.16}$$

and after inserting the (1.15) and the (1.16) into the (1.6), we get:

$$\frac{1}{\pi} = \sqrt{r}kk'^2[(\frac{2}{\pi})^2 K\frac{dK}{dk}] + [\alpha(r) - \sqrt{r}k^2](\frac{2K}{\pi})^2 = \sqrt{r}kk'^2[\frac{1}{2}(\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi})] + [\alpha(r) - \sqrt{r}k^2]mF =$$

$$= \frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk}F + \frac{1}{2}\sqrt{r}kk'^2 m\frac{d\varphi}{dk}\frac{dF}{d\varphi} + [\alpha(r) - \sqrt{r}k^2]mF =$$

$$= \{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\mathrm{F} + (\frac{1}{2}m\sqrt{r}kk'^2\frac{d\varphi}{dk})\frac{dF}{d\varphi}$$ and by using in it the (1.14), that is the:

$F = \sum_{n=0}^{+\infty} a_n\varphi^n$, as well as its derivative: $\frac{dF}{d\varphi} = n\sum_{n=0}^{+\infty} a_n\varphi^{n-1}$, we have:

$$\frac{1}{\pi} = \{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\sum_{n=0}^{+\infty} a_n\varphi^n + n(\frac{1}{2}m\sqrt{r}kk'^2\frac{d\varphi}{dk})\sum_{n=0}^{+\infty} a_n\varphi^{n-1} =$$

$$= \sum_{n=0}^{+\infty} a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\varphi^n + \sum_{n=0}^{+\infty} a_n\varphi^{n-1}\varphi[\frac{m}{\varphi}\frac{1}{2}n\sqrt{r}kk'^2\frac{d\varphi}{dk}] =$$

$$= \frac{1}{\pi} = \sum_{n=0}^{+\infty} a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2] + \frac{1}{2}n\sqrt{r}kk'^2\frac{m}{\varphi}\frac{d\varphi}{dk}\}\varphi^n \tag{1.17}$$

and such a (1.17) just appeared is nothing but the (1.7): $\frac{1}{\pi} = \sum_{k=0}^{\infty}\frac{(4n)!}{(n!)^4}(G + Hn)x^n$, where:

$$\frac{(4n)!}{(n!)^4}G = a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}$$ and

$$\frac{(4n)!}{(n!)^4}H\cdot n = [\frac{1}{2}a_n\sqrt{r}kk'^2\frac{m}{\varphi}\frac{d\varphi}{dk}]n \ .$$

Now, after reminding that in Appendix G we have defined the:

$$G = G_n = (\frac{1}{2kk'})^{\frac{1}{12}} \quad \text{and} \tag{1.18}$$

$$g = g_n = (\frac{k'^2}{2k})^{\frac{1}{12}} \ , \tag{1.19}$$

that are two relations of k and k' , which yield the two above invariants (G and g); invariants as they are functions of two fixed parameters, like k and k', indeed.
Then, by combining those expressions, we also define the following invariants and the following new expressions:

$$J = \frac{(4G^{24}-1)^3}{27G^{24}} = \frac{(4g^{24}+1)^3}{27g^{24}} = \frac{4}{27}\frac{[1-(kk')^2]^3}{(kk')^4} \quad \text{(Absolute Invariant of Klein)} \tag{1.20}$$

(not useful for us, here)

$$k = g^6\sqrt{g^{12}+g^{-12}} - g^{12} = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} - \sqrt{1-\frac{1}{G^{12}}}\right) \tag{1.21}$$

(justified in Appendix G)

$$k' = \sqrt{2k}\,g^6 = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} + \sqrt{1-\frac{1}{G^{12}}}\right) \quad . \tag{1.22}$$

(justified in Appendix G)

Now, let us name r=N in the (1.17), as well as $k=k_N$; as we took the following free choice: $k = k(q) = k(e^{-\pi\sqrt{r}}) = k(e^{-\pi\sqrt{N}}) = k(N) = k_N$, then we have:

$$\frac{1}{\pi} = \sum_{n=0}^{+\infty} a_n \left\{\frac{1}{2}\sqrt{N}k_N k'^2_N \frac{dm}{dk} + m[\alpha(N) - \sqrt{N}k_N^2] + \frac{1}{2}n\sqrt{N}k_N k'^2_N \frac{m}{\varphi}\frac{d\varphi}{dk}\right\}\varphi^n \tag{1.23}$$

(and also the g and G become g_N and G_N) and by stepping back to the (1.6), we have:

$$\frac{1}{\pi} = \sqrt{N}k_N k'^2_N \left[\left(\frac{2}{\pi}\right)^2 K \frac{dK}{dk_N}\right] + [\alpha(N) - \sqrt{N}k_N^2]\left(\frac{2K}{\pi}\right)^2 \tag{1.24}$$

and if we use the (1.9)-(1.13), the (1.8) and the others around it and also the (1.24), then we get all the followings:

$$\boxed{\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{G_N^{12}}\right)^{2n} [\alpha(N) - \sqrt{N}k_N^2 + n\sqrt{N}(k'^2_N - k_N^2)]\right\}} \quad . \tag{1.25}$$

$$\boxed{\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{(-1)^n \left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{g_N^{12}}\right)^{2n} \left[\frac{\alpha(N)}{k'^2_N} + n\sqrt{N}\left(\frac{1+k_N^2}{k'^2_N}\right)\right]\right\}} \tag{1.26}$$

$$\boxed{\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{g_{4N}^{12}}\right)^{2n} \left\{\left[\alpha(N) - \sqrt{N}\frac{k_N^2}{2}\right]\frac{1}{k'_N} + n\sqrt{N}\left(k'_N + \frac{1}{k'_N}\right)\right\}} \tag{1.27}$$

$$\left(\qquad g_{4N}^{12} = 2^{1/4} g_N G_N \qquad \right)$$

$$\boxed{\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \left[\frac{\alpha(N)}{x_N(1+k_N^2)} - \frac{\sqrt{N}}{4g_N^{12}} + n\sqrt{N}\frac{(g_N^{12} - g_N^{-12})}{2}\right] x_N^{2n+1}} \tag{1.28}$$

where $\boxed{x_N = \frac{2}{(g_N^{12} + g_N^{-12})} = \frac{4k_N k'^2_N}{(1+k_N^2)^2}}$ (1.29)

$$\boxed{\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \left[\frac{\alpha(N)}{y_N(k'^2_N - k_N^2)} + \sqrt{N}\frac{k_N^2 G_N^{12}}{2} + n\sqrt{N}\frac{(G_N^{12} + G_N^{-12})}{2}\right] y_N^{2n+1}} \tag{1.30}$$

where $y_N = \frac{2}{(G_N^{12} - G_N^{-12})} = \frac{4k_N k'^2_N}{1 - (2k_N k'_N)^2}$

$$\boxed{\frac{1}{\pi}=\frac{1}{3\sqrt{3}}\sum_{n=0}^{\infty}\frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{(n!)^3}\{2[\alpha(N)-\sqrt{N}k_N^2](4G_N^{24}-1)+\sqrt{N}\sqrt{1-\frac{1}{G_N^{24}}}+2n\sqrt{N}(8G_N^{24}+1)\sqrt{1-\frac{1}{G_N^{24}}}\}(\frac{1}{J_N^{1/2}})^{2n+1}}$$

(1.31)

Now, we provide a complete proof just for the (1.28), as it is the important one, which will drive us to the Series of Pi of Ramanujan; therefore, we write it here again:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{\alpha(N)}{x_N(1+k_N^2)}-\frac{\sqrt{N}}{4g_N^{12}}+n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}]x_N^{2n+1}$$

where $x_N=\frac{2}{(g_N^{12}+g_N^{-12})}=\frac{4k_Nk'^2_N}{(1+k_N^2)^2}$ and from those two we get, of course:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\underline{\alpha(N)\frac{1}{(1+k_N^2)}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}}-\underline{\frac{\sqrt{N}}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}}+\underline{n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}}]$$

(1.32)

<u>proof of the (1.32)</u>:

as usual, we still start with the (1.6) after that it's been completed like the (1.24); we have:

$$\frac{1}{\pi}=\sqrt{N}k_Nk'^2_N[(\frac{2}{\pi})^2K\frac{dK}{dk_N}]+[\alpha(N)-\sqrt{N}k_N^2](\frac{2K}{\pi})^2 \tag{1.33}$$

and if we use the (1.11):

$$[\frac{2}{\pi}K]^2=\frac{1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2)\quad\rightarrow$$

$$\frac{d}{dk_N}(\frac{2K}{\pi})^2=\frac{4}{\pi^2}2K\frac{dK}{dk_N}=\frac{d}{dk_N}[\frac{1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2)]=$$

$$=\frac{d}{dk_N}[\frac{1}{1+k_N^2}\cdot\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}]=\frac{d}{dk_N}[\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{1+k_N^2}(\frac{4k_Nk'^2_N}{(1+k_N^2)^2})^{2n}]=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{d}{dk_N}\{\frac{1}{1+k_N^2}[\frac{4k_Nk'^2_N}{(1+k_N^2)^2}]^{2n}\}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{d}{dk_N}\{\frac{1}{1+k_N^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}\}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{-\frac{2k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+\frac{1}{1+k_N^2}8n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^3}]\}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{(1+k_N^2)^2}\{-2k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+8n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]\};$$

in fact:

$$\frac{d}{dk_N}(\frac{4k_N-4k_N^3)}{(1+k_N^2)^2})^{2n}=\frac{d}{dk_N}[(4k_N-4k_N^3)(1+k_N^2)^{-2}]^{2n}=$$

$$=2n[(4k_N-4k_N^3)(1+k_N^2)^{-2}]^{2n-1}[-4k_N(4k_N-4k_N^3)(1+k_N^2)^{-3}+4(1-3k_N^2)(1+k_N^2)^{-2}]=$$

$$=2n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[-16k_N\frac{(k_N-k_N^3)}{(1+k_N^2)^3}+4\frac{(1-3k_N^2)}{(1+k_N^2)^2}]=$$

$$=2n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-16k_N(k_N-k_N^3)+4(1-3k_N^2)(1+k_N^2)}{(1+k_N^2)^3}]=$$

$$=2n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-16k_N^2+16k_N^4+4+4k_N^2-12k_N^2-12k_N^4}{(1+k_N^2)^3}]=$$

$$=8n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^3}]$$

and finally:

$$(\frac{2}{\pi})^2K\frac{dK}{dk_N}=\frac{1}{2}\frac{d}{dk_N}(\frac{2K}{\pi})^2=\frac{1}{2}\frac{4}{\pi^2}2K\frac{dK}{dk_N}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{(1+k_N^2)^2}\{-k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+4n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1)}{(1+k_N^2)^2}]\} \tag{1.34}$$

so, as: $\frac{1}{\pi}=\sqrt{N}k_Nk'^2_N[(\frac{2}{\pi})^2K\frac{dK}{dk_N}]+[\alpha(N)-\sqrt{N}k_N^2](\frac{2K}{\pi})^2=$

$=\boxed{\alpha(N)(\frac{2K}{\pi})^2}-\boxed{\sqrt{N}k_N^2(\frac{2K}{\pi})^2}+\boxed{\sqrt{N}k_Nk'^2_N[(\frac{2}{\pi})^2K\frac{dK}{dk_N}]}$, then the first block in $\alpha(N)$ is:

$$\boxed{\alpha(N)(\frac{2K}{\pi})^2}=\alpha(N)\frac{1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2)=$$

$=\alpha(N)\frac{1}{1+k_N^2}\cdot\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}$, which is exactly the same block of the (1.32) we are going to prove.

Conversely, regarding the second block in $\sqrt{N}$:

$$-\boxed{\sqrt{N}k_N^2(\frac{2K}{\pi})^2}=\sqrt{N}k_N^2\frac{-1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2)=\sqrt{N}k_N^2\frac{-1}{1+k_N^2}\cdot\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}=$$

$$=\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N^2}{1+k_N^2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}]\frac{1}{4g_N^{12}}\frac{2}{(g_N^{12}+g_N^{-12})}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}]\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}\;; \tag{1.35}$$

in fact: $k_N^2 \frac{1}{1+k_N^2} = \{[\frac{(1+k_N^2)}{2}]\frac{1}{4g_N^{12}}\frac{2}{(g_N^{12}+g_N^{-12})}\}$, as: $\frac{2}{(g_N^{12}+g_N^{-12})} = \frac{4k_N k'^2_N}{(1+k_N^2)^2}$ and, due to the (G.1): $g_N = (\frac{k'^2_N}{2k_N})^{\frac{1}{12}} \to g_N^{12} = (\frac{k'^2_N}{2k_N})$, it follows that:

$$[\frac{(1+k_N^2)}{2}]\frac{1}{4g_N^{12}}\frac{2}{(g_N^{12}+g_N^{-12})} = [\frac{(1+k_N^2)}{2}]\frac{k_N}{2k'^2_N}\frac{4k_N k'^2_N}{(1+k_N^2)^2} = \frac{k_N^2}{(1+k_N^2)} \quad !!!$$

and if we sum to the (1.35) also another contribution in $\sqrt{N}$ coming from the (1.36) of the next point below, on $\frac{dK}{dk_N}$, that is:

$$\sqrt{N}k_N k'^2_N \sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}$$, then we get:

$$-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}} + \sqrt{N}k_N k'^2_N \sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n} =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}} + k_N k'^2_N\frac{k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}\} =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}} + \frac{1}{4}k_N\frac{4k_N k'^2_N}{(1+k_N^2)^2}[\frac{4k_N k'^2_N}{(1+k_N^2)^2}]^{2n}\} =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}} + \frac{1}{4}k_N\frac{2}{g_N^{12}+g_N^{-12}}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}] =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}} + \frac{1}{4}k_N(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}] =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}} + \frac{1}{4}k_N] =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2} + k_N g_N^{12}]$$ and by knowing that:

$g_N = (\frac{k'^2_N}{2k_N})^{\frac{1}{12}}$, then it follows that:

$$......= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2} + k_N(\frac{k'^2_N}{2k_N})] =$$

$$= -\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2} + \frac{k'^2_N}{2}] =$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2}+\frac{(1-k_N^2)}{2}]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}$$, which is exactly the same block of the (1.32) we are going to prove,

At last, regarding the third block containing $\frac{dK}{dk_N}$:

$$\boxed{\sqrt{N}k_N k'^2_N[(\frac{2}{\pi})^2 K\frac{dK}{dk_N}]}=$$

$$=\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{(1+k_N^2)^2}\{-k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+4n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]\}=$$

$$=\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+$$

$$+\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1} \quad (1.36)$$

therefore, in order to sum up things:

-the first block in $\alpha(N)$ has been proved;

-about the block in $\sqrt{N}$, there are two contributions which have been added up and we just got the expression of the (1.32);

-and finally the block in $n\sqrt{N}$ is left, from the residual term in the (1.36), that is:

$$+\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$, which is exactly the term in $n\sqrt{N}$ in the (1.32) which we are going to prove, as by a comparison, we really have that:

$$k_N k'^2_N\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{4k_N k'^2_N}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4\frac{(k_N-k_N^3)}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4\frac{k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{4k_N k'^2_N}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{2}{(g_N^{12}-g_N^{-12})}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{2}{(g_N^{12}+g_N^{-12})}]^{2n}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]=\frac{g_N^{12}-g_N^{-12}}{g_N^{12}+g_N^{-12}} \tag{1.37}$$

but being that for the (G.1): $\frac{1}{g_N^{12}}=\frac{2k_N}{1-k_N^2}$, from which: $g_N^{-24}=\frac{4k_N^2}{(1-k_N^2)^2}$, then it follows that, according to the (1.37):

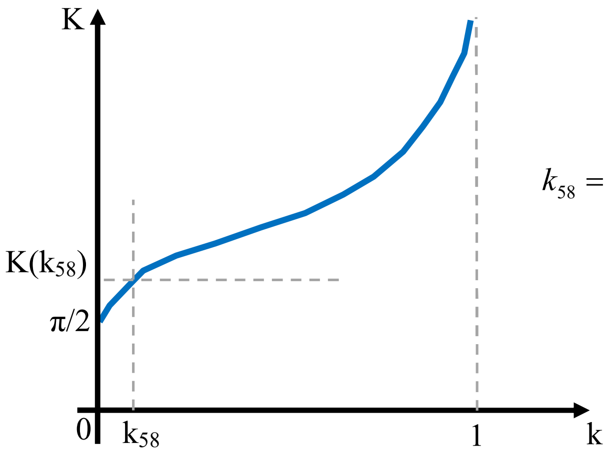
$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]=\frac{g_N^{12}-g_N^{-12}}{g_N^{12}+g_N^{-12}}=\frac{1-\frac{g_N^{-12}}{g_N^{12}}}{1+\frac{g_N^{-12}}{g_N^{12}}}=\frac{1-g_N^{-24}}{1+g_N^{-24}}=\frac{1-\frac{4k_N^2}{(1-k_N^2)^2}}{1+\frac{4k_N^2}{(1-k_N^2)^2}}=\frac{(1-k_N^2)^2-4k_N^2}{(1-k_N^2)^2+4k_N^2}=\frac{1+k_N^4-2k_N^2-4k_N^2}{1+k_N^4-2k_N^2+4k_N^2}=$$

$$=\frac{1+k_N^4-6k_N^2}{1+k_N^4+2k_N^2}=\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}$$ $\rightarrow$ so the (1.37) really yields 1=1, then it is true!!!!!

Now we evaluate the (1.32) for N=58 , that is:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}] \tag{1.38}$$

first of all, by plotting somehow a graph for K(k), through all the available tables of values or through its expansion in series, we get something like:



$k_{58}=0{,}0000255482707576$

Fig. 3.

Then, we defined: $\sqrt{r}=\frac{K(k')}{K(k)}$ (after the (1.4)), which, after being translated into the nomenclature we have reached so far, would sound like: $\sqrt{58}=\frac{K(k'_{58})}{K(k_{58})}=\frac{K(\sqrt{1-k_{58}^2})}{K(k_{58})}$.

Then, through the (C.16), we proved that: $\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}=\frac{(4n)!}{4^{4n}(n!)^4}$; this comes out also by the comparison between the formalism on the Symbols of Pochammer and that of the factorials; such a comparison has been carried out, among things, in Appendix C to get to the (C.6). With all this, the (1.38) becomes:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}]\ ; \tag{1.39}$$

now, we remind the following equalities, justified in the ending part of the Appendix G:

$$\frac{g_{58}^{12}+g_{58}^{-12}}{2}=9801 \tag{1.40}$$

$$k_{58}=0{,}0000255482707576 \tag{1.41}$$

$$\alpha(58)=0{,}3183192742 \tag{1.42}$$

$$\frac{1}{g_{58}^{12}}=\frac{2k_{58}}{1-k_{58}^2}=0{,}00005109654154 \tag{1.43}$$

and: $$\frac{(g_{58}^{12}-g_{58}^{-12})}{2}=9801\ ; \tag{1.44}$$

with such correspondences, that is the (1.40)-----(1.44) and after carrying out some calculations, then it follows that, for the (1.39):

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{0{,}3183192742}{1+6{,}5271413870\cdot10^{-10}}(\frac{1}{9801})^{2n}-\frac{\sqrt{58}}{4}0{,}00005109654154(\frac{1}{9801})^{2n+1}+$$

$$+n\sqrt{58}\cdot9801(\frac{1}{9801})^{2n+1}]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0{,}3183192739-\frac{\sqrt{58}}{4}0{,}00005109654154(\frac{1}{9801})+n\sqrt{58}\cdot9801(\frac{1}{9801})]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0{,}3183192739-9{,}9260194588\cdot10^{-9}+n7{,}6157731058]$$

$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0{,}3183192639+n7{,}6157731058]$; now, after multiplying by the neutral (unit) factor $\frac{\sqrt{8}}{9801}\frac{9801}{\sqrt{8}}\frac{396^{4n}}{396^{4n}}$, we get: $(\frac{396\cdot99}{4}=9801)$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{4n}}{4^{4n}\sqrt{8}}(\frac{4}{396\cdot99})^{2n-1}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{4n}}{4^{4n}\sqrt{8}}(\frac{1}{396})^{2n-1}(\frac{4}{99})^{2n-1}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{2n+1}}{4^{4n}\sqrt{8}}(\frac{4}{99})^{2n-1}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{(99\cdot4)^{2n+1}}{4^{4n}\sqrt{8}}\frac{4^{2n-1}}{99^{2n-1}}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^{2n+1}4^{2n+1}}{4^{4n}\sqrt{8}}\frac{4^{2n-1}}{99^{2n-1}}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^{2}4^{4n}}{4^{4n}\sqrt{8}}[0{,}3183192639+n7{,}6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^2}{\sqrt{8}}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}[\frac{99^2}{\sqrt{8}}0,3183192639...+n\frac{99^2}{\sqrt{8}}7,6157731058...]$$ and finally:

$$\boxed{\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{[1103+26390n]}{396^{4n}}} \tag{1.45}$$

APPENDICES:

APPENDIX A:
Elliptic Integral of the First Kind K(k) (Pendulum – non approximate solution)

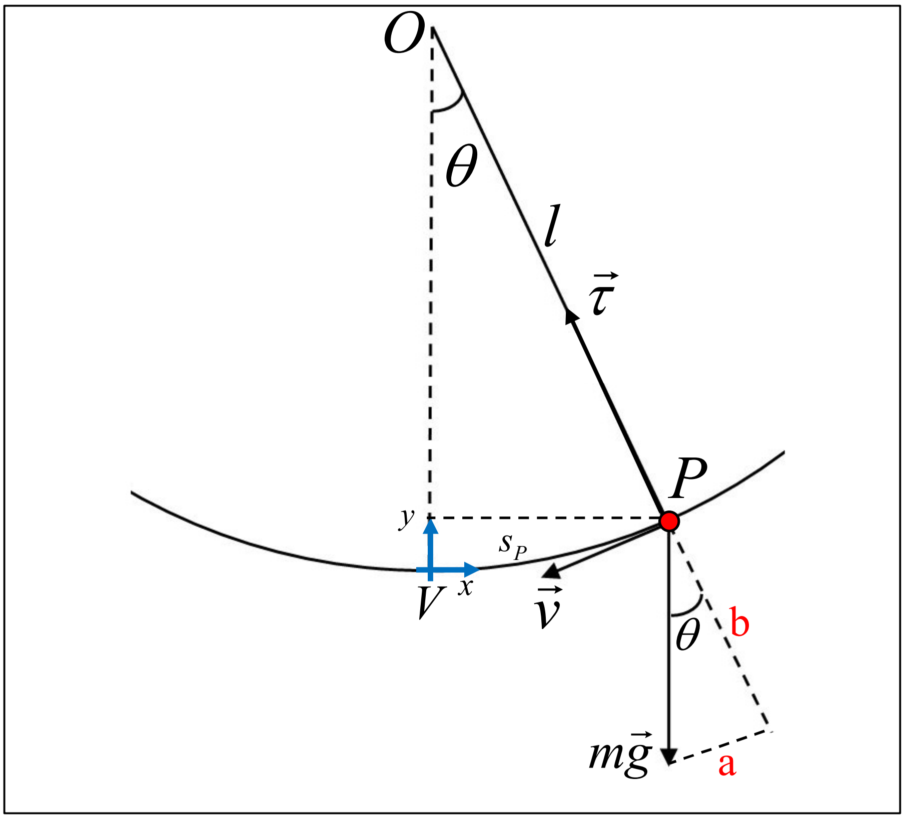


Fig. A.1: Pendulum

A) Intuitive analysis:

With reference to the above figure A.1, we have that on the mass m in P the weight $m\vec{g}$ acts; its components are along **a** and **b**; over **b** (together with the motion) it gives its contribution to make the tension $\vec{\tau}$ of the rope and it doesn't generate any radial motion (OP=const), while over **a** it generates the (tangential) rotational motion:

$mg\sin\theta=-m\frac{dv}{dt}=-m\frac{d}{dt}(\omega l)=-m\frac{d}{dt}(\frac{d\theta}{dt}l)=-ml\frac{d^2\theta}{dt^2}$ (we have the – sign as we have an increase of v as long as there is a decrease of θ). So we get the following:

$$\boxed{\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0} \quad , \qquad \text{(A.1)}$$

In the approximation $\sin\theta \approx \theta$, that is for small angles of oscillation, the (A.1) becomes:

$$\boxed{\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta=0}$$

It's simple to verify, by direct substitution, that the solution is:

$\theta(t)=c\sin(\omega t+d)$,
where c and d can have any values, to be determined according to the initial conditions, and: $\omega=\sqrt{\frac{g}{l}}$. So, as $\omega=\frac{2\pi}{T}$, the period of the oscillations is:

$$\boxed{T=2\pi\sqrt{\frac{l}{g}}} .$$

As an example, if we want a period of 1s, then ℓ=25cm.

So, we have: $\theta'(t)=c\omega\cos(\omega t+d)$ and $\theta''(t)=-c\omega^2\sin(\omega t+d)$.

The angle at which the pendulum was left to oscillate can be calculated when t=0:
$\theta(0)=c\sin d$ (and also: $\theta'(0)=c\omega\cos d$) and if the pendulum is still when we leave it, then $\theta'(0)=c\omega\cos d=0$, so: $d=\pi/2$ and for the initial angle θ_0 we have:

$\theta(0)=\theta_0=c\sin(\frac{\pi}{2})=c$ and finally:

$$\boxed{\theta(t)=\theta_0\sin(\omega t+\frac{\pi}{2})}$$

Pendulum – NON approximate solution.

Let's start again from the (A.1): $\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0$ and set: $\frac{d\theta}{dt}=\eta$, from which:

$\frac{d^2\theta}{dt^2}=\frac{d\eta}{dt}=\frac{d\eta}{d\theta}\frac{d\theta}{dt}=\eta\frac{d\eta}{d\theta}\rightarrow \quad \eta\frac{d\eta}{d\theta}=-\frac{g}{l}\sin\theta \quad \overset{\int}{\rightarrow} \quad \frac{\eta^2}{2}=\frac{g}{l}\cos\theta+const$. Now, as the pendulum is left to oscillate starting from a still state, we have that when $\theta=\theta_0 \rightarrow \eta=0$ and so:

$const=-\frac{g}{l}\cos\theta_0 \quad \rightarrow \quad \eta^2=\frac{2g}{l}(\cos\theta-\cos\theta_0) \quad \rightarrow \quad \frac{d\theta}{dt}=\pm\sqrt{\frac{2g}{l}(\cos\theta-\cos\theta_0)}$.

Now, still with reference to Fig. A.1 and considering just the first quarter of a period, that is from above the point P up to the bottom point V, we see that θ decreases ($d\theta<0$), so $\frac{d\theta}{dt}<0$ and this means that in the last equation only the minus sign must survive, so:

$\frac{d\theta}{dt}=-\sqrt{\frac{2g}{l}(\cos\theta-\cos\theta_0)}$ from which: $t=-\sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{(\cos\theta-\cos\theta_0)}}$. Now, considering that we leave the pendulum at t=0 (and $\theta=\theta_0$) , while at 1/4 of a period (t=T/4) we are at the bottom point (point V→ θ=0), then: $T=4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{(\cos\theta-\cos\theta_0)}}$. Now we recall from the trigonometry that:

$$\cos\theta = 2\sin^2(\frac{\theta}{2}) - 1 \ , \tag{A.2}$$

and it follows that: $T = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{[\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})]}}$. (A.3)

Now, we set: $\sin(\frac{\theta}{2}) = \sin(\frac{\theta_0}{2})\sin\beta$ (A.4)

and we derive, so getting:

$$\frac{1}{2}\cos(\frac{\theta}{2})d\theta = \sin(\frac{\theta_0}{2})(\cos\beta)d\beta \ ; \tag{A.5}$$

if now we set $\sin(\frac{\theta_0}{2}) = k$, the (A.4) becomes: $\sin(\frac{\theta}{2}) = k\sin\beta$ and the (15) yields:

$$\frac{1}{2}\cos(\frac{\theta}{2})d\theta = k(\cos\beta)d\beta \rightarrow$$

$$d\theta = \frac{2k(\cos\beta)d\beta}{\cos(\frac{\theta}{2})} = \frac{2k(\cos\beta)d\beta}{\sqrt{1-\sin^2(\frac{\theta}{2})}} = \frac{2k(\cos\beta)d\beta}{\sqrt{1-k^2\sin^2\beta}} \tag{A.6}$$

Now, from the (A.3) we get: $\theta = 0 \rightarrow \beta = 0$ and $\theta = \theta_0 \rightarrow \beta = \pi/2$, and all this will help us to find the integration ends of the (A.5).

After inserting such a dθ of the (A.6) into the (A.3), we get:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{k(\cos\beta)}{\sqrt{[k^2 - \sin^2(\frac{\theta}{2})]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{k(\cos\beta)}{k\sqrt{[1-\frac{1}{k^2}\sin^2(\frac{\theta}{2})]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} =$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{(\cos\beta)}{\sqrt{[1-\sin^2\beta]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = T$$

. Let us write it again:

$$T = 4\sqrt{\frac{l}{g}}\boxed{\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}} = 4\sqrt{\frac{l}{g}}K(k) \tag{A.7}$$

For small oscillations ($k \cong 0$), we have: $T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}d\beta = 2\pi\sqrt{\frac{l}{g}}$, that is the equation we already found. The integral into the (A.7) is an Elliptic Integral of the First Kind and an exact solution of the (A.7) does not exist; therefore, our solution is:

$$T = 2\pi\sqrt{\frac{l}{g}}[1 + (\frac{1}{2})^2k^2 + (\frac{1}{2}\frac{3}{4})^2k^4 + (\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6 + ...] \qquad (|k| = \left|\sin(\frac{\theta_0}{2})\right| < 1) \tag{A.8}$$

<u>Proof:</u>

after developing in series of Taylor the function $f(x) = (1+x)^z$, through the calculation of all the first derivatives, then the second ones etc, for the point x=0, we get:

$(1+x)^z = 1 + zx + \frac{z(z-1)}{2\cdot 1}x^2 + \frac{z(z-1)(z-2)}{3\cdot 2\cdot 1}x^3 + ...$ and with z=-1/2 , we have:

$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 - \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + ...$; now, with $x = -k^2\sin^2\beta$, the (A.7) yields:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}[1 + \frac{1}{2}k^2\sin^2\beta + \frac{1}{2}\frac{3}{4}k^4\sin^4\beta + ...]d\beta =$$

$$= 4\sqrt{\frac{l}{g}}[\int_0^{\pi/2} \sin^0 \beta d\beta + \frac{1}{2}k^2 \int_0^{\pi/2} \sin^2 \beta d\beta + \frac{1}{2}\frac{3}{4}k^4 \int_0^{\pi/2} \sin^4 \beta d\beta + ...] \quad \text{(A.9)}$$

Now, let's consider the integral $I = \int_0^{\pi/2} \sin^{2n} \beta d\beta$, its value is:

$$I = \int_0^{\pi/2} \sin^{2n} \beta d\beta = \frac{1 \cdot 3 \cdot 5 \cdot ... (2n-1)}{2 \cdot 4 \cdot 6 \cdot ... (2n)} \frac{\pi}{2} \quad \text{(A.10)}$$

and we will provide a proof of it; well, the (A.9) then becomes:

$$T = 4\sqrt{\frac{l}{g}}[\int_0^{\pi/2} \sin^0 \beta d\beta + \frac{1}{2}k^2 \int_0^{\pi/2} \sin^2 \beta d\beta + \frac{1}{2}\frac{3}{4}k^4 \int_0^{\pi/2} \sin^4 \beta d\beta + ...] =$$

$$= 2\pi\sqrt{\frac{l}{g}}[1 + (\frac{1}{2})^2 k^2 + (\frac{1}{2}\frac{3}{4})^2 k^4 + (\frac{1}{2}\frac{3}{4}\frac{5}{6})^2 k^6 + ...] \quad \text{(A.11)}$$

which is nothing but the (A.8).

In order to prove, as we promised, also the (A.10), we can follow two ways; let's start by the roughest one:

1) first method (rough):

we evaluate all the single integrals I(n), for many n values, and we numerically acknowledge that the (A.10) holds; in order to do that, let's consider the formula for the derivative of a product of functions, from which then we can get the formula for the integration by parts: $h(x) = f(x) \cdot g(x) \rightarrow$

$$dh(x) = df(x) \cdot g(x) + f(x) \cdot dg(x) \quad \xrightarrow{\int_A^B}$$

$$\xrightarrow{\int_A^B} \quad \int_A^B dh(x) = \int_A^B d[f(x) \cdot g(x)] = \left| f(x) \cdot g(x) \right|_A^B = \int_A^B g(x)\frac{df(x)}{dx}dx + \int_A^B f(x)\frac{dg(x)}{dx}dx =$$

$$= \int_A^B f'(x)g(x)dx + \int_A^B f(x)g'(x)dx \quad \text{, so:}$$

$$\int_A^B f(x)g'(x)dx = \left| f(x) \cdot g(x) \right|_A^B - \int_A^B f'(x)g(x)dx \quad \text{(A.12)}$$

Now, let's set A=0 and B=π/2 , and also $f(x) = \sin^{n-1}(x)$ and $g'(x) = \sin(x)$, so, for the (A.12):

$$\int_0^{\pi/2} \sin^{n-1}(x)\sin(x)dx = \left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\cos(x)dx =$$

$$= \left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)[1 - \sin^2(x)]dx =$$

$= \left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)dx - (n-1)\int_0^{\pi/2} \sin^n(x)dx$, that is, after that we write it again: $\int_0^{\pi/2} \sin^n(x)dx = \left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)dx - (n-1)\int_0^{\pi/2} \sin^n(x)dx$

and as in both sides we have an integral like $\sin^n x$, we collect them and get:

$n\int_0^{\pi/2} \sin^n(x)dx = \left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)dx$ and at last:

$$\boxed{\int_0^{\pi/2} \sin^n(x)dx = \frac{1}{n}\left| -\sin^{n-1}(x)\cos(x) \right|_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2}(x)dx} \quad \text{(A.13)}$$

For example, the (A.13) can be useful to calculate $\int_0^{\pi/2}\sin^4(x)dx$ once we know $\int_0^{\pi/2}\sin^2(x)dx$ and so on. Anyway, so doing, we get the following results:

$$\int_0^{\pi/2}\sin^0\beta d\beta = \left|\beta\right|_0^{\pi/2} = \frac{\pi}{2}$$

$$\int_0^{\pi/2}\sin^2\beta d\beta = \left|\frac{\beta}{2}-\frac{\sin 2\beta}{4}\right|_0^{\pi/2} = \left|\frac{1}{2}(\beta-\sin\beta\cos\beta)\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^4\beta d\beta = \left|\frac{3\beta}{8}-\frac{\sin 2\beta}{4}+\frac{\sin 4\beta}{32}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^6\beta d\beta = \left|\frac{5\beta}{16}-\frac{15\sin 2\beta}{64}+\frac{3\sin 4\beta}{64}-\frac{\sin 6\beta}{192}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^8\beta d\beta = \left|\frac{35\beta}{128}-\frac{7\sin 2\beta}{32}+\frac{7\sin 4\beta}{128}-\frac{\sin 6\beta}{96}+\frac{\sin 8\beta}{1024}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^{10}\beta d\beta = \left|\frac{63\beta}{256}-\frac{105\sin 2\beta}{512}+\frac{15\sin 4\beta}{256}-\frac{15\sin 6\beta}{1024}+\frac{5\sin 8\beta}{2048}-\frac{\sin 10\beta}{5120}\right|_0^{\pi/2}$$

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…………………………………………………………………………………..
and so on.

2) second method (elegant):

EULER GAMMA FUNCTION

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt \tag{A.14}$$

Of course $\Gamma(1)=1$. Then,

$$\Gamma(n+1) = n\Gamma(n); \tag{A.15}$$

in fact, after an integration by parts:

$\Gamma(z) = e^{-t}\left.\frac{t^z}{z}\right|_0^{\infty} + \frac{1}{z}\int_0^{\infty} e^{-t}t^z dt = 0 + \frac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1)=n\Gamma(n)$, we get:

$$\Gamma(n+1) = n!\ . \tag{A.16}$$

GAUSS INTEGRAL

We have the following two integrals (identical):

$$I = \int_0^{\infty} e^{-\alpha x^2}dx \qquad\qquad I = \int_0^{\infty} e^{-\alpha y^2}dy \tag{A.17}$$

After multiplying them each other: $I^2 = \int_0^{\infty}\int_0^{\infty} e^{-\alpha(x^2+y^2)}dxdy$; now, in polar coordinates: ($x^2+y^2=r^2$, $dS = dxdy = rdrd\theta$), we have:

$$I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-\alpha r^2}rdrd\theta = \frac{\pi}{2}\int_0^{\infty} e^{-\alpha r^2}rdr = -\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^{\infty} = \frac{\pi}{4\alpha}$$, so:

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\ . \tag{A.18}$$

PARTICULAR VALUES OF THE EULER GAMMA FUNCTION

The (A.16) with n=0 yields Γ(1)=0!=1, while with n=1, it gives Γ(2)=1!=1.
Moreover, the (A.14) with z=1/2 becomes:

$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt = \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} dt$; now, by saying that $t=w^2$, we have: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} dt = \int_0^{\infty} e^{-w^2} w^{-1} 2w dw = 2\int_0^{\infty} e^{-w^2} dw \tag{A.19}$$

And according to the (A.18) with α=1, we have: $\Gamma(\frac{1}{2}) = 2\int_0^{\infty} e^{-w^2} dw = \sqrt{\pi}$.

At last, according to the (A.15): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Results: Γ(1)=1, Γ(2)=1, $\Gamma(1/2) = \sqrt{\pi}$, Γ(3/2) = $\frac{\sqrt{\pi}}{2}$. (A.20)

BETA FUNCTION

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \tag{A.21}$$

(convergent for m>0 and n>0)

-We have that: $B(m,n) = B(n,m)$; in fact, by the substitution x=1-y, we have:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^1 (1-y)^{m-1} y^{n-1} dx = \int_0^1 y^{n-1}(1-y)^{m-1} dx = B(n,m)$.

-Moreover, we have that:

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta \ ; \tag{A.22}$$

In fact, by the substitution $x = \sin^2\theta$ into the (A.21), we get:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta =$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$.

-We also have that: $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$; (A.23)

in fact, by the substitution $t = x^2$ (and of course $dt = 2xdx$) into the Γ function:

$\Gamma(m) = \int_0^{\infty} e^{-t} t^{m-1} dt$, we get: $\Gamma(m) = \int_0^{\infty} e^{-t} t^{m-1} dt = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx$ and similarly:

$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy$ and so:

$\Gamma(m)\Gamma(n) = 4(\int_0^{\infty} e^{-x^2} x^{2m-1} dx)(\int_0^{\infty} e^{-y^2} y^{2n-1} dy) = 4\int_0^{\infty}\int_0^{\infty} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dxdy$;

now, if we jump to the polar coordinates ($x = \rho\cos\alpha$, $y = \rho\sin\alpha$), we get:
($dS = dxdy = (\rho d\alpha)d\rho = \rho d\rho d\alpha$)

$\Gamma(m)\Gamma(n) = 4\int_{\alpha=0}^{\pi/2}\int_{\rho=0}^{\infty} \rho^{2(m+n)-1} e^{-\rho^2} (\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha) d\rho d\alpha =$

$$= 4\left(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho\right)\left[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha\right] \; ; \qquad \text{(A.24)}$$

now, let's consider the former integral: $\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho$; we can also see it like that:

$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho = \int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho$ and through the following substitution: $\rho^2 = g$

(and, as a consequence: $d\rho = \frac{1}{2\rho}dg$), we have:

$$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho = \int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho = \int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{2\rho^2}dg = \frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{g}dg =$$

$= \frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)-1}e^{-g}dg = \frac{1}{2}\Gamma(m+n)$ that is, after that we write it again:

$$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho = \frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)-1}e^{-g}dg = \frac{1}{2}\Gamma(m+n) \qquad \text{(A.25)}$$

and the (A.25)+(A.22) into the (A.24) yield:

$$\Gamma(m)\Gamma(n) = 4\left(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho\right)\left[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha\right] = 4\left[\frac{1}{2}\Gamma(m+n)\right]\left[\frac{1}{2}B(m,n)\right] =$$

$= \Gamma(m+n)B(m,n)$, from which, finally: $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

-Moreover, the (A.23) into the (A.22) yields:

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \qquad \text{(A.26)}$$

(m>0 and n>0)

At last, we can get to the (A.10); if into the (A.26) we set (2m-1)=p and (2n-1)=0 , we get:

$\int_0^{\pi/2}\sin^{p}\theta d\theta = \dfrac{\Gamma[\frac{1}{2}(p+1)]\Gamma(\frac{1}{2})}{2\Gamma[\frac{1}{2}(p+2)]}$; now, as we want an even p, we can write: p=2r , so:

$\int_0^{\pi/2}\sin^{2r}\theta d\theta = \dfrac{\Gamma(r+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(r+1)}$ and as the (A.25) holds, that is $\Gamma(n+1) = n\Gamma(n)$ and so does the (A.26), that is $\Gamma(n+1) = n!$, and also the (A.20), that is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have:

$$\int_0^{\pi/2}\sin^{2r}\theta d\theta = \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(r+1)} = \frac{\Gamma(r-\frac{1}{2}+1)\Gamma(\frac{1}{2})}{2\Gamma(r+1)} = \frac{[(r-\frac{1}{2})\Gamma(r-\frac{1}{2})]\Gamma(\frac{1}{2})}{2r!} = \frac{[(r-\frac{1}{2})\Gamma(r-\frac{3}{2}+1)]\Gamma(\frac{1}{2})}{2r!} =$$

$$= \frac{[(r-\frac{1}{2})(r-\frac{3}{2})\Gamma(r-\frac{3}{2})]\Gamma(\frac{1}{2})}{2r!} = \frac{(r-\frac{1}{2})(r-\frac{3}{2}).......\frac{1}{2}\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{2r(r-1).....1} \; ; \qquad \text{(A.27)}$$

In fact, the decomposition among square brackets always gets to $....\frac{1}{2}\Gamma(\frac{1}{2})$; as an example, if r=3 , at a certain point we will have: $....(3-\frac{5}{2})\Gamma(3-\frac{5}{2}) = ...\frac{1}{2}\Gamma(\frac{1}{2})$ and if, as a further example, r=4 , at a certain point we will have: $....(4-\frac{7}{2})\Gamma(4-\frac{7}{2}) = ...\frac{1}{2}\Gamma(\frac{1}{2})$. Now, if we go on with the (A.27), we have: $\int_0^{\pi/2} \sin^{2r}\theta d\theta = \dfrac{(r-\frac{1}{2})(r-\frac{3}{2}).......\frac{1}{2}[\Gamma(\frac{1}{2})]^2}{2r(r-1).....1} = \dfrac{(2r-1)(2r-3).......1}{2r(2r-2).....2}\cdot\dfrac{\pi}{2}$; in fact, we gathered $\frac{1}{2}$ in every round brackets , such as $(r-\frac{1}{2}) = \frac{1}{2}(2r-1)$ and all those $\frac{1}{2}$ went to the denominator, for example by transforming $(r-1)$ into $(2r-2)$. By going on:

$$\boxed{\int_0^{\pi/2} \sin^{2r}\theta d\theta = \frac{(2r-1)(2r-3)(2r-5).......1}{2r(2r-2)(2r-4).....2}\cdot\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdot ...(2r-1)}{2\cdot 4\cdot 6\cdot ...(2r)}\frac{\pi}{2}} \tag{A.28}$$

In fact, if, as an example, r=4, we have:

$$\int_0^{\pi/2} \sin^{2\cdot 4}\theta d\theta = \frac{(2\cdot 4-1)(2\cdot 4-3)(2\cdot 4-5).......1}{(2\cdot 4)(2\cdot 4-2)(2\cdot 4-4).....2}\cdot\frac{\pi}{2} = \frac{[7]\cdot 5\cdot 3.......1}{[8]\cdot 6\cdot 4\cdot .....2}\cdot\frac{\pi}{2} = \frac{[2r-1]\cdot 5\cdot 3.......1}{[2r]\cdot 6\cdot 4\cdot .....2}\cdot\frac{\pi}{2} =$$

$= \dfrac{1\cdot 3\cdot 5\cdot ...(2r-1)}{2\cdot 4\cdot 6\cdot ...(2r)}\dfrac{\pi}{2}$, so the (A.28) is nothing but the (A.10) we wanted to prove.

APPENDIX B:
Elliptic Integral of the Second Kind E(k) (perimeter of the ellipse)

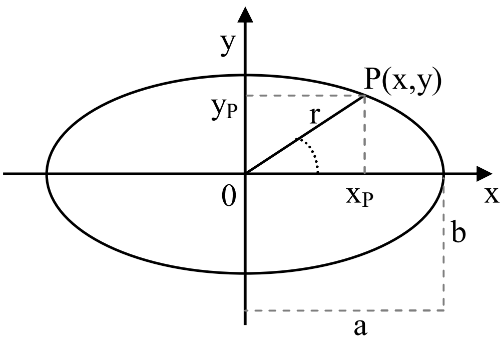

Fig. B.1

The length of an infinitesimal part of an ellipse in the point P(x,y) is obviously given by:

$$dL = \sqrt{(dx)^2 + (dy)^2} \ : \tag{B.1}$$

moreover, we know that the equation of the ellipse is: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, so: $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, and again, after taking the derivative: $\dfrac{dy}{dx} = -\dfrac{b}{a}\dfrac{x}{\sqrt{a^2-x^2}} \rightarrow dy = -\dfrac{b}{a}\dfrac{x}{\sqrt{a^2-x^2}}dx \rightarrow$

$(dy)^2 = \dfrac{b^2}{a^2}\dfrac{x^2}{a^2-x^2}(dx)^2$ and going back to the (B.1):

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + \frac{b^2}{a^2}\frac{x^2}{a^2-x^2}(dx)^2} = dx\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2-x^2}} = dx\sqrt{\frac{a^4 - a^2x^2 + b^2x^2}{a^2(a^2-x^2)}} =$$

$= dx\sqrt{\dfrac{a^4 - x^2(a^2-b^2)}{a^2(a^2-x^2)}} = dx\sqrt{\dfrac{a^2}{(a^2-x^2)} - \dfrac{(a^2-b^2)}{a^2}\dfrac{x^2}{(a^2-x^2)}}$, but we know that: $\dfrac{(a^2-b^2)}{a^2} = e^2$, where “e” is the eccentricity of the ellipse, so it follows that:

$$dL = dx\sqrt{\frac{a^2}{(a^2-x^2)} - \frac{e^2x^2}{(a^2-x^2)}} = dx\sqrt{\frac{a^2-e^2x^2}{(a^2-x^2)}} = dx\sqrt{\frac{1-e^2\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}$$ and so:

$$L = 4\int_0^a \sqrt{\frac{1-e^2\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}dx$$ and putting now $t = \frac{x}{a}$,

(with dx=adt and with t=0 when x=0 and with t=1 when x=a), we get:

$L = 4a\int_0^1 \frac{\sqrt{1-e^2t^2}}{\sqrt{1-t^2}}dx$; at last, putting $t = \sin\theta$,

(from which $dt = \cos\theta d\theta$ and with θ=0 when t=0 and with θ=π/2 when t=1), we get:

$L = 4a\int_0^{\pi/2} \frac{\sqrt{1-e^2\sin^2\theta}}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta = 4a\int_0^{\pi/2} \frac{\sqrt{1-e^2\sin^2\theta}}{\cos\theta}\cos\theta d\theta = 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}d\theta$; now, if we name with k the eccentricity "e" , we have:

$L = 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta = 4aE(k)$, where $\boxed{E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta}$ which is the Elliptic Integral of the Second Kind. Then, as for the (C.7) which has been proved in Appendix C, we have:

$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$, then it follows that, in order to have a real evaluation of the perimeter of an ellipse:

$$L = 4aE(k) = 4a\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n} = 2a\pi\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n} \quad . \qquad \text{(B.2)}$$

Conversely, for what the Elliptic Integral of the First Kind K(k) is concerned, it comes out from the study of the pendulum and see, on this purpose, the Appendix A.

APPENDIX C: The elliptic integrals and the hypergeometric functions.

We have met in the previous Appendices:

$$K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{(Elliptic Integral of the First Kind)} \qquad \text{(C.1)}$$

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\cdot d\theta \quad \text{(Elliptic Integral of the Second Kind)} \qquad \text{(C.2)}$$

They are called elliptic because, for instance, that of the second kind comes out of the calculation of the length of an ellipse.

We saw in Appendix A that:

$K(k) = \int_0^{\pi/2}\sin^0\beta d\beta + \frac{1}{2}k^2\int_0^{\pi/2}\sin^2\beta d\beta + \frac{1}{2}\frac{3}{4}k^4\int_0^{\pi/2}\sin^4\beta d\beta + \ldots$, that is:

$K(k) = \int_0^{\pi/2}(\sin^0\beta + \frac{1}{2}k^2\sin^2\beta + \frac{1}{2}\frac{3}{4}k^4\sin^4\beta + \ldots)d\beta = \int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, so:

$\frac{1}{\sqrt{1-k^2\sin^2\beta}} = (\sin^0\beta + \frac{1}{2}k^2\sin^2\beta + \frac{1}{2}\frac{3}{4}k^4\sin^4\beta + ...) = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta$; in fact, we have:

$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2}$, from which: $\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta = \frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta$ and:

n=0 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta = \frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}k^{2\cdot 0}\sin^{2\cdot 0}\beta = \sin^0\beta = 1$

n=1 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta = \frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}k^{2\cdot 1}\sin^{2\cdot 1}\beta = \frac{1}{4}\frac{(2)!}{(1!)^2}k^2\sin^2\beta = \frac{1}{2}k^2\sin^2\beta$

n=2 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta = \frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}k^{2\cdot 2}\sin^{2\cdot 2}\beta = \frac{1}{16}\frac{(4)!}{(2!)^2}k^2\sin^2\beta = \frac{1}{2}\frac{3}{4}k^2\sin^2\beta$, so, we say that again: $\frac{1}{\sqrt{1-k^2\sin^2\beta}} = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta$. (C.3)

Going back to K(k):

$$K(k) = \int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = \int_0^{\pi/2}\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta d\beta = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta \ , \quad \text{(C.4)}$$

but we have also proved that (see the (A.10) in Appendix A):

$$I = \int_0^{\pi/2}\sin^{2n}\beta d\beta = \frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)}\frac{\pi}{2} = \frac{1}{4^n}\binom{2n}{n}\frac{\pi}{2} \ , \quad \text{(C.5)}$$

as we can prove, through numerical checks, that: $\frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)} = \frac{1}{4^n}\binom{2n}{n}$; in fact, it is really true that: $\frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)} = \frac{1}{4^n}\binom{2n}{n} = \frac{1}{4^n}\frac{(2n)!}{(n!)^2}$; let's see:

$$\frac{1}{4^n}\frac{(2n)!}{(n!)^2} = \frac{1}{2^{2n}}\frac{[2n(2n-1)(2n-2)...]}{[n(n-1)(n-2)....]^2} = \frac{1}{2^{2n-1}}\frac{n(2n-1)(2n-2)...}{n^2(n-1)^2(n-2)^2....} = \frac{1}{2^{2n-1}}\frac{(2n-1)(2n-2)...}{n(n-1)^2(n-2)^2....}$$

and with, as an example, n=3, we have:

$$\frac{1}{2^{2n-1}}\frac{(2n-1)(2n-2)...}{n(n-1)^2(n-2)^2....} = \frac{1}{2^5}\frac{(6-1)(6-2)(6-3)(6-4)(6-5)}{3(3-1)^2(3-2)^2} =$$

$$= \frac{1}{2^5}\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3(2)^2(1)^2} = \frac{1}{2^5}\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 4\cdot 1} = \frac{1}{2^4}\frac{5\cdot 3\cdot 1}{3\cdot 1} = \frac{1}{2\cdot 2^2\cdot 2}\frac{5\cdot 3\cdot 1}{3\cdot 1} = \frac{5\cdot 3\cdot 1}{2\cdot 4\cdot 2\cdot 3\cdot 1} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} \ !!!!!$$

So, by inserting the (C.5) into the (C.4), we get:

$K(k) = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\frac{1}{4^n}\binom{2n}{n}\frac{\pi}{2} = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}$ and by adopting the notation of the hypergeometric functions: ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$, with the:

$(x)_n = x(x+1)...(x+n-1)$

$(\quad = \Gamma(x+n)/\Gamma(x) = \Gamma(x+n-1+1)/\Gamma(x-1+1) = (x+n-1)!/(x-1)! \quad)$

which are the Symbols of Pochhammer (rising factorial), it follows that:

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n} = \frac{\pi}{2}\cdot{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2) \ ; \quad \text{(C.6)}$$

in fact:

$$[\frac{1}{4^n}\binom{2n}{n}]^2=[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2=\frac{(a)_n(b)_n}{(c)_n n!}=\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}=\frac{[\frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1)][\frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1)]}{1(1+1)...(1+n-1)n!}=$$

$$=\frac{[\frac{1}{2}(\frac{1}{2}+1)...(n-\frac{1}{2})]^2}{1(2)...(n)n!}=\frac{[\frac{1}{2}\frac{3}{2}\frac{5}{2}...(n-\frac{1}{2})]^2}{(n!)^2}=\frac{[1\cdot3\cdot5...(2n-1)]^2}{2^{2n}(n!)^2}=\frac{(2n)!}{4^n(n!)^2}\frac{[1\cdot3\cdot5...(2n-1)]}{(2\cdot4\cdot6...2n)}=$$

$$=[\frac{(2n)!}{4^n(n!)^2}]^2 \quad \text{qed.}$$

Now, we define $K'(k)=K(\sqrt{1-k^2})=K(k')$, where $k'=\sqrt{1-k^2}$; there is a symmetry for $k=\frac{1}{\sqrt{2}}$, as in such a case: $K'(\frac{1}{\sqrt{2}})=K(\frac{1}{\sqrt{2}})$.

Now, as well as with the (C.6), we also have that:

$$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=\frac{\pi}{2}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)$$

as from the MacLaurin expansion of $(1-x)^{1/2}$, we have:

$(1-x)^{1/2}=\sum_{n=0}^{\infty}\binom{1/2}{n}(-x)^n=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-...$, from which:

$$(1-k^2\sin^2\beta)^{1/2}=\sum_{n=0}^{\infty}\binom{1/2}{n}(-k^2\sin^2\beta)^n=1-\frac{1}{2}(k\sin\beta)^2-\frac{1}{8}(k\sin\beta)^4-\frac{1}{16}(k\sin\beta)^6-...=$$

$$=(k\sin\beta)^0-\frac{1}{2}(k\sin\beta)^2-\frac{1}{8}(k\sin\beta)^4-\frac{1}{16}(k\sin\beta)^6-...=\sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta=$$

$$=\sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta \quad \text{and then:}$$

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta}\cdot d\beta=\int_0^{\pi/2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta=$$

$$=\sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=\frac{\pi}{2}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2) \quad (C.7)$$

qed.

And, here too, we have: $E'(k)=E(\sqrt{1-k^2})=E(k')$, with $k'=\sqrt{1-k^2}$; we have a symmetry for $k=\frac{1}{\sqrt{2}}$, as, in such a case: $E'(\frac{1}{\sqrt{2}})=E(\frac{1}{\sqrt{2}})$.

-Now, by starting back from the results gotten so far:

$$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=\frac{\pi}{2}\cdot{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2) \quad (C.8)$$

$$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=\frac{\pi}{2}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2). \quad (C.9)$$

Well, for the (C.8) and for the Clausen Product Identity, that is the (C.23) proved here in Appendix C: $[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2={}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)])$ (Clausen Product Identity)

(with: ${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$, definition), we have:

$$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4k^2(1-k^2)]) = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2)) \qquad (C.10)$$

but also: $[{}_2F_1(\frac{1}{2},\frac{1}{2};1;k'^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4k'^2(1-k'^2)]) = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))$

and so:

$$\boxed{[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))} \qquad (C.11)$$

(which is the (C.10)) .

Conversely, for the (simple) Clausen Identity, that is:
${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};x) = [{}_2F_1(a,b;a+b+\frac{1}{2};x)]^2$ which is the (C.19), proved, this one too, in Appendix C, it follows that the (C.11) can be reinterpreted like this:
$[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2)) = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)]^2$, from which:

$$\boxed{\frac{2}{\pi}K(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)} \qquad (C.12)$$

-**<u>Some properties of the hypergeometric functions</u>**:

first of all, we remind the definitions:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \qquad (C.13)$$

$(x)_n = x(x+1)...(x+n-1)$ (rising factorial – symbol of Pochhammer) (C.14)

$${}_3F_2(a,b,c;d,e;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}z^n \qquad (C.15)$$

then, here are some identities that can be directly checked:

$(x)_n = x(x+1)...(x+n-1)$

$(1)_n = 1(1+1)...(1+n-1) = 1\cdot 2\cdot ...n = n!$

$(2)_n = 2(2+1)...(2+n-1) = 2\cdot 3\cdot ...(n+1) = (n+1)!$

$(\frac{3}{2})_n = \frac{3}{2}(\frac{3}{2}+1)...(\frac{3}{2}+n-1) = \frac{3}{2}\frac{5}{2}...(n+\frac{1}{2})$

$\frac{(a)_n}{n!} = (-1)^n\binom{-a}{n}$

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ (from the link between the factorial and the Gamma of Euler)

$\binom{a}{m} = \frac{\Gamma(1+a)}{[m!\Gamma(1+a-m)]}$ (from the link between the factorial and the Gamma of Euler)

$$K(\frac{1}{\sqrt{2}}) = K'(\frac{1}{\sqrt{2}}) = \frac{[\Gamma(1/4)]^2}{4\pi^{1/2}}$$

$$\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3} = \frac{(4n)!}{4^{4n}(n!)^4} \qquad \text{(C.16)}$$

<u>proof of the (C.16)</u>:

we start by underlining some obvious identities on semifactorials (even ones as well as odd ones):

$[(2n+1)!!][2^n][n!] = [(2n+1)(2n+1-2)(2n+1-4)...][2\cdot 2\cdot ...n\_times...2\cdot 2][n(n-1)(n-2)...] =$

$= [(2n+1)(2n-1)(2n-3)...][1][2n2(n-1)2(n-2)...] = (2n+1)2n(2n-1)(2n-2)(2n-3)...2\cdot 1 =$

$= (2n+1)2n(2n-1)(2n-2)(2n-3)...2\cdot 1 = (2n+1)!$, that is: $(2n+1)!!2^n n! = (2n+1)!$, from which:

$\boxed{(2n+1)!! = \frac{(2n+1)!}{2^n n!}\ ;}$ then:

$(2n)!! = 2n(2n-2)(2n-4)...2 = 2n2(n-1)2(n-2) = 2^n n!$, that is: $\boxed{(2n)!! = 2^n n!}$ and, at last:

$[(2n-1)!!][2^n][n!] = [(2n-1)(2n-3)...][2\cdot 2\cdot ...n\_times...2\cdot 2][n(n-1)(n-2)...] =$

$= [(2n-1)(2n-3)...][1][2n2(n-1)2(n-2)...2] = (2n-1)(2n-3)2n(2n-2)(2n-4)...2 = (2n)!$, from

which: $\boxed{(2n-1)!! = \frac{(2n)!}{2^n n!}\ .}$

That being said and by going back to the (C.16), we start by analysing the term $(\frac{1}{2})_n$:

$(\frac{1}{2})_n = \frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1) = \frac{1}{2}\frac{3}{2}\frac{5}{2}...\frac{2n-1}{2} = \frac{(2n-1)!!}{2^n}$; now, we multiply numerator and denominator by the even mid-numbers $(2n)!!$:

$(\frac{1}{2})_n = \frac{(2n-1)!!}{2^n}\frac{(2n)!!}{(2n)!!} = \frac{(2n-1)!!}{2^n}\frac{2^n n!}{2^n n!} = \frac{(2n)!}{2^{2n} n!} = \frac{(2n)!}{4^n n!}$; now, for the other two terms:

$(\frac{1}{4})_n = \frac{1\cdot 5\cdot 9...(4n-3)}{4^n}$ and $(\frac{3}{4})_n = \frac{3\cdot 7\cdot 11...(4n-1)}{4^n}$, from which:

$(\frac{1}{4})_n(\frac{3}{4})_n = \frac{1\cdot 3\cdot 5\cdot 7...(4n-1)}{4^{2n}} = \frac{(4n-1)!!}{4^{2n}}$ and here, too, we multiply numerator and denominator by the even mid-numbers from 2 to 4n, that is $(4n)!!$:

$(\frac{1}{4})_n(\frac{3}{4})_n = \frac{(4n-1)!!}{4^{2n}}\frac{(4n)!!}{(4n)!!} = \frac{(4n-1)!!}{4^{2n}}\frac{2^{2n}(2n)!}{2^{2n}(2n)!} = \frac{(4n-1)!!(4n)!!}{4^{2n}2^{2n}(2n)!} = \frac{(4n)!}{4^{3n}(2n)!}$; totally:

$$\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3} = \frac{\frac{(2n)!}{4^n n!}\frac{(4n)!}{4^{3n}(2n)!}}{(n!)^3} = \frac{(4n)!}{4^{4n}(n!)^4} \quad \text{qed.}$$

-Kummer's Identity-proof/check:

(Gauss quadratic transform) (fast check)

$${}_2F_1(2a,2b;a+b+\frac{1}{2};x) = {}_2F_1(a,b;a+b+\frac{1}{2};4x(1-x))\ ; \qquad \text{(C.17)}$$

we say in advance that:

$(a)_n = a(a+1)...(a+n-1)$, $(a)_0 = 1$, $(a)_1 = a(a+1)...(a+1-1) = a(a+1)...a = a$,

$(a)_2 = a(a+1)...(a+2-1) = a(a+1)...(a+1) = a(a+1)$.

We start from the definition ${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$ and we use it in the (C.17):

$$\sum_{n=0}^{\infty} \frac{(2a)_n (2b)_n}{(a+b+\frac{1}{2})_n n!} x^n = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(a+b+\frac{1}{2})_n n!} [4x(1-x)]^n ;$$

left-hand side:

$$\sum_{n=0}^{\infty} \frac{(2a)_n (2b)_n}{(a+b+\frac{1}{2})_n n!} x^n = \frac{(2a)_0 (2b)_0}{(a+b+\frac{1}{2})_0 0!} x^0 + \frac{(2a)_1 (2b)_1}{(a+b+\frac{1}{2})_1 1!} x^1 + \frac{(2a)_2 (2b)_2}{(a+b+\frac{1}{2})_2 2!} x^2 + ... =$$

$$= \frac{1 \cdot 1}{1} + \frac{(2a)(2b)}{(a+b+\frac{1}{2})} x + \frac{(2a)(2a+1)(2b)(2b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^2 + ... =$$

$$= 1 + \frac{4ab}{(a+b+\frac{1}{2})} x + \frac{16a^2b^2 + 8a^2b + 8ab^2 + 4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^2 + ... \quad ;$$

right-hand side:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(a+b+\frac{1}{2})_n n!} [4x(1-x)]^n =$$

$$= \frac{(a)_0 (b)_0}{(a+b+\frac{1}{2})_0 0!} [4x(1-x)]^0 + \frac{(a)_1 (b)_1}{(a+b+\frac{1}{2})_1 1!} [4x(1-x)]^1 + \frac{(a)_2 (b)_2}{(a+b+\frac{1}{2})_2 2!} [4x(1-x)]^2 + ... =$$

$$= 1 + \frac{(a)(b)}{(a+b+\frac{1}{2})} [4x(1-x)] + \frac{(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} [4x(1-x)]^2 + ... =$$

$$= 1 + \frac{(a)(b)}{(a+b+\frac{1}{2})} (4x - 4x^2) + \frac{(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} [16x^2(1+x^2-2x)] + ... =$$

$$= 1 + \frac{4(a)(b)}{(a+b+\frac{1}{2})} x - \frac{4(a)(b)}{(a+b+\frac{1}{2})} x^2 + \frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^2 + \frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^4 -$$

$$- 2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^3 + ... =$$

$$= 1 + \frac{4(a)(b)}{(a+b+\frac{1}{2})} x + [\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} - \frac{4(a)(b)}{(a+b+\frac{1}{2})}] x^2 - 2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^3 +$$

$$+ \frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})} x^4 + ... = 1 + \frac{4ab}{(a+b+\frac{1}{2})} x + [\frac{16a^2b^2 + 8ab^2 + 8a^2b + 4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}] x^2 -$$

$$-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4+\ldots$$

we check that the equality stands:

$$\boxed{1+\frac{4ab}{(a+b+\frac{1}{2})}x+\frac{16a^2b^2+8a^2b+8ab^2+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^2}+\ldots=\boxed{1+\frac{4ab}{(a+b+\frac{1}{2})}x+[\frac{16a^2b^2+8ab^2+8a^2b+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}]x^2}-$$

$$-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4+\ldots \quad !!!!! \tag{C.18}$$

And so the (C.17) works fine; of course, in the right-hand side of the (C.18) there are further terms, but they would meet their counterparts at the left-hand side if at the left-hand side one considered more terms in the hypergeometric summation, and so on…

-A Clausen's Identity-proof/check:
(fast check)

$${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};x)=[{}_2F_1(a,b;a+b+\frac{1}{2};x)]^2 \; ; \tag{C.19}$$

we start from the definitions ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ and

${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ and we use them in the (C.19):

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b)_n(a+b+\frac{1}{2})_n n!}x^n=[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}x^n]^2 \; ;$$

left-hand side:

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b)_n(a+b+\frac{1}{2})_n n!}x^n=$$

$$=\frac{(2a)_0(2b)_0(a+b)_0}{(2a+2b)_0(a+b+\frac{1}{2})_0 0!}x^0+\frac{(2a)_1(2b)_1(a+b)_1}{(2a+2b)_1(a+b+\frac{1}{2})_1 1!}x^1+\frac{(2a)_2(2b)_2(a+b)_2}{(2a+2b)_2(a+b+\frac{1}{2})_2 2!}x^2+\ldots=$$

$$=1+\frac{(2a)(2b)(a+b)}{(2a+2b)(a+b+\frac{1}{2})}x+\frac{(2a)_2(2b)_2(a+b)_2}{2(2a+2b)_2(a+b+\frac{1}{2})_2}x^2+\ldots$$

right-hand side:

$$[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}x^n]^2=[\frac{(a)_0(b)_0}{(a+b+\frac{1}{2})_0 0!}x^0+\frac{(a)_1(b)_1}{(a+b+\frac{1}{2})_1 1!}x^1+\frac{(a)_2(b)_2}{(a+b+\frac{1}{2})_2 2!}x^2+\ldots]^2=$$

$$=[1+\frac{(a)(b)}{(a+b+\frac{1}{2})}x+\frac{(a)_2(b)_2}{2(a+b+\frac{1}{2})_2}x^2+\ldots]^2 \; ;$$

now we check that equality stands no more with letters, as in the previous case, where we kept a and b letters; we do this because of the huge calculations we would face; instead, we will fix random values, such as a=2 and b=3 :

left-hand side:
(and we remind that we have now enough skills for the calculations of all the $(x)_n$, from the previous proof…)

$$...=1+\frac{(4)(6)(5)}{(10)(\frac{11}{2})}x+\frac{(4)_2(6)_2(5)_2}{2(10)_2(\frac{11}{2})_2}x^2+...=1+\frac{24}{11}x+\frac{20\cdot 42\cdot 30}{2\cdot 110\cdot\frac{143}{4}}x^2+...=1+\frac{24}{11}x+\frac{5040}{1573}x^2+...$$

right-hand side:

$$...=[1+\frac{(2)(3)}{(\frac{11}{2})}x+\frac{(2)_2(3)_2}{2(\frac{11}{2})_2}x^2+...]^2=[1+\frac{(2)(3)}{(\frac{11}{2})}x+\frac{6\cdot 12}{2\frac{143}{4}}x^2+...]^2=[1+\frac{12}{11}x+\frac{144}{143}x^2+...]^2=$$

$$=(1+\frac{12}{11}x+\frac{144}{143}x^2+...)(1+\frac{12}{11}x+\frac{144}{143}x^2+...)=1+\frac{24}{11}x+\frac{5040}{1573}x^2+\frac{3456}{1573}x^3+\frac{20736}{20449}x^4+...$$

from which:

$$\boxed{1+\frac{24}{11}x+\frac{5040}{1573}x^2}+...=\boxed{1+\frac{24}{11}x+\frac{5040}{1573}x^2}+\frac{3456}{1573}x^3+\frac{20736}{20449}x^4+...\quad !!! \tag{C.20}$$

And so the (C.19) works fine; of course, in the right-hand side of the (C.20) there are further terms, but they would meet their counterparts at the left-hand side if at the left-hand side one considered more terms in the hypergeometric summation, and so on…

-Product Identity of Clausen-proof/check:
(Bailey product formula) (fast check)

$${}_2F_1(\frac{1}{4}+a,\frac{1}{4}+b;1+a+b;x)\,{}_2F_1(\frac{1}{4}-a,\frac{1}{4}-b;1-a-b;x)=$$

$$={}_3F_2(\frac{1}{2},\frac{1}{2}+a-b,\frac{1}{2}-a+b;1+a+b,1-a-b;x)\quad ; \tag{C.21}$$

we start from the definitions ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ and

${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ and we use them in the (C.21), with the usual random values, such as a=2 and b=3 :

$${}_2F_1(\frac{1}{4}+2,\frac{1}{4}+3;1+2+3;x)\,{}_2F_1(\frac{1}{4}-2,\frac{1}{4}-3;1-2-3;x)=$$

$$={}_3F_2(\frac{1}{2},\frac{1}{2}+2-3,\frac{1}{2}-2+3;1+2+3,1-2-3;x)\text{ , that is:}$$

$${}_2F_1(\frac{9}{4},\frac{13}{4};6;x)\,{}_2F_1(-\frac{7}{4},-\frac{11}{4};-4;x)={}_3F_2(\frac{1}{2},-\frac{1}{2},\frac{3}{2};6,-4;x);$$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$$

$$\sum_{n=0}^{\infty}\frac{(\frac{9}{4})_n(\frac{13}{4})_n}{(6)_n n!}x^n\sum_{n=0}^{\infty}\frac{(-\frac{7}{4})_n(\frac{11}{4})_n}{(-4)_n n!}x^n=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(-\frac{1}{2})_n(\frac{3}{2})_n}{(6)_n(-4)_n n!}x^n$$

left-hand side:

$$[\frac{(\frac{9}{4})_0(\frac{13}{4})_0}{(6)_0 0!}x^0+\frac{(\frac{9}{4})_1(\frac{13}{4})_1}{(6)_1 1!}x^1+\frac{(\frac{9}{4})_2(\frac{13}{4})_2}{(6)_2 2!}x^2+...][\frac{(-\frac{7}{4})_0(-\frac{11}{4})_0}{(-4)_0 0!}x^n+\frac{(-\frac{7}{4})_1(-\frac{11}{4})_1}{(-4)_1 1!}x^1+\frac{(-\frac{7}{4})_2(-\frac{11}{4})_2}{(-4)_2 2!}x^2+...]=$$

$$=[1+\frac{\frac{9}{4}\frac{13}{4}}{6}x+\frac{\frac{9}{4}(\frac{9}{4}+1)\frac{13}{4}(\frac{13}{4}+1)}{6(6+1)2}x^2+...][1+\frac{\frac{7}{4}\frac{11}{4}}{-4}x+\frac{539}{2048}x^2+...]=$$

$$=[1+\frac{39}{32}x+\frac{8619}{7168}x^2+...][1-\frac{77}{64}x+\frac{539}{2048}x^2+...]=$$

$$=[1-\frac{77}{64}x+\frac{539}{2048}x^2+\frac{39}{32}x-\frac{39}{32}\frac{77}{64}x^2+\frac{39}{32}\frac{539}{2048}x^3+\frac{8619}{7168}x^2-\frac{77}{64}\frac{8619}{7168}x^3+\frac{8619}{7168}\frac{539}{2048}x^4+...]=$$

$$=[1+\frac{1}{64}x-\frac{5}{7168}x^2-\frac{33850392576}{65536}x^3+\frac{4645641}{14680064}x^4+...]=$$

right-hand side:

$$[\frac{(\frac{1}{2})_0(-\frac{1}{2})_0(\frac{3}{2})_0}{(6)_0(-4)_0 0!}x^0+\frac{(\frac{1}{2})_1(-\frac{1}{2})_1(\frac{3}{2})_1}{(6)_1(-4)_1 1!}x^1+\frac{(\frac{1}{2})_2(-\frac{1}{2})_2(\frac{3}{2})_2}{(6)_2(-4)_2 2!}x^2+...]=$$

$$=[1+\frac{-\frac{1}{2}\frac{1}{2}\frac{3}{2}}{-24}x+\frac{-\frac{1}{2}(\frac{1}{2}+1)\frac{1}{2}(-\frac{1}{2}+1)\frac{3}{2}(\frac{3}{2}+1)}{-12(6+1)4(-4+1)}x^2+...]=$$

$$=[1+\frac{1}{64}x-\frac{5}{7168}x^2+...]$$ and after a comparison:

$$\boxed{1+\frac{1}{64}x-\frac{5}{7168}x^2}-\frac{33850392576}{65536}x^3+\frac{4645641}{14680064}x^4+...=\boxed{1+\frac{1}{64}x-\frac{5}{7168}x^2}+...\quad !!! \qquad (C.22)$$

And so the (C.21) works fine; of course, in the right-hand side of the (C.22) there are further terms, but they would meet their counterparts at the left-hand side if at the left-hand side one considered more terms in the hypergeometric summation, and so on…

-And we check also the following:

$$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2={}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)]); \qquad (C.23)$$

-<u>first fast check method</u>:

we start from the definitions ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ and

${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ and apply the values of the (C.23):

$$[\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}x^{2n}]^2=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n(1)_n n!}[4x^2(1-x^2)]^n \;;$$

left-hand side:

$$[\frac{(\frac{1}{2})_0(\frac{1}{2})_0}{(1)_0 0!}x^0+\frac{(\frac{1}{2})_1(\frac{1}{2})_1}{(1)_1 1!}x^2+\frac{(\frac{1}{2})_2(\frac{1}{2})_2}{(1)_2 2!}x^4+...]^2=[1+\frac{1}{4}x^2+\frac{\frac{1}{2}(\frac{1}{2}+1)\frac{1}{2}(\frac{1}{2}+1)}{4}x^4+...]^2=$$

$$[1+\frac{1}{4}x^2+\frac{9}{64}x^4+...]^2=(1+\frac{1}{4}x^2+\frac{9}{64}x^4+...)(1+\frac{1}{4}x^2+\frac{9}{64}x^4+...)=1+\frac{1}{2}x^2+\frac{11}{32}x^4+\frac{18}{256}x^6+\frac{81}{4096}x^8+...$$

right-hand side:

$$\{\frac{(\frac{1}{2})_0(\frac{1}{2})_0(\frac{1}{2})_0}{(1)_0(1)_0 0!}[4x^2(1-x^2)]^0+\frac{(\frac{1}{2})_1(\frac{1}{2})_1(\frac{1}{2})_1}{(1)_1(1)_1 1!}[4x^2(1-x^2)]^1+\frac{(\frac{1}{2})_2(\frac{1}{2})_2(\frac{1}{2})_2}{(1)_2(1)_2 2!}[4x^2(1-x^2)]^2+...\}=$$

$$=\{1+\frac{\frac{1}{2}\frac{1}{2}\frac{1}{2}}{1\cdot1\cdot1}[4x^2(1-x^2)]+\frac{\frac{1}{8}(\frac{1}{2}+1)^3}{2\cdot2\cdot2}16x^4(1+x^4-2x)+...\}=$$

$$=[1+\frac{1}{2}(x^2-x^4)+\frac{27}{32}(x^4+x^8-2x^5)+...]=[1+\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{27}{32}x^4+\frac{27}{32}x^8-\frac{27}{16}x^5+...]=$$

$[1+\frac{1}{2}x^2+\frac{11}{32}x^4-\frac{27}{16}x^5+\frac{27}{32}x^8+...]$ and by comparison:

$$\boxed{1+\frac{1}{2}x^2+\frac{11}{32}x^4}+\frac{18}{256}x^6+\frac{81}{4096}x^8+...=\boxed{1+\frac{1}{2}x^2+\frac{11}{32}x^4}-\frac{27}{16}x^5+\frac{27}{32}x^8+...\quad !!! \qquad (C.24)$$

And so the (C.23) works fine; of course, in the right-hand side of the (C.24) there are further terms, but they would meet their counterparts at the left-hand side if at the left-hand side one considered more terms in the hypergeometric summation, and so on…

-<u>second method based on the use of the (C.17) and of the (C.19)</u>:

we remind the (C.17) in t: ${}_2F_1(2a,2b;a+b+\frac{1}{2};t)={}_2F_1(a,b;a+b+\frac{1}{2};4t(1-t))$ and we fix a=b=1/4, so: ${}_2F_1(\frac{1}{2},\frac{1}{2};1;t)={}_2F_1(\frac{1}{4},\frac{1}{4};1;4t(1-t))$ and through a simple change of variable t=x^2 →

${}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)={}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))$ and by squaring:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2=[{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$; now, as predicted, we consider the (C.19) of Clausen in z:

${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};z)=[{}_2F_1(a,b;a+b+\frac{1}{2};z)]^2$ and obviously still with a=b=1/4:

${}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;z)=[{}_2F_1(\frac{1}{4},\frac{1}{4};1;z)]^2$ and with a change of variable $z=4x^2(1-x^2)$ →

${}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;4x^2(1-x^2))=[{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$, but some lines above these ones, we got:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$, from which the assumption:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)])$ qed.

-The Euler's Transform – proof/check:

$${}_2F_1(a,b;c;x) = (1-x)^{c-b-a} \cdot {}_2F_1(c-a,c-b;c;x) \quad \text{(C.25)}$$

-<u>first method of fast check</u>:

it's known as the equation of autotransformation of the hypergeometric functions;
(an example with a=1 , b=2 and c=4) →

$${}_2F_1(1,2;4;x) = (1-x)^{4-2-1} \cdot {}_2F_1(3,2;4;x)$$

$$[\frac{(1)_0(2)_0}{(4)_0 0!}x^0 + \frac{(1)_1(2)_1}{(4)_1 1!}x^1 + \frac{(1)_2(2)_2}{(4)_2 2!}x^2 + ...] = (1-x)[\frac{(3)_0(2)_0}{(4)_0 0!}x^0 + \frac{(3)_1(2)_1}{(4)_1 1!}x^1 + \frac{(3)_2(2)_2}{(4)_2 2!}x^2 + ...]$$

$$[1 + \frac{1}{2}x + \frac{3}{10}x^2 + ...] = (1-x)[1 + \frac{3}{2}x + \frac{9}{5}x^2 + ...]$$

$$1 + \frac{1}{2}x + \frac{3}{10}x^2 = 1 + \frac{3}{2}x + \frac{9}{5}x^2 - x - \frac{3}{2}x^2 - \frac{9}{5}x^3 + ...$$

$$(\frac{1}{2}x - \frac{3}{2}x + x) + (\frac{3}{10}x^2 - \frac{9}{5}x^2 + \frac{3}{2}x^2) + \frac{9}{5}x^3 + ... = 0$$

$$0x^0 + 0x^2 + \frac{9}{5}x^3 + ... = 0 \ . \quad \text{(C.26)}$$

And so the (C.25) works fine; of course, in the (C.26) there are further terms, but they would meet their exact counterparts if one considered more terms in the hypergeometric summations, and so on…

-<u>second method based on the Transform of Pfaff</u>:
we consider the (C.27) of Euler we prove right here below:

$\int_0^1 \frac{(1-t)^{c-b-1}}{(1-xt)^a} t^{b-1} dt = B(c-b,b) \cdot {}_2F_1(a,b;c;x)$ and we make the change of variable t=1-u (and dt=-du), with the limits of integration which swap one another, but then they go back to their original values, due to the sign of –du:

$$\int_0^1 \frac{u^{c-b-1}}{[1-x(1-u)]^a}(1-u)^{b-1} du = B(c-b,b) \cdot {}_2F_1(a,b;c;x) = (1-x)^{-a} \int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1} du$$

and by reinterpreting the last integral, still over the Euler's integral (C.27), we have:

$\int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1} du = B(c-b,b) \cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})$, from which:

$$B(c-b,b) \cdot {}_2F_1(a,b;c;x) = (1-x)^{-a} \int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1} du = (1-x)^{-a} B(c-b,b) \cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})$$

that is: ${}_2F_1(a,b;c;x) = (1-x)^{-a} \cdot {}_2F_1(a,c-b;c;\frac{x}{x-1}) = (1-x)^{-a} \cdot {}_2F_1(c-b,a;c;\frac{x}{x-1})$ and by applying again the transform of Pfaff to the last one (in $\frac{x}{x-1}=z$) (and of course, this time, after considering the swappings which took place, with (c-b) in place of a) we have:

${}_2F_1(c-b,a;c;z)=(1-z)^{-(c-b)}\cdot{}_2F_1(c-b,c-a;c;\frac{z}{z-1})$ and returning back to x:

$$\left(\left(\frac{x}{x-1}=z \rightarrow 1-z=1-\frac{x}{x-1}=\frac{1}{1-x} \quad \text{and also: } \frac{z}{z-1}=\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=x\right)\right) \rightarrow$$

${}_2F_1(c-b,a;c;\frac{x}{x-1})=(\frac{1}{1-x})^{-(c-b)}\cdot{}_2F_1(c-b,c-a;c;x)=(1-x)^{c-b}\cdot{}_2F_1(c-b,c-a;c;x)$, that is:

${}_2F_1(a,c-b;c;\frac{x}{x-1})=(1-x)^{c-b}\cdot{}_2F_1(c-b,c-a;c;x)$ and as we previously got:

${}_2F_1(a,b;c;x)=(1-x)^{-a}\cdot{}_2F_1(a,c-b;c;\frac{x}{x-1})$, then, from both:

${}_2F_1(a,b;c;x)=(1-x)^{-a}(1-x)^{c-b}\cdot{}_2F_1(c-b,c-a;c;x)$, that is:
(c-b rightly exchangeable in position with c-a)

${}_2F_1(a,b;c;x)=(1-x)^{c-b-a}\cdot{}_2F_1(c-a,c-b;c;x)$ qed.

-Euler'sFormula $\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt=B(c-b,b)\cdot{}_2F_1(a,b;c;z)$ **– proof.**

Here it is: $\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt=B(c-b,b)\cdot{}_2F_1(a,b;c;z)$; (C.27)

in Appendix A, with the (A.23), we saw that the Beta of Euler can be expressed like this:

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\int_0^1x^{m-1}(1-x)^{n-1}dx \quad , \tag{C.28}$$

from which: $B(c-b,b)=\frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)}$ and so the (C.27) becomes:

$$ {}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt \quad . \tag{C.29}$$

So, we consider the function $\frac{1}{(1-zt)^a}=(1-zt)^{-a}$; of course, according to Taylor, we have:

$$(1-zt)^{-a}=\sum_{k=0}^{\infty}\binom{-a}{k}(-1)^k(zt)^k=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^kt^k \tag{C.30}$$

and by inserting the last one in the integral of the (C.27), we get:

$\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt=\int_0^1(1-t)^{c-b-1}t^{b-1}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^kt^kdt=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k\int_0^1(1-t)^{c-b-1}t^{b+k-1}dt$, and for the (C.28) it becomes: $\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k\int_0^1(1-t)^{c-b-1}t^{b+k-1}dt=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\frac{\Gamma(b+k)\Gamma(c-b)}{\Gamma(c+k)}z^k=$

$$=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\frac{\Gamma(b+k)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+k)}z^k=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_kk!}z^k=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\cdot{}_2F_1(a,b;c;z)$$

(as we saw in the above lines that $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$) and so:

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{(1-t)^{c-b-1}}{(1-zt)^a}t^{b-1}dt$$, that is really the (C.29), or the (C.27)!!!

Then, of course: $${}_2F_1(a,b;c;1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1(1-t)^{c-a-b-1}t^{b-1}dt=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}=$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

-Proof of the following hypergeometric equation:

$$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2);\qquad\text{(C.31)}$$

we start from the (C.6): $\frac{2}{\pi}K(k)={}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$ and consider the first quadratic form (Form of Landen): ${}_2F_1(a,b;a-b+1;z)=(1-z)^{1-2b}(1+z)^{2b-1-a}\cdot{}_2F_1(\frac{a-2b+1}{2},\frac{a-2b+2}{2};a-b+1;\frac{4z}{(1+z)^2})$
(it can be checked through the well known method of the comparison of sides, after having adopted the summations to express the two ${}_2F_1$)
and by chosing: a=1/2 , b=1/2 and z=k^2 , we get:

$${}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)=\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{4},\frac{3}{4};1;\frac{4k^2}{(1+k^2)^2})\quad\text{(transf. of Landen);}\qquad\text{(C.32)}$$

then, after using also this further quadratic identity of Gauss, that is:
$${}_2F_1(a,b;a+b+\frac{1}{2};x)={}_2F_1(\frac{a}{2},\frac{b}{2};a+b+\frac{1}{2};4x(1-x))$$
(it can be checked through the well known method of the comparison of sides, after having adopted the summations to express the two ${}_2F_1$)
and with a=1/4 and b=3/4 , we get:
$${}_2F_1(\frac{1}{4},\frac{3}{4};1;x)={}_2F_1(\frac{1}{8},\frac{3}{8};1;4x(1-x));\qquad\text{(C.33)}$$
at this point, we make the substitution:
$x=\frac{4k^2}{(1+k^2)^2}$, from which we also have that: $4x(1-x)=[\frac{4k(1-k^2)}{(1+k^2)^2}]^2=[\frac{4kk'^2}{(1+k^2)^2}]^2$ and due to the (1.29) , we also have that: $\frac{4kk'^2}{(1+k^2)^2}=\frac{2}{(g^{12}+g^{-12})}$, from which: $4x(1-x)=[\frac{2}{(g^{12}+g^{-12})}]^2$ and so the (C.33) becomes:
$${}_2F_1(\frac{1}{4},\frac{3}{4};1;\frac{4k^2}{(1+k^2)^2})={}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2)\qquad\text{(C.34)}$$
and for the (C.32), the (C.34) becomes:
$$\frac{2}{\pi}K(k)\sqrt{1+k^2}={}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2)$$, from which, finally:
$$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2)$$, that is the assumption. qed

-Proof of the following hypergeometric equation:

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{1+k^2} \cdot {}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g^{12}+g^{-12}})^2) \qquad (C.35)$$

which is also the (1.11); then, we consider the (1.8) (which is also the (C.31)):

$\frac{2}{\pi}K(k) = \frac{1}{\sqrt{1+k^2}} \cdot {}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)$ and we use on it the Clausen Product Identity (C.21):

$${}_2F_1(\frac{1}{4}+a,\frac{1}{4}+b;1+a+b;x)\,{}_2F_1(\frac{1}{4}-a,\frac{1}{4}-b;1-a-b;x) =$$

$= {}_3F_2(\frac{1}{2},\frac{1}{2}+a-b,\frac{1}{2}-a+b;1+a+b,1-a-b;x)$ that is: (with $a=-\frac{1}{8}$ and $b=\frac{1}{8}$)

$${}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)\,{}_2F_1(\frac{1}{4}+\frac{1}{8},\frac{1}{4}-\frac{1}{8};1+\frac{1}{8}-\frac{1}{8};(\frac{2}{g^{12}+g^{-12}})^2) =$$

$$= {}_3F_2(\frac{1}{2},\frac{1}{2}-\frac{1}{8}-\frac{1}{8},\frac{1}{2}+\frac{1}{8}+\frac{1}{8};1-\frac{1}{8}+\frac{1}{8},1+\frac{1}{8}-\frac{1}{8};(\frac{2}{g^{12}+g^{-12}})^2) \rightarrow$$

$${}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)\,{}_2F_1(\frac{3}{8},\frac{1}{8};1;(\frac{2}{g^{12}+g^{-12}})^2) = \frac{2}{\pi}K(k)\frac{2}{\pi}K(k) = [\frac{2}{\pi}K(k)]^2 =$$

$= {}_3F_2(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1;(\frac{2}{g^{12}+g^{-12}})^2)$ qed.

APPENDIX D: The Fundamental Legendre's Identity.

$$K(k)E'(k)+K'(k)E(k)-K(k)K'(k) = \frac{\pi}{2} \;; \qquad (D.1)$$

in fact, we have that: $K(k) = \int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, $K'(k) = K(\sqrt{1-k^2}) = \int_0^{\pi/2}\frac{d\beta}{\sqrt{1-(1-k^2)\sin^2\beta}}$

$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta}\cdot d\beta$, $E'(k) = E(\sqrt{1-k^2}) = \int_0^{\pi/2}\sqrt{1-(1-k^2)\sin^2\beta}\cdot d\beta$

but also, obviously: $\frac{dK}{dk} = \int_0^{\pi/2}\frac{k\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta$, $\frac{dE}{dk} = \int_0^{\pi/2}\frac{-k\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta$ and also:

$$\frac{dK}{dk} = \frac{E(k)-k'^2K(k)}{kk'^2}(=\frac{E(k)-(\sqrt{1-k^2})^2K(k)}{k(\sqrt{1-k^2})^2} = \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}) \qquad (D.2)$$

$\frac{dE}{dk} = \frac{E(k)-K(k)}{k}$ (direct check) (D.3)

$\frac{dk'}{dk} = -\frac{k}{k'}$ (direct check, where $k' = \sqrt{1-k^2}$) (D.4)

$\frac{dK'}{dk} = -\frac{E'(k)-k^2K'(k)}{kk'^2}$ (symmetric with respect to the (D.2)) (D.5)

$\frac{dE'}{dk} = \frac{k[K'(k)-E'(k)]}{k'^2}$ (symmetric with respect to the (D.3)) (D.6)

As an example, we provide anyway the proof of the (D.2) and (D.3):

$\frac{dE}{dk} = \frac{E(k)-K(k)}{k}$ proof:

$$\frac{dE}{dk} = \int_0^{\pi/2}\frac{-k\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta = \frac{1}{k}\int_0^{\pi/2}\frac{-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta = \frac{1}{k}\int_0^{\pi/2}\frac{1-1-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta =$$

$$=\frac{1}{k}\int_0^{\pi/2}[\frac{1-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}-\frac{1}{\sqrt{(1-k^2\sin^2\beta)}}]d\beta=\frac{1}{k}\int_0^{\pi/2}[\sqrt{(1-k^2\sin^2\beta)}-\frac{1}{\sqrt{(1-k^2\sin^2\beta)}}]d\beta=$$

$$=\frac{E(k)-K(k)}{k}\ .$$

$\frac{dK}{dk}=\frac{E(k)-k'^2K(k)}{kk'^2}$ proof:

$$\frac{dK}{dk}=\int_0^{\pi/2}\frac{k\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=\frac{-1}{k}\int_0^{\pi/2}\frac{-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=\frac{-1}{k}\int_0^{\pi/2}\frac{-1+1-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=$$

$$=\frac{-1}{k}\int_0^{\pi/2}[\frac{1-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}-\frac{1}{(1-k^2\sin^2\beta)^{3/2}}]d\beta=$$

$$=\frac{-1}{k}\int_0^{\pi/2}\frac{1}{\sqrt{(1-k^2\sin^2\beta)^{3/2}}}d\beta+\frac{1}{k}\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\beta)^{3/2}}d\beta=$$

$=\frac{1}{k}[-K+\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\beta)^{3/2}}d\beta]$ and after a change of variable $\sin^2\beta=x^2$, we have:

$\frac{dK}{dk}=\frac{1}{k}[-K+\int_0^1\frac{1}{(1-k^2x^2)^{3/2}}\frac{1}{\sqrt{1-x^2}}dx]$ and through another change of variable $x^2=y$, we have:

$\frac{dK}{dk}=\frac{1}{k}[-K+\int_0^1\frac{(1-y)^{-\frac{1}{2}}}{(1-k^2y)^{3/2}}\frac{1}{2}y^{-\frac{1}{2}}dy]=\frac{1}{k}[-K+\frac{1}{2}\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-k^2y)^{3/2}}y^{\frac{1}{2}-1}dy]$, but as in the Appendix A we have shown, as the Beta of Euler, the identity: $B(m,n)=\int_0^1(1-x)^{n-1}x^{m-1}dx$, then we can write that: $B(\frac{1}{2},\frac{1}{2})=\int_0^1(1-y)^{\frac{1}{2}-1}y^{\frac{1}{2}-1}dx$ and being the following integral true:

$\int_0^1\frac{(1-y)^{c-b-1}}{(1-ty)^a}y^{b-1}dy=B(c-b,b)\cdot{}_2F_1(a,b;c;t)$ (see the (C.27)), it follows that:

$\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-ty)^{3/2}}y^{\frac{1}{2}-1}dy=B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;t)$ and so:

$$\frac{dK}{dk}=\frac{1}{k}[-K+\frac{1}{2}\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-k^2y)^{3/2}}y^{\frac{1}{2}-1}dy]=\frac{1}{k}[-K+\frac{1}{2}B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;k^2)]\qquad(t=k^2)$$

and as through an Euler's Transform (see the (C.25)) we have:

${}_2F_1(a,b;c;x)=(1-x)^{c-b-a}\cdot{}_2F_1(c-a,c-b;c;x)$, then, being, among things, that $B(\frac{1}{2},\frac{1}{2})=\pi$, it follows that: $\frac{dK}{dk}=\frac{1}{k}[-K+\frac{1}{2}B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;k^2)]=\frac{1}{k}[-K+\frac{\pi}{2}(1-k^2)^{-1}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)]=$

$=\frac{1}{k}[-K+\frac{1}{(1-k^2)}\frac{\pi}{2}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)]=\frac{1}{k}[-K+\frac{1}{k'^2}E(k)]=\frac{E(k)-k'^2K(k)}{kk'^2}$,

after having considered the (C.9). qed

As a proof of the (D.1) , we start trying to get the derivative of it:

$$\frac{dK(k)}{dk}E'(k)+K(k)\frac{dE'(k)}{dk}+\frac{dK'(k)}{dk}E(k)+K'(k)\frac{dE(k)}{dk}-\frac{dK(k)}{dk}K'(k)-K(k)\frac{dK'(k)}{dk}=0$$

from which:

$$\frac{E(k)-k'^2K(k)}{kk'^2}E'(k)+K(k)\frac{k[K'(k)-E'(k)]}{k'^2}-\frac{E'(k)-k^2K'(k)}{kk'^2}E(k)+K'(k)\frac{E(k)-K(k)}{k}-$$
$$-\frac{E(k)-k'^2K(k)}{kk'^2}K'(k)+K(k)\frac{E'(k)-k^2K'(k)}{kk'^2}=0$$
$$\frac{E(k)-k'^2K(k)}{kk'^2}[E'(k)-K'(k)]-\frac{E'(k)-k^2K'(k)}{kk'^2}[E(k)-K(k)]+K(k)\frac{k[K'(k)-E'(k)]}{k'^2}+$$
$$+K'(k)\frac{E(k)-K(k)}{k}=0$$
$$[E(k)-k'^2K(k)][E'(k)-K'(k)]-[E'(k)-k^2K'(k)][E(k)-K(k)]-k^2K(k)[E'(k)-K'(k)]+$$
$$+k'^2K'(k)[E(k)-K(k)]=0$$
$$[E(k)-k'^2K(k)-k^2K(k)][E'(k)-K'(k)]+[k'^2K'(k)-E'(k)+k^2K'(k)][E(k)-K(k)]=0$$

$$[E(k)E'(k)-k'^2K(k)E'(k)-k^2K(k)E'(k)-E(k)K'(k)+k'^2K(k)K'(k)+k^2K(k)K'(k)]+$$
$$+[k'^2K'(k)E(k)-E'(k)E(k)+k^2K'(k)E(k)-k'^2K'(k)K(k)+E'(k)K(k)-k^2K'(k)K(k)]=0$$

$$[\cancel{E(k)E'(k)}+K(k)E'(k)(-k'^2-k^2+1)-E(k)K'(k)+\cancel{K(k)K'(k)(k^2+k'^2)}]+$$
$$+[K'(k)E(k)(k'^2+k^2)-\cancel{E'(k)E(k)}-\cancel{K'(k)K(k)(k'^2+k^2)}]=0$$
$$(((\,(-k'^2-k^2+1)=0\,)))\qquad(((\,k'^2+k^2=1\,)))$$

$$[-E(k)K'(k)]+$$
$$+[K'(k)E(k)]=0\quad\rightarrow\quad 0=0\ .\quad \text{qed}$$

For what the value π/2 is concerned, let's try to write the (D.1) in the limit case of k=0: $K(0)E'(0)+K'(0)E(0)-K(0)K'(0)=const$, that is:

$$\frac{\pi}{2}1+(\int_0^{\pi/2}\frac{d\beta}{\cos\beta})(\frac{\pi}{2})-\frac{\pi}{2}(\int_0^{\pi/2}\frac{d\beta}{\cos\beta})=\frac{\pi}{2}\quad \text{qed}$$

Now, we also prove the following one:

$$\frac{1}{\pi}=-\frac{2}{\pi^2}kk'^2\frac{d}{dk}(KK')\ ;\qquad\text{(D.7)}$$

in fact, from the (D.2) we get: $E=kk'^2\frac{dK}{dk}+k'^2K$, while from the (D.5) we get: $E'=-kk'^2\frac{dK'}{dk}+k^2K'$, which, if put into the (D.1), yields:

$$K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2}=K(-kk'^2\frac{dK'}{dk}+k^2K')+K'(kk'^2\frac{dK}{dk}+k'^2K)-KK'$$
$$-kk'^2K\frac{dK'}{dk}+k^2KK'+kk'^2K'\frac{dK}{dk}+k'^2KK'-KK'=\frac{\pi}{2}$$
$$-kk'^2K\frac{dK'}{dk}+kk'^2K'\frac{dK}{dk}+k^2KK'+k'^2KK'-KK'=\frac{\pi}{2}$$
$$-kk'^2K\frac{dK'}{dk}+kk'^2K'\frac{dK}{dk}+k^2KK'(k^2+k'^2-1)=\frac{\pi}{2}$$, but $(k^2+k'^2-1)=0$, from which:

$$kk'^2(K'\frac{dK}{dk}-K\frac{dK'}{dk})=\frac{\pi}{2}\quad\rightarrow$$

$$\rightarrow (K'\frac{dK}{dk}-K\frac{dK'}{dk})=\frac{\pi}{2}\frac{1}{kk'^2} \quad \text{(D.8)}$$

$$\rightarrow (K'\frac{dK(-k)}{d(-k)}-K\frac{dK'}{dk})=\frac{\pi}{2}\frac{1}{kk'^2}$$, but:

K(-k)=K(k) , from which: $[K'\frac{dK(-k)}{d(-k)}-K\frac{dK'}{dk}]=[K'\frac{dK(k)}{d(-k)}-K\frac{dK'}{dk}]=[-K'\frac{dK(k)}{dk}-K\frac{dK'}{dk}]=$

$$=-\frac{d}{dk}(KK')=\frac{\pi}{2}\frac{1}{kk'^2}\rightarrow \frac{d}{dk}(KK')=-\frac{\pi}{2}\frac{1}{kk'^2}\rightarrow$$

$$\rightarrow \frac{1}{\pi}=-\frac{2}{\pi^2}kk'^2\frac{d}{dk}(KK') . \quad \text{(D.9)}$$

APPENDIX E: The Jacobi Triple Product.

Here it is:

$$\sum_{n=-\infty}^{\infty}x^n q^{n^2}=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n}) \quad \text{(E.1)}$$

-proof:

let us define the function:

$$F(\sqrt{x})=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})=(1+qx)(1+\frac{q}{x})(1+xq^3)(1+\frac{q^3}{x})(1+xq^5)(1+\frac{q^5}{x}).... \quad \text{(E.2)}$$

from which, for $F(q\sqrt{x})$ (of course, at the right-hand side of the (E.2), we carry out the replacement of x($=(\sqrt{x})^2$) with xq^2):

$$F(q\sqrt{x})=\prod_{n=1}^{\infty}(1+xq^{2n+1})(1+x^{-1}q^{2n-3})=(1+q^3x)(1+\frac{1}{xq})(1+xq^5)(1+\frac{q}{x})(1+xq^7)(1+\frac{q^3}{x})....; \quad \text{(E.3)}$$

now, getting the (E.3) divided by the (E.2), we get:

$$\frac{F(q\sqrt{x})}{F(\sqrt{x})}=\frac{(1+q^3x)(1+\frac{1}{xq})(1+xq^5)(1+\frac{q}{x})(1+xq^7)(1+\frac{q^3}{x})....}{(1+qx)(1+\frac{q}{x})(1+xq^3)(1+\frac{q^3}{x})(1+xq^5)(1+\frac{q^5}{x})....}=\boxed{[[[\cancel{\frac{(1+xq^7)}{(1+\frac{q^5}{x})}}]]]}\frac{(1+\frac{1}{xq})}{(1+qx)}=\frac{(1+\frac{1}{xq})}{(1+qx)}$$

as the term between square brackets disappears with the adding at numerator and denominator of the two couples of terms which appear in the products when n=4 and 5 (as we stopped at n=3, that are just three couples of products) and such two couples of products more, about n=4 and 5, are:

$$\frac{[(1+xq^9)(1+x^{-1}q^5)][(1+xq^{11})(1+x^{-1}q^7)]}{[(1+xq^7)(1+x^{-1}q^7)][(1+xq^9)(1+x^{-1}q^9)]}=\boxed{\frac{(1+x^{-1}q^5)}{(1+xq^7)}}\cdot[[\cancel{\frac{(1+xq^9)(1+xq^{11})(1+x^{-1}q^7)}{(1+x^{-1}q^7)(1+xq^9)(1+x^{-1}q^9)}}]]$$

where the ratio in the red frame appears and deletes the previous one, indeed, in the green frame, and so on. Therefore, by summing things up, we have obtained:

$$\frac{F(q\sqrt{x})}{F(\sqrt{x})}=\frac{(1+\frac{1}{xq})}{(1+qx)}$$ and by resolving the numerator, we get: $$\frac{F(q\sqrt{x})}{F(\sqrt{x})}=\frac{(\frac{1+qx}{xq})}{(1+qx)}=\frac{1}{xq}$$, from which:

$F(\sqrt{x}) = xqF(q\sqrt{x})$. (E.4)

Now, we define the following:

$$G(\sqrt{x}) = F(\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n}) \tag{E.5}$$

$$G(q\sqrt{x}) = F(q\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n}) \tag{E.6}$$

Now, using the (E.4) into the (E.6), we get:

$G(q\sqrt{x}) = \dfrac{F(\sqrt{x})}{xq}\prod_{n=1}^{\infty}(1-q^{2n}) = \dfrac{G(\sqrt{x})}{xq}$, that is:

$$G(\sqrt{x}) = xqG(q\sqrt{x}) \tag{E.7}$$

Now, we notice that $F(\sqrt{x})$ is an even function, as if we remove $\sqrt{x}$ and put $-\sqrt{x}$, then $F(\sqrt{x})$ stays the same, from which also the $G(\sqrt{x})$, due to the (E.5), is even.

Now, we imagine to develop $G(\sqrt{x})$ in series of Laurent, but, if we prefer, without involving the concept of series of Laurent, we can assign to $G(\sqrt{x})$ the most general expansion, which includes also terms whose exponent is negative, that is:

$$G(\sqrt{x}) = \sum_{m=-\infty}^{+\infty} a_m x^m \tag{E.8}$$

then, the (E.4) requires that:

$$\sum_{m=-\infty}^{+\infty} a_m x^m (= \sum_{m=-\infty}^{+\infty} a_m[(\sqrt{x})^2]^m) = xq\sum_{m=-\infty}^{+\infty} a_m[(q\sqrt{x})^2]^m = \sum_{m=-\infty}^{+\infty} a_m q^{2m+1}x^{m+1} \quad . \tag{E.9}$$

Now, at the right-hand side of the (E.9) we can rename the index in this way: m → m'-1 , from which:

$$\sum_{m=-\infty}^{+\infty} a_m x^m = \sum_{m'=-\infty}^{+\infty} a_{m'-1}q^{2(m'-1)+1}x^{(m'-1)+1} = \sum_{m'=-\infty}^{+\infty} a_{m'-1}q^{2m'-1}x^{m'} \tag{E.10}$$

($m'-1=-\infty$ is the same as $m'=-\infty$, as well as $m'-1=+\infty$ is the same as $m'=+\infty$, from which $\sum_{m'-1=-\infty}^{+\infty} = \sum_{m'=-\infty}^{+\infty}$)

and also renaming in the last right-hand block of the (E.10), m' back into m, then we get:

$\sum_{m=-\infty}^{+\infty} a_m x^m = \sum_{m=-\infty}^{+\infty} a_{m-1}q^{2m-1}x^m$, from which we deduce that: $a_m = a_{m-1}q^{2m-1}$, from which:

$a_1 = a_0 q$

$a_2 = a_1 q^3 = a_0 q\ q^3 = a_0 q^4 = a_0 q^{2^2}$

$a_3 = a_2 q^5 = a_0 q^{2^2} q^5 = a_0 q^9 = a_0 q^{3^2}$

………………………………

$a_m = a_0 q^{m^2}$

so $G(\sqrt{x}) = \sum_{m=-\infty}^{+\infty} a_m x^m = a_0 \sum_{m=-\infty}^{+\infty} q^{m^2} x^m$ (E.11)

and from the last one we have that for $\sqrt{x}=1$, that is for x=1:

$G(1) = a_0 \sum_{m=-\infty}^{+\infty} q^{m^2} 1^m = a_0 \sum_{m=-\infty}^{+\infty} q^{m^2}$ and when m=0 → $q^{m^2} = q^{0^2} = 1$ and so a_0 is together with no q, or rather it is with a q^0 (=1), that is, in the end, no q; but we also notice that:

$$F(\sqrt{x}=1)=F(1)=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})\Big|_{x=1} =\prod_{n=1}^{\infty}(1+q^{2n-1})(1+q^{2n-1})=\prod_{n=1}^{\infty}(1+q^{2n-1})^2 \quad \text{and:}$$

$$G(\sqrt{x}=1)=G(1)=F(1)\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1+q^{2n-1})^2\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n})$$

and the argument in the last right-hand block of the last one is:

$$(1+q^{2n-1})^2(1-q^{2n})=(1+q^{2(2n-1)}+2q^{2n-1})(1-q^{2n})=1+q^{2(2n-1)}+2q^{2n-1}-q^{2n}(1+q^{2(2n-1)}+2q^{2n-1})=$$
$$=1+q^{2(2n-1)}+2q^{2n-1}-q^{2n}-q^{2n}q^{2(2n-1)}-q^{2n}2q^{2n-1}=1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1}=$$
$$=1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} \quad \text{e:}$$

with n=1 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1}=1+q^2+2q\ -q^2-q^4-2q^3$

with n=2 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1}=1+q^6+2q^3-q^4-q^{10}-2q^7$

with n=3 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1}=1+q^{10}+2q^5-q^6-q^{16}-2q^{11}$

....

therefore, the product of the first two, that is:

$(1+q^2+2q\ -q^2-q^4-2q^3)(1+q^6+2q^3-q^4-q^{10}-2q^7)=1+......=1\cdot q^0+......$

yields indeed, as a numerical term (that is the one together with q^0) the number 1 and so the (E.2), (E.5) and (E.11)-with a_0=1, tell us that:

$$F(\sqrt{x})=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})$$

$$G(\sqrt{x})=F(\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$

$$G(\sqrt{x})=\sum_{m=-\infty}^{+\infty}q^{m^2}x^m$$

from which:

$$\sum_{m=-\infty}^{+\infty}q^{m^2}x^m=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$, that is exactly the (E.1) qed.

APPENDIX F: The Jacobi Theta Functions.

Here they are:

$$\theta_1(q)=0\ ,$$
$$\theta_2(q)=......+q^{(-\frac{3}{2})^2}+q^{(-\frac{1}{2})^2}+q^{(\frac{1}{2})^2}+q^{(\frac{3}{2})^2}+q^{(\frac{5}{2})^2}+.........\ ,$$
$$\theta_3(q)=......+q^{(-2)^2}+q^{(-1)^2}+1+q^{1^2}+q^{2^2}+q^{3^2}+.........\ , \qquad \text{(F.1)}$$
$$\theta_4(q)=......+q^{(-2)^2}-q^{(-1)^2}+1-q^{1^2}+q^{2^2}-q^{3^2}+.........\ .$$

Relationships among the Jacobi Theta Functions and the Elliptic Integral of the First Kind K(k):
by setting:

$$q=e^{-\pi\frac{K(k')}{K(k)}}$$ (q is called "nome", k is the parameter, which we already defined) , (F.2)

it follows that:

$$k=\frac{\theta_2^2(q)}{\theta_3^2(q)}\ , \qquad \text{(F.3)}$$

$$k'=\sqrt{1-k^2}=\frac{\theta_4^2(q)}{\theta_3^2(q)}=\frac{\theta_3^2(-q)}{\theta_3^2(q)} \qquad \text{(F.4)}$$

$(\to kk' = \dfrac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)})$ and (F.5)

$$K(k) = \frac{\pi}{2}\theta_3^2(q)\ ; \tag{F.6}$$

modular function $\lambda(q) = k^2(q) = [\dfrac{\theta_2(q)}{\theta_3(q)}]^4$. (F.7)

-<u>Proofs</u>:

first of all, we can write the Jacobi Theta Functions also in the following way:

$$\theta_2(q) = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n})\ , \tag{F.8}$$

$$\theta_3(q) = \prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n})\ , \tag{F.9}$$

$$\theta_4(q) = \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n})\ ; \tag{F.10}$$

in fact, by using the Jacobi Triple Product (see the Appendix E , eq. (E.1)):

$$\sum_{n=-\infty}^{\infty}x^n q^{n^2} = \prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$

and setting x=1 , we really get $\theta_3(q)$:

$$\theta_3(q) = \sum_{n=-\infty}^{\infty}1^n q^{n^2} = \prod_{n=1}^{\infty}(1+1q^{2n-1})(1+1^{-1}q^{2n-1})(1-q^{2n}) = \prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n})\ ;$$

conversely, setting x=-1 , we get $\theta_4(q)$:

$$\theta_4(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1})(1-q^{2n})\ .$$

At last, by setting x=q:

$\displaystyle\sum_{n=-\infty}^{\infty}q^n q^{n^2} = \prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n})$ and by multiplying side to side by $q^{\frac{1}{4}}$:

$\displaystyle q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^n q^{n^2} = q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n})$, that is:

$$\theta_2(q) = \sum_{n=-\infty}^{\infty}q^{\frac{1}{4}}q^n q^{n^2} = \sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2} = q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n}) =$$

$$= q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n-2})(1-q^{2n}) = q^{\frac{1}{4}}[(1+q^2)(1+q^0)(1-q^2)][(1+q^4)(1+q^2)(1-q^4)]\cdot$$

$$\cdot[(1+q^6)(1+q^4)(1-q^6)]\ldots = q^{\frac{1}{4}}[(1+q^2)\underline{2}(1-q^2)][(1+q^4)\underline{(1+q^2)}(1-q^4)][(1+q^6)\underline{(1+q^4)}(1-q^6)]\underline{\ldots} =$$

$$= \underline{2}q^{\frac{1}{4}}[(1+q^2)\underline{(1+q^2)}(1-q^2)][(1+q^4)\underline{(1+q^4)}(1-q^4)][(1+q^6)\underline{(1+q^6)}(1-q^6)]\ldots =$$

$$= 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n})(1-q^{2n}) = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n}) = \theta_2(q)\ .$$

So, by summing things up, we also have that:

$$\theta_2(q)=\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} \tag{F.11}$$

$$\theta_3(q)=\sum_{n=-\infty}^{\infty} q^{n^2} \tag{F.12}$$

$$\theta_4(q)=\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \tag{F.13}$$

And we have: $\theta_4^2(q)=\theta_3^2(-q)$. (F.14)

And now we remind that:

$$\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1})=1 \tag{F.15}$$

in fact, $(1+q^{2n})$ has an exponent 2n and 2n are even numbers and nothing else, while $(1+q^{2n-1})$ has an exponent 2n-1 and 2n-1 are odd numbers; therefore, the expression which includes bot evens and odds is $(1+q^n)$, so:

$\prod_{n=1}^{\infty}[\underline{(1+q^{2n})(1+q^{2n-1})}](1-q^{2n-1})=\prod_{n=1}^{\infty}[\underline{(1+q^n)}](1-q^{2n-1})$, but also in this last case we can say that $(1-q^{2n-1})$ has an odd exponent and so, for what we have just said, it can be expressed as $(1-q^n)/(1-q^{2n})$, from which:

$$\prod_{n=1}^{\infty}[(1+q^n)](1-q^{2n-1})=\prod_{n=1}^{\infty}[(1+q^n)]\frac{(1-q^n)}{(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{(1+q^n)(1-q^n)}{(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n})}=1 \quad \text{qed}$$

APPENDIX G: On the invariants of Ramanujan.

By defining the invariants: $G=G_N=(\frac{1}{2kk'})^{\frac{1}{12}}$ and $g=g_N=(\frac{k'^2}{2k})^{\frac{1}{12}}$, (G.1)

(invariants as they depend on fixed parameters k and k' ; then, we can use N or n…)
we have that:

$$\prod_{k=1,3,5,...}^{\infty}(1+e^{-k\pi\sqrt{n}})=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}G_N \quad \text{e} \tag{G.2}$$

$$\prod_{k=1,3,5,...}^{\infty}(1-e^{-k\pi\sqrt{n}})=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N \tag{G.3}$$

in fact, with $q=e^{-k\pi\sqrt{n}}$, these two products are the same as:

$(1+q)(1+q^3)(1+q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}}$ and $(1-q)(1-q^3)(1-q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}}$, because, for what the former is concerned, and also by reminding that:

$K(k)=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, $k'=\sqrt{1-k^2}$, and also:

$q=e^{-\pi\frac{K(k')}{K(k)}}$ $(\frac{K'}{K}=\sqrt{n}=\sqrt{N}\equiv\sqrt{r})$ (our personal choices we took at the beginning, in the reasonings we carried out right after the (1.4)…)

and the already proved $kk' = \dfrac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)}$ (see Appendix F , eq. (F.5)), then, by making explicit the last one:

$$kk' = \frac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n})^{2}\prod_{n=1}^{\infty}(1-q^{2n-1})^4(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1-q^{2n})^4} =$$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\cancel{(1-q^{2n})^2}(1-q^{2n-1})^4\cancel{(1-q^{2n})^2}}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8\cancel{(1-q^{2n})^4}} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8} =$$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8}\,\frac{(1+q^{2n-1})^4}{(1+q^{2n-1})^4} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}[(1+q^{2n})^4(1+q^{2n-1})^4](1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4}$$, but:

$[(1+q^{2n})^4(1+q^{2n-1})^4] = (1+q^n)^4$ because of the matter of the even and odd exponents (we already met) involved by the only n, from which:

$$kk' = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4}$$ and that, for the same reason, we also have that:

$(1-q^{2n-1})^4 = \dfrac{(1-q^n)^4}{(1-q^{2n})^4}$, then: $kk' = \dfrac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4[\frac{(1-q^n)^4}{(1-q^{2n})^4}]}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4} =$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^n)^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4(1-q^{2n})^4} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^{2n})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4(1-q^{2n})^4} =$$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}1}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4} = \frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \frac{kk'}{4q^{\frac{1}{2}}} = \frac{1}{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \frac{4q^{\frac{1}{2}}}{kk'} = \prod_{n=1}^{\infty}(1+q^{2n-1})^{12} \rightarrow$$

$\rightarrow \sqrt[12]{\dfrac{4q^{\frac{1}{2}}}{kk'}} = \sqrt[12]{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \dfrac{2^{\frac{1}{6}}q^{\frac{1}{24}}}{k^{\frac{1}{12}}k'^{\frac{1}{12}}} = \prod_{n=1}^{\infty}(1+q^{2n-1}) = (1+q)(1+q^3)(1+q^5)......$, from which the assumption is matched, that is: $(1+q)(1+q^3)(1+q^5)...... = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}}$ qed .

A similar reasoning stands for the latter, as we prove it here below.

In the meantime, through the following definitions:

$$G = G_N = (\frac{1}{2kk'})^{\frac{1}{12}} \quad , \quad g = g_N = (\frac{k'^2}{2k})^{\frac{1}{12}} \rightarrow k = g^6\sqrt{g^{12}+\frac{1}{g^{12}}} - g^{12} \quad , \quad k' = \sqrt{2k}\,g^6 \quad , \qquad \text{(G.4)}$$

we have: $(1+q)(1+q^3)(1+q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}}=2^{\frac{1}{6}}e^{\frac{-\pi\sqrt{n}}{24}}[2^{\frac{1}{12}}G_N]=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}G_N$

$(1-q)(1-q^3)(1-q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}}=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N$

More specifically, we said, regarding the latter, that being still: $k=\frac{\theta_2^2(q)}{\theta_3^2(q)}$, $k'=\frac{\theta_4^2(q)}{\theta_3^2(q)}$ and $\theta_4^2(q)=\theta_3^2(-q)$, it follows that: $\frac{k}{k'^2}=\frac{\frac{\theta_2^2(q)}{\theta_3^2(q)}}{[\frac{\theta_4^2(q)}{\theta_3^2(q)}]^2}=\frac{\theta_2^2(q)\theta_3^2(q)}{\theta_4^4(q)}=\frac{\theta_2^2(q)\theta_3^2(q)}{\theta_3^4(-q)}$ and by making this explicit: $\frac{k}{k'^2}=\frac{\theta_2^2(q)\theta_3^2(q)}{\theta_3^4(-q)}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n})^2\prod_{n=1}^{\infty}(1+q^{2n-1})^4(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8(1-q^{2n})^4}=$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\prod_{n=1}^{\infty}(1+q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\prod_{n=1}^{\infty}(1+q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}\frac{(1+q^{2n-1})^4}{(1+q^{2n-1})^4}=$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{n})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{n})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}\frac{(1-q^{2n-1})^4}{(1-q^{2n-1})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{n})^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}=$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{n})^4(1-q^{n})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}(1-q^{2n})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^{2n})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}(1-q^{2n})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}1}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}=\frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}$$, that is:

$$\frac{k}{k'^2}=\frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}\rightarrow\frac{k'^2}{k}4q^{\frac{1}{2}}=\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}\rightarrow$$

$\rightarrow(\frac{k'^2}{k})^{\frac{1}{12}}2^{\frac{1}{6}}q^{\frac{1}{24}}=\prod_{n=1}^{\infty}(1-q^{2n-1})=(1-q)(1-q^3)(1-q^5)......$, from which we get the assumption, that is: $(1-q)(1-q^3)(1-q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}}=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N$ qed.

By summing things up, from the definitions (G.1) at the beginning of the Appendix, that is from: $G=G_N=(\frac{1}{2kk'})^{\frac{1}{12}}$ and $g=g_N=(\frac{k'^2}{2k})^{\frac{1}{12}}$, with an N=58 (a free choice, but not complulsory, which will lead us to very useful results) and by using the arbitrary choices (also useful, as well)

$\frac{K'}{K}=\sqrt{r}=\sqrt{N}=\sqrt{58}=\frac{K'_{58}}{K_{58}}=\boxed{\frac{K(\sqrt{1-k_{58}^2})}{K(k_{58})}=\sqrt{58}}$, and also the $q=e^{-\pi\frac{K(k')}{K(k)}}=e^{-\pi\sqrt{N}}$ which was a choice from the beginning, as well, it follows a k_{58} (equal to $k_{58}=0{,}0000255482707576$), that is the (1.41), that we can evaluate here in the following way:

we remind the $k = \frac{\theta_2^2(q)}{\theta_3^2(q)}$ just proved right in the above lines, and also the (F.11) and (F.12): $\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$ and $\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ and also the $q_{58} = e^{-\pi\sqrt{58}} = e^{-23,9249512120} = 4,0794633669 \cdot 10^{-11}$, from which:

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{[\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}]^2}{[\sum_{n=-\infty}^{\infty} q^{n^2}]^2} = \frac{[\ldots + q^{(-3+\frac{1}{2})^2} + q^{(-2+\frac{1}{2})^2} + q^{(-1+\frac{1}{2})^2} + q^{(0+\frac{1}{2})^2} + q^{(1+\frac{1}{2})^2} + q^{(2+\frac{1}{2})^2} + \ldots]^2}{[\ldots + q^{(-3)^2} + q^{(-2)^2} + q^{(-1)^2} + q^{0^2} + q^{1^2} + q^{2^2} + q^{3^2} + \ldots]^2} =$$

$$= \frac{[\ldots + q^{(-3+\frac{1}{2})^2} + q^{(-2+\frac{1}{2})^2} + q^{(-1+\frac{1}{2})^2} + q^{(0+\frac{1}{2})^2} + q^{(1+\frac{1}{2})^2} + q^{(2+\frac{1}{2})^2} + \ldots]^2}{[\ldots + q^{(-3)^2} + q^{(-2)^2} + q^{(-1)^2} + q^{0^2} + q^{1^2} + q^{2^2} + q^{3^2} + \ldots]^2} = \frac{[\ldots + q^{\frac{25}{4}} + q^{\frac{9}{4}} + q^{\frac{1}{4}} + q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \ldots]^2}{[\ldots + q^9 + q^4 + +q + 1 + q + q^4 + q^9 + \ldots]^2} =$$

$$= \frac{[2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \ldots]^2}{[1 + 2q + 2q^4 + 2q^9 + \ldots]^2} = \frac{4[q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \ldots]^2}{4[\frac{1}{2} + q + q^4 + q^9 + \ldots]^2} = \frac{[q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \ldots]^2}{[\frac{1}{2} + q + q^4 + q^9 + \ldots]^2}$$ and then:

$$k_{58} = \frac{[(4,0794633669 \cdot 10^{-11})^{\frac{1}{4}} + (4,0794633669 \cdot 10^{-11})^{\frac{9}{4}} + (4,0794633669 \cdot 10^{-11})^{\frac{25}{4}} + \ldots]^2}{[\frac{1}{2} + (4,0794633669 \cdot 10^{-11}) + (4,0794633669 \cdot 10^{-11})^4 + (4,0794633669 \cdot 10^{-11})^9 + \ldots]^2} \cong$$

$$\cong \frac{[(4,0794633669 \cdot 10^{-11})^{\frac{1}{4}} + \sim 0 + \sim 0 + \ldots]^2}{[\frac{1}{2} + (4,0794633669 \cdot 10^{-11}) + \sim 0 + \sim 0 + \ldots]^2} \cong \frac{[(4,0794633669 \cdot 10^{-11})^{\frac{1}{4}}]^2}{[\frac{1}{2} + (4,0794633669 \cdot 10^{-11})]^2} = \frac{\sqrt{4,0794633669 \cdot 10^{-11}}}{\frac{1}{4}} =$$

$$= 4\sqrt{4,0794633669 \cdot 10^{-11}} = 4\sqrt{40,794633669} \cdot 10^{-6} = 25,5482707576 \cdot 10^{-6} = 0,0000255482707576\,!$$

that is: $k_{58} = 0,0000255482707576$. (accredited value: $k_{58} = (\sqrt{2}-1)^6 (13\sqrt{58} - 99)$)

At this point, once we have determined the k_{58} , we have, in a consequential way:

$g = g_{58} = (\frac{k'^2_{58}}{2k_{58}})^{\frac{1}{12}} = (\frac{1-k^2_{58}}{2k_{58}})^{\frac{1}{12}} = 2,2784213480$ (accredited value: $g_{58} = (\frac{2}{\sqrt{29}-5})^{1/2} \cong 2,279$),

that is the (1.42), and also: $\frac{g_{58}^{12} + g_{58}^{-12}}{2} = 9785,3981139349$ (accredited value: $9801 = (\frac{1}{x_{58}})$), that is the (1.40) (please notice that if in our counts, carried out here, we used more decimal figures after the point, instead of using, as an example, just 2,718 as "e", then we wouldn't have stopped just at the simple value 9785,3981139349 here presented, but we would have easily obtained the accredited value of $9.800,99997449 \cong 9801$), and also: $\frac{1}{g_{58}^{12}} = \frac{2k_{58}}{1-k_{58}^2} = 0,00005109654154$ (accredited value: $\frac{1}{g_{58}^{12}} = (\frac{\sqrt{29}-5}{2})^6$), that is the (1.44); then, the: $g_{58}^6 = 139,8956617510$ (accredited value: $g_{58}^6 = (\frac{2}{\sqrt{29}-5})^3$), which is the (1.45), the $\frac{(g_{58}^{12} - g_{58}^{-12})}{2} = 9785,3981139349$ (accredited value: 9801), which is the (1.46) and, as a consequence of the (1.5), that is of the $\alpha(58) = \frac{\pi}{(2K_{58})^2} - \sqrt{58}(\frac{E_{58}}{K_{58}} - 1)$, at last we have, as an example, by using the values from tables (or from the (C.8) and (C.9)) of K_{58} and E_{58} , (but also through different ways), the

$\alpha(58) \cong 0{,}3183192742$ (in the end, it is: $1/\pi$) (accredited value: $\alpha(58) = 3g_{58}^6 k_{58}(33\sqrt{29} - 148)$), which is the (1.43).

Instead, still regarding α(58), in a more elegant way, after involving the (H.11), that is:

$$\alpha(r) = \frac{\frac{1}{\pi} - \sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty} n^2(-1)^{n^2} e^{-\pi \cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty} (-1)^{n^2} e^{-\pi \cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty} e^{-\pi \cdot n^2\sqrt{r}})^4}$$, we have: $(e^{-\pi \cdot n^2\sqrt{58}} = e^{-23{,}9249512120 n^2})$

$$\alpha(58) = \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{\sum_{n=-\infty}^{\infty} n^2(-1)^{n^2} e^{-\pi \cdot n^2\sqrt{58}}}{\sum_{n=-\infty}^{\infty} (-1)^{n^2} e^{-\pi \cdot n^2\sqrt{58}}}}{(\sum_{n=-\infty}^{\infty} e^{-\pi \cdot n^2\sqrt{58}})^4} = \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{\sum_{n=-\infty}^{\infty} n^2(-1)^{n^2} e^{-23{,}9249512120 n^2}}{\sum_{n=-\infty}^{\infty} (-1)^{n^2} e^{-23{,}9249512120 n^2}}}{(\sum_{n=-\infty}^{\infty} e^{-23{,}9249512120 n^2})^4} =$$

$$= \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{[0 + 2\cdot 1^2(-1)^{1^2} e^{-23{,}9249512120\cdot 1^2} + 2\cdot 2^2(-1)^{2^2} e^{-23{,}9249512120\cdot 2^2} + \ldots]}{[1 + (-1)^{1^2} e^{-23{,}9249512120\cdot 1^2} + (-1)^{2^2} e^{-23{,}9249512120\cdot 2^2} + \ldots]}}{(1 + e^{-23{,}9249512120\cdot 1^2} + e^{-23{,}9249512120\cdot 2^2} + \ldots)^4} =$$

$$= \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{[-2e^{-23{,}9249512120} + 8e^{-95{,}6998048480} + \ldots]}{[1 - e^{-23{,}9249512120} + e^{-95{,}6998048480} + \ldots]}}{(1 + e^{-23{,}9249512120} + e^{-95{,}6998048480} + \ldots)^4} =$$

$$= \frac{\frac{1}{\pi} - 30{,}4630924234\frac{(-2\cdot 4{,}0794633669\cdot 10^{-11} + 8\cdot 2{,}7695687502\cdot 10^{-42} + \ldots)}{(1 - 4{,}0794633669\cdot 10^{-11} + 2{,}7695687502\cdot 10^{-42} + \ldots)}}{(1 + 4{,}0794633669\cdot 10^{-11} + 2{,}7695687502\cdot 10^{-42} + \ldots)^4} \cong$$

$$\cong \frac{\frac{1}{\pi} - 30{,}4630924234\frac{(-2\cdot 4{,}0794633669\cdot 10^{-11})}{1}}{1} \cong \frac{1}{\pi} + 248{,}5461391674\cdot 10^{-11} \cong \frac{1}{\pi} \cong 0{,}3183192742$$

that is: $\boxed{\alpha(58) \cong 0{,}3183192742\ .}$

APPENDIX H: On the convergence to 1/π of the function α(k).

(in order to prove this convergence, we use the first raw method and then the second fine method)

-first raw method:

now, regarding the convergence to $\frac{1}{\pi}$ of $\alpha(r)$ (the (1.4) which follows the (1.3)), we initially show it through a raw method, based on a pure calculation by means of the expansions of the elliptic integrals K and E; on this purpose, we remind that:

$$\alpha(r) = \frac{E'(k)}{K(k)} - \frac{\pi}{[2K(k)]^2}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$ (Elliptic Integral of the First Kind-Appendix C)

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\cdot d\theta$$ (Elliptic Integral of the Second Kind -Appendix C)

$K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2}$ (Fundamental Legendre's Identity-Appendix D)

$k'=\sqrt{1-k^2}$ (parameters k and k') .

-So, we have already seen that, in case of $k=k'=\frac{1}{\sqrt{2}}$ $\rightarrow$ $\boxed{\alpha(\frac{1}{\sqrt{2}})=0{,}5\sim 1/\pi}$ (see the (1.3)).

-If k=1 (that is a limit value) → (K(1)= ∞ and E(1)=1) (and so k'=0) →

$$K(1)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\sin^2\theta}}=\int_0^{\pi/2}\frac{d\theta}{\cos\theta}=\ln[\tan(\frac{\pi}{4}+\frac{\theta}{2})]\Big|_0^{\pi/2}=\ln[\tan(\frac{\pi}{4}+\frac{\pi}{4})]-\ln[\tan(\frac{\pi}{4})]=$$

$$=\ln\tan\frac{\pi}{2}-\ln\tan\frac{\pi}{4}=\ln\frac{\tan\frac{\pi}{2}}{\tan\frac{\pi}{4}}=\infty$$

$E(1)=\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\cdot d\theta=\int_0^{\pi/2}\cos\theta d\theta=\sin\theta\Big|_0^{\pi/2}=1$, and then:

$$K'(1)=K(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

$$E'(1)=E(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

from which: $\boxed{\alpha(1)=\frac{E'(1)}{K(1)}-\frac{\pi}{[2K(1)]^2}=\frac{\frac{\pi}{2}}{\infty}-\frac{\pi}{[2\infty]^2}=0}$, and in fact such a result doesn't match $\frac{1}{\pi}$.

-Conversely, if k=0 (K(0)= π/2 and E(0)= π/2) (and so k'=1) →

$$K(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

$E(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$, and then:

$$K'(0)=K(1)=\infty$$

$E'(0)=E(1)=1$ and so: $\boxed{\alpha(0)=\frac{E'(0)}{K(0)}-\frac{\pi}{[2K(0)]^2}=\frac{1}{\frac{\pi}{2}}-\frac{\pi}{[2\frac{\pi}{2}]^2}=\frac{2}{\pi}-\frac{1}{\pi}=\frac{1}{\pi}}$ (ok!).

-If k=0,8 (and so $k'=\sqrt{1-(0{,}8)^2}=0{,}6$) and after considering the (C.6) and the (C.7) , that are:

$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$ and $E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$, it follows that:

$$K(0{,}8)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}8)^{2n}\cong\frac{\pi}{2}\{[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}8)^{2\cdot 0}+[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}8)^{2\cdot 1}+[\frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}]^2(0{,}8)^{2\cdot 2}+...\}\cong$$

$\cong 1{,}57\{1+[\frac{1}{4}\frac{2}{1}]^2(0{,}8)^2+[\frac{1}{4^2}\frac{(4)!}{2^2}]^2(0{,}8)^4+...\}\cong 1{,}57(1+0{,}16+0{,}05759+...)\cong\boxed{1{,}912}$, or by using the expression given in the Appendix A (see the (A.9)):

$$(K=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}=\frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...])$$

it follows that: $K(0{,}8)=\frac{\pi}{2}[1+(\frac{1}{2})^2(0{,}8)^2+(\frac{1}{2}\frac{3}{4})^2(0{,}8)^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2(0{,}8)^6+...]\cong$

$$\cong 1{,}57[1+\frac{0{,}64}{4}+\frac{9}{64}0{,}4096+\frac{225}{2304}0{,}2621+...]=1{,}57[1+0{,}16+0{,}0576+0{,}0256+...]\cong\boxed{1{,}952}$$

As we don't have plenty of time to carry out too long calculations and use too many terms, we will consider as a good value the following one from the table: $\boxed{1{,}995.}$

Then: ($E(0{,}8)\boxed{\cong 1{,}276}$, a table value), but by carrying out some calculations:

$$E(0{,}8)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}8)^{2n}=$$

$$=\frac{\pi}{2}[(-\frac{1}{2\cdot 0-1})[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}8)^{2\cdot 0}+(-\frac{1}{2\cdot 1-1})[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}8)^{2\cdot 1}+(-\frac{1}{2\cdot 2-1})[\frac{1}{4^2}\frac{(4)!}{(2!)^2}]^2(0{,}8)^{2\cdot 2}+...]=$$

$$=1{,}57[1-\frac{1}{4}0{,}64-0{,}0469\cdot 0{,}4096+...]=1{,}57[1-0{,}16-0{,}0192+...]\cong\boxed{1{,}289}$$

and then: (a table value: $\boxed{1{,}751}$)

$$K'(0{,}8)=K(0{,}6)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}6)^{2n}\cong$$

$$\cong\frac{\pi}{2}\{[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}6)^{2\cdot 0}+[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}6)^{2\cdot 1}+[\frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}]^2(0{,}6)^{2\cdot 2}+...\}\cong$$

$$\cong 1{,}57\{1+[\frac{1}{4}\frac{2}{1}]^2(0{,}6)^2+[\frac{1}{4^2}\frac{(4)!}{2^2}]^2(0{,}6)^4+...\}\cong 1{,}57\{1+\frac{1}{4}0{,}36+[\frac{3}{8}]^2 0{,}1296+...\}\cong$$

$\cong 1{,}57(1+0{,}09+0{,}0182+...)\cong\boxed{1{,}74}$, or by using the (A.9) in the Appendix A:

$$(K=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}=\frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...])$$

$$K'(0{,}8)=K(0{,}6)=\frac{\pi}{2}[1+(\frac{1}{2})^2(0{,}6)^2+(\frac{1}{2}\frac{3}{4})^2(0{,}6)^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2(0{,}6)^6+...]\cong$$

$$\cong 1{,}57[1+\frac{0{,}36}{4}+\frac{9}{64}0{,}1296+\frac{225}{2304}0{,}0466+...]=1{,}57[1+0{,}09+0{,}0182+0{,}00455+...]\cong\boxed{1{,}747}\ ;$$

then: (a table value: $\boxed{1{,}418}$) $\quad E'(0{,}8)=E(0{,}6)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}6)^{2n}=$

$$=\frac{\pi}{2}[(-\frac{1}{2\cdot 0-1})[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}6)^{2\cdot 0}+(-\frac{1}{2\cdot 1-1})[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}6)^{2\cdot 1}+(-\frac{1}{2\cdot 2-1})[\frac{1}{4^2}\frac{(4)!}{(2!)^2}]^2(0{,}6)^{2\cdot 2}+...]=$$

$$=1{,}57[1-\frac{1}{4}0{,}36-0{,}0469\cdot 0{,}1296+...]=1{,}57[1-0{,}09-0{,}00607+...]\cong\boxed{1{,}419}$$

and, finally:

$$\boxed{\alpha(0{,}8)=\frac{E'(0{,}8)}{K(0{,}8)}-\frac{\pi}{[2K(0{,}8)]^2}=\frac{E(0{,}6)}{K(0{,}8)}-\frac{\pi}{[2K(0{,}8)]^2}=\frac{1{,}418}{1{,}995}-\frac{\pi}{[2\cdot 1{,}995]^2}=0{,}711-0{,}197=0{,}514}$$

or the inverse, which is 1,946 , which is obviously worse, in the comparison with 1/π, with respect to the case of k=0,707.

-Conversely, by staying closer to the zero:

if k=0,2 (and so $k'=\sqrt{1-(0,2)^2}\cong 0{,}98$) $\rightarrow$: (a table value: $\boxed{1{,}587}$)

$K(0{,}2)\cong\frac{\pi}{2}\sum_{n=0}^{2}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}2)^{2n}\cong\boxed{1{,}586}$ and:

(a table value: $\boxed{1{,}555}$)

$E(0{,}2)\cong\frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}2)^{2n}\cong\boxed{1{,}554}$

and: (a table value: $\boxed{1{,}05}$)

$E'(0{,}2)=E(0{,}98)\cong\frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}98)^{2n}\cong\boxed{1{,}125}$ and finally:

$$\boxed{\alpha(0{,}2)=\frac{E'(0{,}2)}{K(0{,}2)}-\frac{\pi}{[2K(0{,}2)]^2}=\frac{E(0{,}98)}{K(0{,}2)}-\frac{\pi}{[2K(0{,}2)]^2}=\frac{1{,}05}{1{,}587}-\frac{\pi}{[2\cdot 1{,}587]^2}=0{,}6616-0{,}312=0{,}3117}$$

whose inverse is 3,21, a bit closer to π.

-And by staying even closer to zero:

if k=0,1 (and so $k'=\sqrt{1-(0{,}1)^2}\cong 0{,}995$) $\rightarrow$ (a table value: $\boxed{1{,}575}$)

$K(0{,}1)\cong\frac{\pi}{2}\sum_{n=0}^{2}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}1)^{2n}\cong\boxed{1{,}574}$ and:

(a table value: $\boxed{1{,}567}$) $E(0{,}1)\cong\frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}1)^{2n}\cong\boxed{1{,}566}$

and then: (a table value: $\boxed{\sim 1{,}004}$)

$E'(0{,}1)=E(0{,}995)\cong\frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}995)^{2n}\cong\boxed{1{,}11}$

and finally:

$$\boxed{\alpha(0{,}1)=\frac{E'(0{,}1)}{K(0{,}1)}-\frac{\pi}{[2K(0{,}1)]^2}=\frac{E(0{,}995)}{K(0{,}1)}-\frac{\pi}{[2K(0{,}1)]^2}=\frac{1{,}004}{1{,}575}-\frac{\pi}{[2\cdot 1{,}575]^2}=0{,}6375-0{,}3164=0{,}3211}$$

whose inverse is 3,114 , so closer to π.

-<u>second fine method</u>:

we predicted, right after the (1.5), that:

$$\frac{1}{\pi}=\theta_3^4\alpha(r)+\sqrt{r}\,4q\frac{\dot{\theta}_4}{\theta_4}, \tag{H.1}$$

as the following one was standing:

$$\frac{4K}{\pi^2}(E-K) = 4q\frac{d}{dq}\ln\theta_4 = 4q\frac{\dot{\theta}_4}{\theta_4}, \qquad \text{(H.2)}$$

(logarithmic derivative of the Jacobi Theta function) whose proof is here reported:

by comparing the (F.4) ($k' = \sqrt{1-k^2} = \frac{\theta_4^2(q)}{\theta_3^2(q)}$) to the (F.6) ($K(k) = \frac{\pi}{2}\theta_3^2(q)$) we get: $\theta_4(q) = \sqrt{\frac{2Kk'}{\pi}}$ and after taking its logarithm, we get:

$$\ln\theta_4(q) = \ln\left(\frac{2Kk'}{\pi}\right)^{1/2} = \frac{1}{2}\ln\frac{2}{\pi} + \frac{1}{2}\ln K + \frac{1}{2}\ln k' \qquad \text{(H.3)}$$

and after recalling the equation (F.2) of the modular nome q: $q = e^{-\pi\frac{K(k')}{K(k)}} = e^{-\pi\frac{K'}{K}}$, we evaluate its derivative:

$$\frac{dq}{dk} = -\pi e^{-\pi\frac{K'}{K}}\frac{d}{dk}\frac{K'}{K} = -\pi e^{-\pi\frac{K'}{K}}\left(K'\frac{d}{dk}K^{-1} + K^{-1}\frac{dK'}{dk}\right) = \frac{\pi q}{K^2}\left(K'\frac{dK}{dk} - K\frac{dK'}{dk}\right), \qquad \text{(H.4)}$$

but from the (D.8) we have: $(K'\frac{dK}{dk} - K\frac{dK'}{dk}) = \frac{\pi}{2}\frac{1}{kk'^2}$, and if we put this one in the (H.4), we get:

$$\frac{dq}{dk} = \frac{\pi q}{K^2}\left(K'\frac{dK}{dk} - K\frac{dK'}{dk}\right) = \frac{\pi^2 q}{2kk'^2K^2}\ . \qquad \text{(H.5)}$$

Now, from the (H.5) we obviously have:

$$q\frac{d}{dq} = \frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\ ; \qquad \text{(H.6)}$$

now, according to the (D.2), we have: $\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2}$, while for the (D.4) we have: $\frac{dk'}{dk} = -\frac{k}{k'}$,

from which, for the (H.3): $4q\frac{d}{dq}\ln\theta_4 = 4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\ln\theta_4 = 4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\ln\left(\frac{2Kk'}{\pi}\right)^{1/2} =$

$$= 4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\left(\frac{1}{2}\ln\frac{2}{\pi}\right) + 4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\left(\frac{1}{2}\ln K\right) + 4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\left(\frac{1}{2}\ln k'\right) =$$

$$= 0 + 4\frac{kk'^2K}{\pi^2}\frac{dK}{dk} + 4\frac{kk'K^2}{\pi^2}\frac{dk'}{dk} = 4\frac{kk'^2K}{\pi^2}\frac{(E-k'^2K)}{kk'^2} - 4\frac{k^2K^2}{\pi^2} = \frac{4K}{\pi^2}(E-k'^2K) - 4\frac{k^2K^2}{\pi^2} \qquad \text{(H.7)}$$

and being that: $k'^2 = 1-k^2$, it follows that: $k'^2K = K - Kk^2$ and so the (H.7) yields:

$4q\frac{d}{dq}\ln\theta_4 = \frac{4K}{\pi^2}(E-k'^2K) - 4\frac{k^2K^2}{\pi^2} = \frac{4K}{\pi^2}(E-K+Kk^2) - 4\frac{k^2K^2}{\pi^2} = \frac{4K}{\pi^2}(E-K)$, that is:

$4q\frac{d}{dq}\ln\theta_4 = \frac{4K}{\pi^2}(E-K)$, which is nothing but the (H.2)! qed.

Now, from the (H.1) we get:

$$\alpha(r) = \frac{\frac{1}{\pi} - \sqrt{r}\,4q\frac{\dot{\theta}_4}{\theta_4}}{\theta_3^4} \qquad \text{(H.8)}$$

and by reminding that from a comparison between the (F.9) and the (F.10), it turns out that $\theta_4(q) = \theta_3(-q)$, that is: $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \sum_{n=-\infty}^{\infty}(-q)^{n^2}$ and by also considering the expressions found in the Appendix F for all the θ_i ((F.11)---(F.14)), we have:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\frac{d}{dq}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\frac{d}{dq}\sum_{n=-\infty}^{\infty}(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=$$

$$=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}[(-1)^{n^2}(q)^{n^2}]}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}(q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}n^2q^{n^2-1}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2q^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=$$

$$=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}\text{, so:}\qquad\text{(H.9)}$$

$$\boxed{\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}}\ .\qquad\text{(H.10)}$$

Now, due to the $q=e^{-\pi\frac{K'}{K}}=e^{-\pi\sqrt{r}}$ just recalled, we have:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{r}})^4}\qquad\text{(H.11)}$$

and due to the fact that when r→∞ , then $e^{-\pi\sqrt{r}}$ drops to zero very fast (**from which comes the high effectiveness of the Ramanujan's Formula…**), then:

$$\lim_{r\to\infty}\alpha(r)=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{r}})^4}=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{[\sum_{n=-\infty,n\neq o}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}]}{[\sum_{n=-\infty,n\neq o}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}]}}{(\sum_{n=-\infty,n\neq o}^{\infty}e^{-\pi\cdot n^2\sqrt{r}}+e^{-\pi\cdot 0^2\sqrt{r}})^4}=$$

$$=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{[\sum_{n=-\infty,n\neq o}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+0]}{[\sum_{n=-\infty,n\neq o}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+1]}}{(\sum_{n=-\infty,n\neq o}^{\infty}e^{-\pi\cdot n^2\sqrt{r}}+1)^4}\to\frac{\frac{1}{\pi}-\sqrt{\infty(slow)}\cdot 4\frac{[0(fast)+0]}{[0+1]}}{(0+1)^4}\to\frac{1}{\pi}\quad !!! \qquad \text{(H.12)}$$

Now, if we also want to have a real evaluation of the **rapidity of convergence** of α(∞) to 1/π, then we start back from the (H.10) and remind that in the limit of r→∞ , we have that q→0 and so:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{0+2\sum_{n=1}^{\infty}n^2(-q)^{n^2}}{1+2\sum_{n=1}^{\infty}(-q)^{n^2}}}{(1+2\sum_{n=1}^{\infty}q^{n^2})^4}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{2(-q)^1+8(-q)^4+\ldots}{1+2(-q)^1+2(-q)^4+\ldots}}{[1+2q^1+2q^4+\ldots]^4}\cong$$

$$\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{(-2q+8q^4+\ldots)}{1-2q+2q^4+\ldots}}{[1+2q+\ldots]^4}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{(-2q+\ldots)}{1-2q+\ldots}}{(1+8q+\ldots)}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4(-2q+\ldots)(1+2q+\ldots)}{(1+8q+\ldots)}\cong$$

$$\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4(-2q-4q^2+\ldots)}{(1+8q+\ldots)}\cong[\frac{1}{\pi}-\sqrt{r}\,8(-q-2q^2+\ldots)](1-8q+\ldots)\cong$$

$$\cong\frac{1}{\pi}(1-8q+\ldots)-\sqrt{r}\,8(-q-2q^2+\ldots)(1-8q+\ldots)\cong\frac{1}{\pi}-\frac{8q}{\pi}-\sqrt{r}\,8(-q+\ldots)(1-8q+\ldots)\cong$$

$$\cong\frac{1}{\pi}-\frac{8q}{\pi}-\sqrt{r}\,8(-q+8q^2+\ldots)\cong\frac{1}{\pi}-\frac{8q}{\pi}+8q\sqrt{r}+\ldots\cong\frac{1}{\pi}+8(\sqrt{r}-\frac{1}{\pi})q$$, da cui:

$$\boxed{\alpha(r)-\frac{1}{\pi}\to 8(\sqrt{r}-\frac{1}{\pi})q=8(\sqrt{r}-\frac{1}{\pi})e^{-\pi\sqrt{r}}\to 0}\quad !!!\ \text{qed.} \qquad \text{(H.13)}$$

(a very fast tendency, as well as fast is the convergence of the Pi series of Ramanujan, as $e^{-\pi\sqrt{r}}$ overwhelms very fast $\sqrt{r}$, when r→∞)

APPENDIX I: On the Ramanujan's summation of all naturals (-1/12).

1 – The Riemann Zeta Function – definition:

$$\boxed{\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}} \tag{1.1}$$

obviuusly, out of reasons of convergence, s>1 (in case s is real) or Re(s)>1 in case s is complex. Nevertheless, we have values of $\zeta(s)$ also for s<1 and here we want to highlight the value: $\zeta(-1)=-\frac{1}{12}$, which is official for the Riemann Zeta Function!
The series (1.1) wouldn't converge for $s\leq 1$, but it is possible to extend the domain through other functions, as well as, for instance, the geometric series $S(x)=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+...$ converges only for $x\in]-1,+1[$, although, if we want to see this series as the Taylor expansion of $f(x)=\frac{1}{1-x}$, then S(x) can have its definition as extended over all the domain of f(x), which has values everywhere, except for x=1.
And the same happens to $\zeta(s)$.

2 – The relationship with prime numbers – Euler Product Formula:

We remind that: $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+...$; now, by multiplying both sides by $\frac{1}{2^s}$, we have:

$$\frac{1}{2^s}\zeta(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+...; \tag{2.1}$$

Now, by subtracting one from the other, we get:

$$\zeta(s)-\frac{1}{2^s}\zeta(s)=\zeta(s)(1-\frac{1}{2^s})=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{15^s}+...; \tag{2.2}$$

Now, we multiply both sides by $\frac{1}{3^s}$:

$$\frac{1}{3^s}(1-\frac{1}{2^s})\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+... \tag{2.3}$$

and by subtracting now the (2.3) from the (2.2), we get:

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{25^s}+... \tag{2.4}$$

By repeating such a method, let's multiply both sides of the (2.4) by $\frac{1}{5^s}$, from which:

$$\frac{1}{5^s}(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{35^s}+\frac{1}{55^s}+... \tag{2.5}$$

and again, by subtracting the (2.5) from the (2.4):

$$(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{29^s}+... \quad . \tag{2.6}$$

We can already see, in the (2.6), that the dominating term is the 1, to which smaller and smaller terms are added; and we can also notice that the right side of the (2.6) is less dense than that of the (2.4), which, as well, has got less heavy terms than that of the (2.2), so, after going on, at a certain point we will get to the unit and that's it:

$$.......(1-\frac{1}{13^s})(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1\ , \qquad (2.7)$$

and we notice that at the left side of the (2.7) all prime numbers show up. Therefore, by cropping $\zeta(s)$ out of the (2.7), we have:

$$\zeta(s)=\frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})(1-\frac{1}{7^s})(1-\frac{1}{11^s})(1-\frac{1}{13^s}).......}=\prod_p(1-p^{-s})^{-1}\ \text{, so:}$$

$$\boxed{\zeta(s)=\prod_p(1-p^{-s})^{-1}}\quad \text{(p prime number)} \qquad (2.8)$$

which is the **Euler Product Formula**, indeed, which links the Riemann Z to the prime numbers. And one also notice that prime numbers are infinite, although the proof by Euclid that primes are infinite always keeps on being the most fashinating:
Euclid's theorem is a fundamental proof in mathematics demonstrating that there is an infinite number of prime numbers. Established in 300 B.C., it uses a "proof by contradiction" to show that no matter how many primes you list, there will always be another one. How the Proof Works Euclid's brilliant logical argument follows these simple steps: The Assumption: Assume there are only a finite number of prime numbers in existence. The Construction: Multiply all these assumed primes together, so getting N and then add 1; The Contradiction: This new number cannot be evenly divided by any prime on your list, because dividing it will always leave a remainder of exactly 1. Therefore, such a number must either:
-Be a prime number itself (that wasn't on your list).
-Be a composite number divisible by a prime that was not included in your original list.
The Conclusion: In either case, your original list was incomplete. Because you can always find a new prime this way, the list of primes must go on forever.

Fake prime generators: let us consider the primes generated by the following formula: $p_n = n^2 - 79n + 1601$; well, for $0 \le n \le 79$ the p_n are all prime, but with n=80, already no more; in fact: $p_{80}=41^2$!!!!

Still regarding the relationship between the Riemann Zeta Function and the primes, let's name $\pi(N)$ the number of prime numbers one can meet in counting up to N: for N=1000, one encounters approximately 170 primes ($\pi(1000)\approx 170$); then: $\pi(10^6)\approx 80.000$, $\pi(10^9)\approx 51\cdot 10^6$, $\pi(10^{12})\approx 38\cdot 10^9$ and so on. Therefore, it seems that (**Prime Numbers Theorem**): $\pi(N)\approx \frac{N}{\ln N}$.

Moreover, let us consider the integral log function: $\int_0^x \frac{1}{\ln t}dt$ and let us define: $Li(x)=\int_2^x \frac{1}{\ln t}dt$; we have: $Li(x)=\int_2^x \frac{1}{\ln t}dt \approx \pi(x) \cong \frac{x}{\ln x}$ (number of primes within x); in fact, by integrating by parts:

$$Li(x)=\int_2^x [\frac{1}{\ln t}]\cdot\{1\}\cdot dt = \left|[\frac{1}{\ln t}]\{t\}\right|_2^x - \int_2^x [(-1)\frac{1}{\ln^2 t}\frac{1}{t}]\{t\}dt = \frac{x}{\ln x}-\frac{2}{\ln 2}+\int_2^x \frac{1}{\ln^2 t}dt\ ,$$

so, for not so small x:

$$\int_2^x \frac{1}{\ln t}dt - \cancel{\int_2^x \frac{1}{\ln^2 t}dt} \approx \int_2^x \frac{1}{\ln t}dt \approx Li(x) \approx \frac{x}{\ln x} - \cancel{\frac{2}{\ln 2}} \approx \frac{x}{\ln x} \approx \pi(x)\ .$$

(in fact, the surface subtended by $\frac{1}{\ln^2 t}$ gets negligible, as long as t increases, with respect to that subtended by $\frac{1}{\ln t}$)

After that, let us reexamine the Euler Product Formula (2.8): $\zeta(s)=\prod_p (1-p^{-s})^{-1}$ and let us execute the logarithm of both sides: $\ln\zeta(s)=-\sum_p (1-\frac{1}{p^s})=-\int_1^\infty \ln(1-\frac{1}{x^s})d\pi(x)$, after having reinterpreted the summation as an integral and after having considered that in counting up to a generic x, we have $\pi(x)$ primes. Now, by integrating by parts, we have:

$$\ln\zeta(s)=-\int_1^\infty \ln(1-\frac{1}{x^s})d\pi(x)=-\ln(1-\frac{1}{x^s})\pi(x)\Big|_1^\infty+\int_1^\infty \pi(x)\cdot d\ln(1-\frac{1}{x^s})=$$

$$=-[\ln(1-\frac{1}{\infty^s})\pi(\infty)-\ln(1-\frac{1}{1^s})\pi(1)]+\int_1^\infty \pi(x)\cdot\frac{sx^{-s-1}}{(1-\frac{1}{x^s})}dx=[0\cdot\infty-(-\infty\cdot 0)]+s\int_1^\infty \frac{\pi(x)}{x(x^s-1)}dx=$$

$$\boxed{=\ln\zeta(s)=s\int_1^\infty \frac{\pi(x)}{x(x^s-1)}dx\,.}$$

Or rather, as an alternative, $\ln\zeta(s)=-\sum_p (1-\frac{1}{p^s})=-\sum_{n=2}^\infty[\pi(n)-\pi(n-1)]\ln(1-\frac{1}{n^s})=$
$=-\sum_{n=2}^\infty \pi(n)[\ln(1-\frac{1}{n^s})-\ln(1-\frac{1}{(n-1)^s})]=\sum_{n=2}^\infty \pi(n)[\int_n^{n+1}\frac{s}{x(x^s-1)}dx]=s\int_2^\infty \frac{\pi(n)}{x(x^s-1)}dx]$, which is, in practice, the result we already got.

-<u>complex values of s</u>:

If in the Riemann Zeta $\zeta(s)=\sum_{n=1}^\infty \frac{1}{n^s}$ we consider complex values of s, such as s=a+ib , then we remind in advance that that function has as zeroes also the complex conjugates s* of s , that are the s*=a-ib . And at this point we provide some complex zeroes of Zeta:
s_1~(1/2)+i14,135 , s_2~(1/2)+i21,022 , s_3~(1/2)+i25,011 and, of course, also the s_i* are .
As a matter of fact, as an example, by using s_1, we have:

$$\zeta(\frac{1}{2}+i14{,}135)=\sum_{n=1}^\infty \frac{1}{n^s}=\frac{1}{1^{(\frac{1}{2}+i14.135)}}+\frac{1}{2^{(\frac{1}{2}+i14{,}135)}}+\frac{1}{3^{(\frac{1}{2}+i14{,}135)}}+...=\frac{1}{1^{\frac{1}{2}}1^{i14{,}135}}+\frac{1}{2^{\frac{1}{2}}2^{i14{,}135}}+\frac{1}{3^{\frac{1}{2}}3^{i14{,}135}}+...=$$

$$=\frac{1}{e^{i14{,}135\ln 1}}+\frac{1}{2^{\frac{1}{2}}e^{i14{,}135\ln 2}}+\frac{1}{3^{\frac{1}{2}}e^{i14{,}135\ln 3}}+...=1+\frac{1}{1{,}41e^{i9{,}8}}+\frac{1}{1{,}73e^{i15{,}53}}+...=$$

$$=1+\frac{1}{1{,}41(\cos 9{,}8+i\sin 9{,}8)}+\frac{1}{1{,}73(\cos 15{,}53+i\sin 15{,}53)}+...=$$

$$=1+\frac{(-0{,}93+i0{,}37)}{1{,}41(-0{,}93-i0{,}37)(-0{,}93+i0{,}37)}+\frac{(-0{,}98-i0{,}177)}{1{,}73(-0{,}98+i0{,}177)(-0{,}98-i0{,}177)}+...=$$

$$=1+\frac{(-0{,}93+i0{,}37)}{1{,}41\cdot 1{,}0018}+\frac{(-0{,}98-i0{,}177)}{1{,}73\cdot 0{,}9917}+...=1+\frac{(-0{,}93+i0{,}37)}{1{,}41}+\frac{(-0{,}98-i0{,}177)}{1{,}72}+...=$$

$= 1 - 0{,}66 + i0{,}262 - 0{,}570 - i0{,}103 + ... = 1 - 1{,}23 + i0{,}159 \approx 0 + i0 = \underline{0}$ (after having considered all the approximations had in providing the values of the zeroes, and also the low number of terms we have used).

And after analyzing the three complex zeroes here provided, we have also the Riemann Hypothesis: all the non trivial zeroes have got the real part equal to 1/2.

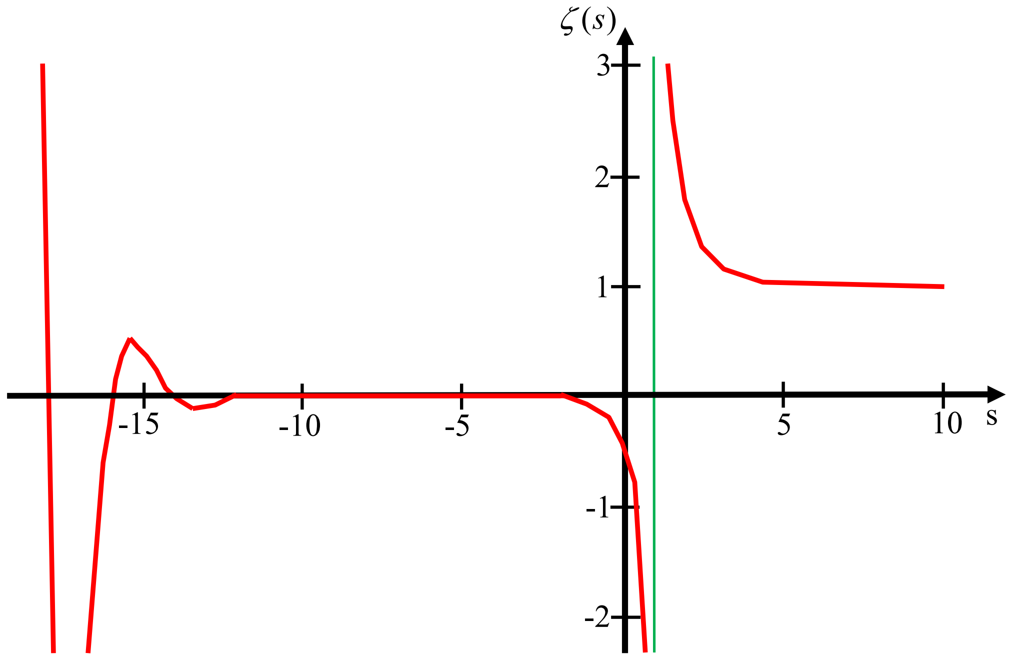

Fig. 2.1: The Riemann Zeta Function $\zeta(s)$ graph in a wide range of s.

3 – Extension of the domain of the Riemann Zeta Function:

a) through the η(s) series:

let us consider the eta series:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + ... \qquad (3a.1)$$

which converges for s>1, and we easily notice that we can see it in the following way:

$$\eta(s) = (1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + ...) = (1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ...) - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + ...) =$$

$$= \zeta(s) - 2\frac{1}{2^s}\zeta(s) = \zeta(s)(1 - \frac{1}{2^{s-1}}) \rightarrow$$

$$\boxed{\zeta(s) = \eta(s)(1 - \frac{1}{2^{s-1}})^{-1}} \qquad (3a.2)$$

Then, as $\eta(1/2) \cong 0{,}6$, it follows that $\zeta(1/2) = \eta(1/2)(1 - \frac{1}{2^{s-1}})^{-1} = 0{,}6(1 - \frac{1}{2^{-1/2}})^{-1} \cong -1{,}46$ and so we have obtained a $\zeta(s)$ value corresponding to a value (just over the zero) of s, that is (1/2), which is out of the range of convergence of the series (1.1) , that is s>1.

b) <u>through the Bernoulli Numbers</u>:

with reference to the Subappendix A, where Bernoulli Numbers B_i are introduced and provided, we have:

$$\boxed{\zeta(2n)=\frac{2^{2n-1}}{(2n)!}|B_{2n}|\pi^{2n}} \quad \text{and} \tag{3b.1}$$

$$\boxed{\zeta(1-n)=(-1)^{n+1}\frac{B_n}{n} \quad .} \tag{3b.2}$$

from which, for instance:

n=2 $\longrightarrow$ $\zeta(1-2)=\zeta(-1)=(-1)^3\frac{B_2}{2}=-\frac{1}{12}$, n=3 $\longrightarrow$ $\zeta(1-3)=\zeta(-2)=(-1)^4\frac{B_3}{3}=0$,

n=4 $\longrightarrow$ $\zeta(1-4)=\zeta(-3)=(-1)^5\frac{B_4}{4}=\frac{1}{120}$, etc.

<u>Deduction</u>:

We know that $p(x)=\frac{\sin x}{x}$ is zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, so:

$$p(x)=\frac{\sin x}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})..........=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2}).....=\prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2] \ .$$

If we say $\alpha=\frac{x^2}{\pi^2}$, then:

$$p(\alpha)=\prod_{n=1}^{\infty}[1-\frac{\alpha}{n^2}]. \tag{3b.3}$$

$$(p(0)=\prod_{n=1}^{\infty}[1-\frac{0}{n^2}]=\prod_{n=1}^{\infty}1=1)$$

We also know from Taylor that: $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.....$ and so:

$$p(x)=\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+..... \tag{3b.4}$$

Now, Taylor over the (3b.3) yields:

$$p(\alpha)=\prod_{n=1}^{\infty}[1-\frac{\alpha}{n^2}]=p(0)+p'(0)\alpha+\frac{p''(0)}{2!}\alpha^2+\frac{p'''(0)}{3!}\alpha^3+... \tag{3b.5}$$

$\frac{d}{d\alpha}\ln p(\alpha)=\frac{1}{p(\alpha)}p'(\alpha)$, so: $\quad p'(\alpha)=p(\alpha)\frac{d}{d\alpha}\ln p(\alpha)=p(\alpha)\frac{d}{d\alpha}[\ln\prod(1-\frac{\alpha}{n^2})]=$

$$=p(\alpha)\frac{d}{d\alpha}[\sum\ln(1-\frac{\alpha}{n^2})]=p(\alpha)\sum\frac{d}{d\alpha}\ln(1-\frac{\alpha}{n^2})=p(\alpha)\sum\frac{-\frac{1}{n^2}}{(1-\frac{\alpha}{n^2})}=p(\alpha)\sum\frac{1}{(\alpha-n^2)} \ .$$

When $\alpha = 0$, $p'(0) = p(0)\sum \frac{1}{(-n^2)} = -\sum \frac{1}{n^2}$ and according to the (3b.4) and (3b.5), this must be $-\frac{x^2}{3!}$ and also $p'(0)\alpha = p'(0)\frac{x^2}{\pi^2}$, so: $-\frac{x^2}{\pi^2}\sum \frac{1}{n^2} = -\frac{x^2}{3!}$ and $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. As the Riemann Zeta Function is so defined: $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, then: $\boxed{\zeta(2) = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.}$

(and $p'(0) = p(0)\sum \frac{1}{(-n^2)} = -\frac{\pi^2}{6}$)

Now, let's calculate $p''(\alpha)$: as $p'(\alpha) = p(\alpha)\sum \frac{1}{(\alpha - n^2)}$, then:

$p''(\alpha) = p'(\alpha)\sum \frac{1}{(\alpha - n^2)} - p(\alpha)\sum \frac{1}{(\alpha - n^2)^2}$, so: $p''(0) = -p'(0)\sum \frac{1}{n^2} - p(0)\sum \frac{1}{n^4} = \frac{\pi^4}{36} - \sum \frac{1}{n^4}$

but according to (3b.4) and (3b.5), $\frac{p''(0)}{2!}\alpha^2 = \frac{x^4}{5!}$, so: $\frac{1}{2}(\frac{\pi^4}{36} - \sum \frac{1}{n^4})\frac{x^4}{\pi^4} = \frac{x^4}{5!}$ and finally:

$\boxed{\zeta(4) = \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.}$ (and $p''(0) = \frac{\pi^4}{36} - \sum \frac{1}{n^4} = \frac{\pi^4}{36} - \frac{\pi^4}{90}$) .

By going on, we get $p'''(\alpha)$ starting from $p''(\alpha)$:

$$p'''(\alpha) = \frac{d}{d\alpha}p''(\alpha) = \frac{d}{d\alpha}[p'(\alpha)\sum \frac{1}{(\alpha - n^2)} - p(\alpha)\sum \frac{1}{(\alpha - n^2)^2}] =$$

$$= p''(\alpha)\sum \frac{1}{(\alpha - n^2)} - p'(\alpha)\sum \frac{1}{(\alpha - n^2)^2} - p'(\alpha)\sum \frac{1}{(\alpha - n^2)^2} + 2p(\alpha)\sum \frac{1}{(\alpha - n^2)^3} =$$

$$= p''(\alpha)\sum \frac{1}{(\alpha - n^2)} - 2p'(\alpha)\sum \frac{1}{(\alpha - n^2)^2} + 2p(\alpha)\sum \frac{1}{(\alpha - n^2)^3}$$ and for $\alpha=0$:

$$p'''(0) = -p''(0)\sum \frac{1}{n^2} - 2p'(0)\sum \frac{1}{n^4} - 2p(0)\sum \frac{1}{n^6} = -(\frac{\pi^4}{36} - \frac{\pi^4}{90})\frac{\pi^2}{6} + 2\frac{\pi^2}{6}\frac{\pi^4}{90} - 2 \cdot 1 \cdot \sum \frac{1}{n^6} =$$

$$= -\frac{\pi^2}{6}\frac{\pi^4}{36} + 3\frac{\pi^2}{6}\frac{\pi^4}{90} - 2\sum \frac{1}{n^6} = \frac{\pi^2}{6}(\frac{\pi^4}{30} - \frac{\pi^4}{36}) - 2\sum \frac{1}{n^6} = \frac{\pi^6}{1080} - 2\sum \frac{1}{n^6}$$, but the usual comparison between the two previous Taylor expansions requires: $\frac{p'''(0)}{3!}\alpha^3 = -\frac{x^6}{7!}$, from which:

$$\frac{1}{6}(\frac{\pi^6}{1080} - 2\sum \frac{1}{n^6})\frac{x^6}{\pi^6} = -\frac{x^6}{7!} \rightarrow \frac{1}{6480} - \frac{1}{3}\frac{1}{\pi^6}\sum \frac{1}{n^6} = -\frac{1}{5040} \rightarrow \sum \frac{1}{n^6} = 3\pi^6 \frac{11520}{32659200} = \frac{\pi^6}{945},$$

that is: $\boxed{\zeta(6) = \sum_1^\infty \frac{1}{n^6} = \frac{\pi^6}{945} .}$

By summing things up: $\zeta(2) = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$, $\zeta(4) = \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, $\zeta(6) = \sum_1^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$ and by reminding the Bernoulli Numbers introduced in the Subappendix A, that are:
$B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$ etc, we deduce that:

$$\boxed{\zeta(2n) = \frac{2^{2n-1}}{(2n)!}|B_{2n}|\pi^{2n} .} \qquad (3b.6)$$

Moreover, if we remind some values of $\zeta(s)$, that are the following:

$\zeta(-1)=-\frac{1}{12}$, $\zeta(-2)=0$, $\zeta(-3)=\frac{1}{120}$, $\zeta(0)=-\frac{1}{2}$, $\zeta(\frac{3}{2})=2{,}612$, $\zeta(2)=\frac{\pi^2}{6}=1{,}645$, $\zeta(\frac{5}{2})=1{,}341$, $\zeta(3)=1{,}202$, $\zeta(\frac{7}{2})=1{,}127$, $\zeta(5)=1{,}0369$, $\zeta(7)=1{,}0083$, $\zeta(4)=\frac{\pi^4}{90}=1{,}0823$, $\zeta(6)=\frac{\pi^6}{945}=1{,}0173$, then we understand also the following one, still involving the Bernoulli Numbers:

$$\boxed{\zeta(1-n)=(-1)^{n+1}\frac{B_n}{n}\,;} \tag{3b.7}$$

in fact, as an example, by taking n=1, we have:

$\zeta(1-1)=\zeta(0)=(-1)^{1+1}\frac{(-1/2)}{1}=-\frac{1}{2}$ (a true thing) and, as another example, for n=2:

$\zeta(1-2)=\zeta(-1)=(-1)^{2+1}\frac{B_2}{2}=-1\frac{(1/6)}{2}=-\frac{1}{12}$, that is: $\boxed{\zeta(-1)=-\frac{1}{12}}$, (a true thing) , that is again that notorious -1/12 !!!

c) <u>through the **Functional Equation**</u>:

the Functional Equation provides $\zeta(1-s)$ in terms of $\zeta(s)$ and it allows us to get down to s=0 and even below zero; so, if for instance we already know $\zeta(16)$, then we can get $\zeta(-15)$.

Functional Equation with the factorial:

$$\zeta(1-s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)(s-1)!\zeta(s)\ . \tag{3c.1}$$

We have that s and (1-s) are here interchangeable, from which:

$$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})(-s)!\zeta(1-s)\ . \tag{3c.2}$$

Functional Equation with the Euler's Gamma Function ($\Gamma(s+1)=s!$):

$$\boxed{\zeta(1-s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)\Gamma(s)\zeta(s)} \quad \text{e} \tag{3c.3}$$

$$\boxed{\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)} \tag{3c.4}$$

Well, we start from the Euler's Gamma Function definition in x': $\Gamma(s)=\int_0^\infty e^{-x'}x'^{s-1}dx'$; if in it we put (1/2)s in place of s and $n^2\pi x$ in place of x' ($x'=n^2\pi x \rightarrow dx'=n^2\pi dx$), then we have:

$$\Gamma(\frac{1}{2}s)=\int_0^\infty e^{-n^2\pi x}(n^2\pi x)^{\frac{1}{2}s-1}(n^2\pi)dx=\int_0^\infty e^{-n^2\pi x}(n^2\pi)^{\frac{1}{2}s-1}x^{\frac{1}{2}s-1}(n^2\pi)dx=(n^2\pi)^{\frac{1}{2}s}\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx,$$

from which: $\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx=\frac{\Gamma(\frac{1}{2}s)}{n^s\pi^{\frac{1}{2}s}}$; now, by applying the summation operator, we get:

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx=\int_0^{\infty} x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty}e^{-n^2\pi x}dx=\sum_{n=1}^{\infty}\frac{\Gamma(\frac{1}{2}s)}{n^s\pi^{\frac{1}{2}s}}=\frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}}\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}}\zeta(s)$$, from which:

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{\frac{1}{2}s}}=\int_0^{\infty} x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty}e^{-n^2\pi x}dx \quad . \tag{3c.5}$$

Now, by naming $f(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$, it follows that, for the (3c.5):

$$\zeta(s)=\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\int_0^{\infty} x^{\frac{1}{2}s-1}f(x)dx \quad . \tag{3c.6}$$

Now, here below, in the Subappendix D, we will soon show the following:

$$\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty}e^{-n^2\pi\frac{1}{x}} \tag{3c.7}$$

and it can be decomposed like that:

$$\sum_{n=-\infty}^{-1}e^{-n^2\pi x}+e^{-n^2\pi 0}+\sum_{n=1}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}(\sum_{n=-\infty}^{-1}e^{-n^2\pi\frac{1}{x}}+e^{-n^2\pi 0}+\sum_{n=1}^{\infty}e^{-n^2\pi\frac{1}{x}})$$, that is:

$f(x)+1+f(x)=\frac{1}{\sqrt{x}}[f(\frac{1}{x})+1+f(\frac{1}{x})]$ and so: $2f(x)+1=\frac{1}{\sqrt{x}}[2f(\frac{1}{x})+1]$, from which, again:

$f(x)=\frac{1}{2\sqrt{x}}[2f(\frac{1}{x})+1]-\frac{1}{2}$ and so the (3c.6) yields:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\int_0^{\infty} x^{\frac{1}{2}s-1}f(x)dx=\int_0^{1} x^{\frac{1}{2}s-1}f(x)dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\int_0^{1} x^{\frac{1}{2}s-1}\{\frac{1}{2\sqrt{x}}[2f(\frac{1}{x})+1]-\frac{1}{2}\}dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\int_0^{1} x^{\frac{1}{2}s-1}\frac{1}{2\sqrt{x}}dx+\int_0^{1} x^{\frac{1}{2}s-1}(-\frac{1}{2})dx+\int_0^{1} x^{\frac{1}{2}s-1}\frac{1}{\sqrt{x}}f(\frac{1}{x})dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{2}\int_0^{1} x^{\frac{1}{2}s-1-\frac{1}{2}}dx-\frac{1}{s}\left|x^{\frac{1}{2}s-1+1}\right|_0^1+\int_0^{1} x^{\frac{1}{2}s-1-\frac{1}{2}}f(\frac{1}{x})dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{2(\frac{1}{2}s-\frac{3}{2}+1)}\left|x^{\frac{1}{2}s-\frac{3}{2}+1}\right|_0^1-\frac{1}{s}+\int_0^{1} x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{(s-1)}-\frac{1}{s}+\int_0^{1} x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx=\frac{1}{s(s-1)}+\int_0^{1} x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^{\infty} x^{\frac{1}{2}s-1}f(x)dx \quad . \tag{3c.8}$$

Now, in the integral $\int_0^{1} x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx$ of the (3c.8) we carry out the following change of variable:

$y=\frac{1}{x}\rightarrow\ dy=-\frac{1}{x^2}dx\rightarrow\ dx=-\frac{1}{y^2}dy$ and x=(0 , 1) yields y=(∞ , 1) $\rightarrow\ \int_0^{1} x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx=$

$$=\int_{\infty}^{1}(\frac{1}{y})^{\frac{1}{2}s-\frac{3}{2}}f(y)(-\frac{1}{y^2}dy)=-\int_{\infty}^{1}(y)^{\frac{-1}{2}s+\frac{3}{2}}f(y)y^{-2}dy=-\int_{\infty}^{1}(y)^{\frac{-1}{2}s-\frac{1}{2}}f(y)dy=\int_1^{\infty} x^{\frac{-1}{2}s-\frac{1}{2}}f(x)dx \tag{3c.9}$$

and from the (3c.9) inserted into the (3c.8), we get:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty x^{\frac{-1}{2}s-\frac{1}{2}}f(x)dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \quad ,$$

that is: $$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \quad . \tag{3c.10}$$

Now we notice that the right side of the (3c.10) doesn't change when we exchange (s) with (1-s), from which, after carrying out such an exchange, the (3c.10) yields:

$$\pi^{\frac{-1}{2}+\frac{1}{2}s}\Gamma(\frac{1-s}{2})\zeta(1-s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \ ; \tag{3c.11}$$

now, after a comparison between the (3c.10) and the (3c.11), we finally get:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\pi^{\frac{-1}{2}+\frac{1}{2}s}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{3c.12}$$

and we easily notice that by exchanging s with (1-s) also in the (3c.12), we get the (3c.12) itself.
Now, we write again $\zeta(1-s)$ and $\zeta(s)$ from the (3c.12); we have:

$$\zeta(1-s)=\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})}\zeta(s) \tag{3c.13}$$

$$\zeta(s)=\pi^{\frac{-1}{2}+s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2}s)}\zeta(1-s) \tag{3c.14}$$

and we also remind the Reflection Formula $\frac{2}{\Gamma(s)\Gamma(1-s)}=\frac{2\sin s\pi}{\pi}$ (eq. (C.1) in the Subappendix C) and we write it for $\frac{1-s}{2}$ first, in place of s and later also for $\frac{s}{2}$ in place of s, so getting:

$$\frac{2}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}=\frac{2\sin(\frac{1-s}{2}\pi)}{\pi}$$ (and here, now, we multiply the left side up and down by $\Gamma(\frac{1}{2}s)$)

$$\frac{2}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}=\frac{2\sin(\frac{\pi s}{2})}{\pi}$$ (and here, now, we multiply the left side up and down by $\Gamma(\frac{1-s}{2})$)

so having:

$$\frac{2\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)}=\frac{2\sin(\frac{1-s}{2}\pi)}{\pi}\rightarrow\boxed{\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})}}=\frac{\sin(\frac{1-s}{2}\pi)}{\pi}\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)$$

$$\frac{2\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})}=\frac{2\sin(\frac{\pi s}{2})}{\pi}\rightarrow\boxed{\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2}s)}}=\frac{\sin(\frac{\pi s}{2})}{\pi}\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})$$

and now we insert the two quantities which are in the two blue frames respectively in the (3c.13) and (3c.14), so having:

$$\zeta(1-s)=\pi^{\frac{1}{2}-s}\frac{\sin(\frac{1-s}{2}\pi)}{\pi}\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)\zeta(s)=\pi^{\frac{-1}{2}-s}\boxed{\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)}\sin(\frac{1-s}{2}\pi)\zeta(s) \qquad (3c.15)$$

$$\zeta(s)=\pi^{\frac{-1}{2}+s}\frac{\sin(\frac{\pi s}{2})}{\pi}\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})\zeta(1-s)=\pi^{\frac{-3}{2}+s}\boxed{\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})}\sin(\frac{\pi s}{2})\zeta(1-s) \qquad (3c.16)$$

Now, we remind the Legendre Duplication Formula: $\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$ (eq. (E.1) in the Subappendix E) and by putting in it, in place of s, $\frac{s}{2}$ first and then also $(\frac{1}{2}-\frac{s}{2})$, we get the following:

$$s\to\frac{s}{2} \qquad \to \qquad \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})=2^{1-s}\pi^{\frac{1}{2}}\Gamma(s)$$

$$s\to(\frac{1}{2}-\frac{s}{2}) \qquad \to \qquad \Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})=2^{s}\pi^{\frac{1}{2}}\Gamma(1-s)$$

and we use these two respectively in the green frames of the (3c.15) and (3c.16), so obtaining:

$$\zeta(1-s)=\pi^{\frac{-1}{2}-s}[2^{1-s}\pi^{\frac{1}{2}}\Gamma(s)]\sin(\frac{1-s}{2}\pi)\zeta(s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)\Gamma(s)\zeta(s)$$

$$\zeta(s)=\pi^{\frac{-3}{2}+s}[2^{s}\pi^{\frac{1}{2}}\Gamma(1-s)]\sin(\frac{\pi s}{2})\zeta(1-s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

which are nothing but the (3c.3) and the (3c.4)!!!!!

d) through this **weird** proof:

$$\boxed{S=\sum_{n=1}^{\infty}n=1+2+3+4+5+..........=-\frac{1}{12}} \qquad (3d.1)$$

(here S would be the $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ in the (1.1) where s is forced to -1)

Proof:

$S_1=\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+1-1+........=1-(1-1+1-1+........)=1-S_1$, so: $S_1=\frac{1}{2}$.

$S_2=\sum_{n=0}^{\infty}(-1)^n(n+1)=1-2+3-4+5-6+........$; moreover: $2S_2=S_2+S_2$, that is:

$$\begin{array}{ll} 1-2+3-4+5-6+..............+ & S_2+ \\ \quad 1-2+3-4+5-6+........= & S_2= \\ \hline 1-1+1-1+1-1+........ & S_1 \end{array}$$

so: $2S_2=S_1=\frac{1}{2}$, so: $S_2=\frac{1}{4}$. Moreover: $S-4S=-3S$ and: $-4S=-4-8-12-16-.....$, so:

$$\begin{array}{ll} 1+2+3+4+5+6+..............+ & S+ \\ \quad -4 \quad -8 \quad -12-..............= & (-4S)= \\ \hline 1-2+3-4+5-6+........ & S_2 \end{array}$$

, then: $S-4S=-3S=S_2=\frac{1}{4}$ and finally:

$\boxed{S = -\frac{1}{12}}$. ($\zeta(-1) = -\frac{1}{12}$, and unfortunately this is true for the $\zeta(s)$ of Riemann)

e) through this further reasoning:

but such a weird result (S=-1/12) shows up again also through a different way; in fact, let us consider the Euler's Gamma Function $\Gamma(s)$:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \qquad (3e.1)$$

where, obviously: $\Gamma(1) = 1$; we have:

$$\Gamma(n+1) = n\Gamma(n); \qquad (3e.2)$$

in fact, after an integration by parts:

$\Gamma(s) = e^{-t} \frac{t^s}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-t} t^s dt = 0 + \frac{1}{s}\Gamma(s+1)$, and by iterating the (3e.2) ($\Gamma(n+1) = n\Gamma(n)$), we get: $\Gamma(n+1) = n!$. Now, let us consider the Riemann Zeta Function:

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \quad ; \qquad (3e.3)$$

we can write, after having considered the Euler's Gamma Function ($\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$), that:

Γ of Euler written in t': $\Gamma(s) = \int_0^\infty e^{-t'} t'^{(s-1)} dt' \rightarrow$ (change of variable): t'=nt $\rightarrow$ dt'=ndt $\rightarrow$

$\rightarrow \Gamma(s) = \int_0^\infty e^{-t'} t'^{(s-1)} dt' = \int_0^\infty e^{-nt} (nt)^{(s-1)} n dt = n^s \int_0^\infty e^{-nt} t^{(s-1)} dt \rightarrow \frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-nt} t^{(s-1)} dt \rightarrow$

$\rightarrow \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \frac{1}{\Gamma(s)}\int_0^\infty e^{-nt} t^{s-1} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{n=1}^\infty e^{-nt} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{n=1}^\infty (e^{-t})^n dt$, but as the following Taylor expansion holds: $\frac{x}{1-x} = \sum_{n=1}^\infty x^n$, it follows that: $\frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^\infty (e^{-t})^n$ and so:

$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{n=1}^\infty (e^{-t})^n dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{1-e^{-t}} dt$; now, as also the following Taylor expansion holds: $e^x = \frac{1}{n!}\sum_{n=0}^\infty x^n$, then it follows that:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{1-e^{-t}} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{1-(1-t+\frac{t^2}{2}-\frac{t^3}{6}+...)} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{t-\frac{t^2}{2}+\frac{t^3}{6}-...} dt =$$

$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{t(1-\frac{t}{2}+\frac{t^2}{6}-...)} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\frac{e^{-t}}{1-[(\frac{t}{2}-\frac{t^2}{6})+...]} dt$ and as the following Taylor expansion holds: $\frac{1}{1-x} = 1+x+x^2+x^3+... = \sum_{n=0}^\infty x^n$, then it follows that: (x $\rightarrow (\frac{t}{2}-\frac{t^2}{6})$)

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\frac{e^{-t}}{1-[(\frac{t}{2}-\frac{t^2}{6})+...]} dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}[1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}+...)]dt =$$

$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(1+\frac{t}{2}+\frac{t^2}{12}+...)dt=\frac{1}{\Gamma(s)}[\int_0^\infty t^{s-2}e^{-t}dt+\int_0^\infty \frac{t}{2}t^{s-2}e^{-t}dt+\int_0^\infty \frac{t^2}{12}t^{s-2}e^{-t}dt+...]=$$

$$=\frac{1}{\Gamma(s)}[\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\frac{1}{12}\Gamma(s+1)+...] \tag{3e.4}$$

and being that, due to the (3e.2): $\Gamma(s+1)=s\Gamma(s)$ and, moreover, also the last one, with $s-1$ in place of s, which yields: $\Gamma(s)=(s-1)\Gamma(s-1)$, it follows that:

$$\zeta(s)=\frac{1}{\Gamma(s)}[\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\frac{1}{12}\Gamma(s+1)+...]=\frac{1}{\Gamma(s)}[\frac{\Gamma(s)}{(s-1)}+\frac{1}{2}\Gamma(s)+\frac{1}{12}s\Gamma(s)+...]=[\frac{1}{(s-1)}+\frac{1}{2}+\frac{1}{12}s+...]$$

so: $\zeta(s)=\frac{1}{(s-1)}+\frac{1}{2}+\frac{1}{12}s+...$ and with $s=-1$ $\rightarrow$

$$\zeta(-1)=\sum_{n=1}^{\infty}\frac{1}{n^{-1}}=\sum_{n=1}^{\infty}n=1+2+3+4+...=\frac{1}{(-1-1)}+\frac{1}{2}+\frac{1}{12}(-1)+...=-\frac{1}{12} \quad (*) \tag{3e.5}$$

therefore:

$$\boxed{\sum_{n=1}^{\infty}n=1+2+3+4+...=-\frac{1}{12}}$$

and here we go again!

(*): the (3e.5) is a result which is apparently incomplete, as there would be further terms represented by the **(+ …)** , but all these terms we neglected are all null; in fact, they would be born into the (3e.4); so, let us analyze a couple of the next terms we neglected:

$$[1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}+...)] \rightarrow [1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}\boxed{-\frac{t^3}{6}}+\boxed{\frac{t^4}{36}})] \rightarrow$$

$$\rightarrow......\boxed{+\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(-\frac{t^3}{6})dt}+\boxed{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(\frac{t^4}{36})dt}=-\frac{1}{6}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+1}e^{-t}dt+\frac{1}{36}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+2}e^{-t}dt=$$

$$=\frac{-\Gamma(s+2)}{6\Gamma(s)}+\frac{\Gamma(s+3)}{36\Gamma(s)}=\frac{-(s+1)\Gamma(s+1)}{6\Gamma(s)}+\frac{(s+2)\Gamma(s+2)}{36\Gamma(s)}=\frac{-(s+1)\Gamma(s+1)}{6\Gamma(s)}+\frac{(s+2)(s+1)\Gamma(s+1)}{36\Gamma(s)},$$

which, still when $s=-1$, is 0.

SUBAPPENDICES:

SUBAPPENDIX A: The Bernoulli Numbers.

We consider the following function:

$$f(x)=\frac{x}{e^x-1} \tag{A.1}$$

whose limit for x→0 exists (De L'Hopital) and the same thing can be stated for its derivatives. Then, by expanding it with Taylor around zero, we get:

$$f(x)=\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}, \tag{A.2}$$

with $B_n=f^{(n)}(0)$ and $B_0=1$, $B_1=-1/2$, $B_2=1/6$ etc, which are the Bernoulli Numbers. In order to have a method of calculation for such numbers, we notice that from the (A.1) and (A.2) we have:

$$x=(e^x-1)\frac{x}{e^x-1}=(e^x-1)\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=(\frac{1}{n!}\sum_{n=0}^{\infty}x^n-1)\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=(x+\frac{x^2}{2!}+\frac{x^3}{3!}+...)(B_0+B_1x+B_2\frac{x^2}{2!}+B_3\frac{x^2}{3!}+...)=$$
$$=x[B_0+(\frac{B_0}{2!}+B_1)x+(\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!})x^2+(\frac{B_0}{4!}+\frac{B_1}{3!}+\frac{B_2}{2!2!}+\frac{B_3}{3!})x^3+(\frac{B_0}{5!}+\frac{B_1}{4!}+\frac{B_2}{3!2!}+\frac{B_3}{2!3!}+\frac{B_4}{4!})x^4+...]$$

and as in the left side there is only the x, while in the last side on the right there is $B_0x+...$, then it follows that:

$$B_0=1\ ,$$
$$\frac{B_0}{2!}+B_1=0\ \longrightarrow\ B_1=-\frac{1}{2} \tag{A.3}$$
$$\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!}=0\ \longrightarrow\ B_2=\frac{1}{6}$$
……………….

and so on.Moreover, all the (A.3) can be summarized like that:

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k=0\,,\qquad (\text{with } \binom{n}{k}=\frac{n!}{(n-k)!k!}) \tag{A.4}$$

Then, if we consider, for instance, the last of all the (A.3), that is the $\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!}=0$, then we will notice that:

$$\frac{B_2}{2!}=-(\frac{B_0}{3!}+\frac{B_1}{2!})=-(\frac{1}{3!}B_0+\frac{1}{2!}B_1)=-[\frac{1}{3!}\binom{3}{0}B_0+\frac{1}{3!}B_1\binom{3}{1}]=-\frac{1}{3!}\sum_{k=0}^{2-1}\binom{3}{k}B_k$$, that is:

$$B_2=-\frac{2!}{3!}\sum_{k=0}^{2-1}\binom{3}{k}B_k=-\frac{1}{3}\sum_{k=0}^{2-1}\binom{3}{k}B_k$$ and, in general:

$$\boxed{B_n=-\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k} \tag{A.5}$$

through which it is possible to calculate recursively and more simply all the B_i :
$B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$ etc and we can also state that all the odd ones, starting from 3, are null.

SUBAPPENDIX B: The Euler's Γ (Gamma) Function.

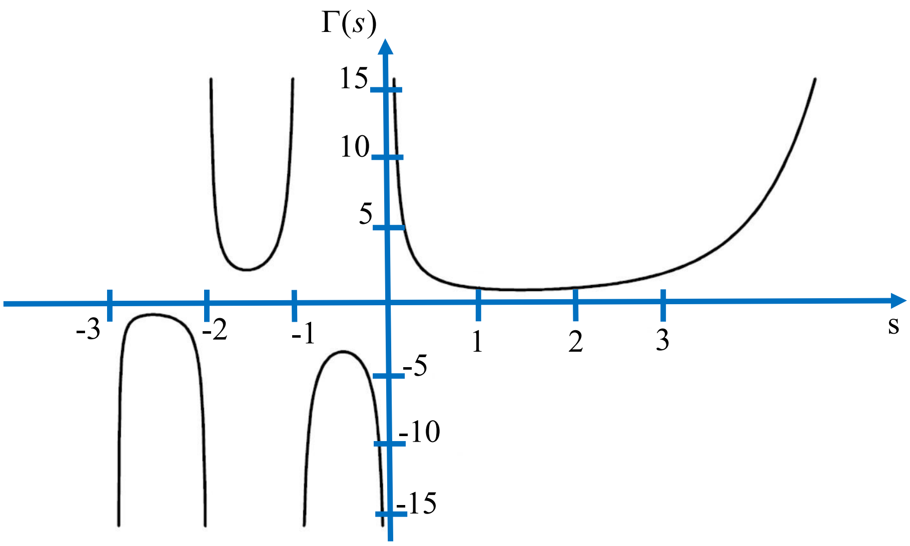

Fig. B.1: The Euler's Gamma Function graph.

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt \tag{B.1}$$

Obviously, $\Gamma(1)=1$. Then,

$$\Gamma(n+1)=n\Gamma(n); \tag{B.2}$$

in fact, after an integration by parts:

$\Gamma(z)=e^{-t}\left.\dfrac{t^z}{z}\right|_0^\infty+\dfrac{1}{z}\int_0^\infty e^{-t}t^z dt=0+\dfrac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1)=n\Gamma(n)$, we get:

$$\Gamma(n+1)=n!\ . \tag{B.3}$$

THE GAUSS INTEGRAL

Let us consider these two integrals (identical ones):

$$I=\int_0^\infty e^{-\alpha x^2}dx \qquad I=\int_0^\infty e^{-\alpha y^2}dy \tag{B.4}$$

By multiplying one with the other: $I^2=\int_0^\infty\int_0^\infty e^{-\alpha(x^2+y^2)}dxdy$ and, in polar coordinates: ($x^2+y^2=r^2$, $dS=dxdy=rdrd\theta$), we have:

$I^2=\int_0^{\pi/2}\int_0^\infty e^{-\alpha r^2}rdrd\theta=\dfrac{\pi}{2}\int_0^\infty e^{-\alpha r^2}rdr=-\dfrac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty=\dfrac{\pi}{4\alpha}$, so:

$$I=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\ . \tag{B.5}$$

PARTICULAR VALUES OF THE EULER'S GAMMA FUNCTION

The (B.3) with n=0 yields Γ(1)=0!=1, while with n=1, yields Γ(2)=1!=1.
Moreover, the (B.1) with z=1/2 becomes:

$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$; now, by chosing t=w^2, we have: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt=\int_0^\infty e^{-w^2}w^{-1}2wdw=2\int_0^\infty e^{-w^2}dw \tag{B.6}$$

and according to the (B.5) with α=1, we have: $\Gamma(\frac{1}{2})=2\int_0^\infty e^{-w^2}dw=\sqrt{\pi}$.

At last, for the (B.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\dfrac{\sqrt{\pi}}{2}$

To sum things up: Γ(1)=1, Γ(2)=1, $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(3/2)=\dfrac{\sqrt{\pi}}{2}$. (B.7)

SUBAPPENDIX C: THE REFLECTION FORMULA.

Here it is: $$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}\ . \tag{C.1}$$

In fact, we know that $\dfrac{\sin x}{x}$ is zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, so:

$\frac{\sin x}{x} = (1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi}).........= (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2}).....= \prod_{k=1}^{\infty}[1-(\frac{x}{k\pi})^2]$, and so:

$$\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \tag{C.2}$$

Moreover, a factorial can be seen also like that:

$n! = \lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}\cdot\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n}$; in fact, the denominator of the first limit is:

$(n+1)...(n+k) = \frac{(n+k)!}{n!}$, while from both numerators we can collect: $k!(k+1)...(k+n) = (k+n)!$,

so just n! is left . Then, we see that the second limit is 1: $\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n} = 1$, as it

reduces to the form $\lim_{k\to\infty}\frac{k^n}{k^n}$. Therefore: $n! = \lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}$. As $\Gamma(x+1) = x!$, then we have:

$\Gamma(x+1) = x! = \lim_{k\to\infty}\frac{k!k^x}{(x+1)...(x+k)}$ and after having divided both numerator and denominator by k!:

$$\Gamma(x+1) = \lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \tag{C.3}$$

Now, let's introduce the constant $\gamma_k = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}-\ln k$, and set:

($\gamma = \lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}(\sum_{n=1}^{k}\frac{1}{k}-\ln k) \approx 0{,}55721$, Euler-Mascheroni Constant);

on the basis of that, the (C.3) becomes: $\Gamma(x+1) = e^{-\gamma x}\lim_{k\to\infty}\frac{e^x}{(1+x)}\frac{e^{x/2}}{(1+x/2)}....\frac{e^{x/k}}{(1+x/k)}$. (C.4)

In fact, all the e^x, $e^{x/2}$,…, $e^{x/k}$ cancel with the terms in $e^{-\gamma x}$ and the term $k^x = e^{(\ln k)x}$ also comes from $e^{-\gamma x}$. Moreover, as $\Gamma(x+1) = x\Gamma(x)$, the (C.4) becomes:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}} = xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \tag{C.5}$$

Now, as $\Gamma(y+1) = y\Gamma(y)$, it follows that $\Gamma(y) = \frac{1}{y}\Gamma(y+1)$ and after having replaced y by –x:

$\Gamma(-x) = -\frac{1}{x}\Gamma(1-x)$ and so: $\Gamma(1-x) = -x\Gamma(-x)$. Let us evaluate $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x) = [x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}] = [A]\cdot[B] = \frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} , \tag{C.6}$$

where A is the (C.5) inverted and B is the (C.4) with x swapped with –x. So, the (C.6) is:

$\Gamma(x)\Gamma(1-x) = \frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, and according to the (C.2) (($\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) we have:

$\frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi}$ which is exactly the (C.1).

SUBAPPENDIX D: THE EQUATION $\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi\frac{1}{x}}$.

Such an equation has been useful above (eq. (3c.7)) for the proof of the Functional Equation. Well, now, after simplifying "a lot" things, in a heuristic way we consider a function f(t) and we make its Fourier Transform:

$f(p) = \int_{-\infty}^{\infty} f(t)e^{-ipt}dt$ $\quad$ ($p \equiv \omega = 2\pi f \;\; \rightarrow \;\; f \Rightarrow n \;\; \rightarrow \;\; p = 2\pi n$) , from which:

$f(n) = \int_{-\infty}^{\infty} f(t)e^{-2\pi \,\mathrm{int}}dt$ $\quad$ and we also apply the summation operator to both sides:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)e^{-2\pi \,\mathrm{int}}dt \; . \tag{D.1}$$

Now, in the (D.1) we consider a $f(t) = e^{-\pi \cdot t^2/x}$, from which, due to the (D.1):

$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)e^{-2\pi \,\mathrm{int}}dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi \cdot t^2/x} e^{-2\pi \,\mathrm{int}}dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt$, that is:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt \tag{D.2}$$

and being $f(t) = e^{-\pi \cdot t^2/x}$, then, by considering the same frame also for f(n), that is: $f(n) = e^{-\pi \cdot n^2/x}$ we will have from the (D.2) that:

$$\sum_{n=-\infty}^{\infty} e^{-\pi \cdot n^2/x} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt \tag{D.3}$$

Now, we take the integral in the (D.3): $I = \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt$ and we carry out the replacement $t = \sqrt{x}u$:

($u = \frac{t}{\sqrt{x}}$, $dt = \sqrt{x}du$ and t=(-∞,+∞) >>> u=(-∞,+∞) $\rightarrow$

$I = \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt = \sqrt{x}\int_{-\infty}^{\infty} e^{-\pi \cdot u^2 - 2\pi in\sqrt{x}u}du = \sqrt{x}\int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2 - \pi n^2 x}du = \sqrt{x}e^{-\pi n^2 x}\int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2}du$;

now, by the change $v = u + in\sqrt{x}$ (dv=du and the integration ends don't obviously change), we have: $I = \sqrt{x}e^{-\pi n^2 x}\int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2}du = \sqrt{x}e^{-\pi n^2 x}\int_{-\infty}^{\infty} e^{-\pi v^2}dv = 2\sqrt{x}e^{-\pi n^2 x}\int_{0}^{\infty} e^{-\pi v^2}dv$, as we are dealing with an integrand which is an even function. Now, after a second change: $r = \pi v^2$

($dr = 2\pi v dv$, $dv = \frac{dr}{2\pi v} = \frac{dr}{2\pi\sqrt{\frac{r}{\pi}}} = \frac{dr}{2\sqrt{\pi r}}$ and same ends), we have:

$I = 2\sqrt{x}e^{-\pi n^2 x}\int_{0}^{\infty} e^{-\pi v^2}dv = 2\sqrt{x}e^{-\pi n^2 x}\int_{0}^{\infty} e^{-r}\frac{dr}{2\sqrt{\pi r}} = \sqrt{\frac{x}{\pi}}e^{-\pi n^2 x}\int_{0}^{\infty} e^{-r}r^{-\frac{1}{2}}dr = \sqrt{\frac{x}{\pi}}e^{-\pi n^2 x}\Gamma(\frac{1}{2}) = \sqrt{\frac{x}{\pi}}e^{-\pi n^2 x}\sqrt{\pi} =$

$= \sqrt{x}e^{-\pi n^2 x}$, and so the (D.3) yields:

$\sum_{n=-\infty}^{\infty} e^{-\pi \cdot n^2/x} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\pi \cdot t^2/x)-2\pi \,\mathrm{int}}dt = \sqrt{x}\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$, from which: $\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty} e^{-\pi \cdot n^2/x}$, that is the assertion. Being $\theta(z) = \sum_{n=-\infty}^{\infty} e^{-n^2 z}$ the Jacobi Theta Function, we can write also:

$\theta(\pi x) = \frac{1}{\sqrt{x}}\theta(\frac{\pi}{x})$.

SUBAPPENDIX E: The Legendre Duplication Formula.

$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$ (Legendre Duplication Formula) (E.1)

-<u>First proof</u>:

In the Subappendix C we saw that: $\Gamma(x+1) = \lim_{k\to\infty}\frac{k!k^x}{(x+1)...(x+k)}$ and so if we set x=s-1, we get:

$$\Gamma(s) = \lim_{k\to\infty}\frac{k!k^{(s-1)}}{(s)...(s-1+k)} = \lim_{k\to\infty}\frac{1\cdot 2\cdot ....(k-1)kk^{(s-1)}}{(s)...(s-1+k)} = \lim_{k\to\infty}\frac{1\cdot 2\cdot ....(k-1)}{(s)(s+1)...(s-1+k)}k^s . \quad \text{(E.2)}$$

Now, by using the fact that $\lim_{k\to\infty}\frac{k}{(s+k)} = 1$, we can add in the (E.2) k at the numerator and (s+k) at the denominator, so having:

$$\Gamma(s) = \lim_{k\to\infty}\frac{1\cdot 2\cdot ....(k-1)k}{(s)(s+1)...(s-1+k)(s+k)}k^s = \lim_{k\to\infty}\frac{k!}{(s)(s+1)...(s+k)}k^s \quad \text{(E.3)}$$

Now, if in the (E.3) we set s→s+(1/2) , then we get:

$\Gamma(s+\frac{1}{2}) = \lim_{k\to\infty}\frac{k!}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}k^{s+\frac{1}{2}}$ while if we set s→2s , we get:

$\Gamma(2s) = \lim_{k\to\infty}\frac{k!}{(2s)...(2s+k)}k^{2s}$ and in the last one we also set k→2k , as the limit is for k which tends to the infinity, so 2k does, as well, and all stays into the proportions: $\Gamma(2s) = \lim_{k\to\infty}\frac{(2k)!}{(2s)...(2s+2k)}(2k)^{2s}$;

at this point, let us evaluate the quantity $\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}$:

$$\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)} = \lim_{k\to\infty}[\frac{k!}{(s)...(s+k)}k^s\frac{k!}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}k^{s+\frac{1}{2}}\frac{(2s+0)...(2s+2k)}{(2k)!(2k)^{2s}}] =$$

$$= \lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}}{(2k)!}\frac{1}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}\frac{2^{2k+1}(s)...(s+k)}{2^{2s}(s)...(s+k)}] =$$

$$= \frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}2^{2k+1}}{(2k)!}\frac{1}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}] = \frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}2^{2k+1}}{k(2k)!}\frac{k}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}] =$$

$= \frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2 2^{2k+1}}{(2k)!k^{\frac{1}{2}}}\frac{k}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}] = \frac{1}{2^{2s}}const$, that is:

$$\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)} = \frac{1}{2^{2s}}const \quad , \quad \text{(E.4)}$$

where $const = \lim_{k\to\infty}\{\frac{(k!)^2 2^{2k+1}}{(2k)!k^{\frac{1}{2}}}\frac{(k)}{(s+\frac{1}{2})...[s+\frac{1}{2}+(k)]}\}$; in order to find the const , we set into the

(E.4) $s=\frac{1}{2}$: $\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(2\frac{1}{2})} = \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(1)} = \Gamma(\frac{1}{2}) = \sqrt{\pi} = \frac{1}{2^{2\frac{1}{2}}}const = \frac{1}{2}const$, from which:

$const = 2\sqrt{\pi}$ and so the (E.4) yields: $\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)} = \frac{1}{2^{2s}}2\sqrt{\pi} = 2^{1-2s}\sqrt{\pi}$, that is exactly the (E.1), qed.

-<u>Second proof</u>:

-Preamble on the Euler Beta Function:

here it is: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ (E.5)

(convergent for m>0 and n>0)

-We have that: $B(m,n) = B(n,m)$; in fact, by the substitution x=1-y, we have:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^1 (1-y)^{m-1}y^{n-1}dx = \int_0^1 y^{n-1}(1-y)^{m-1}dx = B(n,m)$.

-Moreover, we have that:

$B(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$; (E.6)

In fact, by the substitution $x=\sin^2\theta$ into the (E.5), we get:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}2\sin\theta\cos\theta d\theta =$

$= 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$.

-We also have that: $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$; (E.7)

in fact, by the substitution $t = x^2$ (and of course $dt = 2xdx$) into the Γ function:

$\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dt$, we get: $\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dt = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$ and similarly:

$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy$ and so:

$\Gamma(m)\Gamma(n) = 4(\int_0^\infty e^{-x^2}x^{2m-1}dx)(\int_0^\infty e^{-y^2}y^{2n-1}dy) = 4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dxdy$;

now, if we jump to the polar coordinates ($x=\rho\cos\alpha$, $y=\rho\sin\alpha$), we get:

($dS = dxdy = (\rho d\alpha)d\rho = \rho d\rho d\alpha$)

$\Gamma(m)\Gamma(n) = 4\int_{\alpha=0}^{\pi/2}\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\rho d\alpha =$

$= 4(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho)[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha]$; (E.8)

now, let's consider the former integral: $\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho$; we can also see it like that:

$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho$ and through the following substitution: $\rho^2=g$

(and, as a consequence: $d\rho=\frac{1}{2\rho}dg$), we have:

$$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{2\rho^2}dg=\frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{g}dg=$$

$=\frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)-1}e^{-g}dg=\frac{1}{2}\Gamma(m+n)$ that is, after that we write it again:

$$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)-1}e^{-g}dg=\frac{1}{2}\Gamma(m+n) \tag{E.9}$$

and the (E.9)+(E.6) into the (E.8) yield:

$$\Gamma(m)\Gamma(n)=4(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho)[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha]=4[\frac{1}{2}\Gamma(m+n)][\frac{1}{2}B(m,n)]=$$

$=\Gamma(m+n)B(m,n)$, from which, finally: $B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

-Moreover, the (E.7) into the (E.6) yields:

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\frac{1}{2}B(m,n)=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \tag{E.10}$$

(m>0 and n>0)

After that preamble, we write the Beta Function $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ for z_1 and z_2 :

$B(z_1,z_2)=\int_0^1 x^{z_1-1}(1-x)^{z_2-1}dx$ and with $z_1 = z_2$: $B(z,z)=\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)}=\int_0^1 x^{z-1}(1-x)^{z-1}dx$ and by the substitution $x=\frac{1+u}{2}$ (and $dx=\frac{1}{2}du$ and x=0 → u=-1 + x=1 → u=1) we have:

$B(z,z)=\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)}=\int_0^1 x^{z-1}(1-x)^{z-1}dx=\frac{1}{2}\int_{-1}^1(\frac{1+u}{2})^{z-1}(\frac{1-u}{2})^{z-1}du=\frac{1}{2^{2z-1}}\int_{-1}^1(1-u^2)^{z-1}du$, from which:

$$2^{2z-1}\Gamma(z)\Gamma(z)=\Gamma(2z)\int_{-1}^1(1-u^2)^{z-1}du=2\Gamma(2z)\int_0^1(1-u^2)^{z-1}du\,. \tag{E.11}$$

(the integration ends have changed, as we have an integrand which is an even function)

Now, by carrying out the substitution $x=u^2$ in the previous Euler Beta Function in z_1 , z_2 (and dx=2udu and x=0 → u=0 + x=1 → u=1) we have:

$B(z_1,z_2)=\int_0^1 x^{z_1-1}(1-x)^{z_2-1}dx=2\int_0^1 uu^{2z_1-2}(1-u^2)^{z_2-1}du$ and by setting $z_1=\frac{1}{2}$ and $z_2=z$, we have:

$$B(\frac{1}{2},z)=\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(\frac{1}{2}+z)}=2\int_0^1(1-u^2)^{z-1}du \tag{E.12}$$

and by using now the (E.12) into the (E.11), we get:

$2^{2z-1}\Gamma(z)\Gamma(z)=\Gamma(2z)\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(\frac{1}{2}+z)}$, from which: $\Gamma(z)\Gamma(\frac{1}{2}+z)=2^{-2z+1}\Gamma(2z)\Gamma(\frac{1}{2})$ and by reminding that $\Gamma(\frac{1}{2})=\sqrt{\pi}$, it follows that: $\Gamma(z)\Gamma(\frac{1}{2}+z)=2^{-2z+1}\Gamma(2z)\sqrt{\pi}$, which is nothing but the (E.1), that is the assertion.

SERIE DEL PI GRECO DI RAMANUJAN- DIMOSTRAZIONE

Leonardo ***Rubino***
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Abstract: serie del Pi greco di Ramanujan – dimostrazione, ed altri argomenti di interesse in Appendice I ("i").

$$\boxed{\pi = 1/[\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\cdot\frac{(1103+26390n)}{396^{4n}}]}$$

1 – Nozioni generali:

Le prime cento cifre decimali di Pi greco:
π=3,1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899 8628034825 3421170679

Magari 3000 anni fa si faceva prima a prendere un pezzo di corda lunga quanto il diametro 2R, adagiandolo sulla circonferenza e vedendo che ci volevano 3 corde e un pezzo (virgola 14) per coprire tutta la circonferenza appunto. Ma noi invece partiamo dall'equazione di un cerchio con centro nell'origine (Fig. 1): $x^2 + y^2 = R^2$

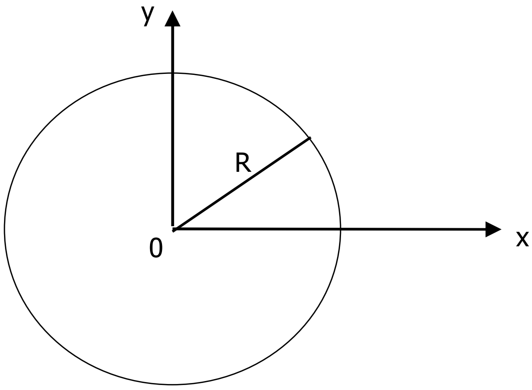


Fig. 1

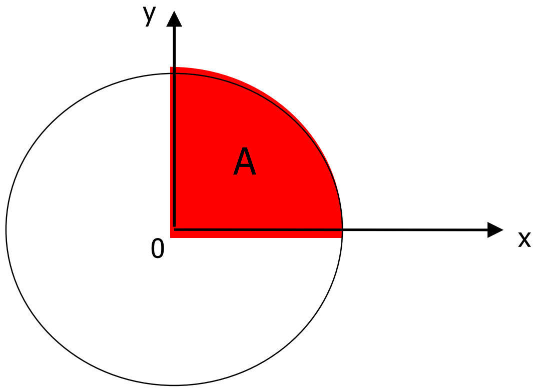


Fig. 2

Se ora al raggio diamo valore R=1, l'equazione implicita diventerà: $x^2 + y^2 = 1$. L'arco di cerchio del primo quadrante sarà espresso da una y ricavata dall'equazione precedente:

$y=+\sqrt{1-x^2}$. Sviluppando quest'ultima in serie di Taylor/MacLaurin, si ha:

$$(f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(0)\cdot x^k)$$

$$\sqrt{1-x^2}=(1-x^2)^{1/2}=1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6+\frac{5}{128}x^8-\frac{7}{256}x^{10}-\frac{21}{1024}x^{12}-.........$$

L'area A sottesa da tale curva (quella rossa in Fig. 2, dove R=1) varrà:

$$A=\frac{1}{4}\pi\cdot R^2=\frac{1}{4}\pi\cdot 1^2=\frac{1}{4}\pi=\int_0^1\sqrt{1-x^2}dx=\int_0^1(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-\frac{7}{256}x^{10}-\frac{21}{1024}x^{12}-...)dx=$$

$$=\left|x-\frac{1}{6}x^3-\frac{1}{40}x^5-\frac{1}{112}x^7-\frac{5}{1152}x^9-\frac{7}{2816}x^{11}-\frac{21}{13312}x^{13}-...\right|_0^1=1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}-\frac{21}{13312}-...$$

da cui: $\boxed{\pi=4(1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}-\frac{21}{13312}-...)}$

Utilizzando ora solo i 7 termini contenuti in quest'ultima formula per π , otteniamo, giusto come esempio, π = 4 x 0,791001164633977….. = 3,164004658535908….. , che fornisce **solo la prima cifra decimale esatta**; si dice che questa è una serie che converge molto lentamente.

-<u>serie di Gregory-Leibniz</u>:

consideriamo il seguente integrale: $\int_0^x\frac{1}{1+x^2}=\arctan x$ e consideriamo anche lo sviluppo di MacLaurin dell'integrando: $\frac{1}{1+x^2}=1-x^2+x^4-...=\sum_{n=0}^{\infty}(-1)^n x^{2n}$, da cui:

$\arctan x=\int_0^x\frac{1}{1+x^2}=\int_0^x(1-x^2+x^4-...)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ e se poniamo x=1, si avrà: $\arctan 1=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\quad\rightarrow\quad\pi=4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...)$; per ottenere 7 cifre corrette di Pi greco dopo la virgola, **occorrono 5.000.000 di termini!!**

-<u>sulla convergenza della serie di Ramanujan</u>:

considerando anche solo il primo termine (n=0) della serie di Ramanujan:

$$\pi=1/[\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\cdot\frac{(1103+26390n)}{396^{4n}}] \tag{1.1}$$

si ottiene: $\pi=1/[\frac{\sqrt{8}}{9801}\frac{(4\cdot 0)!}{(0!)^4}\cdot\frac{(1103+26390\cdot 0)}{396^{4\cdot 0}}]=\frac{9801}{1103\sqrt{8}}\cong 3,14159273....$, **dunque già solo col primo termine abbiamo ben sei cifre decimali esatte!!!**

Considerando invece anche il secondo termine, si ottiene:

$\pi=1/[\frac{\sqrt{8}}{9801}\sum_{n=0}^{2}\frac{(4n)!}{(n!)^4}\cdot\frac{(1103+26390n)}{396^{4n}}]\cong 3,1415926535897938779.....$, **ossia già ben 15 cifre decimali esatte; se ne sono dunque aggiunte altre 9!!!**

Infine, con anche il terzo termine, si ha:

$\pi=1/[\frac{\sqrt{8}}{9801}\sum_{n=0}^{3}\frac{(4n)!}{(n!)^4}\cdot\frac{(1103+26390n)}{396^{4n}}]\cong 3,14159265358979323846264 9065.....$, **ossia altre 8 cifre, per un totale di 23 cifre totali!!!** E così via.

Stabiliamo ora quella che sarà l'**ossatura** della serie del Pi di Ramanujan; in Appendice G abbiamo introdotto gli integrali ellittici:

$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ (Integrale Ellittico di prima specie)

$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\cdot d\theta$ (Integrale Ellittico di seconda specie)

ed in Appendice D abbiamo anche introdotto la Relazione Fondamentale di Legendre: $K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2}$ ed abbiamo anche detto che, denominando il parametro $k'=\sqrt{1-k^2}$, allora definiamo K'(k)=K(k') ed E'(k)=E(k') e, dunque, quando il "parametro" $k=k'=\frac{1}{\sqrt{2}}$, allora, ovviamente, K=K' ed E=E' e la Relazione Fondamentale di Legendre, che corrisponde a:

$$KE'+K'E-KK'=\frac{\pi}{2} \qquad (1.2)$$

diventa, in quel caso: $2KE-K^2=\frac{\pi}{2}$, ossia:

$$\frac{E}{K}-\frac{\pi}{4K^2}=\frac{1}{2}=0{,}5\cong 0{,}3183\boxed{\approx\frac{1}{\pi}}\ . \qquad (1.3)$$

Ora, fuori dal caso $k=k'=\frac{1}{\sqrt{2}}$, valutiamo lo stesso la quantità nell'equazione precedente, ossia la (1.3), che, per similitudine, indichiamo questa volta così:

$$\alpha(r)=\frac{E'(k)}{K(k)}-\frac{\pi}{[2K(k)]^2}\ . \qquad (1.4)$$

Adesso, nessuno ci impedisce di considerare il caso (che vedremo essere vantaggioso) in cui il rapporto tra K' e K sia la radice di un tal numero r: $\sqrt{r}=\frac{K(k')}{K(k)}$; e, tra l'altro, come peraltro abbiamo già fatto anche in Appendice F, consideriamo (nessuno ce lo impedisce) il parametro k come funzione di un tal q (detto "nome"), ossia k=k(q) , dove decidiamo (scelta nostra) di analizzare la casistica in cui esso abbia la forma $q=e^{-\pi\sqrt{r}}$.

(a livello di nomenclatura, vanno bene $\sqrt{r}$, così come $\sqrt{n}$ oppure anche $\sqrt{N}$ …)

Bene, in Appendice H abbiamo dato prova che α(r) della (1.4) tende proprio ad 1/π quando k→0 ; inoltre, dal momento che tale tendenza è piuttosto rapida, ne discende il motivo per cui la serie del Pi di Ramanujan è molto rapida. E, sempre in Appendice H, abbiamo visto che per k→0 si ha che K'(0)=K(1)=∞ e $K(0)=\frac{\pi}{2}$, da cui: $\sqrt{r}=\frac{K(k')}{K(k)}=\frac{\infty}{\pi/2}=\infty$ e, dunque: $\lim_{k\to 0}\alpha(k)=\lim_{r\to\infty}\alpha(r)=\frac{1}{\pi}$.

Inoltre, sempre per Legendre e per l'espressione di $\sqrt{r}$, si ha che:

$$\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\left(\frac{E}{K}-1\right)\ ; \qquad (1.5)$$

infatti, partendo proprio dalla (1.4):

$\alpha(r)=\frac{E'}{K}-\frac{\pi}{(2K)^2}=\frac{4E'K-\pi}{(2K)^2}$, ma per Legendre, ossia per la (1.2), si ha:

$4E'K=-4K'E+4KK'+2\pi$, da cui:

$$\alpha(r)=\frac{4KK'-4EK'+\pi}{(2K)^2}=\frac{4K\sqrt{r}K-4E\sqrt{r}K+\pi}{(2K)^2}=\sqrt{r}-\frac{E\sqrt{r}}{K}+\frac{\pi}{(2K)^2}=\frac{\pi}{(2K)^2}+\sqrt{r}(1-\frac{E}{K})\quad \text{cvd.}$$

Tra parentesi, riscriviamo un attimo la precedente, ossia la (1.5): $\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{1}{K}(E-K)$ e ricordando che, per la (F.6), si ha che $K(k)=\frac{\pi}{2}\theta_3^2(q)$, allora: $\frac{1}{\pi\theta_3^4(q)}=\frac{\pi}{4K^2}$ e allora la (1.5) diventa: $\alpha(r)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{1}{K}(E-K)=\frac{\pi}{(2K)^2}-\sqrt{r}\frac{4K^2}{\pi^2}\frac{\pi^2}{4K^2}\frac{1}{K}(E-K)=$

$$=\frac{1}{\pi\theta_3^4(q)}-\sqrt{r}\frac{4K}{\pi^2}\frac{1}{\theta_3^4(q)}(E-K)\ \rightarrow\ \theta_3^4(q)\alpha(r)=\frac{1}{\pi}-\sqrt{r}\frac{4K}{\pi^2}(E-K)\ \rightarrow$$

$\frac{1}{\pi}=\theta_3^4(q)\alpha(r)+\sqrt{r}\frac{4K}{\pi^2}(E-K)=\theta_3^4(q)\alpha(r)+\sqrt{r}4q\frac{d}{dq}\ln\theta_4(q)=\theta_3^4\alpha(r)+\sqrt{r}4q\frac{d}{dq}\frac{\dot{\theta}_4}{\theta_4}$, ossia:

$\frac{1}{\pi}=\theta_3^4\alpha(r)+\sqrt{r}4q\frac{\dot{\theta}_4}{\theta_4}$, che sarebbe la (H.1), là dimostrata (in Appendice H), dove viene dimostrato anche che: $\frac{4K}{\pi^2}(E-K)=4q\frac{d}{dq}\ln\theta_4=4q\frac{\dot{\theta}_4}{\theta_4}$, che sarebbe la (H.2), equazione, peraltro, che abbiamo appena utilizzato. E con la (H.11) abbiamo là anche ridimostrato la convergenza veloce di α(r) ad 1/π (r→∞), con una tendenza, secondo la (H.13) là dimostrata, del tipo:

$$\alpha(r)-\frac{1}{\pi}\rightarrow 8(\sqrt{r}-\frac{1}{\pi})q=8(\sqrt{r}-\frac{1}{\pi})e^{-\pi\sqrt{r}}\rightarrow 0\quad !!!$$

(tendenza molto veloce, come, conseguentemente, veloce è la convergenza della serie del Pi di Ramanujan, poiché $e^{-\pi\sqrt{r}}$ sopraffà molto velocemente $\sqrt{r}$, per r→∞)

E ricordando (dall'Appendice D) che: $\frac{dK}{dk}=\frac{E-k'^2K}{kk'^2}$, $\frac{dE}{dk}=\frac{E-K}{k}$, da cui:

$E=kk'^2\frac{dK}{dk}+k'^2K$, con $E'=-kk'^2\frac{dK'}{dk}+k^2K'$, si ha infine:

$$\alpha(r)=\frac{\pi}{(2K)^2}+\sqrt{r}(1-\frac{E}{K})=\frac{1}{\pi}(\frac{\pi}{2K})^2+\sqrt{r}(1-\frac{kk'^2\frac{dK}{dk}+k'^2K}{K})=$$

$$=\frac{1}{\pi}(\frac{\pi}{2K})^2+\sqrt{r}[(1-k'^2)-kk'^2\frac{1}{K}\frac{dK}{dk}]=\frac{1}{\pi}(\frac{\pi}{2K})^2-\sqrt{r}(kk'^2\frac{1}{K}\frac{dK}{dk}-k^2)=\alpha(r)\text{, sicché:}$$

$$\boxed{\frac{1}{\pi}=\sqrt{r}kk'^2[(\frac{2}{\pi})^2K\frac{dK}{dk}]+[\alpha(r)-\sqrt{r}k^2](\frac{2K}{\pi})^2\ .}\qquad (1.6)$$

Ricordiamo in anticipo che il nostro target è quello di giungere ad una forma generica di questo tipo: $\frac{1}{\pi}=\sum_{k=0}^{\infty}\frac{(4n)!}{(n!)^4}(G+Hn)x^n$ (x=f(k) e G ed H che sono delle costanti) , (1.7)

poiché questa, infatti, è la forma della serie del Pi di Ramanujan che stiamo cercando di dimostrare.

Prendendo ora in esame la (C.12) in Appendice C: $\frac{2}{\pi}K(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)$ che esprime l'integrale ellittico K in funzione della funzione ipergeometrica, segue che, per le proprietà delle ipergeometriche appunto, si hanno anche le seguenti, simili:

$$\frac{2}{\pi}K(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2) = \frac{1}{k'}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;\frac{-(2kk')^2}{k'^6}) = \frac{1}{k'}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;-(\frac{2k}{k'^2})^2)$$

$$\frac{2}{\pi}K(k) = \frac{1}{\sqrt{k'}}\cdot{}_2F_1(\frac{1}{4},\frac{1}{4};1;-(\frac{k^2}{2k'})^2)$$

$$\boxed{\frac{2}{\pi}K(k) = \frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)} \qquad (1.8)$$

$$\frac{2}{\pi}K(k) = \frac{1}{\sqrt{k'^2-k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;-(\frac{2}{G^{12}+G^{-12}})^2)$$

$$\frac{2}{\pi}K(k) = \frac{1}{[1-kk'^2]^{1/4}}\cdot{}_2F_1(\frac{1}{12},\frac{5}{12};1;\frac{1}{J})$$

dimostrazione: (**dimostrazione completa della sola (1.8), poiché è solo essa che giocherà, per noi, un ruolo fondamentale nel prossimo step**): vedere la (C.31) in Appendice C.

Similmente, prendendo invece in esame la (C.11): $[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))$ segue che, per le proprietà delle ipergeometriche, si hanno anche le seguenti, simili:

$$[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2)) = \frac{1}{k'^2}\cdot{}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{2k}{k'^2})^2) \qquad (1.9)$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{k'}\cdot{}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{k^2}{2k'})^2) \qquad (1.10)$$

$$\boxed{[\frac{2}{\pi}K(k)]^2 = \frac{1}{1+k^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g^{12}+g^{-12}})^2)} \qquad (1.11)$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{k'^2-k^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-(\frac{2}{G^{12}+G^{-12}})^2) \qquad (1.12)$$

$$[\frac{2}{\pi}K(k)]^2 = \frac{1}{\sqrt{k'^2-k^2}}\cdot{}_3F_2(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;\frac{1}{J}) \qquad (1.13)$$

dimostrazione: (**dimostrazione completa della sola (1.11), poiché è solo essa che giocherà, per noi, un ruolo fondamentale nei prossimi steps**): vedere la (C.35) in Appendice C.

A questo punto, il punto chiave è l'osservare che sia le espressioni trovate dei vari $\frac{2}{\pi}K(k)$ che quelle dei vari $[\frac{2}{\pi}K(k)]^2$ hanno un coefficiente numerico in k (m(k)), tipo $\frac{1}{\sqrt{k'^2-k^2}}$ ed una funzione di una sommatoria data dalla ipergeometrica ${}_iF_j$ $\left({}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n\right)$, ossia una: $F(\varphi(k)) = F(\sum_{n=0}^{+\infty}a'_n\varphi^n) = \sum_{n=0}^{+\infty}a_n\varphi^n$, (1.14) cioè, ad esempio:

$$[\frac{2}{\pi}K(k)]^2 = m(k)\cdot\sum_{n=0}^{+\infty}a_n\varphi^n \qquad (1.15)$$

e derivando:

$$(\frac{2}{\pi})^2 2K\frac{dK}{dk} = [\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi}]$$, ossia:

$$(\frac{2}{\pi})^2 K\frac{dK}{dk} = \frac{1}{2}[\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi}] \quad , \qquad (1.16)$$

ed inserendo allora la (1.15) e la (1.16) nella (1.6), otteniamo:

$$\frac{1}{\pi} = \sqrt{r}kk'^2[(\frac{2}{\pi})^2 K\frac{dK}{dk}] + [\alpha(r) - \sqrt{r}k^2](\frac{2K}{\pi})^2 = \sqrt{r}kk'^2[\frac{1}{2}(\frac{dm}{dk}F + m\frac{d\varphi}{dk}\frac{dF}{d\varphi})] + [\alpha(r) - \sqrt{r}k^2]mF =$$

$$= \frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk}F + \frac{1}{2}\sqrt{r}kk'^2 m\frac{d\varphi}{dk}\frac{dF}{d\varphi} + [\alpha(r) - \sqrt{r}k^2]mF =$$

$$= \{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\mathrm{F} + (\frac{1}{2}m\sqrt{r}kk'^2\frac{d\varphi}{dk})\frac{dF}{d\varphi}$$ ed utilizzando in quest'ultima la (1.14),

ossia la: $F = \sum_{n=0}^{+\infty}a_n\varphi^n$, nonché la sua derivata: $\frac{dF}{d\varphi} = n\sum_{n=0}^{+\infty}a_n\varphi^{n-1}$, si ha:

$$\frac{1}{\pi} = \{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\sum_{n=0}^{+\infty}a_n\varphi^n + n(\frac{1}{2}m\sqrt{r}kk'^2\frac{d\varphi}{dk})\sum_{n=0}^{+\infty}a_n\varphi^{n-1} =$$

$$= \sum_{n=0}^{+\infty}a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}\varphi^n + \sum_{n=0}^{+\infty}a_n\varphi^{n-1}\varphi[\frac{m}{\varphi}\frac{1}{2}n\sqrt{r}kk'^2\frac{d\varphi}{dk}] =$$

$$= \frac{1}{\pi} = \sum_{n=0}^{+\infty}a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2] + \frac{1}{2}n\sqrt{r}kk'^2\frac{m}{\varphi}\frac{d\varphi}{dk}\}\varphi^n \qquad (1.17)$$

e questa (1.17) appena ottenuta altro non è che la (1.7): $\frac{1}{\pi} = \sum_{k=0}^{\infty}\frac{(4n)!}{(n!)^4}(G + Hn)x^n$, con:

$$\frac{(4n)!}{(n!)^4}G = a_n\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk} + m[\alpha(r) - \sqrt{r}k^2]\}$$ e

$$\frac{(4n)!}{(n!)^4}H\cdot n = [\frac{1}{2}a_n\sqrt{r}kk'^2\frac{m}{\varphi}\frac{d\varphi}{dk}]n \quad .$$

Adesso, ricordiamo che in Appendice G abbiamo definito le:

$$G = G_n = (\frac{1}{2kk'})^{\frac{1}{12}} \quad \text{e} \qquad (1.18)$$

$$g = g_n = (\frac{k'^2}{2k})^{\frac{1}{12}} \quad , \qquad (1.19)$$

ossia due relazioni di k e k' , che danno i due invarianti di cui sopra (G e g); invarianti poiché funzioni di due parametri fissabili, ossia appunto k e k' .

Combinando poi quelle espressioni, si definiscono anche i seguenti invarianti e le seguenti espressioni ulteriori:

$$J = \frac{(4G^{24} - 1)^3}{27G^{24}} = \frac{(4g^{24} + 1)^3}{27g^{24}} = \frac{4}{27}\frac{[1 - (kk')^2]^3}{(kk')^4} \quad \text{(Invariante Assoluto di Klein)} \qquad (1.20)$$

(a noi non utile, in questa sede)

$$k = g^6\sqrt{g^{12}+g^{-12}} - g^{12} = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} - \sqrt{1-\frac{1}{G^{12}}}\right) \qquad (1.21)$$

(giustificata in Appendice G)

$$k' = \sqrt{2k}\,g^6 = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} + \sqrt{1-\frac{1}{G^{12}}}\right) \quad . \qquad (1.22)$$

(giustificata in Appendice G)

Denominando ora r=N nella (1.17), nonché $k=k_N$, a causa del fatto che, come scegliemmo volontariamente più sopra: $k = k(q) = k(e^{-\pi\sqrt{r}}) = k(e^{-\pi\sqrt{N}}) = k(N) = k_N$, si ha:

$$\frac{1}{\pi} = \sum_{n=0}^{+\infty} a_n \left\{\frac{1}{2}\sqrt{N}k_N k'^2_N \frac{dm}{dk} + m[\alpha(N) - \sqrt{N}k_N^2] + \frac{1}{2}n\sqrt{N}k_N k'^2_N \frac{m}{\varphi}\frac{d\varphi}{dk}\right\}\varphi^n \qquad (1.23)$$

(ed anche le g e G diventano g_N e G_N) e tornando anche un passo indietro, alla (1.6), si ha:

$$\frac{1}{\pi} = \sqrt{N}k_N k'^2_N \left[\left(\frac{2}{\pi}\right)^2 K \frac{dK}{dk_N}\right] + [\alpha(N) - \sqrt{N}k_N^2]\left(\frac{2K}{\pi}\right)^2 \qquad (1.24)$$

e se usiamo le (1.9)-(1.13), la (1.8) ed attigue e la (1.24), si ottengono tutte le seguenti:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\left\{\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{G_N^{12}}\right)^{2n}[\alpha(N) - \sqrt{N}k_N^2 + n\sqrt{N}(k'^2_N - k_N^2)]\right\} \quad . \qquad (1.25)$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\left\{(-1)^n\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{g_N^{12}}\right)^{2n}\left[\frac{\alpha(N)}{k'^2_N} + n\sqrt{N}\left(\frac{1+k_N^2}{k'^2_N}\right)\right]\right\} \qquad (1.26)$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3 \left(\frac{1}{g_{4N}^{12}}\right)^{2n}\left\{\left[\alpha(N) - \sqrt{N}\frac{k_N^2}{2}\right]\frac{1}{k'_N} + n\sqrt{N}\left(k'_N + \frac{1}{k'_N}\right)\right\} \qquad (1.27)$$

$$\left(\qquad g_{4N}^{12} = 2^{1/4} g_N G_N \qquad \right)$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3}\left[\frac{\alpha(N)}{x_N(1+k_N^2)} - \frac{\sqrt{N}}{4g_N^{12}} + n\sqrt{N}\frac{(g_N^{12} - g_N^{-12})}{2}\right]x_N^{2n+1} \qquad (1.28)$$

dove $$x_N = \frac{2}{(g_N^{12} + g_N^{-12})} = \frac{4k_N k'^2_N}{(1+k_N^2)^2} \qquad (1.29)$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3}\left[\frac{\alpha(N)}{y_N(k'^2_N - k_N^2)} + \sqrt{N}\frac{k_N^2 G_N^{12}}{2} + n\sqrt{N}\frac{(G_N^{12} + G_N^{-12})}{2}\right]y_N^{2n+1} \qquad (1.30)$$

dove $$y_N = \frac{2}{(G_N^{12} - G_N^{-12})} = \frac{4k_N k'^2_N}{1-(2k_N k'_N)^2}$$

$$\boxed{\frac{1}{\pi}=\frac{1}{3\sqrt{3}}\sum_{n=0}^{\infty}\frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{(n!)^3}\{2[\alpha(N)-\sqrt{N}k_N^2](4G_N^{24}-1)+\sqrt{N}\sqrt{1-\frac{1}{G_N^{24}}}+2n\sqrt{N}(8G_N^{24}+1)\sqrt{1-\frac{1}{G_N^{24}}}\}(\frac{1}{J_N^{1/2}})^{2n+1}}$$

(1.31)

Diamo ora dimostrazione completa solo della (1.28), poiché essa è quella per noi importante, che ci condurrà alla Serie del Pi di Ramanujan; dunque, riscriviamola:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{\alpha(N)}{x_N(1+k_N^2)}-\frac{\sqrt{N}}{4g_N^{12}}+n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}]x_N^{2n+1}$$

con $x_N=\frac{2}{(g_N^{12}+g_N^{-12})}=\frac{4k_Nk'^2_N}{(1+k_N^2)^2}$ e dalle due si ottiene, ovviamente:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\underline{\alpha(N)\frac{1}{(1+k_N^2)}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}}-\underline{\frac{\sqrt{N}}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}}+\underline{n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}}]$$

(1.32)

<u>dimostrazione della (1.32)</u>:

come consuetudine, dunque, partiamo sempre dalla (1.6) completata a mo' della (1.24); si ha:

$$\frac{1}{\pi}=\sqrt{N}k_Nk'^2_N[(\frac{2}{\pi})^2K\frac{dK}{dk_N}]+[\alpha(N)-\sqrt{N}k_N^2](\frac{2K}{\pi})^2 \qquad (1.33)$$

e se usiamo la (1.11):

$$[\frac{2}{\pi}K]^2=\frac{1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2) \quad\rightarrow$$

$$\frac{d}{dk_N}(\frac{2K}{\pi})^2=\frac{4}{\pi^2}2K\frac{dK}{dk_N}=\frac{d}{dk_N}[\frac{1}{1+k_N^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g_N^{12}+g_N^{-12}})^2)]=$$

$$=\frac{d}{dk_N}[\frac{1}{1+k_N^2}\cdot\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}]=\frac{d}{dk_N}[\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{1+k_N^2}(\frac{4k_Nk'^2_N}{(1+k_N^2)^2})^{2n}]=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{d}{dk_N}\{\frac{1}{1+k_N^2}[\frac{4k_Nk'^2_N}{(1+k_N^2)^2}]^{2n}\}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{d}{dk_N}\{\frac{1}{1+k_N^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}\}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{-\frac{2k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+\frac{1}{1+k_N^2}8n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^3}]\}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{(1+k_N^2)^2}\{-2k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+8n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]\};$$

infatti:

$$\frac{d}{dk_N}(\frac{4k_N-4k_N^3}{(1+k_N^2)^2})^{2n}=\frac{d}{dk_N}[(4k_N-4k_N^3)(1+k_N^2)^{-2}]^{2n}=$$

$$=2n[(4k_N-4k_N^3)(1+k_N^2)^{-2}]^{2n-1}[-4k_N(4k_N-4k_N^3)(1+k_N^2)^{-3}+4(1-3k_N^2)(1+k_N^2)^{-2}]=$$

$$=2n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[-16k_N\frac{(k_N-k_N^3)}{(1+k_N^2)^3}+4\frac{(1-3k_N^2)}{(1+k_N^2)^2}]=$$

$$= 2n[4(k_N - k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-16k_N(k_N - k_N^3) + 4(1-3k_N^2)(1+k_N^2)}{(1+k_N^2)^3}] =$$

$$= 2n[4(k_N - k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-16k_N^2 + 16k_N^4 + 4 + 4k_N^2 - 12k_N^2 - 12k_N^4}{(1+k_N^2)^3}] =$$

$$= 8n[4(k_N - k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2 + k_N^4 + 1}{(1+k_N^2)^3}]$$

e infine:

$$(\frac{2}{\pi})^2 K \frac{dK}{dk_N} = \frac{1}{2}\frac{d}{dk_N}(\frac{2K}{\pi})^2 = \frac{1}{2}\frac{4}{\pi^2} 2K \frac{dK}{dk_N} =$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} \frac{1}{(1+k_N^2)^2} \{-k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n} + 4n[4(k_N - k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2 + k_N^4 + 1)}{(1+k_N^2)^2}]\} \tag{1.34}$$

essendo dunque che: $\frac{1}{\pi} = \sqrt{N} k_N k'^2_N [(\frac{2}{\pi})^2 K \frac{dK}{dk_N}] + [\alpha(N) - \sqrt{N} k_N^2](\frac{2K}{\pi})^2 =$

$= \boxed{\alpha(N)(\frac{2K}{\pi})^2} - \boxed{\sqrt{N} k_N^2 (\frac{2K}{\pi})^2} + \boxed{\sqrt{N} k_N k'^2_N [(\frac{2}{\pi})^2 K \frac{dK}{dk_N}]}$, allora il primo blocco in $\alpha(N)$ è:

$$\boxed{\alpha(N)(\frac{2K}{\pi})^2} = \alpha(N)\frac{1}{1+k_N^2} \cdot {}_3F_2(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; 1, 1; (\frac{2}{g_N^{12} + g_N^{-12}})^2) =$$

$= \alpha(N) \frac{1}{1+k_N^2} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} (\frac{2}{g_N^{12} + g_N^{-12}})^{2n}$, che è proprio lo stesso blocco della (1.32) che vogliamo dimostrare.

Riguardo invece il secondo blocco in $\sqrt{N}$:

$$-\boxed{\sqrt{N} k_N^2 (\frac{2K}{\pi})^2} = \sqrt{N} k_N^2 \frac{-1}{1+k_N^2} \cdot {}_3F_2(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; 1, 1; (\frac{2}{g_N^{12} + g_N^{-12}})^2) = \sqrt{N} k_N^2 \frac{-1}{1+k_N^2} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} (\frac{2}{g_N^{12} + g_N^{-12}})^{2n} =$$

$$= \sqrt{N} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} \frac{-k_N^2}{1+k_N^2} (\frac{2}{g_N^{12} + g_N^{-12}})^{2n} =$$

$$= -\sqrt{N} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} [\frac{(1+k_N^2)}{2}] \frac{1}{4g_N^{12}} \frac{2}{(g_N^{12} + g_N^{-12})} (\frac{2}{g_N^{12} + g_N^{-12}})^{2n}$$

$$= -\sqrt{N} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} [\frac{(1+k_N^2)}{2}] \frac{1}{4g_N^{12}} \frac{2^{2n+1}}{(g_N^{12} + g_N^{-12})^{2n+1}} ; \tag{1.35}$$

infatti: $k_N^2 \frac{1}{1+k_N^2} = \{[\frac{(1+k_N^2)}{2}] \frac{1}{4g_N^{12}} \frac{2}{(g_N^{12} + g_N^{-12})}\}$, poiché essendo che: $\frac{2}{(g_N^{12} + g_N^{-12})} = \frac{4k_N k'^2_N}{(1+k_N^2)^2}$ e,

per la (G.1): $g_N = (\frac{k'^2_N}{2k_N})^{\frac{1}{12}} \rightarrow g_N^{12} = (\frac{k'^2_N}{2k_N})$, segue che:

$$[\frac{(1+k_N^2)}{2}]\frac{1}{4g_N^{12}}\frac{2}{(g_N^{12}+g_N^{-12})}=[\frac{(1+k_N^2)}{2}]\frac{k_N}{2k'^2_N}\frac{4k_N k'^2_N}{(1+k_N^2)^2}=\frac{k_N^2}{(1+k_N^2)} \ \ !!!$$

e se si somma algebricamente alla (1.35) anche un altro contributo in $\sqrt{N}$ proveniente dalla (1.36) del punto successivo qui sotto, su $\frac{dK}{dk_N}$, ossia:

$$\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}$$, si ottiene:

$$-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}+\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}+k_N k'^2_N\frac{k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}\}=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\{\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}+\frac{1}{4}k_N\frac{4k_N k'^2_N}{(1+k_N^2)^2}[\frac{4k_N k'^2_N}{(1+k_N^2)^2}]^{2n}\}=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}+\frac{1}{4}k_N\frac{2}{g_N^{12}+g_N^{-12}}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n}]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}\frac{2^{2n+1}}{(g_N^{12}+g_N^{-12})^{2n+1}}+\frac{1}{4}k_N(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}[\frac{(1+k_N^2)}{2}\frac{1}{4g_N^{12}}+\frac{1}{4}k_N]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2}+k_N g_N^{12}]$$ e sapendo noi che: $g_N=(\frac{k'^2_N}{2k_N})^{\frac{1}{12}}$,

segue che: $......=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2}+k_N(\frac{k'^2_N}{2k_N})]=$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2}+\frac{k'^2_N}{2}]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}[\frac{(1+k_N^2)}{2}+\frac{(1-k_N^2)}{2}]=$$

$$=-\sqrt{N}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}\frac{1}{4g_N^{12}}$$, che è proprio lo stesso blocco della (1.32) che vogliamo dimostrare.

Riguardo infine il terzo blocco contenente $\frac{dK}{dk_N}$:

$$\boxed{\sqrt{N}k_N k'^2_N[(\frac{2}{\pi})^2 K\frac{dK}{dk_N}]}=$$

$$=\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{1}{(1+k_N^2)^2}\{-k_N[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+4n[4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]\}=$$

$$=\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}\frac{-k_N}{(1+k_N^2)^2}[\frac{4k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n}+$$

$$+\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1} \qquad (1.36)$$

dunque, per riassumere:

-il primo blocco in $\alpha(N)$ è stato dimostrato;

-riguardo il blocco in $\sqrt{N}$, vi sono due contributi che sono stati sommati algebricamente ed abbiamo ottenuto proprio l'espressione della (1.32);

-e resta infine un blocco in $n\sqrt{N}$ dato dal termine residuo della (1.36), ossia:

$$+\sqrt{N}k_N k'^2_N\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}=$$

$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}n\sqrt{N}\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$, che è proprio il termine in $n\sqrt{N}$ nella (1.32) che ci proponiamo di dimostrare, poiché, dal confronto, si ha proprio che:

$$k_N k'^2_N\frac{4}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4(k_N-k_N^3)(1+k_N^2)^{-2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{4k_N k'^2_N}{(1+k_N^2)^2}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4\frac{(k_N-k_N^3)}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][4\frac{k_N(1-k_N^2)}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{4k_N k'^2_N}{(1+k_N^2)^2}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$\frac{2}{(g_N^{12}+g_N^{-12})}[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{2}{(g_N^{12}-g_N^{-12})}]^{2n-1}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}][\frac{2}{(g_N^{12}+g_N^{-12})}]^{2n}=\frac{(g_N^{12}-g_N^{-12})}{2}(\frac{2}{g_N^{12}+g_N^{-12}})^{2n+1}$$

$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]=\frac{g_N^{12}-g_N^{-12}}{g_N^{12}+g_N^{-12}} \qquad (1.37)$$

ma essendo che, per la (G.1): $\frac{1}{g_N^{12}}=\frac{2k_N}{1-k_N^2}$, da cui: $g_N^{-24}=\frac{4k_N^2}{(1-k_N^2)^2}$, segue che, per la (1.37):

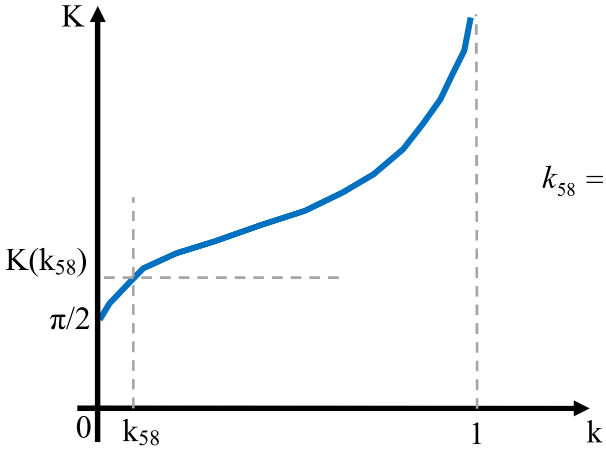
$$[\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}]=\frac{g_N^{12}-g_N^{-12}}{g_N^{12}+g_N^{-12}}=\frac{1-\frac{g_N^{-12}}{g_N^{12}}}{1+\frac{g_N^{-12}}{g_N^{12}}}=\frac{1-g_N^{-24}}{1+g_N^{-24}}=\frac{1-\frac{4k_N^2}{(1-k_N^2)^2}}{1+\frac{4k_N^2}{(1-k_N^2)^2}}=\frac{(1-k_N^2)^2-4k_N^2}{(1-k_N^2)^2+4k_N^2}=\frac{1+k_N^4-2k_N^2-4k_N^2}{1+k_N^4-2k_N^2+4k_N^2}=$$

$$=\frac{1+k_N^4-6k_N^2}{1+k_N^4+2k_N^2}=\frac{-6k_N^2+k_N^4+1}{(1+k_N^2)^2}$$ $\rightarrow$ ossia la (1.37) dà proprio 1=1, ossia è vera!!!!!

Valutiamo ora la (1.32) per N=58 , ossia:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}] \tag{1.38}$$

innanzitutto, graficando in qualche modo K(k), tramite le tabelle disponibili di valori o tramite il suo sviluppo in serie, si ottiene un qualcosa del genere:



$k_{58}=0{,}0000255482707576$

Fig. 3.

Definimmo poi: $\sqrt{r}=\frac{K(k')}{K(k)}$ (dopo la (1.4)), che, tradotta nella nomenclatura raggiunta attualmente, sarebbe: $\sqrt{58}=\frac{K(k'_{58})}{K(k_{58})}=\frac{K(\sqrt{1-k_{58}^2})}{K(k_{58})}$.

Poi, con la (C.16) abbiamo dimostrato che: $\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}=\frac{(4n)!}{4^{4n}(n!)^4}$; ciò scaturisce anche dal confronto tra il formalismo sui Simboli di Pochammer e quello dei fattoriali; confronto che è stato effettuato, ad esempio in Appendice C per giungere alla (C.6). Con ciò, la (1.38) diviene:

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}]\ ; \tag{1.39}$$

presentiamo adesso le seguenti uguaglianze, giustificate a fine Appendice G:

$$\frac{g_{58}^{12}+g_{58}^{-12}}{2}=9801 \tag{1.40}$$

$$k_{58}=0{,}0000255482707576 \tag{1.41}$$

$$\alpha(58)=0{,}3183192742 \tag{1.42}$$

$$\frac{1}{g_{58}^{12}}=\frac{2k_{58}}{1-k_{58}^2}=0{,}00005109654154 \tag{1.43}$$

e: $\frac{(g_{58}^{12}-g_{58}^{-12})}{2}=9801$; (1.44)

con tali corrispondenze, ossia le (1.40)-----(1.44) e facendo due conti, segue, per la (1.39):

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{\alpha(58)}{(1+k_{58}^2)}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n}-\frac{\sqrt{58}}{4g_{58}^{12}}\frac{2^{2n+1}}{(g_{58}^{12}+g_{58}^{-12})^{2n+1}}+n\sqrt{58}\frac{(g_{58}^{12}-g_{58}^{-12})}{2}(\frac{2}{g_{58}^{12}+g_{58}^{-12}})^{2n+1}]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}[\frac{0,3183192742}{1+6,5271413870\cdot10^{-10}}(\frac{1}{9801})^{2n}-\frac{\sqrt{58}}{4}0,00005109654154(\frac{1}{9801})^{2n+1}+$$
$$+n\sqrt{58}\cdot9801(\frac{1}{9801})^{2n+1}]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0,3183192739-\frac{\sqrt{58}}{4}0,00005109654154(\frac{1}{9801})+n\sqrt{58}\cdot9801(\frac{1}{9801})]$$

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0,3183192739-9,9260194588\cdot10^{-9}+n7,6157731058]$$

$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}(\frac{1}{9801})^{2n}[0,3183192639+n7,6157731058]$; ora, moltiplicando per il fattore neutro (unitario) $\frac{\sqrt{8}}{9801}\frac{9801}{\sqrt{8}}\frac{396^{4n}}{396^{4n}}$, si ottiene: $(\frac{396\cdot99}{4}=9801)$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{4n}}{4^{4n}\sqrt{8}}(\frac{4}{396\cdot99})^{2n-1}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{4n}}{4^{4n}\sqrt{8}}(\frac{1}{396})^{2n-1}(\frac{4}{99})^{2n-1}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{396^{2n+1}}{4^{4n}\sqrt{8}}(\frac{4}{99})^{2n-1}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{(99\cdot4)^{2n+1}}{4^{4n}\sqrt{8}}\frac{4^{2n-1}}{99^{2n-1}}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^{2n+1}4^{2n+1}}{4^{4n}\sqrt{8}}\frac{4^{2n-1}}{99^{2n-1}}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^2 4^{4n}}{4^{4n}\sqrt{8}}[0,3183192639+n7,6157731058]$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}\frac{99^2}{\sqrt{8}}[0,3183192639+n7,6157731058]$$

$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{1}{396^{4n}}[\frac{99^2}{\sqrt{8}}0,3183192639...+n\frac{99^2}{\sqrt{8}}7,6157731058...]$ e infine:

$$\boxed{\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{+\infty}\frac{(4n)!}{(n!)^4}\frac{[1103+26390\text{n}]}{396^{4n}}} \quad (1.45)$$

APPENDICI:

APPENDICE A:
Integrale Ellittico di Prima Specie K(k) (Pendolo – soluzione non approssimata)

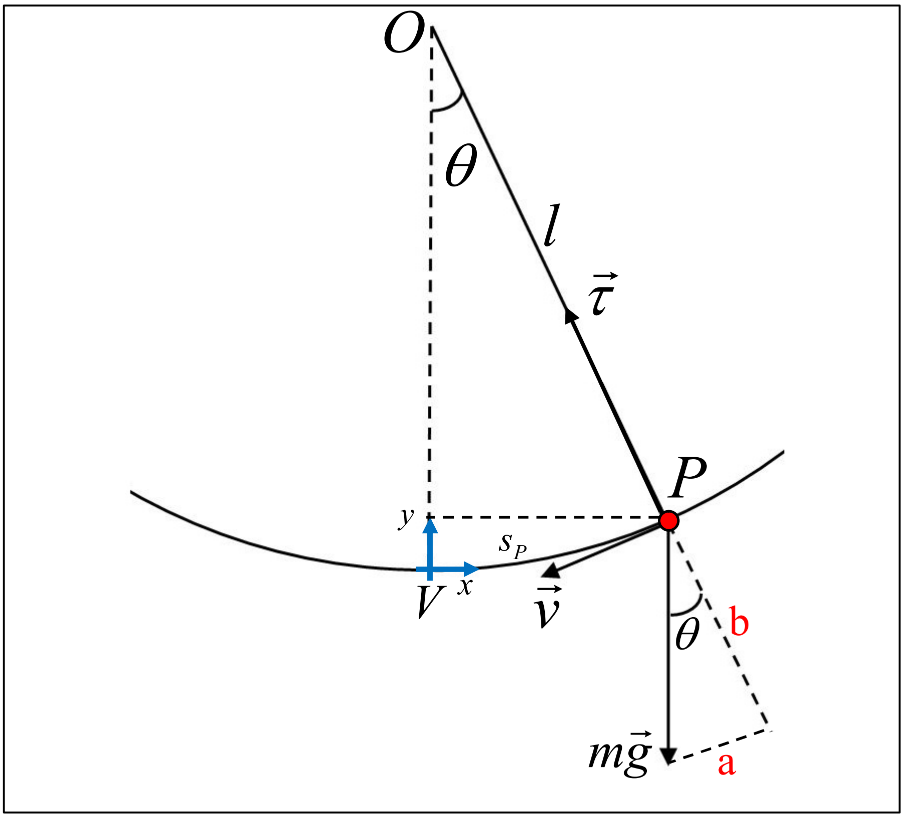


Fig. A.1: Pendolo

A) <u>Analisi intuitiva:</u>
Con riferimento alla figura A.1 qui sopra, si ha che sulla massa m in P agisce la forza peso $m\vec{g}$; essa si scompone su **a** e **b**; su **b** (insieme con il moto) va a contribuire alla tensione $\vec{\tau}$ del filo e non genera moto radiale (OP=const), mentre su **a** va a generare il moto (tangenziale) di rotazione:

$$mg\sin\theta = -m\frac{dv}{dt} = -m\frac{d}{dt}(\omega l) = -m\frac{d}{dt}(\frac{d\theta}{dt}l) = -ml\frac{d^2\theta}{dt^2}$$

(vi è il segno – poiché al diminuire di θ si ha un aumento di v)

Si ottiene così la seguente: $\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$. (A.1)

Nell'approssimazione $\sin\theta \approx \theta$, ossia per angoli di oscillazione piccoli, la (A.1) diventa:

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0}$$

Come è semplice verificare per sostituzione diretta, la soluzione è:
$\theta(t) = c\sin(\omega t + d)$,

con c e d qualsiasi, da determinare in base alle condizioni iniziali, e: $\omega = \sqrt{\frac{g}{l}}$.

Allora, visto che $\omega = \frac{2\pi}{T}$, il periodo delle oscillazioni è:

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}\ .$$

Se si vuole ad esempio un periodo di 1s, allora ℓ=25cm.

Si ha, dunque: $\theta'(t) = c\omega\cos(\omega t + d)$ e $\theta''(t) = -c\omega^2\sin(\omega t + d)$.

L'angolo con cui il pendolo è stato abbandonato alle oscillazioni lo si ha ovviamente per t=0: $\theta(0) = c\sin d$ (nonché: $\theta'(0) = c\omega\cos d$) e se nel momento in cui abbandono il pendolo esso è fermo, allora $\theta'(0) = c\omega\cos d = 0$, da cui: $d = \pi/2$ e per l'angolo iniziale θ_0 abbiamo dunque:

$\theta(0) = \theta_0 = c\sin(\frac{\pi}{2}) = c$ e infine:

$$\boxed{\theta(t) = \theta_0\sin(\omega t + \frac{\pi}{2})}$$

Pendolo – soluzione NON approssimata.

Ripartiamo dalla (A.1): $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$ e poniamo: $\frac{d\theta}{dt} = \eta$, da cui:

$\frac{d^2\theta}{dt^2} = \frac{d\eta}{dt} = \frac{d\eta}{d\theta}\frac{d\theta}{dt} = \eta\frac{d\eta}{d\theta} \rightarrow \quad \eta\frac{d\eta}{d\theta} = -\frac{g}{l}\sin\theta \quad \xrightarrow{\int} \quad \frac{\eta^2}{2} = \frac{g}{l}\cos\theta + const$. Ora, dal momento che il pendolo viene lasciato oscillare partendo da fermo, si ha che quando $\theta=\theta_0 \rightarrow \eta=0$ e allora: $const = -\frac{g}{l}\cos\theta_0 \quad \rightarrow \quad \eta^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) \quad \rightarrow \quad \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$.

Adesso, sempre con riferimento alla Fig. A.1, considerando il primo quarto di periodo, ossia da sopra il punto P fino al punto di valle V , vediamo che θ va a diminuire (dθ<0), da cui $\frac{d\theta}{dt} < 0$ e ciò significa che nell'ultima equazione deve rimanere unicamente il segno meno, ossia:

$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$ da cui: $t = -\sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}$. Osservando adesso che rilasciamo il pendolo a t=0 (e $\theta=\theta_0$) , mentre ad 1/4 di periodo (t=T/4) siamo a valle (punto V→ θ=0), allora: $T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}$. Ricordando ora, dalla trigonometria, che:

$$\cos\theta = 2\sin^2(\frac{\theta}{2}) - 1\ , \tag{A.2}$$

segue che: $$T = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{[\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})]}}\ . \tag{A.3}$$

Poniamo ora: $$\sin(\frac{\theta}{2}) = \sin(\frac{\theta_0}{2})\sin\beta \tag{A.4}$$

e deriviamo, ottenendo:

$$\frac{1}{2}\cos(\frac{\theta}{2})d\theta = \sin(\frac{\theta_0}{2})(\cos\beta)d\beta\ ; \tag{A.5}$$

ponendo ora $\sin(\frac{\theta_0}{2}) = k$, la (A.4) diventa: $\sin(\frac{\theta}{2}) = k\sin\beta$ e la (A.5) fornisce:

$$\frac{1}{2}\cos(\frac{\theta}{2})d\theta = k(\cos\beta)d\beta \rightarrow$$

$$d\theta = \frac{2k(\cos\beta)d\beta}{\cos(\frac{\theta}{2})} = \frac{2k(\cos\beta)d\beta}{\sqrt{1-\sin^2(\frac{\theta}{2})}} = \frac{2k(\cos\beta)d\beta}{\sqrt{1-k^2\sin^2\beta}} \qquad (A.6)$$

Adesso, dalla (A.3) si evince che: $\theta = 0 \rightarrow \beta = 0$ e $\theta = \theta_0 \rightarrow \beta = \pi/2$, fatti questi che ci permetteranno di decidere gli estremi di integrazione della (A.5).
Inserendo tale dθ della (A.6) nella (A.3), si ha:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{k(\cos\beta)}{\sqrt{[k^2-\sin^2(\frac{\theta}{2})]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{k(\cos\beta)}{k\sqrt{[1-\frac{1}{k^2}\sin^2(\frac{\theta}{2})]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} =$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{(\cos\beta)}{\sqrt{[1-\sin^2\beta]}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = T$$. Riscriviamola:

$$T = 4\sqrt{\frac{l}{g}}\boxed{\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}} = 4\sqrt{\frac{l}{g}}K(k) \quad \text{c.v.d.} \qquad (A.7)$$

Per piccole oscillazioni ($k \cong 0$), si ha: $T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}d\beta = 2\pi\sqrt{\frac{l}{g}}$, ossia la formula già ottenuta.

L'integrale K(k) nella (A.7) è un Integrale Ellittico (di prima specie) e la soluzione esatta non esiste; la nostra soluzione dunque è:

$$T = 2\pi\sqrt{\frac{l}{g}}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...] \qquad (|k| = \left|\sin(\frac{\theta_0}{2})\right| < 1) \qquad (A.8)$$

<u>Dimostrazione:</u>
sviluppando in serie di Taylor la funzione $f(x) = (1+x)^z$, calcolando le varie derivate prime, seconde ecc, nel punto x=0, si ottiene:

$(1+x)^z = 1 + zx + \frac{z(z-1)}{2\cdot1}x^2 + \frac{z(z-1)(z-2)}{3\cdot2\cdot1}x^3 + ...$ e con z=-1/2 , si ha:

$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 - \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + ...$; ora, con $x = -k^2\sin^2\beta$, la (A.7) fornisce:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}[1+\frac{1}{2}k^2\sin^2\beta+\frac{1}{2}\frac{3}{4}k^4\sin^4\beta+...]d\beta =$$

$$= 4\sqrt{\frac{l}{g}}[\int_0^{\pi/2}\sin^0\beta d\beta + \frac{1}{2}k^2\int_0^{\pi/2}\sin^2\beta d\beta + \frac{1}{2}\frac{3}{4}k^4\int_0^{\pi/2}\sin^4\beta d\beta + ...] \qquad (A.9)$$

Consideriamo ora l'integrale $I = \int_0^{\pi/2}\sin^{2n}\beta d\beta$, esso vale:

$$I = \int_0^{\pi/2}\sin^{2n}\beta d\beta = \frac{1\cdot3\cdot5\cdot...(2n-1)}{2\cdot4\cdot6\cdot...(2n)}\frac{\pi}{2} \qquad (A.10)$$

e di quest'ultima andremo a fornire la dimostrazione; bene, la (A.9) allora diventa:

$$T = 4\sqrt{\frac{l}{g}}[\int_0^{\pi/2}\sin^0\beta d\beta + \frac{1}{2}k^2\int_0^{\pi/2}\sin^2\beta d\beta + \frac{1}{2}\frac{3}{4}k^4\int_0^{\pi/2}\sin^4\beta d\beta + ...] =$$

$$= 2\pi\sqrt{\frac{l}{g}}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...] \qquad \text{(A.11)}$$

che altro non è che la (A.8).

Per dimostrare, come promesso, anche la (A.10), possiamo seguire due strade; iniziamo con la più grezza:

1) primo metodo (grezzo):

si valutano i singoli integrali I(n), per i vari n, e si constata che numericamente sussiste appunto la (A.10); per fare ciò, ricordiamo preliminarmente la formula per la derivazione di un prodotto di funzioni, da cui si ricava poi la formula per la integrazione per parti: $h(x) = f(x)\cdot g(x) \rightarrow$

$$dh(x) = df(x)\cdot g(x) + f(x)\cdot dg(x) \quad \xrightarrow{\int_A^B}$$

$$\xrightarrow{\int_A^B} \quad \int_A^B dh(x) = \int_A^B d[f(x)\cdot g(x)] = \left|f(x)\cdot g(x)\right|_A^B = \int_A^B g(x)\frac{df(x)}{dx}dx + \int_A^B f(x)\frac{dg(x)}{dx}dx =$$

$= \int_A^B f'(x)g(x)dx + \int_A^B f(x)g'(x)dx$, da cui:

$$\int_A^B f(x)g'(x)dx = \left|f(x)\cdot g(x)\right|_A^B - \int_A^B f'(x)g(x)dx \qquad \text{(A.12)}$$

Poniamo ora A=0 e B=π/2 , nonché $f(x) = \sin^{n-1}(x)$ e $g'(x) = \sin(x)$, da cui, per la (A.12):

$$\int_0^{\pi/2}\sin^{n-1}(x)\sin(x)dx = \left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)\cos(x)dx =$$

$$= \left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)[1-\sin^2(x)]dx =$$

$= \left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)dx - (n-1)\int_0^{\pi/2}\sin^{n}(x)dx$, ossia, riscrivendola:

$\int_0^{\pi/2}\sin^{n}(x)dx = \left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)dx - (n-1)\int_0^{\pi/2}\sin^{n}(x)dx$ ed essendo presente in entrambi i membri un integrale di $\sin^n x$, raccogliamo, ottenendo:

$n\int_0^{\pi/2}\sin^{n}(x)dx = \left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)dx$ e infine:

$$\boxed{\int_0^{\pi/2}\sin^{n}(x)dx = \frac{1}{n}\left|-\sin^{n-1}(x)\cos(x)\right|_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2}\sin^{n-2}(x)dx} \qquad \text{(A.13)}$$

La (A.13) può ad esempio esserci utile per calcolare $\int_0^{\pi/2}\sin^4(x)dx$ una volta noto $\int_0^{\pi/2}\sin^2(x)dx$ e così via. In ogni caso, così facendo, otterremmo i seguenti risultati:

$$\int_0^{\pi/2}\sin^0\beta d\beta = \left|\beta\right|_0^{\pi/2} = \frac{\pi}{2}$$

$$\int_0^{\pi/2}\sin^2\beta d\beta = \left|\frac{\beta}{2} - \frac{\sin 2\beta}{4}\right|_0^{\pi/2} = \left|\frac{1}{2}(\beta - \sin\beta\cos\beta)\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^4\beta d\beta = \left|\frac{3\beta}{8} - \frac{\sin 2\beta}{4} + \frac{\sin 4\beta}{32}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^6\beta d\beta = \left|\frac{5\beta}{16} - \frac{15\sin 2\beta}{64} + \frac{3\sin 4\beta}{64} - \frac{\sin 6\beta}{192}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^8\beta d\beta=\left|\frac{35\beta}{128}-\frac{7\sin2\beta}{32}+\frac{7\sin4\beta}{128}-\frac{\sin6\beta}{96}+\frac{\sin8\beta}{1024}\right|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^{10}\beta d\beta=\left|\frac{63\beta}{256}-\frac{105\sin2\beta}{512}+\frac{15\sin4\beta}{256}-\frac{15\sin6\beta}{1024}+\frac{5\sin8\beta}{2048}-\frac{\sin10\beta}{5120}\right|_0^{\pi/2}$$

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e così via.

2) secondo metodo (elegante):

FUNZIONE GAMMA DI EULERO

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt \qquad \text{(A.14)}$$

Naturalmente, $\Gamma(1)=1$. Poi,

$$\Gamma(n+1)=n\Gamma(n); \qquad \text{(A.15)}$$

infatti, dopo una integrazione per parti:

$\Gamma(z)=e^{-t}\left.\frac{t^z}{z}\right|_0^\infty+\frac{1}{z}\int_0^\infty e^{-t}t^z dt=0+\frac{1}{z}\Gamma(z+1)$ ed iterando la $\Gamma(n+1)=n\Gamma(n)$, otteniamo:

$$\Gamma(n+1)=n! \quad . \qquad \text{(A.16)}$$

INTEGRALE DI GAUSS

Consideriamo i due integrali (uguali tra loro):

$$I=\int_0^\infty e^{-\alpha x^2}dx \qquad I=\int_0^\infty e^{-\alpha y^2}dy \qquad \text{(A.17)}$$

Moltiplicandoli tra loro: $I^2=\int_0^\infty\int_0^\infty e^{-\alpha(x^2+y^2)}dxdy$ e, in coordinate polari:

($x^2+y^2=r^2, dS=dxdy=rdrd\theta$), si ha:

$I^2=\int_0^{\pi/2}\int_0^\infty e^{-\alpha r^2}rdrd\theta=\frac{\pi}{2}\int_0^\infty e^{-\alpha r^2}rdr=-\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty=\frac{\pi}{4\alpha}$, perciò:

$$I=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad . \qquad \text{(A.18)}$$

VALORI PARTICOLARI DELLA FUNZIONE GAMMA DI EULERO

La (A.16) con n=0 fornisce Γ(1)=0!=1, mentre con n=1, fornisce Γ(2)=1!=1.

Inoltre, la (A.14) con z=1/2 diventa:

$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$; adesso, ponendo t=w^2, si ha: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt=\int_0^\infty e^{-w^2}w^{-1}2wdw=2\int_0^\infty e^{-w^2}dw \qquad \text{(A.19)}$$

e per la (A.18) con α=1, si ha: $\Gamma(\frac{1}{2})=2\int_0^\infty e^{-w^2}dw=\sqrt{\pi}$.

Infine, per la (A.15): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Per riassumere: Γ(1)=1, Γ(2)=1, $\Gamma(1/2)=\sqrt{\pi}$, Γ(3/2) = $\frac{\sqrt{\pi}}{2}$. (A.20)

FUNZIONE BETA

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{A.21}$$

(convergente per m>0 ed n>0)

-Si ha che: $B(m,n)=B(n,m)$; infatti, con la sostituzione x=1-y, si ha:

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^1 (1-y)^{m-1}y^{n-1}dx=\int_0^1 y^{n-1}(1-y)^{m-1}dx=B(n,m).$$

-Si ha inoltre che:

$$B(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta \;; \tag{A.22}$$

infatti, usando la sostituzione $x=\sin^2\theta$ nella (A.21), si ottiene:

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}2\sin\theta\cos\theta d\theta=$$

$$=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta \;.$$

-Si ha anche che: $B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$; (A.23)

infatti, con la sostituzione $t=x^2$ (e ovviamente $dt=2xdx$) nella funzione Γ:

$\Gamma(m)=\int_0^\infty e^{-t}t^{m-1}dt$, si ottiene: $\Gamma(m)=\int_0^\infty e^{-t}t^{m-1}dt=2\int_0^\infty e^{-x^2}x^{2m-1}dx$ e similmente:

$\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$ e dunque:

$$\Gamma(m)\Gamma(n)=4(\int_0^\infty e^{-x^2}x^{2m-1}dx)(\int_0^\infty e^{-y^2}y^{2n-1}dy)=4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dxdy;$$

passando ora alle coordinate polari ($x=\rho\cos\alpha$, $y=\rho\sin\alpha$), si ottiene:

($dS=dxdy=(\rho d\alpha)d\rho=\rho d\rho d\alpha$)

$$\Gamma(m)\Gamma(n)=4\int_{\alpha=0}^{\pi/2}\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\rho d\alpha=$$

$$=4(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho)[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha] \;; \tag{A.24}$$

prendiamo ora il primo integrale: $\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho$; possiamo anche vederlo così:

$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho$ ed effettuando ora la seguente sostituzione: $\rho^2=g$

(e, di conseguenza: $d\rho=\frac{1}{2\rho}dg$), si ha:

$$\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}\rho^{2(m+n)}\frac{1}{\rho}e^{-\rho^2}d\rho=\int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{2\rho^2}dg=\frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)}e^{-g}\frac{1}{g}dg=$$

$=\frac{1}{2}\int_{\rho=0}^{\infty}g^{(m+n)-1}e^{-g}dg=\frac{1}{2}\Gamma(m+n)$ ossia, riscrivendola:

$$\int_{\rho=0}^{\infty} \rho^{2(m+n)-1} e^{-\rho^2} d\rho = \frac{1}{2}\int_{\rho=0}^{\infty} g^{(m+n)-1} e^{-g} dg = \frac{1}{2}\Gamma(m+n) \qquad (A.25)$$

e le (A.25)+(A.22) nella (A.24) ci danno:

$$\Gamma(m)\Gamma(n) = 4(\int_{\rho=0}^{\infty} \rho^{2(m+n)-1} e^{-\rho^2} d\rho)[\int_{\alpha=0}^{\pi/2} (\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha] = 4[\frac{1}{2}\Gamma(m+n)][\frac{1}{2}B(m,n)] =$$

$= \Gamma(m+n)B(m,n)$, da cui, finalmente: $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

-La (A.23) nella (A.22) fornisce inoltre:

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \qquad (A.26)$$

(m>0 ed n>0)

Infine, possiamo finalmente giungere alla (A.10); se nella (A.26) poniamo (2m-1)=p e (2n-1)=0 , si ha:

$\displaystyle\int_0^{\pi/2} \sin^p \theta d\theta = \frac{\Gamma[\frac{1}{2}(p+1)]\Gamma(\frac{1}{2})}{2\Gamma[\frac{1}{2}(p+2)]}$; volendo adesso un p pari, possiamo scrivere: p=2r , da cui:

$\displaystyle\int_0^{\pi/2} \sin^{2r} \theta d\theta = \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(r+1)}$ e valendo la (A.25), cioè $\Gamma(n+1) = n\Gamma(n)$ e poi anche la (A.26), ossia $\Gamma(n+1) = n!$, nonché le (A.20), ossia $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, si ha:

$$\int_0^{\pi/2} \sin^{2r} \theta d\theta = \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(r+1)} = \frac{\Gamma(r-\frac{1}{2}+1)\Gamma(\frac{1}{2})}{2\Gamma(r+1)} = \frac{[(r-\frac{1}{2})\Gamma(r-\frac{1}{2})]\Gamma(\frac{1}{2})}{2r!} = \frac{[(r-\frac{1}{2})\Gamma(r-\frac{3}{2}+1)]\Gamma(\frac{1}{2})}{2r!} =$$

$$= \frac{[(r-\frac{1}{2})(r-\frac{3}{2})\Gamma(r-\frac{3}{2})]\Gamma(\frac{1}{2})}{2r!} = \frac{(r-\frac{1}{2})(r-\frac{3}{2}).......\frac{1}{2}\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{2r(r-1).....1} ; \qquad (A.27)$$

infatti, la scomposizione tra parentesi quadre approda sempre a $....\frac{1}{2}\Gamma(\frac{1}{2})$; ad esempio, se r=3 , ad un certo punto si avrà: $....(3-\frac{5}{2})\Gamma(3-\frac{5}{2}) = ...\frac{1}{2}\Gamma(\frac{1}{2})$ e se, come ulteriore esempio, r=4 , ad un certo punto si avrà: $....(4-\frac{7}{2})\Gamma(4-\frac{7}{2}) = ...\frac{1}{2}\Gamma(\frac{1}{2})$. Proseguendo dunque con la (A.27), abbiamo:

$\displaystyle\int_0^{\pi/2} \sin^{2r} \theta d\theta = \frac{(r-\frac{1}{2})(r-\frac{3}{2}).......\frac{1}{2}[\Gamma(\frac{1}{2})]^2}{2r(r-1).....1} = \frac{(2r-1)(2r-3).......1}{2r(2r-2).....2}\cdot\frac{\pi}{2}$; infatti, si è raccolto $\frac{1}{2}$ in ogni parentesi tonda , tipo $(r-\frac{1}{2}) = \frac{1}{2}(2r-1)$ e tutti questi $\frac{1}{2}$ sono andati a denominatore, ad esempio trasformando $(r-1)$ in $(2r-2)$. Continuando:

$$\boxed{\int_0^{\pi/2} \sin^{2r} \theta d\theta = \frac{(2r-1)(2r-3)(2r-5).......1}{2r(2r-2)(2r-4).....2}\cdot\frac{\pi}{2} = \frac{1\cdot3\cdot5\cdot...(2r-1)}{2\cdot4\cdot6\cdot...(2r)}\frac{\pi}{2}} \qquad (A.28)$$

infatti, se, ad esempio, r=4, si ha:

$$\int_0^{\pi/2} \sin^{2\cdot 4}\theta d\theta = \frac{(2\cdot 4-1)(2\cdot 4-3)(2\cdot 4-5).......1}{(2\cdot 4)(2\cdot 4-2)(2\cdot 4-4).....2}\cdot\frac{\pi}{2} = \frac{[7]\cdot 5\cdot 3.......1}{[8]\cdot 6\cdot 4\cdot .....2}\cdot\frac{\pi}{2} = \frac{[2r-1]\cdot 5\cdot 3.......1}{[2r]\cdot 6\cdot 4\cdot .....2}\cdot\frac{\pi}{2} =$$

$= \frac{1\cdot 3\cdot 5\cdot ...(2r-1)}{2\cdot 4\cdot 6\cdot ...(2r)}\frac{\pi}{2}$, dunque la (A.28) altro non è che la (A.10) che ci proponevamo di dimostrare.

APPENDICE B:
Integrale Ellittico di Seconda Specie E(k) (perimetro dell'ellisse)

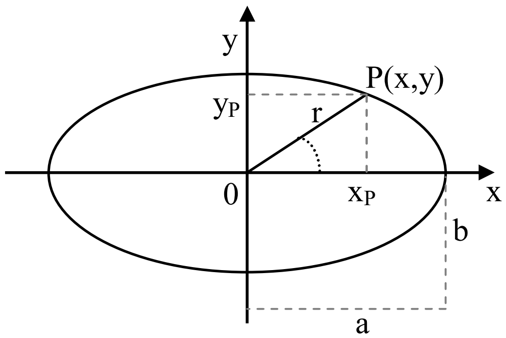

Fig. B.1

La lunghezza di un tratto infinitesimo di ellisse nel punto P(x,y) è ovviamente data da:

$$dL = \sqrt{(dx)^2 + (dy)^2} \ : \qquad \text{(B.1)}$$

inoltre, sappiamo che l'equazione dell'ellisse è: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, da cui: $y=\frac{b}{a}\sqrt{a^2-x^2}$, da cui ancora, per derivazione: $\frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}} \quad\rightarrow\quad dy = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}dx \quad\rightarrow\quad (dy)^2 = \frac{b^2}{a^2}\frac{x^2}{a^2-x^2}(dx)^2$ e ritornando alla (B.1):

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + \frac{b^2}{a^2}\frac{x^2}{a^2-x^2}(dx)^2} = dx\sqrt{1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}} = dx\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2(a^2-x^2)}} =$$

$= dx\sqrt{\frac{a^4-x^2(a^2-b^2)}{a^2(a^2-x^2)}} = dx\sqrt{\frac{a^2}{(a^2-x^2)}-\frac{(a^2-b^2)}{a^2}\frac{x^2}{(a^2-x^2)}}$, ma sapendo che: $\frac{(a^2-b^2)}{a^2}=e^2$, dove "e" è l'eccentricità dell'ellisse, segue che:

$$dL = dx\sqrt{\frac{a^2}{(a^2-x^2)}-\frac{e^2x^2}{(a^2-x^2)}} = dx\sqrt{\frac{a^2-e^2x^2}{(a^2-x^2)}} = dx\sqrt{\frac{1-e^2\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}$$ e allora:

$L = 4\int_0^a \sqrt{\frac{1-e^2\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}dx$ e ponendo ora $t=\frac{x}{a}$,

(con dx=adt e con t=0 quando x=0 e con t=1 quando x=a), si ottiene:

$L = 4a\int_0^1 \frac{\sqrt{1-e^2t^2}}{\sqrt{1-t^2}}dx$; infine, ponendo $t=\sin\theta$, (da cui $dt=\cos\theta d\theta$ e con θ=0 quando t=0 e con θ=π/2 quando t=1), si ottiene:

$$L=4a\int_0^{\pi/2}\frac{\sqrt{1-e^2\sin^2\theta}}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta=4a\int_0^{\pi/2}\frac{\sqrt{1-e^2\sin^2\theta}}{\cos\theta}\cos\theta d\theta=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}d\theta\,;$$

chiamando ora k l'eccentricità "e" , si ha:

$L=4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta=4aE(k)$, con $\boxed{E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta}$ che è l'Integrale Ellittico di Seconda Specie. Essendo poi che per la (C.7) dimostrata in Appendice C si ha:

$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$, segue che, per una valutazione concreta del perimetro dell'ellisse:

$$L=4aE(k)=4a\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=2a\pi\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}\quad. \tag{B.2}$$

Riguardo invece l'Integrale Ellittico di Prima Specie K(k), esso discende dallo studio del pendolo e si veda, a tal proposito, l'Appendice A.

APPENDICE C: Gli integrali ellittici e le funzioni ipergeometriche.

Abbiamo dunque incontrato, nelle Appendici precedenti:

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad\text{(Integrale Ellittico di prima specie)} \tag{C.1}$$

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\cdot d\theta\quad\text{(Integrale Ellittico di seconda specie)} \tag{C.2}$$

Si chiamano ellittici perché, ad esempio quello di seconda specie deriva dal calcolo del perimetro dell'ellisse.

Abbiamo visto in Appendice A che:

$K(k)=\int_0^{\pi/2}\sin^0\beta d\beta+\frac{1}{2}k^2\int_0^{\pi/2}\sin^2\beta d\beta+\frac{1}{2}\frac{3}{4}k^4\int_0^{\pi/2}\sin^4\beta d\beta+...$, cioè:

$K(k)=\int_0^{\pi/2}(\sin^0\beta+\frac{1}{2}k^2\sin^2\beta+\frac{1}{2}\frac{3}{4}k^4\sin^4\beta+...)d\beta=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, ossia:

$\frac{1}{\sqrt{1-k^2\sin^2\beta}}=(\sin^0\beta+\frac{1}{2}k^2\sin^2\beta+\frac{1}{2}\frac{3}{4}k^4\sin^4\beta+...)=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta$; infatti, si ha che:

$\binom{2n}{n}=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{(n!)^2}$, da cui: $\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta=\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta$ e:

n=0 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta=\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}k^{2\cdot 0}\sin^{2\cdot 0}\beta=\sin^0\beta=1$

n=1 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta=\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}k^{2\cdot 1}\sin^{2\cdot 1}\beta=\frac{1}{4}\frac{(2)!}{(1!)^2}k^2\sin^2\beta=\frac{1}{2}k^2\sin^2\beta$

n=2 $\rightarrow$ $\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta=\frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}k^{2\cdot 2}\sin^{2\cdot 2}\beta=\frac{1}{16}\frac{(4)!}{(2!)^2}k^2\sin^2\beta=\frac{1}{2}\frac{3}{4}k^2\sin^2\beta$, quindi, ribadiamolo: $\frac{1}{\sqrt{1-k^2\sin^2\beta}}=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta$. (C.3)

Tornando a K(k):

$$K(k)=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}=\int_0^{\pi/2}\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta d\beta=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta\ , \quad (C.4)$$

ma abbiamo anche dimostrato che (vedere la (A.10) in Appendice A):

$$I=\int_0^{\pi/2}\sin^{2n}\beta d\beta=\frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)}\frac{\pi}{2}=\frac{1}{4^n}\binom{2n}{n}\frac{\pi}{2}\ , \quad (C.5)$$

poiché si può constatare con verifiche numeriche che: $\frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)}=\frac{1}{4^n}\binom{2n}{n}$; infatti, è proprio vero che: $\frac{1\cdot 3\cdot 5\cdot ...(2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)}=\frac{1}{4^n}\binom{2n}{n}=\frac{1}{4^n}\frac{(2n)!}{(n!)^2}$; vediamo un attimo:

$$\frac{1}{4^n}\frac{(2n)!}{(n!)^2}=\frac{1}{2^{2n}}\frac{[2n(2n-1)(2n-2)...]}{[n(n-1)(n-2)....]^2}=\frac{1}{2^{2n-1}}\frac{n(2n-1)(2n-2)...}{n^2(n-1)^2(n-2)^2....}=\frac{1}{2^{2n-1}}\frac{(2n-1)(2n-2)...}{n(n-1)^2(n-2)^2....}$$

e con, ad esempio, n=3, si ha: $\frac{1}{2^{2n-1}}\frac{(2n-1)(2n-2)...}{n(n-1)^2(n-2)^2....}=\frac{1}{2^5}\frac{(6-1)(6-2)(6-3)(6-4)(6-5)}{3(3-1)^2(3-2)^2}=$

$$=\frac{1}{2^5}\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3(2)^2(1)^2}=\frac{1}{2^5}\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 4\cdot 1}=\frac{1}{2^4}\frac{5\cdot 3\cdot 1}{3\cdot 1}=\frac{1}{2\cdot 2^2\cdot 2}\frac{5\cdot 3\cdot 1}{3\cdot 1}=\frac{5\cdot 3\cdot 1}{2\cdot 4\cdot 2\cdot 3\cdot 1}=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\ !!!!!$$

Inserendo allora la (C.5) nella (C.4), si ottiene:

$K(k)=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}k^{2n}\frac{1}{4^n}\binom{2n}{n}\frac{\pi}{2}=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}$ ed adottando ora la notazione delle funzioni ipergeometriche: ${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$, con gli

$(x)_n=x(x+1)...(x+n-1)$

$(\quad =\Gamma(x+n)/\Gamma(x)=\Gamma(x+n-1+1)/\Gamma(x-1+1)=(x+n-1)!/(x-1)!\quad)$

che sono i Simboli di Pochhammer (fattoriale crescente), segue che:

$$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}=\frac{\pi}{2}\cdot{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)\ ; \quad (C.6)$$

infatti:

$$[\frac{1}{4^n}\binom{2n}{n}]^2=[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2=\frac{(a)_n(b)_n}{(c)_n n!}=\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}=\frac{[\frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1)][\frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1)]}{1(1+1)...(1+n-1)n!}=$$

$$=\frac{[\frac{1}{2}(\frac{1}{2}+1)...(n-\frac{1}{2})]^2}{1(2)...(n)n!}=\frac{[\frac{1}{2}\frac{3}{2}\frac{5}{2}...(n-\frac{1}{2})]^2}{(n!)^2}=\frac{[1\cdot 3\cdot 5...(2n-1)]^2}{2^{2n}(n!)^2}=\frac{(2n)!}{4^n(n!)^2}\frac{[1\cdot 3\cdot 5...(2n-1)]}{(2\cdot 4\cdot 6...2n)}=$$

$=[\frac{(2n)!}{4^n(n!)^2}]^2$ cvd.

Definiamo ora $K'(k) = K(\sqrt{1-k^2}) = K(k')$, con $k' = \sqrt{1-k^2}$; si ha simmetria per $k = \frac{1}{\sqrt{2}}$, poiché, in tal caso: $K'(\frac{1}{\sqrt{2}}) = K(\frac{1}{\sqrt{2}})$.

Ora, in analogia con la (C.6), si ha anche che:

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\binom{2n}{n}]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2 k^{2n} = \frac{\pi}{2}\cdot {}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)$$

poiché dallo sviluppo di MacLaurin di $(1-x)^{1/2}$, si ha:

$(1-x)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-x)^n = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots$, da cui:

$$(1-k^2\sin^2\beta)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-k^2\sin^2\beta)^n = 1 - \frac{1}{2}(k\sin\beta)^2 - \frac{1}{8}(k\sin\beta)^4 - \frac{1}{16}(k\sin\beta)^6 - \ldots =$$

$$= (k\sin\beta)^0 - \frac{1}{2}(k\sin\beta)^2 - \frac{1}{8}(k\sin\beta)^4 - \frac{1}{16}(k\sin\beta)^6 - \ldots = \sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\binom{2n}{n}k^{2n}\sin^{2n}\beta =$$

$= \sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta$ e allora:

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta}\cdot d\beta = \int_0^{\pi/2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\sin^{2n}\beta =$$

$$= \sum_{n=0}^{\infty}(-\frac{1}{2n-1})\frac{1}{4^n}\frac{(2n)!}{(n!)^2}k^{2n}\int_0^{\pi/2}\sin^{2n}\beta d\beta = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2 k^{2n} = \frac{\pi}{2}\cdot {}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2) \quad \text{(C.7)}$$

c.v.d.

E, anche qui, si ha: $E'(k) = E(\sqrt{1-k^2}) = E(k')$, con $k' = \sqrt{1-k^2}$; si ha simmetria per $k = \frac{1}{\sqrt{2}}$, poiché, in tal caso: $E'(\frac{1}{\sqrt{2}}) = E(\frac{1}{\sqrt{2}})$.

-Ripartendo ora dai risultati qui ottenuti, ossia:

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\binom{2n}{n}]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2 k^{2n} = \frac{\pi}{2}\cdot {}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2) \quad \text{(C.8)}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\binom{2n}{n}]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2 k^{2n} = \frac{\pi}{2}\cdot {}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)\,. \quad \text{(C.9)}$$

Bene, per la (C.8) e per l'Identità Prodotto di Clausen, ossia la (C.23) dimostrata qui in Appendice C: $[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)])$ (Identità Prodotto di Clausen)

(con: ${}_3F_2(a,b,c;d,e;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$, definizione), si ha:

$$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4k^2(1-k^2)]) = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2)) \quad \text{(C.10)}$$

ma anche: $[{}_2F_1(\frac{1}{2},\frac{1}{2};1;k'^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4k'^2(1-k'^2)]) = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))$

e, dunque:

$$\boxed{[\frac{2}{\pi}K(k)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))} \quad \text{(C.11)}$$

(che sarebbe la (C.10)) .

Invece, per l'Identità (semplice) di Clausen, ossia la:
${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};x)=[{}_2F_1(a,b;a+b+\frac{1}{2};x)]^2$ che è la (C.19), dimostrata anch'essa qui in Appendice C, segue che la (C.11) può essere così reinterpretata:
$[\frac{2}{\pi}K(k)]^2={}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(4k^2k'^2))=[{}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)]^2$, da cui:

$$\boxed{\frac{2}{\pi}K(k)={}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2)} \qquad (C.12)$$

-**Alcune proprietà delle funzioni ipergeometriche**:

ricordiamo innanzitutto le definizioni:

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \qquad (C.13)$$

$(x)_n=x(x+1)...(x+n-1)$ (fattoriale crescente-simbolo di Pochhammer) (C.14)

$${}_3F_2(a,b,c;d,e;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}z^n \qquad (C.15)$$

poi, ecco alcune identità di verifica diretta:

$(x)_n=x(x+1)...(x+n-1)$

$(1)_n=1(1+1)...(1+n-1)=1\cdot 2\cdot ...n=n!$

$(2)_n=2(2+1)...(2+n-1)=2\cdot 3\cdot ...(n+1)=(n+1)!$

$(\frac{3}{2})_n=\frac{3}{2}(\frac{3}{2}+1)...(\frac{3}{2}+n-1)=\frac{3}{2}\frac{5}{2}...(n+\frac{1}{2})$

$\frac{(a)_n}{n!}=(-1)^n\binom{-a}{n}$

$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ (dal legame tra fattoriale e Gamma di Eulero)

$\binom{a}{m}=\frac{\Gamma(1+a)}{[m!\Gamma(1+a-m)]}$ (dal legame tra fattoriale e Gamma di Eulero)

$K(\frac{1}{\sqrt{2}})=K'(\frac{1}{\sqrt{2}})=\frac{[\Gamma(1/4)]^2}{4\pi^{1/2}}$

$$\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}=\frac{(4n)!}{4^{4n}(n!)^4} \qquad (C.16)$$

<u>dimostrazione della (C.16)</u>:
iniziamo col sottolineare alcune identità ovvie sui semifattoriali (sia pari che dispari):

$[(2n+1)!!][2^n][n!]=[(2n+1)(2n+1-2)(2n+1-4)...][2\cdot 2\cdot ...n\_times...2\cdot 2][n(n-1)(n-2)...]=$

$=[(2n+1)(2n-1)(2n-3)...][1][2n2(n-1)2(n-2)...]=(2n+1)2n(2n-1)(2n-2)(2n-3)...2\cdot 1=$

$= (2n+1)2n(2n-1)(2n-2)(2n-3)...2\cdot 1 = (2n+1)!$, ossia: $(2n+1)!!2^n n! = (2n+1)!$, da cui:

$\boxed{(2n+1)!! = \frac{(2n+1)!}{2^n n!}\ ;}$ poi:

$(2n)!! = 2n(2n-2)(2n-4)...2 = 2n2(n-1)2(n-2) = 2^n n!$, ossia: $\boxed{(2n)!! = 2^n n!}$ e, per ultimo:

$$[(2n-1)!!][2^n][n!] = [(2n-1)(2n-3)...][2\cdot 2\cdot ...n\_times...2\cdot 2][n(n-1)(n-2)...] =$$
$$= [(2n-1)(2n-3)...][1][2n2(n-1)2(n-2)...2] = (2n-1)(2n-3)2n(2n-2)(2n-4)...2 = (2n)!,$$ da cui:

$\boxed{(2n-1)!! = \frac{(2n)!}{2^n n!}\ .}$

Detto ciò, tornando alla (C.16), iniziamo con l'analizzare il termine $(\frac{1}{2})_n$:

$(\frac{1}{2})_n = \frac{1}{2}(\frac{1}{2}+1)...(\frac{1}{2}+n-1) = \frac{1}{2}\frac{3}{2}\frac{5}{2}...\frac{2n-1}{2} = \frac{(2n-1)!!}{2^n}$; adesso moltiplichiamo numeratore e denominatore per i numeri pari intermedi $(2n)!!$:

$(\frac{1}{2})_n = \frac{(2n-1)!!}{2^n}\frac{(2n)!!}{(2n)!!} = \frac{(2n-1)!!}{2^n}\frac{2^n n!}{2^n n!} = \frac{(2n)!}{2^{2n} n!} = \frac{(2n)!}{4^n n!}$; passando agli altri due termini:

$(\frac{1}{4})_n = \frac{1\cdot 5\cdot 9...(4n-3)}{4^n}$ e $(\frac{3}{4})_n = \frac{3\cdot 7\cdot 11...(4n-1)}{4^n}$, da cui:

$(\frac{1}{4})_n(\frac{3}{4})_n = \frac{1\cdot 3\cdot 5\cdot 7...(4n-1)}{4^{2n}} = \frac{(4n-1)!!}{4^{2n}}$ ed anche qui moltiplichiamo numeratore e denominatore per i numeri pari da 2 a 4n, ossia $(4n)!!$:

$(\frac{1}{4})_n(\frac{3}{4})_n = \frac{(4n-1)!!}{4^{2n}}\frac{(4n)!!}{(4n)!!} = \frac{(4n-1)!!}{4^{2n}}\frac{2^{2n}(2n)!}{2^{2n}(2n)!} = \frac{(4n-1)!!(4n)!!}{4^{2n}2^{2n}(2n)!} = \frac{(4n)!}{4^{3n}(2n)!}$; in totale:

$$\frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3} = \frac{\frac{(2n)!}{4^n n!}\frac{(4n)!}{4^{3n}(2n)!}}{(n!)^3} = \frac{(4n)!}{4^{4n}(n!)^4}$$ cvd.

-Identità di Kummer-dimostrazione/verifica:
(trasformazione quadratica di Gauss) (verifica veloce)

$${}_2F_1(2a,2b;a+b+\frac{1}{2};x) = {}_2F_1(a,b;a+b+\frac{1}{2};4x(1-x))\ ; \qquad \text{(C.17)}$$

premettiamo che: $(a)_n = a(a+1)...(a+n-1)$, $(a)_0 = 1$, $(a)_1 = a(a+1)...(a+1-1) = a(a+1)...a = a$, $(a)_2 = a(a+1)...(a+2-1) = a(a+1)...(a+1) = a(a+1)$.

Partiamo dalla definizione ${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ ed applichiamola alla (C.17):

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n}{(a+b+\frac{1}{2})_n n!}x^n = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}[4x(1-x)]^n\ ;$$

primo membro:

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n}{(a+b+\frac{1}{2})_n n!}x^n=\frac{(2a)_0(2b)_0}{(a+b+\frac{1}{2})_0 0!}x^0+\frac{(2a)_1(2b)_1}{(a+b+\frac{1}{2})_1 1!}x^1+\frac{(2a)_2(2b)_2}{(a+b+\frac{1}{2})_2 2!}x^2+\ldots=$$

$$=\frac{1\cdot 1}{1}+\frac{(2a)(2b)}{(a+b+\frac{1}{2})}x+\frac{(2a)(2a+1)(2b)(2b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^2+\ldots=$$

$$=1+\frac{4ab}{(a+b+\frac{1}{2})}x+\frac{16a^2b^2+8a^2b+8ab^2+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^2+\ldots \quad ;$$

secondo membro:

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}[4x(1-x)]^n=$$

$$=\frac{(a)_0(b)_0}{(a+b+\frac{1}{2})_0 0!}[4x(1-x)]^0+\frac{(a)_1(b)_1}{(a+b+\frac{1}{2})_1 1!}[4x(1-x)]^1+\frac{(a)_2(b)_2}{(a+b+\frac{1}{2})_2 2!}[4x(1-x)]^2+\ldots=$$

$$=1+\frac{(a)(b)}{(a+b+\frac{1}{2})}[4x(1-x)]+\frac{(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}[4x(1-x)]^2+\ldots=$$

$$=1+\frac{(a)(b)}{(a+b+\frac{1}{2})}(4x-4x^2)+\frac{(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}[16x^2(1+x^2-2x)]+\ldots=$$

$$=1+\frac{4(a)(b)}{(a+b+\frac{1}{2})}x-\frac{4(a)(b)}{(a+b+\frac{1}{2})}x^2+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^2+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4-$$

$$-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+\ldots=$$

$$=1+\frac{4(a)(b)}{(a+b+\frac{1}{2})}x+[\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}-\frac{4(a)(b)}{(a+b+\frac{1}{2})}]x^2-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+$$

$$+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4+\ldots=1+\frac{4ab}{(a+b+\frac{1}{2})}x+[\frac{16a^2b^2+8ab^2+8a^2b+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}]x^2-$$

$$-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4+\ldots$$

verifichiamo la sussistenza dell'uguaglianza:

$$\boxed{1+\frac{4ab}{(a+b+\frac{1}{2})}x+\frac{16a^2b^2+8a^2b+8ab^2+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^2}+\ldots=\boxed{1+\frac{4ab}{(a+b+\frac{1}{2})}x+[\frac{16a^2b^2+8ab^2+8a^2b+4ab}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}]x^2}-$$

$$-2\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^3+\frac{16(a)(a+1)(b)(b+1)}{2(a+b+\frac{1}{2})(a+b+\frac{3}{2})}x^4+\dots\quad !!!!! \tag{C.18}$$

E dunque la (C.17) funziona; ovviamente, al secondo membro della (C.18) vi sono altri termini, che però troverebbero il loro equivalente al primo membro se al primo membro si fossero considerati più termini della sommatoria ipergeometrica, e così via…

-Una Identità di Clausen-dimostrazione/verifica:
(verifica veloce)

$${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};x)=[{}_2F_1(a,b;a+b+\frac{1}{2};x)]^2\ ; \tag{C.19}$$

partiamo dalle definizioni ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ e ${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ ed applichiamole alla (C.19):

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b)_n(a+b+\frac{1}{2})_n n!}x^n=[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}x^n]^2\ ;$$

primo membro:

$$\sum_{n=0}^{\infty}\frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b)_n(a+b+\frac{1}{2})_n n!}x^n=$$

$$=\frac{(2a)_0(2b)_0(a+b)_0}{(2a+2b)_0(a+b+\frac{1}{2})_0 0!}x^0+\frac{(2a)_1(2b)_1(a+b)_1}{(2a+2b)_1(a+b+\frac{1}{2})_1 1!}x^1+\frac{(2a)_2(2b)_2(a+b)_2}{(2a+2b)_2(a+b+\frac{1}{2})_2 2!}x^2+\dots=$$

$$=1+\frac{(2a)(2b)(a+b)}{(2a+2b)(a+b+\frac{1}{2})}x+\frac{(2a)_2(2b)_2(a+b)_2}{2(2a+2b)_2(a+b+\frac{1}{2})_2}x^2+\dots$$

secondo membro:

$$[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+\frac{1}{2})_n n!}x^n]^2=[\frac{(a)_0(b)_0}{(a+b+\frac{1}{2})_0 0!}x^0+\frac{(a)_1(b)_1}{(a+b+\frac{1}{2})_1 1!}x^1+\frac{(a)_2(b)_2}{(a+b+\frac{1}{2})_2 2!}x^2+\dots]^2=$$

$$=[1+\frac{(a)(b)}{(a+b+\frac{1}{2})}x+\frac{(a)_2(b)_2}{2(a+b+\frac{1}{2})_2}x^2+\dots]^2\ ;$$

verifichiamo la sussistenza dell'uguaglianza non più a livello letterale, come nel caso precedente, ossia mantenendo a e b letterali, a causa delle dimensioni eccessive delle espressioni che si ottengono, ma sceglieremo dei valori casuali, tipo a=2 e b=3 :
primo membro:
(e ricordiamo che abbiamo già imparato bene a calcolare i vari $(x)_n$ dalla dimostrazione precedente…)

$$\dots=1+\frac{(4)(6)(5)}{(10)(\frac{11}{2})}x+\frac{(4)_2(6)_2(5)_2}{2(10)_2(\frac{11}{2})_2}x^2+\dots=1+\frac{24}{11}x+\frac{20\cdot 42\cdot 30}{2\cdot 110\cdot\frac{143}{4}}x^2+\dots=1+\frac{24}{11}x+\frac{5040}{1573}x^2+\dots$$

secondo membro:

$$...=[1+\frac{(2)(3)}{(\frac{11}{2})}x+\frac{(2)_2(3)_2}{2(\frac{11}{2})_2}x^2+...]^2=[1+\frac{(2)(3)}{(\frac{11}{2})}x+\frac{6\cdot 12}{2\frac{143}{4}}x^2+...]^2=[1+\frac{12}{11}x+\frac{144}{143}x^2+...]^2=$$

$$=(1+\frac{12}{11}x+\frac{144}{143}x^2+...)(1+\frac{12}{11}x+\frac{144}{143}x^2+...)=1+\frac{24}{11}x+\frac{5040}{1573}x^2+\frac{3456}{1573}x^3+\frac{20736}{20449}x^4+...$$

da cui:

$$\boxed{1+\frac{24}{11}x+\frac{5040}{1573}x^2}+...=\boxed{1+\frac{24}{11}x+\frac{5040}{1573}x^2}+\frac{3456}{1573}x^3+\frac{20736}{20449}x^4+... \quad !!! \qquad (C.20)$$

E dunque la (C.19) funziona; ovviamente, al secondo membro della (C.20) vi sono altri termini, che però troverebbero il loro equivalente al primo membro se al primo membro si fossero considerati più termini della sommatoria ipergeometrica, e così via…

-Identità Prodotto di Clausen-dimostrazione/verifica:

(formula di prodotto di Bailey) (verifica veloce)

$${}_2F_1(\frac{1}{4}+a,\frac{1}{4}+b;1+a+b;x)\,{}_2F_1(\frac{1}{4}-a,\frac{1}{4}-b;1-a-b;x)=$$

$$={}_3F_2(\frac{1}{2},\frac{1}{2}+a-b,\frac{1}{2}-a+b;1+a+b,1-a-b;x)\ ; \qquad (C.21)$$

partiamo dalle definizioni ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ e ${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ ed applichiamole alla (C.21), con i soliti valori casuali, tipo a=2 e b=3 :

$${}_2F_1(\frac{1}{4}+2,\frac{1}{4}+3;1+2+3;x)\,{}_2F_1(\frac{1}{4}-2,\frac{1}{4}-3;1-2-3;x)=$$

$$={}_3F_2(\frac{1}{2},\frac{1}{2}+2-3,\frac{1}{2}-2+3;1+2+3,1-2-3;x)$$, ossia:

$${}_2F_1(\frac{9}{4},\frac{13}{4};6;x)\,{}_2F_1(-\frac{7}{4},-\frac{11}{4};-4;x)={}_3F_2(\frac{1}{2},-\frac{1}{2},\frac{3}{2};6,-4;x);$$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$$

$$\sum_{n=0}^{\infty}\frac{(\frac{9}{4})_n(\frac{13}{4})_n}{(6)_n n!}x^n\sum_{n=0}^{\infty}\frac{(-\frac{7}{4})_n(\frac{11}{4})_n}{(-4)_n n!}x^n=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(-\frac{1}{2})_n(\frac{3}{2})_n}{(6)_n(-4)_n n!}x^n$$

primo membro:

$$[\frac{(\frac{9}{4})_0(\frac{13}{4})_0}{(6)_0 0!}x^0+\frac{(\frac{9}{4})_1(\frac{13}{4})_1}{(6)_1 1!}x^1+\frac{(\frac{9}{4})_2(\frac{13}{4})_2}{(6)_2 2!}x^2+...][\frac{(-\frac{7}{4})_0(-\frac{11}{4})_0}{(-4)_0 0!}x^n+\frac{(-\frac{7}{4})_1(-\frac{11}{4})_1}{(-4)_1 1!}x^1+\frac{(-\frac{7}{4})_2(-\frac{11}{4})_2}{(-4)_2 2!}x^2+...]=$$

$$=[1+\frac{\frac{9}{4}\frac{13}{4}}{6}x+\frac{\frac{9}{4}(\frac{9}{4}+1)\frac{13}{4}(\frac{13}{4}+1)}{6(6+1)2}x^2+...][1+\frac{\frac{7}{4}\frac{11}{4}}{-4}x+\frac{539}{2048}x^2+...]=$$

$$=[1+\frac{39}{32}x+\frac{8619}{7168}x^2+...][1-\frac{77}{64}x+\frac{539}{2048}x^2+...]=$$

$$=[1-\frac{77}{64}x+\frac{539}{2048}x^2+\frac{39}{32}x-\frac{39}{32}\frac{77}{64}x^2+\frac{39}{32}\frac{539}{2048}x^3+\frac{8619}{7168}x^2-\frac{77}{64}\frac{8619}{7168}x^3+\frac{8619}{7168}\frac{539}{2048}x^4+...]=$$
$$=[1+\frac{1}{64}x-\frac{5}{7168}x^2-\frac{33850392576}{65536}x^3+\frac{4645641}{14680064}x^4+...]=$$

secondo membro:

$$[\frac{(\frac{1}{2})_0(-\frac{1}{2})_0(\frac{3}{2})_0}{(6)_0(-4)_0 0!}x^0+\frac{(\frac{1}{2})_1(-\frac{1}{2})_1(\frac{3}{2})_1}{(6)_1(-4)_1 1!}x^1+\frac{(\frac{1}{2})_2(-\frac{1}{2})_2(\frac{3}{2})_2}{(6)_2(-4)_2 2!}x^2+...]=$$
$$=[1+\frac{-\frac{1}{2}\frac{1}{2}\frac{3}{2}}{-24}x+\frac{-\frac{1}{2}(\frac{1}{2}+1)\frac{1}{2}(-\frac{1}{2}+1)\frac{3}{2}(\frac{3}{2}+1)}{-12(6+1)4(-4+1)}x^2+...]=$$
$$=[1+\frac{1}{64}x-\frac{5}{7168}x^2+...]$$ e confrontando:

$$\boxed{1+\frac{1}{64}x-\frac{5}{7168}x^2}-\frac{33850392576}{65536}x^3+\frac{4645641}{14680064}x^4+...=\boxed{1+\frac{1}{64}x-\frac{5}{7168}x^2}+... \quad !!! \tag{C.22}$$

E dunque la (C.21) funziona; ovviamente, al secondo membro della (C.22) vi sono altri termini, che però troverebbero il loro equivalente al primo membro se al primo membro si fossero considerati più termini della sommatoria ipergeometrica, e così via…

-E verifichiamo anche la seguente:

$$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2={}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)])\,; \tag{C.23}$$

-<u>primo metodo di verifica veloce</u>:

partiamo dalle definizioni ${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ e ${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$ ed

applichiamo i valori della (C.23):

$$[\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}x^{2n}]^2=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n(1)_n n!}[4x^2(1-x^2)]^n\ ;$$

primo membro:

$$[\frac{(\frac{1}{2})_0(\frac{1}{2})_0}{(1)_0 0!}x^0+\frac{(\frac{1}{2})_1(\frac{1}{2})_1}{(1)_1 1!}x^2+\frac{(\frac{1}{2})_2(\frac{1}{2})_2}{(1)_2 2!}x^4+...]^2=[1+\frac{1}{4}x^2+\frac{\frac{1}{2}(\frac{1}{2}+1)\frac{1}{2}(\frac{1}{2}+1)}{4}x^4+...]^2=$$
$$[1+\frac{1}{4}x^2+\frac{9}{64}x^4+...]^2=(1+\frac{1}{4}x^2+\frac{9}{64}x^4+...)(1+\frac{1}{4}x^2+\frac{9}{64}x^4+...)=1+\frac{1}{2}x^2+\frac{11}{32}x^4+\frac{18}{256}x^6+\frac{81}{4096}x^8+...$$

secondo membro:

$$\{\frac{(\frac{1}{2})_0(\frac{1}{2})_0(\frac{1}{2})_0}{(1)_0(1)_0 0!}[4x^2(1-x^2)]^0+\frac{(\frac{1}{2})_1(\frac{1}{2})_1(\frac{1}{2})_1}{(1)_1(1)_1 1!}[4x^2(1-x^2)]^1+\frac{(\frac{1}{2})_2(\frac{1}{2})_2(\frac{1}{2})_2}{(1)_2(1)_2 2!}[4x^2(1-x^2)]^2+...\}=$$
$$=\{1+\frac{\frac{1}{2}\frac{1}{2}\frac{1}{2}}{1\cdot 1\cdot 1}[4x^2(1-x^2)]+\frac{\frac{1}{8}(\frac{1}{2}+1)^3}{2\cdot 2\cdot 2}16x^4(1+x^4-2x)+...\}=$$

$= [1+\frac{1}{2}(x^2-x^4)+\frac{27}{32}(x^4+x^8-2x^5)+...] = [1+\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{27}{32}x^4+\frac{27}{32}x^8-\frac{27}{16}x^5+...] =$

$[1+\frac{1}{2}x^2+\frac{11}{32}x^4-\frac{27}{16}x^5+\frac{27}{32}x^8+...]$ e confrontando:

$$\boxed{1+\frac{1}{2}x^2+\frac{11}{32}x^4}+\frac{18}{256}x^6+\frac{81}{4096}x^8+... = \boxed{1+\frac{1}{2}x^2+\frac{11}{32}x^4}-\frac{27}{16}x^5+\frac{27}{32}x^8+... \quad !!! \qquad \text{(C.24)}$$

E dunque la (C.23) funziona; ovviamente, al secondo membro della (C.24) vi sono altri termini, che però troverebbero il loro equivalente preciso al primo membro se in entrambi i membri si fossero considerati più termini delle sommatorie ipergeometriche, e così via…

-<u>secondo metodo basato sull'utilizzo della (C.17) e della (C.19)</u>:

ricordiamo la (C.17) in t: ${}_2F_1(2a,2b;a+b+\frac{1}{2};t) = {}_2F_1(a,b;a+b+\frac{1}{2};4t(1-t))$ e fissiamo a=b=1/4, da cui: ${}_2F_1(\frac{1}{2},\frac{1}{2};1;t) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;4t(1-t))$ e col banale cambio di variabile $t=x^2$ →

${}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))$ ed elevando al quadrato:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$; adesso, come anticipato, prendiamo la (C.19) di Clausen in z:

${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};z) = [{}_2F_1(a,b;a+b+\frac{1}{2};z)]^2$ ed ovviamente sempre con a=b=1/4:

${}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;z) = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;z)]^2$ e col cambio di variabile $z=4x^2(1-x^2)$ →

${}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;4x^2(1-x^2)) = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$, ma poco fa abbiamo ottenuto che:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = [{}_2F_1(\frac{1}{4},\frac{1}{4};1;4x^2(1-x^2))]^2$, da cui la tesi:

$[{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)]^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;[4x^2(1-x^2)])$ cvd.

-Trasformazione di Eulero – dimostrazione/verifica:

$${}_2F_1(a,b;c;x) = (1-x)^{c-b-a} \cdot {}_2F_1(c-a,c-b;c;x) \qquad \text{(C.25)}$$

-<u>primo metodo di verifica veloce</u>:

detta equazione di autotrasformazione delle funzioni ipergeometriche;
(esempio con a=1 , b=2 e c=4) →

${}_2F_1(1,2;4;x) = (1-x)^{4-2-1} \cdot {}_2F_1(3,2;4;x)$

$[\frac{(1)_0(2)_0}{(4)_0 0!}x^0+\frac{(1)_1(2)_1}{(4)_1 1!}x^1+\frac{(1)_2(2)_2}{(4)_2 2!}x^2+...] = (1-x)[\frac{(3)_0(2)_0}{(4)_0 0!}x^0+\frac{(3)_1(2)_1}{(4)_1 1!}x^1+\frac{(3)_2(2)_2}{(4)_2 2!}x^2+...]$

$[1+\frac{1}{2}x+\frac{3}{10}x^2+...] = (1-x)[1+\frac{3}{2}x+\frac{9}{5}x^2+...]$

$$1+\frac{1}{2}x+\frac{3}{10}x^2=1+\frac{3}{2}x+\frac{9}{5}x^2-x-\frac{3}{2}x^2-\frac{9}{5}x^3+...$$

$$(\frac{1}{2}x-\frac{3}{2}x+x)+(\frac{3}{10}x^2-\frac{9}{5}x^2+\frac{3}{2}x^2)+\frac{9}{5}x^3+...=0$$

$$0x^0+0x^2+\frac{9}{5}x^3+...=0\ . \qquad \text{(C.26)}$$

E dunque la (C.25) funziona; ovviamente, nella (C.26) vi sono altri termini, che però troverebbero il loro equivalente preciso se si fossero considerati più termini delle sommatorie ipergeometriche, e così via…

-<u>secondo metodo basato sulla Trasformazione di Pfaff</u>:
consideriamo la (C.27) di Eulero dimostrata appena qui sotto:

$\int_0^1 \frac{(1-t)^{c-b-1}}{(1-xt)^a}t^{b-1}dt=B(c-b,b)\cdot {}_2F_1(a,b;c;x)$ ed effettuiamo il cambio di variabile t=1-u (e dt=-du), con i limiti di integrazione che si invertono, ma poi si ripristinano grazie al segno di –du:

$$\int_0^1 \frac{u^{c-b-1}}{[1-x(1-u)]^a}(1-u)^{b-1}du=B(c-b,b)\cdot {}_2F_1(a,b;c;x)=(1-x)^{-a}\int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1}du$$

e reinterpretando l'ultimo integrale sempre sulla base dell'integrale di Eulero (C.27), si ha:

$\int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1}du=B(c-b,b)\cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})$, da cui:

$$B(c-b,b)\cdot {}_2F_1(a,b;c;x)=(1-x)^{-a}\int_0^1 \frac{u^{c-b-1}}{(1-\frac{x}{x-1}u)^a}(1-u)^{b-1}du=(1-x)^{-a}B(c-b,b)\cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})$$

ossia: ${}_2F_1(a,b;c;x)=(1-x)^{-a}\cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})=(1-x)^{-a}\cdot {}_2F_1(c-b,a;c;\frac{x}{x-1})$ ed applicando nuovamente la trasformazione di Pfaff a quest'ultima (in $\frac{x}{x-1}=z$) (ed ovviamente, questa volta, visti gli scambi, con (c-b) nel ruolo di a) si ha:

${}_2F_1(c-b,a;c;z)=(1-z)^{-(c-b)}\cdot {}_2F_1(c-b,c-a;c;\frac{z}{z-1})$ e ritornando di nuovo in x:

(($\frac{x}{x-1}=z \rightarrow 1-z=1-\frac{x}{x-1}=\frac{1}{1-x}$ ed anche: $\frac{z}{z-1}=\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=x$)) $\rightarrow$

${}_2F_1(c-b,a;c;\frac{x}{x-1})=(\frac{1}{1-x})^{-(c-b)}\cdot {}_2F_1(c-b,c-a;c;x)=(1-x)^{c-b}\cdot {}_2F_1(c-b,c-a;c;x)$, ossia:

${}_2F_1(a,c-b;c;\frac{x}{x-1})=(1-x)^{c-b}\cdot {}_2F_1(c-b,c-a;c;x)$ e visto che prima ottenemmo che:

${}_2F_1(a,b;c;x)=(1-x)^{-a}\cdot {}_2F_1(a,c-b;c;\frac{x}{x-1})$, allora, dalle due:

${}_2F_1(a,b;c;x)=(1-x)^{-a}(1-x)^{c-b}\cdot {}_2F_1(c-b,c-a;c;x)$, ossia:
(c-b legittimamente scambiabile di posizione con c-a)

${}_2F_1(a,b;c;x)=(1-x)^{c-b-a}\cdot {}_2F_1(c-a,c-b;c;x)$ cvd.

-Formula di Eulero $\int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt = B(c-b,b) \cdot {}_2F_1(a,b;c;z)$ **– dimostrazione.**

Eccola: $\int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt = B(c-b,b) \cdot {}_2F_1(a,b;c;z)$; (C.27)

in Appendice A, con la (A.23), abbiamo visto che la Beta di Eulero è così esprimibile:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad , \tag{C.28}$$

da cui: $B(c-b,b) = \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)}$ e dunque la (C.27) diviene:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt \quad . \tag{C.29}$$

Consideriamo allora la funzione $\frac{1}{(1-zt)^a} = (1-zt)^{-a}$; si ha, ovviamente, per Taylor:

$$(1-zt)^{-a} = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k (zt)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k t^k \tag{C.30}$$

ed inserendo quest'ultima nell'integrale della (C.27) si ottiene:

$\int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt = \int_0^1 (1-t)^{c-b-1} t^{b-1} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k t^k dt = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \int_0^1 (1-t)^{c-b-1} t^{b+k-1} dt$, che per la (C.28)

diventa: $\int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \int_0^1 (1-t)^{c-b-1} t^{b+k-1} dt = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \frac{\Gamma(b+k)\Gamma(c-b)}{\Gamma(c+k)} z^k =$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \frac{\Gamma(b+k)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+k)} z^k = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \cdot {}_2F_1(a,b;c;z)$$

(poiché vedemmo più su che $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$) e allora:

${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{(1-t)^{c-b-1}}{(1-zt)^a} t^{b-1} dt$, ossia proprio la (C.29), ossia la (C.27)!!!

Ovviamente, poi: ${}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{c-a-b-1} t^{b-1} dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} =$

$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}$.

-Dimostrazione della seguente relazione ipergeometrica:

$$\frac{2}{\pi} K(k) = \frac{1}{\sqrt{1+k^2}} \cdot {}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; \left(\frac{2}{g^{12}+g^{-12}}\right)^2\right); \tag{C.31}$$

partiamo dalla (C.6): $\frac{2}{\pi} K(k) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ e consideriamo la prima forma quadratica (Forma di Landen): ${}_2F_1(a,b;a-b+1;z) = (1-z)^{1-2b}(1+z)^{2b-1-a} \cdot {}_2F_1(\frac{a-2b+1}{2}, \frac{a-2b+2}{2}; a-b+1; \frac{4z}{(1+z)^2})$

(verificabile anche con l'ormai collaudato metodo del confronto tra i due membri, dopo aver adottato le sommatorie per esprimere le due ${}_2F_1$)

e scegliendo: a=1/2 , b=1/2 e z=k^2 , si ottiene:

$${}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)=\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{4},\frac{3}{4};1;\frac{4k^2}{(1+k^2)^2}) \quad \text{(trasf. di Landen);} \qquad (C.32)$$

usando poi anche quest'altra identità quadratica di Gauss, ossia la:

$${}_2F_1(a,b;a+b+\frac{1}{2};x)={}_2F_1(\frac{a}{2},\frac{b}{2};a+b+\frac{1}{2};4x(1-x))$$

(verificabile anch'essa con l'ormai collaudato metodo del confronto tra i due membri, dopo aver adottato le sommatorie per esprimere le due ${}_2F_1$)

e con a=1/4 e b=3/4 , si ottiene:

$${}_2F_1(\frac{1}{4},\frac{3}{4};1;x)={}_2F_1(\frac{1}{8},\frac{3}{8};1;4x(1-x))\,; \qquad (C.33)$$

a questo punto effettuiamo la sostituzione:

$x=\frac{4k^2}{(1+k^2)^2}$, da cui segue anche che: $4x(1-x)=[\frac{4k(1-k^2)}{(1+k^2)^2}]^2=[\frac{4kk'^2}{(1+k^2)^2}]^2$ e per la (1.29) , si ha anche: $\frac{4kk'^2}{(1+k^2)^2}=\frac{2}{(g^{12}+g^{-12})}$, da cui: $4x(1-x)=[\frac{2}{(g^{12}+g^{-12})}]^2$ e quindi la (C.33) diventa:

$${}_2F_1(\frac{1}{4},\frac{3}{4};1;\frac{4k^2}{(1+k^2)^2})={}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2) \qquad (C.34)$$

e per la (C.32), la (C.34) diventa:

$\frac{2}{\pi}K(k)\sqrt{1+k^2}={}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2)$, da cui, infine:

$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;[\frac{2}{(g^{12}+g^{-12})}]^2)$, ossia l'asserto. cvd

-Dimostrazione della seguente relazione ipergeometrica:

$$[\frac{2}{\pi}K(k)]^2=\frac{1}{1+k^2}\cdot{}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{2}{g^{12}+g^{-12}})^2) \qquad (C.35)$$

che è anche la (1.11); si consideri allora la (1.8) (che è poi la (C.31)):

$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\cdot{}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)$ e si applichi l'Identità Prodotto di Clausen (C.21):

$${}_2F_1(\frac{1}{4}+a,\frac{1}{4}+b;1+a+b;x)\,{}_2F_1(\frac{1}{4}-a,\frac{1}{4}-b;1-a-b;x)=$$

$={}_3F_2(\frac{1}{2},\frac{1}{2}+a-b,\frac{1}{2}-a+b;1+a+b,1-a-b;x)$ ossia: (con $a=-\frac{1}{8}$ e $b=\frac{1}{8}$)

$${}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)\,{}_2F_1(\frac{1}{4}+\frac{1}{8},\frac{1}{4}-\frac{1}{8};1+\frac{1}{8}-\frac{1}{8};(\frac{2}{g^{12}+g^{-12}})^2)=$$

$$={}_3F_2(\frac{1}{2},\frac{1}{2}-\frac{1}{8}-\frac{1}{8},\frac{1}{2}+\frac{1}{8}+\frac{1}{8};1-\frac{1}{8}+\frac{1}{8},1+\frac{1}{8}-\frac{1}{8};(\frac{2}{g^{12}+g^{-12}})^2) \;\rightarrow$$

$${}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)\,{}_2F_1(\frac{3}{8},\frac{1}{8};1;(\frac{2}{g^{12}+g^{-12}})^2)=\frac{2}{\pi}K(k)\frac{2}{\pi}K(k)=[\frac{2}{\pi}K(k)]^2=$$

$={}_3F_2(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1;(\frac{2}{g^{12}+g^{-12}})^2)$ cvd.

APPENDICE D: La Relazione Fondamentale di Legendre.

$$K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2} \quad ; \tag{D.1}$$

infatti, si ha che: $K(k)=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, $K'(k)=K(\sqrt{1-k^2})=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-(1-k^2)\sin^2\beta}}$

$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta}\cdot d\beta$, $E'(k)=E(\sqrt{1-k^2})=\int_0^{\pi/2}\sqrt{1-(1-k^2)\sin^2\beta}\cdot d\beta$

nonché, ovviamente: $\frac{dK}{dk}=\int_0^{\pi/2}\frac{k\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta$, $\frac{dE}{dk}=\int_0^{\pi/2}\frac{-k\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta$ ed anche:

$$\frac{dK}{dk}=\frac{E(k)-k'^2K(k)}{kk'^2}\left(=\frac{E(k)-(\sqrt{1-k^2})^2K(k)}{k(\sqrt{1-k^2})^2}=\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right) \tag{D.2}$$

$$\frac{dE}{dk}=\frac{E(k)-K(k)}{k} \quad \text{(verifica diretta)} \tag{D.3}$$

$$\frac{dk'}{dk}=-\frac{k}{k'} \quad \text{(verifica diretta, dove } k'=\sqrt{1-k^2}\text{)} \tag{D.4}$$

$$\frac{dK'}{dk}=-\frac{E'(k)-k^2K'(k)}{kk'^2} \quad \text{(simmetrica rispetto alla (D.2))} \tag{D.5}$$

$$\frac{dE'}{dk}=\frac{k[K'(k)-E'(k)]}{k'^2} \quad \text{(simmetrica rispetto alla (D.3))} \tag{D.6}$$

A titolo esemplificativo, diamo comunque prova delle (D.2) e (D.3):

$\frac{dE}{dk}=\frac{E(k)-K(k)}{k}$ dimostrazione:

$$\frac{dE}{dk}=\int_0^{\pi/2}\frac{-k\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta=\frac{1}{k}\int_0^{\pi/2}\frac{-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta=\frac{1}{k}\int_0^{\pi/2}\frac{1-1-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}d\beta=$$

$$=\frac{1}{k}\int_0^{\pi/2}\left[\frac{1-k^2\sin^2\beta}{\sqrt{(1-k^2\sin^2\beta)}}-\frac{1}{\sqrt{(1-k^2\sin^2\beta)}}\right]d\beta=\frac{1}{k}\int_0^{\pi/2}\left[\sqrt{(1-k^2\sin^2\beta)}-\frac{1}{\sqrt{(1-k^2\sin^2\beta)}}\right]d\beta=$$

$$=\frac{E(k)-K(k)}{k} \; .$$

$\frac{dK}{dk}=\frac{E(k)-k'^2K(k)}{kk'^2}$ dimostrazione:

$$\frac{dK}{dk}=\int_0^{\pi/2}\frac{k\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=\frac{-1}{k}\int_0^{\pi/2}\frac{-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=\frac{-1}{k}\int_0^{\pi/2}\frac{-1+1-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}d\beta=$$

$$=\frac{-1}{k}\int_0^{\pi/2}\left[\frac{1-k^2\sin^2\beta}{(1-k^2\sin^2\beta)^{3/2}}-\frac{1}{(1-k^2\sin^2\beta)^{3/2}}\right]d\beta=$$

$$=\frac{-1}{k}\int_0^{\pi/2}\frac{1}{\sqrt{(1-k^2\sin^2\beta)}^{3/2}}d\beta+\frac{1}{k}\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\beta)^{3/2}}d\beta=$$

$=\frac{1}{k}\left[-K+\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\beta)^{3/2}}d\beta\right]$ ed effettuando il cambio di variabile $\sin^2\beta=x^2$, si ha:

$\frac{dK}{dk}=\frac{1}{k}\left[-K+\int_0^{1}\frac{1}{(1-k^2x^2)^{3/2}}\frac{1}{\sqrt{1-x^2}}dx\right]$ e con un ulteriore cambio di variabile $x^2=y$, si ha:

$\frac{dK}{dk}=\frac{1}{k}[-K+\int_0^1\frac{(1-y)^{-\frac{1}{2}}}{(1-k^2y)^{3/2}}\frac{1}{2}y^{-\frac{1}{2}}dy]=\frac{1}{k}[-K+\frac{1}{2}\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-k^2y)^{3/2}}y^{\frac{1}{2}-1}dy]$, ma visto che in Appendice A abbiamo dato alla funzione Beta di Eulero l'espressione:

$B(m,n)=\int_0^1(1-x)^{n-1}x^{m-1}dx$, allora possiamo scrivere che: $B(\frac{1}{2},\frac{1}{2})=\int_0^1(1-y)^{\frac{1}{2}-1}y^{\frac{1}{2}-1}dx$

e valendo l'eguaglianza: $\int_0^1\frac{(1-y)^{c-b-1}}{(1-ty)^a}y^{b-1}dy=B(c-b,b)\cdot{}_2F_1(a,b;c;t)$ (vedere la (C.27)), segue

che: $\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-ty)^{3/2}}y^{\frac{1}{2}-1}dy=B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;t)$ e, dunque:

$$\frac{dK}{dk}=\frac{1}{k}[-K+\frac{1}{2}\int_0^1\frac{(1-y)^{\frac{1}{2}-1}}{(1-k^2y)^{3/2}}y^{\frac{1}{2}-1}dy]=\frac{1}{k}[-K+\frac{1}{2}B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;k^2)] \qquad (\ t=k^2\ )$$

ed essendo che con una Trasformazione di Eulero (vedere la (C.25)) si ha:

${}_2F_1(a,b;c;x)=(1-x)^{c-b-a}\cdot{}_2F_1(c-a,c-b;c;x)$, allora, essendo tra l'altro che $B(\frac{1}{2},\frac{1}{2})=\pi$, segue

che: $\frac{dK}{dk}=\frac{1}{k}[-K+\frac{1}{2}B(\frac{1}{2},\frac{1}{2})\cdot{}_2F_1(\frac{3}{2},\frac{1}{2};1;k^2)]=\frac{1}{k}[-K+\frac{\pi}{2}(1-k^2)^{-1}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)]=$

$=\frac{1}{k}[-K+\frac{1}{(1-k^2)}\frac{\pi}{2}\cdot{}_2F_1(-\frac{1}{2},\frac{1}{2};1;k^2)]=\frac{1}{k}[-K+\frac{1}{k'^2}E(k)]=\frac{E(k)-k'^2K(k)}{kk'^2}$,

dopo aver contemplato la (C.9). c.v.d.

Come dimostrazione della (D.1) , proviamo a fare la derivata della stessa:

$$\frac{dK(k)}{dk}E'(k)+K(k)\frac{dE'(k)}{dk}+\frac{dK'(k)}{dk}E(k)+K'(k)\frac{dE(k)}{dk}-\frac{dK(k)}{dk}K'(k)-K(k)\frac{dK'(k)}{dk}=0$$

da cui:

$$\frac{E(k)-k'^2K(k)}{kk'^2}E'(k)+K(k)\frac{k[K'(k)-E'(k)]}{k'^2}-\frac{E'(k)-k^2K'(k)}{kk'^2}E(k)+K'(k)\frac{E(k)-K(k)}{k}-$$

$$-\frac{E(k)-k'^2K(k)}{kk'^2}K'(k)+K(k)\frac{E'(k)-k^2K'(k)}{kk'^2}=0$$

$$\frac{E(k)-k'^2K(k)}{kk'^2}[E'(k)-K'(k)]-\frac{E'(k)-k^2K'(k)}{kk'^2}[E(k)-K(k)]+K(k)\frac{k[K'(k)-E'(k)]}{k'^2}+$$

$$+K'(k)\frac{E(k)-K(k)}{k}=0$$

$$[E(k)-k'^2K(k)][E'(k)-K'(k)]-[E'(k)-k^2K'(k)][E(k)-K(k)]-k^2K(k)[E'(k)-K'(k)]+$$

$$+k'^2K'(k)[E(k)-K(k)]=0$$

$$[E(k)-k'^2K(k)-k^2K(k)][E'(k)-K'(k)]+[k'^2K'(k)-E'(k)+k^2K'(k)][E(k)-K(k)]=0$$

$$[E(k)E'(k)-k'^2K(k)E'(k)-k^2K(k)E'(k)-E(k)K'(k)+k'^2K(k)K'(k)+k^2K(k)K'(k)]+$$

$$+[k'^2K'(k)E(k)-E'(k)E(k)+k^2K'(k)E(k)-k'^2K'(k)K(k)+E'(k)K(k)-k^2K'(k)K(k)]=0$$

$$[\cancel{E(k)E'(k)}+K(k)E'(k)(-k'^2-k^2+1)-E(k)K'(k)+\cancel{K(k)K'(k)(k^2+k'^2)}]+$$

$$+[K'(k)E(k)(k'^2+k^2)-\cancel{E'(k)E(k)}-\cancel{K'(k)K(k)(k'^2+k^2)}]=0$$

$(((\,(-k'^2-k^2+1)=0\,)))$ $\quad (((\,k'^2+k^2=1\,)))$

$[-E(k)K'(k)]+$
$+[K'(k)E(k)]=0 \quad \rightarrow$ 0=0 . c.v.d.

Per quanto riguarda poi il valore di π/2, proviamo a scrivere la (D.1) nel caso limite di k=0:
$K(0)E'(0)+K'(0)E(0)-K(0)K'(0)=const$, ossia:

$$\frac{\pi}{2}1+(\int_0^{\pi/2}\frac{d\beta}{\cos\beta})(\frac{\pi}{2})-\frac{\pi}{2}(\int_0^{\pi/2}\frac{d\beta}{\cos\beta})=\frac{\pi}{2} \quad \text{c.v.d.}$$

Diamo ora anche dimostrazione della seguente equazione:

$$\frac{1}{\pi}=-\frac{2}{\pi^2}kk'^2\frac{d}{dk}(KK')\ ; \tag{D.7}$$

infatti, dalla (D.2) si ottiene: $E=kk'^2\frac{dK}{dk}+k'^2K$, mentre dalla (D.5) si ha: $E'=-kk'^2\frac{dK'}{dk}+k^2K'$, che sostituite nella (D.1) forniscono:

$$K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\frac{\pi}{2}=K(-kk'^2\frac{dK'}{dk}+k^2K')+K'(kk'^2\frac{dK}{dk}+k'^2K)-KK'$$

$$-kk'^2K\frac{dK'}{dk}+k^2KK'+kk'^2K'\frac{dK}{dk}+k'^2KK'-KK'=\frac{\pi}{2}$$

$$-kk'^2K\frac{dK'}{dk}+kk'^2K'\frac{dK}{dk}+k^2KK'+k'^2KK'-KK'=\frac{\pi}{2}$$

$-kk'^2K\frac{dK'}{dk}+kk'^2K'\frac{dK}{dk}+k^2KK'(k^2+k'^2-1)=\frac{\pi}{2}$, ma $(k^2+k'^2-1)=0$, da cui:

$$kk'^2(K'\frac{dK}{dk}-K\frac{dK'}{dk})=\frac{\pi}{2} \quad \rightarrow$$

$$\rightarrow (K'\frac{dK}{dk}-K\frac{dK'}{dk})=\frac{\pi}{2}\frac{1}{kk'^2} \tag{D.8}$$

$\rightarrow (K'\frac{dK(-k)}{d(-k)}-K\frac{dK'}{dk})=\frac{\pi}{2}\frac{1}{kk'^2}$, ma:

K(-k)=K(k) , da cui: $[K'\frac{dK(-k)}{d(-k)}-K\frac{dK'}{dk}]=[K'\frac{dK(k)}{d(-k)}-K\frac{dK'}{dk}]=[-K'\frac{dK(k)}{dk}-K\frac{dK'}{dk}]=$

$$=-\frac{d}{dk}(KK')=\frac{\pi}{2}\frac{1}{kk'^2} \rightarrow \frac{d}{dk}(KK')=-\frac{\pi}{2}\frac{1}{kk'^2} \rightarrow$$

$$\rightarrow \frac{1}{\pi}=-\frac{2}{\pi^2}kk'^2\frac{d}{dk}(KK')\ . \tag{D.9}$$

APPENDICE E: Il Prodotto Triplo di Jacobi.

Eccolo:

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n}) \qquad \text{(E.1)}$$

-<u>dimostrazione</u>:

definiamo la funzione

$$F(\sqrt{x}) = \prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1}) = (1+qx)(1+\frac{q}{x})(1+xq^3)(1+\frac{q^3}{x})(1+xq^5)(1+\frac{q^5}{x}).... \qquad \text{(E.2)}$$

da cui, per $F(q\sqrt{x})$ (si sostituisce ovviamente, al secondo membro della (E.2), x$(=(\sqrt{x})^2)$ con xq^2):

$$F(q\sqrt{x}) = \prod_{n=1}^{\infty}(1+xq^{2n+1})(1+x^{-1}q^{2n-3}) = (1+q^3x)(1+\frac{1}{xq})(1+xq^5)(1+\frac{q}{x})(1+xq^7)(1+\frac{q^3}{x})....; \qquad \text{(E.3)}$$

facendo ora la (E.3) diviso la (E.2), si ottiene:

$$\frac{F(q\sqrt{x})}{F(\sqrt{x})} = \frac{(1+q^3x)(1+\frac{1}{xq})(1+xq^5)(1+\frac{q}{x})(1+xq^7)(1+\frac{q^3}{x})....}{(1+qx)(1+\frac{q}{x})(1+xq^3)(1+\frac{q^3}{x})(1+xq^5)(1+\frac{q^5}{x})....} = \boxed{[[[\cancel{\frac{(1+xq^7)}{(1+\frac{q^5}{x})}}]]]\frac{(1+\frac{1}{xq})}{(1+qx)}} = \frac{(1+\frac{1}{xq})}{(1+qx)}$$

poiché il termine tra le parentesi quadre scompare con l'aggiunta a numeratore e denominatore delle due coppie di termini che compaiono nelle produttorie quando n=4 e 5 (poiché noi ci eravamo fermati ad n=3, ossia solo tre coppie di prodotti) e tali due coppie di prodotti aggiuntivi degli n=4 e 5 sono:

$$\frac{[(1+xq^9)(1+x^{-1}q^5)][(1+xq^{11})(1+x^{-1}q^7)]}{[(1+xq^7)(1+x^{-1}q^7)][(1+xq^9)(1+x^{-1}q^9)]} = \boxed{\frac{(1+x^{-1}q^5)}{(1+xq^7)}} \cdot [[\frac{(1+xq^9)\cancel{(1+xq^{11})}\cancel{(1+x^{-1}q^7)}}{\cancel{(1+x^{-1}q^7)}\cancel{(1+xq^9)}(1+x^{-1}q^9)}]]$$

tra cui compare la frazione nel riquadro rosso che cancella appunto quella precedente del riquadro verde, e così via. Dunque, riassumendo, abbiamo ottenuto che:

$$\frac{F(q\sqrt{x})}{F(\sqrt{x})} = \frac{(1+\frac{1}{xq})}{(1+qx)}$$ e svolgendo il numeratore, si ottiene: $$\frac{F(q\sqrt{x})}{F(\sqrt{x})} = \frac{(\frac{1+qx}{xq})}{(1+qx)} = \frac{1}{xq}$$, da cui:

$$F(\sqrt{x}) = xqF(q\sqrt{x}) \ . \qquad \text{(E.4)}$$

Definiamo ora le seguenti:

$$G(\sqrt{x}) = F(\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n}) \qquad \text{(E.5)}$$

$$G(q\sqrt{x}) = F(q\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n}) \qquad \text{(E.6)}$$

Ora, usando la (E.4) nella (E.6), si ottiene:

$$G(q\sqrt{x}) = \frac{F(\sqrt{x})}{xq}\prod_{n=1}^{\infty}(1-q^{2n}) = \frac{G(\sqrt{x})}{xq}$$, ossia:

$$G(\sqrt{x}) = xqG(q\sqrt{x}) \qquad \text{(E.7)}$$

Notiamo ora che $F(\sqrt{x})$ è una funzione pari, poiché se si sostituisce $\sqrt{x}$ con $-\sqrt{x}$, $F(\sqrt{x})$ resta immutata, da cui anche $G(\sqrt{x})$, per la (E.5), è pari.
Immaginiamo ora di sviluppare $G(\sqrt{x})$ in serie di Laurent, ma, se preferiamo, senza scomodare il concetto di serie di Laurent, possiamo attribuire a $G(\sqrt{x})$ uno sviluppo il più generale possible, che includa dunque anche termini ad esponente negativo, ossia:

$$G(\sqrt{x}) = \sum_{m=-\infty}^{+\infty} a_m x^m \tag{E.8}$$

la (E.4) richiede allora che:

$$\sum_{m=-\infty}^{+\infty} a_m x^m (= \sum_{m=-\infty}^{+\infty} a_m [(\sqrt{x})^2]^m) = xq \sum_{m=-\infty}^{+\infty} a_m [(q\sqrt{x})^2]^m = \sum_{m=-\infty}^{+\infty} a_m q^{2m+1} x^{m+1} \quad . \tag{E.9}$$

Adesso, a secondo membro della (E.9) possiamo ribattezzare l'indice così: m → m'-1 , da cui:

$$\sum_{m=-\infty}^{+\infty} a_m x^m = \sum_{m'=-\infty}^{+\infty} a_{m'-1} q^{2(m'-1)+1} x^{(m'-1)+1} = \sum_{m'=-\infty}^{+\infty} a_{m'-1} q^{2m'-1} x^{m'} \tag{E.10}$$

($m'-1=-\infty$ equivale a $m'=-\infty$, così come $m'-1=+\infty$ equivale a $m'=+\infty$, da cui $\sum\limits_{m'-1=-\infty}^{+\infty} = \sum\limits_{m'=-\infty}^{+\infty}$)

e ribattezzando, nell'ultimo membro della (E.10), m' di nuovo in m, si ottiene:

$\sum\limits_{m=-\infty}^{+\infty} a_m x^m = \sum\limits_{m=-\infty}^{+\infty} a_{m-1} q^{2m-1} x^m$, da cui deduciamo che: $a_m = a_{m-1} q^{2m-1}$, da cui:

$$a_1 = a_0 q$$

$$a_2 = a_1 q^3 = a_0 q \; q^3 = a_0 q^4 = a_0 q^{2^2}$$

$$a_3 = a_2 q^5 = a_0 q^{2^2} q^5 = a_0 q^9 = a_0 q^{3^2}$$

……………………………………

$$a_m = a_0 q^{m^2}$$

dunque $$G(\sqrt{x}) = \sum_{m=-\infty}^{+\infty} a_m x^m = a_0 \sum_{m=-\infty}^{+\infty} q^{m^2} x^m \tag{E.11}$$

e da quest'ultima si ha che per $\sqrt{x}=1$, e cioè per x=1:

$G(1) = a_0 \sum\limits_{m=-\infty}^{+\infty} q^{m^2} 1^m = a_0 \sum\limits_{m=-\infty}^{+\infty} q^{m^2}$ e quando m=0 → $q^{m^2} = q^{0^2} = 1$ e a_0 non è dunque accompagnato da alcun q, o meglio è accompagnato da q^0 (=1), ossia appunto nessun q; ma notiamo altresì che:

$$F(\sqrt{x}=1) = F(1) = \prod_{n=1}^{\infty} (1+xq^{2n-1})(1+x^{-1}q^{2n-1})\Big|_{x=1} = \prod_{n=1}^{\infty} (1+q^{2n-1})(1+q^{2n-1}) = \prod_{n=1}^{\infty} (1+q^{2n-1})^2 \quad \text{e:}$$

$$G(\sqrt{x}=1) = G(1) = F(1)\prod_{n=1}^{\infty}(1-q^{2n}) = \prod_{n=1}^{\infty}(1+q^{2n-1})^2 \prod_{n=1}^{\infty}(1-q^{2n}) = \prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n})$$

e l'argomento nell'ultimo membro di quest'ultima è:

$$(1+q^{2n-1})^2(1-q^{2n}) = (1+q^{2(2n-1)}+2q^{2n-1})(1-q^{2n}) = 1+q^{2(2n-1)}+2q^{2n-1}-q^{2n}(1+q^{2(2n-1)}+2q^{2n-1}) =$$
$$= 1+q^{2(2n-1)}+2q^{2n-1}-q^{2n}-q^{2n}q^{2(2n-1)}-q^{2n}2q^{2n-1} = 1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} =$$
$$= 1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} \quad \text{e:}$$

con n=1 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} = 1+q^2+2q-q^2-q^4-2q^3$

con n=2 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} = 1+q^6+2q^3-q^4-q^{10}-2q^7$

con n=3 → $1+q^{4n-2}+2q^{2n-1}-q^{2n}-q^{6n-2}-2q^{4n-1} = 1+q^{10}+2q^5-q^6-q^{16}-2q^{11}$

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quindi il prodotto dei primi due, ossia:

$$(1+q^2+2q\ \ -q^2-q^4-2q^3)(1+q^6+2q^3-q^4-q^{10}-2q^7)=1+......=1\cdot q^0+......$$

dà appunto, come termine numerico (ossia quello accompagnato da q^0) il numero 1 e allora le (E.2), (E.5) ed (E.11)-con $a_0=1$, ci dicono che:

$$F(\sqrt{x})=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})$$

$$G(\sqrt{x})=F(\sqrt{x})\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$

$$G(\sqrt{x})=\sum_{m=-\infty}^{+\infty}q^{m^2}x^m$$

da cui:

$$\sum_{m=-\infty}^{+\infty}q^{m^2}x^m=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$, ossia proprio la (E.1) cvd.

APPENDICE F: Le Funzioni Theta di Jacobi.

Eccole:

$$\theta_1(q)=0\ ,$$

$$\theta_2(q)=......+q^{(-\frac{3}{2})^2}+q^{(-\frac{1}{2})^2}+q^{(\frac{1}{2})^2}+q^{(\frac{3}{2})^2}+q^{(\frac{5}{2})^2}+.........\ ,$$

$$\theta_3(q)=......+q^{(-2)^2}+q^{(-1)^2}+1+q^{1^2}+q^{2^2}+q^{3^2}+.........\ , \tag{F.1}$$

$$\theta_4(q)=......+q^{(-2)^2}-q^{(-1)^2}+1-q^{1^2}+q^{2^2}-q^{3^2}+.........\ .$$

Imparentamento delle Funzioni Theta di Jacobi con l'Integrale Ellittico di Prima Specie K(k): ponendo:

$$q=e^{-\pi\frac{K(k')}{K(k)}}$$ (q è detto "nome", k è il parametro, che già definimmo) , (F.2)

segue che:

$$k=\frac{\theta_2^2(q)}{\theta_3^2(q)}\ , \tag{F.3}$$

$$k'=\sqrt{1-k^2}=\frac{\theta_4^2(q)}{\theta_3^2(q)}=\frac{\theta_3^2(-q)}{\theta_3^2(q)} \tag{F.4}$$

$$(\rightarrow\ kk'=\frac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)})\ \text{e} \tag{F.5}$$

$$K(k)=\frac{\pi}{2}\theta_3^2(q)\ ; \tag{F.6}$$

funzione modulare $\lambda(q)=k^2(q)=[\frac{\theta_2(q)}{\theta_3(q)}]^4$. (F.7)

-<u>Dimostrazioni</u>:

innanzitutto possiamo scrivere le Funzioni Theta di Jacobi anche nel seguente modo:

$$\theta_2(q)=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n})\ , \tag{F.8}$$

$$\theta_3(q)=\prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n})\ , \tag{F.9}$$

$$\theta_4(q)=\prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n}) \ ; \qquad \text{(F.10)}$$

infatti, utilizzando il Prodotto Triplo di Jacobi (vedere l'Appendice E , eq. (E.1)):

$$\sum_{n=-\infty}^{\infty}x^n q^{n^2}=\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1})(1-q^{2n})$$

e ponendo x=1 si ottiene proprio $\theta_3(q)$:

$$\theta_3(q)=\sum_{n=-\infty}^{\infty}1^n q^{n^2}=\prod_{n=1}^{\infty}(1+1q^{2n-1})(1+1^{-1}q^{2n-1})(1-q^{2n})=\prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n}) \ ;$$

ponendo invece x=-1 , si ottiene $\theta_4(q)$:

$$\theta_4(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1})(1-q^{2n}) \ .$$

Per ultimo, ponendo x=q:

$$\sum_{n=-\infty}^{\infty}q^n q^{n^2}=\prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n})$$ e moltiplicando m.a.m. per $q^{\frac{1}{4}}$:

$$q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^n q^{n^2}=q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n})$$, ossia:

$$\theta_2(q)=\sum_{n=-\infty}^{\infty}q^{\frac{1}{4}}q^n q^{n^2}=\sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2}=q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+qq^{2n-1})(1+q^{-1}q^{2n-1})(1-q^{2n})=$$

$$=q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n-2})(1-q^{2n})=q^{\frac{1}{4}}[(1+q^2)(1+q^0)(1-q^2)][(1+q^4)(1+q^2)(1-q^4)]\cdot$$

$$\cdot[(1+q^6)(1+q^4)(1-q^6)]...=q^{\frac{1}{4}}[(1+q^2)2(1-q^2)][(1+q^4)(1+q^2)(1-q^4)][(1+q^6)(1+q^4)(1-q^6)]...=$$

$$=2q^{\frac{1}{4}}[(1+q^2)(1+q^2)(1-q^2)][(1+q^4)(1+q^4)(1-q^4)][(1+q^6)(1+q^6)(1-q^6)]...=$$

$$=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n})(1-q^{2n})=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n})=\theta_2(q) \ .$$

Dunque, riassumendo, si ha anche che:

$$\theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2} \qquad \text{(F.11)}$$

$$\theta_3(q)=\sum_{n=-\infty}^{\infty}q^{n^2} \qquad \text{(F.12)}$$

$$\theta_4(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} \qquad \text{(F.13)}$$

E si ha: $\theta_4^2(q)=\theta_3^2(-q)$. (F.14)

E ricordiamo ora che:

$$\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1})=1 \qquad \text{(F.15)}$$

infatti, $(1+q^{2n})$ ha esponente 2n e 2n sono numeri pari e basta, mentre $(1+q^{2n-1})$ ha esponente 2n-1 e 2n-1 sono numeri dispari; allora l'espressione che contempla sia i pari che i dispari è $(1+q^n)$, dunque:

$$\prod_{n=1}^{\infty}[\underline{(1+q^{2n})(1+q^{2n-1})}](1-q^{2n-1})=\prod_{n=1}^{\infty}[\underline{(1+q^{n})}](1-q^{2n-1})$$, ma anche in quest'ultimo caso possiamo dire che $(1-q^{2n-1})$ ha esponente dispari e dunque, per quanto appena detto, può essere espresso come $(1-q^{n})/(1-q^{2n})$, da cui:

$$\prod_{n=1}^{\infty}[(1+q^{n})](1-q^{2n-1})=\prod_{n=1}^{\infty}[(1+q^{n})]\frac{(1-q^{n})}{(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{(1+q^{n})(1-q^{n})}{(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n})}=1 \quad \text{c.v.d.}$$

APPENDICE G: Sugli invarianti di Ramanujan.

Definendo gli invarianti: $G=G_N=(\frac{1}{2kk'})^{\frac{1}{12}}$ e $g=g_N=(\frac{k'^2}{2k})^{\frac{1}{12}}$, (G.1)

(invarianti poiché dipendono da parametri fissi k e k' ; poi, si può usare N oppure n…)
si ha che:

$$\prod_{k=1,3,5,...}^{\infty}(1+e^{-k\pi\sqrt{n}})=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}G_N \quad \text{e} \tag{G.2}$$

$$\prod_{k=1,3,5,...}^{\infty}(1-e^{-k\pi\sqrt{n}})=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N \tag{G.3}$$

infatti, con $q=e^{-k\pi\sqrt{n}}$, queste due produttorie equivalgono ovviamente a scrivere che:

$(1+q)(1+q^3)(1+q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}}$ e $(1-q)(1-q^3)(1-q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}}$, poiché, per quanto riguarda ad esempio la prima delle due e ricordando che:

$K(k)=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$, $k'=\sqrt{1-k^2}$, nonché:

$q=e^{-\pi\frac{K(k')}{K(k)}}$ $(\frac{K'}{K}=\sqrt{n}=\sqrt{N}\equiv\sqrt{r})$ (scelte nostre personali che facemmo all'inizio, nei ragionamenti fatti appena dopo la (1.4)…)

e la già dimostrata $kk'=\frac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)}$ (vedere Appendice F , eq. (F.5)), allora, esplicitando quest'ultima:

$$kk'=\frac{\theta_2^2(q)\theta_4^2(q)}{\theta_3^4(q)}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n})^{2}\prod_{n=1}^{\infty}(1-q^{2n-1})^4(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1-q^{2n})^4}=$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\cancel{(1-q^{2n})^2}(1-q^{2n-1})^4\cancel{(1-q^{2n})^2}}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8\cancel{(1-q^{2n})^4}}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8}=$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8}\frac{(1+q^{2n-1})^4}{(1+q^{2n-1})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}[(1+q^{2n})^4(1+q^{2n-1})^4](1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4}$$, ma:

$[(1+q^{2n})^4(1+q^{2n-1})^4]=(1+q^n)^4$ per la questione già vista degli esponenti pari e dispari, contemplati dal solo n , da cui:

$$kk' = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4}$$ ed essendo che, per lo stesso motivo, si ha anche che:

$(1-q^{2n-1})^4 = \frac{(1-q^n)^4}{(1-q^{2n})^4}$, allora: $kk' = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4[\frac{(1-q^n)^4}{(1-q^{2n})^4}]}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4} =$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^n)^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4(1-q^{2n})^4} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^{2n})^4}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4(1-q^{2n})^4} =$$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}1}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8(1+q^{2n-1})^4} = \frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \frac{kk'}{4q^{\frac{1}{2}}} = \frac{1}{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \frac{4q^{\frac{1}{2}}}{kk'} = \prod_{n=1}^{\infty}(1+q^{2n-1})^{12} \rightarrow$$

$\rightarrow \sqrt[12]{\frac{4q^{\frac{1}{2}}}{kk'}} = \sqrt[12]{\prod_{n=1}^{\infty}(1+q^{2n-1})^{12}} \rightarrow \frac{2^{\frac{1}{6}}q^{\frac{1}{24}}}{k^{\frac{1}{12}}k'^{\frac{1}{12}}} = \prod_{n=1}^{\infty}(1+q^{2n-1}) = (1+q)(1+q^3)(1+q^5)......$, da cui l'asserto, ossia: $(1+q)(1+q^3)(1+q^5)...... = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}}$ cvd .

Ragionamento simile per la seconda, come dimostriamo qui sotto.

Intanto, con le seguenti definizioni:

$$G = G_N = (\frac{1}{2kk'})^{\frac{1}{12}} \quad , \quad g = g_N = (\frac{k'^2}{2k})^{\frac{1}{12}} \rightarrow k = g^6\sqrt{g^{12}+\frac{1}{g^{12}}} - g^{12} \quad , \quad k' = \sqrt{2k}g^6 \quad , \qquad \text{(G.4)}$$

si ha: $(1+q)(1+q^3)(1+q^5)...... = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{\frac{-1}{12}} = 2^{\frac{1}{6}}e^{\frac{-\pi\sqrt{n}}{24}}[2^{\frac{1}{12}}G_N] = 2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}G_N$

$(1-q)(1-q^3)(1-q^5)...... = 2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}} = 2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N$

Più nello specifico, dicevamo, riguardo la seconda, essendo sempre che: $k = \frac{\theta_2^2(q)}{\theta_3^2(q)}$, $k' = \frac{\theta_4^2(q)}{\theta_3^2(q)}$ e

$\theta_4^2(q) = \theta_3^2(-q)$, segue che: $\frac{k}{k'^2} = \frac{\frac{\theta_2^2(q)}{\theta_3^2(q)}}{[\frac{\theta_4^2(q)}{\theta_3^2(q)}]^2} = \frac{\theta_2^2(q)\theta_3^2(q)}{\theta_4^4(q)} = \frac{\theta_2^2(q)\theta_3^2(q)}{\theta_3^4(-q)}$ ed esplicitando:

$$\frac{k}{k'^2} = \frac{\theta_2^2(q)\theta_3^2(q)}{\theta_3^4(-q)} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4(1-q^{2n})^2\prod_{n=1}^{\infty}(1+q^{2n-1})^4(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8(1-q^{2n})^4} =$$

$$= \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\prod_{n=1}^{\infty}(1+q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8} = \frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{2n})^4\prod_{n=1}^{\infty}(1+q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}\frac{(1+q^{2n-1})^4}{(1+q^{2n-1})^4} =$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^8}\frac{(1-q^{2n-1})^4}{(1-q^{2n-1})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}=$$

$$=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^n)^4(1-q^n)^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}(1-q^{2n})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^{2n})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}(1-q^{2n})^4}=\frac{4q^{\frac{1}{2}}\prod_{n=1}^{\infty}1}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}=\frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}$$, ossia:

$$\frac{k}{k'^2}=\frac{4q^{\frac{1}{2}}}{\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}}\ \rightarrow\ \frac{k'^2}{k}4q^{\frac{1}{2}}=\prod_{n=1}^{\infty}(1-q^{2n-1})^{12}\ \rightarrow$$

$$\rightarrow\ (\frac{k'^2}{k})^{\frac{1}{12}}2^{\frac{1}{6}}q^{\frac{1}{24}}=\prod_{n=1}^{\infty}(1-q^{2n-1})=(1-q)(1-q^3)(1-q^5)......$$, da cui l'asserto, ossia:

$$(1-q)(1-q^3)(1-q^5)......=2^{\frac{1}{6}}q^{\frac{1}{24}}(k)^{\frac{-1}{12}}k'^{\frac{1}{6}}=2^{\frac{1}{4}}e^{\frac{-\pi\sqrt{n}}{24}}g_N \quad \text{cvd.}$$

Riassumendo, dalle definizioni (G.1) ad inizio Appendice, ossia dalle:

$G=G_N=(\frac{1}{2kk'})^{\frac{1}{12}}$ e $g=g_N=(\frac{k'^2}{2k})^{\frac{1}{12}}$, con un N=58 (scelta libera, ma non obbligata, che ci porterà a risultati molto convenienti) ed utilizzando le scelte arbitrarie (ed anche loro convenienti)

$\frac{K'}{K}=\sqrt{r}=\sqrt{N}=\sqrt{58}=\frac{K'_{58}}{K_{58}}=\boxed{\frac{K(\sqrt{1-k_{58}^2})}{K(k_{58})}=\sqrt{58}}$, nonché la $q=e^{-\pi\frac{K(k')}{K(k)}}=e^{-\pi\sqrt{N}}$ che anche

facemmo ad inizio Appendice, ne discende un k_{58} (pari a $k_{58}=0{,}0000255482707576$), ossia la (1.41), che possiamo qui valutare nel seguente modo:

ricordiamo la $k=\frac{\theta_2^2(q)}{\theta_3^2(q)}$ dimostrata appena qui sopra, nonché le (F.11) ed (F.12):

$\theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2}$ e $\theta_3(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$ ed anche la $q_{58}=e^{-\pi\sqrt{58}}=e^{-23,9249512120}=4{,}0794633669\cdot 10^{-11}$, da cui:

$$k=\frac{\theta_2^2(q)}{\theta_3^2(q)}=\frac{[\sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2}]^2}{[\sum_{n=-\infty}^{\infty}q^{n^2}]^2}=\frac{[...+q^{(-3+\frac{1}{2})^2}+q^{(-2+\frac{1}{2})^2}+q^{(-1+\frac{1}{2})^2}+q^{(0+\frac{1}{2})^2}+q^{(1+\frac{1}{2})^2}+q^{(2+\frac{1}{2})^2}+...]^2}{[...+q^{(-3)^2}+q^{(-2)^2}+q^{(-1)^2}+q^{0^2}+q^{1^2}+q^{2^2}+q^{3^2}+...]^2}=$$

$$=\frac{[...+q^{(-3+\frac{1}{2})^2}+q^{(-2+\frac{1}{2})^2}+q^{(-1+\frac{1}{2})^2}+q^{(0+\frac{1}{2})^2}+q^{(1+\frac{1}{2})^2}+q^{(2+\frac{1}{2})^2}+...]^2}{[...+q^{(-3)^2}+q^{(-2)^2}+q^{(-1)^2}+q^{0^2}+q^{1^2}+q^{2^2}+q^{3^2}+...]^2}=\frac{[...+q^{\frac{25}{4}}+q^{\frac{9}{4}}+q^{\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{9}{4}}+q^{\frac{25}{4}}+...]^2}{[...+q^9+q^4++q+1+q+q^4+q^9+...]^2}=$$

$$=\frac{[2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+...]^2}{[1+2q+2q^4+2q^9+...]^2}=\frac{4[q^{\frac{1}{4}}+q^{\frac{9}{4}}+q^{\frac{25}{4}}+...]^2}{4[\frac{1}{2}+q+q^4+q^9+...]^2}=\frac{[q^{\frac{1}{4}}+q^{\frac{9}{4}}+q^{\frac{25}{4}}+...]^2}{[\frac{1}{2}+q+q^4+q^9+...]^2}$$ e allora:

$$k_{58}=\frac{[(4{,}0794633669\cdot 10^{-11})^{\frac{1}{4}}+(4{,}0794633669\cdot 10^{-11})^{\frac{9}{4}}+(4{,}0794633669\cdot 10^{-11})^{\frac{25}{4}}+...]^2}{[\frac{1}{2}+(4{,}0794633669\cdot 10^{-11})+(4{,}0794633669\cdot 10^{-11})^4+(4{,}0794633669\cdot 10^{-11})^9+...]^2}\cong$$

$$\cong\frac{[(4{,}0794633669\cdot 10^{-11})^{\frac{1}{4}}+\sim 0+\sim 0+...]^2}{[\frac{1}{2}+(4{,}0794633669\cdot 10^{-11})+\sim 0+\sim 0+...]^2}\cong\frac{[(4{,}0794633669\cdot 10^{-11})^{\frac{1}{4}}]^2}{[\frac{1}{2}+(4{,}0794633669\cdot 10^{-11})]^2}=\frac{\sqrt{4{,}0794633669\cdot 10^{-11}}}{\frac{1}{4}}=$$

$$=4\sqrt{4{,}0794633669\cdot 10^{-11}}=4\sqrt{40{,}794633669}\cdot 10^{-6}=25{,}5482707576\cdot 10^{-6}=0{,}0000255482707576\,!$$

ossia: $k_{58}=0{,}0000255482707576$. (valore accreditato: $k_{58}=(\sqrt{2}-1)^6(13\sqrt{58}-99)$)

A questo punto, determinato il k_{58} , si ha, in modo consequenziale:

$g=g_{58}=(\frac{k'^2_{58}}{2k_{58}})^{\frac{1}{12}}=(\frac{1-k^2_{58}}{2k_{58}})^{\frac{1}{12}}=2{,}2784213480$ (valore accreditato: $g_{58}=(\frac{2}{\sqrt{29}-5})^{1/2}\cong 2{,}279$),

ossia la (1.42), nonché: $\frac{g^{12}_{58}+g^{-12}_{58}}{2}=9785{,}3981139349$ (valore accreditato: $9801=(\frac{1}{x_{58}})$), ossia la (1.40) (e si noti che se nei nostri conteggi qui effettuati avessimo usato più decimali, tipo, ad esempio, il non considerare semplicemente 2,718 come "e", non ci saremmo fermati al valore simbolico 9785,3981139349 qui presentato sommariamente, ma saremmo arrivati tranquillamente al valore accreditato di $9.800{,}99997449\cong 9801$), nonché:

$\frac{1}{g^{12}_{58}}=\frac{2k_{58}}{1-k^2_{58}}=0{,}00005109654154$ (valore accreditato: $\frac{1}{g^{12}_{58}}=(\frac{\sqrt{29}-5}{2})^6$), ossia la (1.44) , poi la:

$g^6_{58}=139{,}8956617510$ (valore accreditato: $g^6_{58}=(\frac{2}{\sqrt{29}-5})^3$), che è la (1.45), la

$\frac{(g^{12}_{58}-g^{-12}_{58})}{2}=9785{,}3981139349$ (valore accreditato: 9801), che è la (1.46) e, come conseguenza della (1.5), ossia della $\alpha(58)=\frac{\pi}{(2K_{58})^2}-\sqrt{58}(\frac{E_{58}}{K_{58}}-1)$, si ha infine, ad esempio coi valori tabellari (o ricavati dalle (C.8) e (C.9)) di K_{58} ed E_{58} , (ma anche per altra via), la $\alpha(58)\cong 0{,}3183192742$ (di fatto: 1/π) (valore accreditato: $\alpha(58)=3g^6_{58}k_{58}(33\sqrt{29}-148)$), che è la (1.43).

Invece, sempre riguardo α(58), in modo molto più elegante, invocando la (H.11), ossia la:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{r}})^4}$$

, si ha: $(e^{-\pi\cdot n^2\sqrt{58}}=e^{-23{,}9249512120n^2})$

$$\alpha(58)=\frac{\frac{1}{\pi}-4\sqrt{58}\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{58}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{58}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{58}})^4}=\frac{\frac{1}{\pi}-4\sqrt{58}\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-23{,}9249512120n^2}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-23{,}9249512120n^2}}}{(\sum_{n=-\infty}^{\infty}e^{-23{,}9249512120n^2})^4}=$$

$$= \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{[0+2\cdot 1^2(-1)^{1^2}e^{-23,9249512120\cdot 1^2}+2\cdot 2^2(-1)^{2^2}e^{-23,9249512120\cdot 2^2}+...]}{[1+(-1)^{1^2}e^{-23,9249512120\cdot 1^2}+(-1)^{2^2}e^{-23,9249512120\cdot 2^2}+...]}}{(1+e^{-23,9249512120\cdot 1^2}+e^{-23,9249512120\cdot 2^2}+...)^4} =$$

$$= \frac{\frac{1}{\pi} - 4\sqrt{58}\frac{[-2e^{-23,9249512120}+8e^{-95,6998048480}+...]}{[1-e^{-23,9249512120}+e^{-95,6998048480}+...]}}{(1+e^{-23,9249512120}+e^{-95,6998048480}+...)^4} =$$

$$= \frac{\frac{1}{\pi} - 30,4630924234\frac{(-2\cdot 4,0794633669\cdot 10^{-11}+8\cdot 2,7695687502\cdot 10^{-42}+...)}{(1-4,0794633669\cdot 10^{-11}+2,7695687502\cdot 10^{-42}+...)}}{(1+4,0794633669\cdot 10^{-11}+2,7695687502\cdot 10^{-42}+...)^4} \cong$$

$$\cong \frac{\frac{1}{\pi} - 30,4630924234\frac{(-2\cdot 4,0794633669\cdot 10^{-11})}{1}}{1} \cong \frac{1}{\pi} + 248,5461391674\cdot 10^{-11} \cong \frac{1}{\pi} \cong 0,3183192742$$

ossia: $\boxed{\alpha(58) \cong 0,3183192742\ .}$

APPENDICE H: Sulla convergenza ad 1/π della funzione α(k).

(per la dimostrazione della convergenza usiamo un primo metodo grezzo e poi un secondo metodo più fine)

-primo metodo (grezzo):

Riguardo ora l'eventuale convergenza a $\frac{1}{\pi}$ di $\alpha(r)$ (la (1.4) che vuole ricalcare la (1.3)), esponiamola dapprima con un metodo grezzo, di puro calcolo basato sugli sviluppi in serie che esprimono gli integrali ellittici K ed E; a tal proposito, ricordiamo che:

$$\alpha(r) = \frac{E'(k)}{K(k)} - \frac{\pi}{[2K(k)]^2}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$ (Integrale Ellittico di Prima Specie-Appendice C)

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\cdot d\theta$$ (Integrale Ellittico di Seconda Specie-Appendice C)

$$K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \frac{\pi}{2}$$ (Relazione Fondamentale di Legendre-Appendice D)

$k' = \sqrt{1-k^2}$ (parametri k e k') .

-Abbiamo dunque già verificato che, nel caso di $k = k' = \frac{1}{\sqrt{2}}$ → $\boxed{\alpha(\frac{1}{\sqrt{2}})=0,5\sim 1/\pi}$ (vedere la (1.3)).

-Se k=1 (ossia un valore limite) → (K(1)= ∞ ed E(1)=1) (e quindi k'=0) →

$$K(1) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{\cos\theta} = \ln[\tan(\frac{\pi}{4}+\frac{\theta}{2})]\Big|_0^{\pi/2} = \ln[\tan(\frac{\pi}{4}+\frac{\pi}{4})] - \ln[\tan(\frac{\pi}{4})] =$$

$$= \ln\tan\frac{\pi}{2} - \ln\tan\frac{\pi}{4} = \ln\frac{\tan\frac{\pi}{2}}{\tan\frac{\pi}{4}} = \infty$$

$E(1) = \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\cdot d\theta = \int_0^{\pi/2} \cos\theta d\theta = \sin\theta\Big|_0^{\pi/2} = 1$, e poi:

$$K'(1)=K(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

$$E'(1)=E(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

da cui: $\boxed{\alpha(1)=\frac{E'(1)}{K(1)}-\frac{\pi}{[2K(1)]^2}=\frac{\frac{\pi}{2}}{\infty}-\frac{\pi}{[2\infty]^2}=0}$, e infatti questo risultato non combacia con $\frac{1}{\pi}$.

-Se però k=0 (K(0)= π/2 ed E(0)= π/2) (e quindi k'=1) →

$$K(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

$E(0)=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$, e poi:

$$K'(0)=K(1)=\infty$$

$E'(0)=E(1)=1$ e dunque: $\boxed{\alpha(0)=\frac{E'(0)}{K(0)}-\frac{\pi}{[2K(0)]^2}=\frac{1}{\frac{\pi}{2}}-\frac{\pi}{[2\frac{\pi}{2}]^2}=\frac{2}{\pi}-\frac{1}{\pi}=\frac{1}{\pi}}$ (risultato ok).

-Se k=0,8 (e quindi $k'=\sqrt{1-(0,8)^2}=0,6$) e considerando le (C.6) e (C.7) , ossia le:

$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$ ed $E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2k^{2n}$, segue che:

$$K(0,8)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0,8)^{2n}\cong\frac{\pi}{2}\{[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0,8)^{2\cdot 0}+[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0,8)^{2\cdot 1}+[\frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}]^2(0,8)^{2\cdot 2}+...\}\cong$$

$\cong 1,57\{1+[\frac{1}{4}\frac{2}{1}]^2(0,8)^2+[\frac{1}{4^2}\frac{(4)!}{2^2}]^2(0,8)^4+...\}\cong 1,57(1+0,16+0,05759+...)\cong\boxed{1,912}$, oppure, usando l'espressione data in Appendice A (vedere la (A.9)):

$$(K=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}=\frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...])$$

segue che: $K(0,8)=\frac{\pi}{2}[1+(\frac{1}{2})^2(0,8)^2+(\frac{1}{2}\frac{3}{4})^2(0,8)^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2(0,8)^6+...]\cong$

$$\cong 1,57[1+\frac{0,64}{4}+\frac{9}{64}0,4096+\frac{225}{2304}0,2621+...]=1,57[1+0,16+0,0576+0,0256+...]\cong\boxed{1,952}$$

Noi, però, che non abbiamo troppo tempo per fare calcoli troppo lunghi ed utilizzare troppi termini, considereremo come buono un valore di tabella: $\boxed{1,995.}$

Poi: ($E(0,8)\cong\boxed{1,276\ ,}$ valore di tabella), ma facendo comunque qualche calcolo esemplificativo:

$$E(0,8)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0,8)^{2n}=$$

$$=\frac{\pi}{2}[(-\frac{1}{2\cdot 0-1})[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0,8)^{2\cdot 0}+(-\frac{1}{2\cdot 1-1})[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0,8)^{2\cdot 1}+(-\frac{1}{2\cdot 2-1})[\frac{1}{4^2}\frac{(4)!}{(2!)^2}]^2(0,8)^{2\cdot 2}+...]=$$

$$=1,57[1-\frac{1}{4}0,64-0,0469\cdot 0,4096+...]=1,57[1-0,16-0,0192+...]\cong\boxed{1,289}$$

e poi: (valore di tabella: $\boxed{1,751}$)

$$K'(0{,}8)=K(0{,}6)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}6)^{2n}\cong$$

$$\cong\frac{\pi}{2}\{[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}6)^{2\cdot 0}+[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}6)^{2\cdot 1}+[\frac{1}{4^2}\frac{(2\cdot 2)!}{(2!)^2}]^2(0{,}6)^{2\cdot 2}+...\}\cong$$

$$\cong 1{,}57\{1+[\frac{1}{4}\frac{2}{1}]^2(0{,}6)^2+[\frac{1}{4^2}\frac{(4)!}{2^2}]^2(0{,}6)^4+...\}\cong 1{,}57\{1+\frac{1}{4}0{,}36+[\frac{3}{8}]^2 0{,}1296+...\}\cong$$

$\cong 1{,}57(1+0{,}09+0{,}0182+...)\cong\boxed{1{,}74}$, oppure, usando la (A.9) di Appendice A:

$$(K=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}=\frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1}{2}\frac{3}{4})^2k^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2k^6+...])$$

$$K'(0{,}8)=K(0{,}6)=\frac{\pi}{2}[1+(\frac{1}{2})^2(0{,}6)^2+(\frac{1}{2}\frac{3}{4})^2(0{,}6)^4+(\frac{1}{2}\frac{3}{4}\frac{5}{6})^2(0{,}6)^6+...]\cong$$

$$\cong 1{,}57[1+\frac{0{,}36}{4}+\frac{9}{64}0{,}1296+\frac{225}{2304}0{,}0466+...]=1{,}57[1+0{,}09+0{,}0182+0{,}00455+...]\cong\boxed{1{,}747}\quad;$$

poi: (valore di tabella: $\boxed{1{,}418}$) $E'(0{,}8)=E(0{,}6)=\frac{\pi}{2}\sum_{n=0}^{\infty}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}6)^{2n}=$

$$=\frac{\pi}{2}[(-\frac{1}{2\cdot 0-1})[\frac{1}{4^0}\frac{(2\cdot 0)!}{(0!)^2}]^2(0{,}6)^{2\cdot 0}+(-\frac{1}{2\cdot 1-1})[\frac{1}{4^1}\frac{(2\cdot 1)!}{(1!)^2}]^2(0{,}6)^{2\cdot 1}+(-\frac{1}{2\cdot 2-1})[\frac{1}{4^2}\frac{(4)!}{(2!)^2}]^2(0{,}6)^{2\cdot 2}+...]=$$

$$=1{,}57[1-\frac{1}{4}0{,}36-0{,}0469\cdot 0{,}1296+...]=1{,}57[1-0{,}09-0{,}00607+...]\cong\boxed{1{,}419}$$

e, infine:

$$\boxed{\alpha(0{,}8)=\frac{E'(0{,}8)}{K(0{,}8)}-\frac{\pi}{[2K(0{,}8)]^2}=\frac{E(0{,}6)}{K(0{,}8)}-\frac{\pi}{[2K(0{,}8)]^2}=\frac{1{,}418}{1{,}995}-\frac{\pi}{[2\cdot 1{,}995]^2}=0{,}711-0{,}197=0{,}514}$$

o l'inverso è 1,946 , che è ovviamente peggiorativo, nel confronto con 1/π, rispetto al caso di k=0,707.

-Stando invece più vicini allo zero:

se k=0,2 (e quindi $k'=\sqrt{1-(0{,}2)^2}\cong 0{,}98$) $\rightarrow$: (valore di tabella: $\boxed{1{,}587}$)

$K(0{,}2)\cong\frac{\pi}{2}\sum_{n=0}^{2}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}2)^{2n}\cong\boxed{1{,}586}$ e:

(valore di tabella: $\boxed{1{,}555}$)

$$E(0{,}2)\cong\frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}2)^{2n}\cong\boxed{1{,}554}$$

e: (valore di tabella: 1,05)

$E'(0{,}2)=E(0{,}98)\cong\frac{\pi}{2}\sum_{n=0}^{\boxed{2}}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0{,}98)^{2n}\cong\boxed{1{,}125}$ e, infine:

$$\boxed{\alpha(0{,}2)=\frac{E'(0{,}2)}{K(0{,}2)}-\frac{\pi}{[2K(0{,}2)]^2}=\frac{E(0{,}98)}{K(0{,}2)}-\frac{\pi}{[2K(0{,}2)]^2}=\frac{1{,}05}{1{,}587}-\frac{\pi}{[2\cdot 1{,}587]^2}=0{,}6616-0{,}312=0{,}3117}$$

il cui inverso è 3,21, già più simile a π.

-E stando ancora più vicini allo zero:

se k=0,1 (e quindi $k' = \sqrt{1-(0,1)^2} \cong 0,995$) → (valore di tabella: $\boxed{1,575}$)

$$K(0,1) \cong \frac{\pi}{2}\sum_{n=0}^{2}[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0,1)^{2n} \cong \boxed{1,574} \quad \text{e:}$$

(valore di tabella: $\boxed{1,567}$) $E(0,1) \cong \frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0,1)^{2n} \cong \boxed{1,566}$

e poi: (valore di tabella: $\boxed{\sim 1,004}$)

$$E'(0,1) = E(0,995) \cong \frac{\pi}{2}\sum_{n=0}^{2}(-\frac{1}{2n-1})[\frac{1}{4^n}\frac{(2n)!}{(n!)^2}]^2(0,995)^{2n} \cong \boxed{1,11}$$

e, infine:

$$\boxed{\alpha(0,1) = \frac{E'(0,1)}{K(0,1)} - \frac{\pi}{[2K(0,1)]^2} = \frac{E(0,995)}{K(0,1)} - \frac{\pi}{[2K(0,1)]^2} = \frac{1,004}{1,575} - \frac{\pi}{[2\cdot 1,575]^2} = 0,6375 - 0,3164 = 0,3211}$$

il cui inverso è 3,114 , ossia vicino a π.

-secondo metodo (fine):

abbiamo preannunciato, appena dopo la (1.5), che:

$$\frac{1}{\pi} = \theta_3^4\alpha(r) + \sqrt{r}\,4q\frac{\dot{\theta}_4}{\theta_4}, \tag{H.1}$$

poiché valeva la seguente:

$$\frac{4K}{\pi^2}(E-K) = 4q\frac{d}{dq}\ln\theta_4 = 4q\frac{\dot{\theta}_4}{\theta_4}, \tag{H.2}$$

(derivata logaritmica della funzione theta di Jacobi) di cui qui però diamo qui <u>dimostrazione</u>:

incrociando la (F.4) ($k' = \sqrt{1-k^2} = \frac{\theta_4^2(q)}{\theta_3^2(q)}$) con la (F.6) ($K(k) = \frac{\pi}{2}\theta_3^2(q)$) si ottiene: $\theta_4(q) = \sqrt{\frac{2Kk'}{\pi}}$

ed effettuandone il logaritmo, si ottiene:

$$\ln\theta_4(q) = \ln(\frac{2Kk'}{\pi})^{1/2} = \frac{1}{2}\ln\frac{2}{\pi} + \frac{1}{2}\ln K + \frac{1}{2}\ln k' \tag{H.3}$$

e ricordando l'espressione (F.2) del nome modulare q: $q = e^{-\pi\frac{K(k')}{K(k)}} = e^{-\pi\frac{K'}{K}}$, ne valutiamo la derivata:

$$\frac{dq}{dk} = -\pi e^{-\pi\frac{K'}{K}}\frac{d}{dk}\frac{K'}{K} = -\pi e^{-\pi\frac{K'}{K}}(K'\frac{d}{dk}K^{-1} + K^{-1}\frac{dK'}{dk}) = \frac{\pi q}{K^2}(K'\frac{dK}{dk} - K\frac{dK'}{dk}), \tag{H.4}$$

ma dalla (D.8) si ha che: $(K'\frac{dK}{dk} - K\frac{dK'}{dk}) = \frac{\pi}{2}\frac{1}{kk'^2}$, che inserita nella (H.4) fornisce:

$$\frac{dq}{dk} = \frac{\pi q}{K^2}(K'\frac{dK}{dk} - K\frac{dK'}{dk}) = \frac{\pi^2 q}{2kk'^2K^2}\ . \tag{H.5}$$

Ora, dalla (H.5) si ha, ovviamente: $q\frac{d}{dq} = \frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}$; (H.6) adesso, per la (D.2) si ha: $\frac{dK}{dk}=\frac{E-k'^2K}{kk'^2}$, mentre per la (D.4) si ha: $\frac{dk'}{dk}=-\frac{k}{k'}$, da cui, per la (H.3): $4q\frac{d}{dq}\ln\theta_4=4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\ln\theta_4=4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}\ln(\frac{2Kk'}{\pi})^{1/2}=$

$$=4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}(\frac{1}{2}\ln\frac{2}{\pi})+4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}(\frac{1}{2}\ln K)+4\frac{2kk'^2K^2}{\pi^2}\frac{d}{dk}(\frac{1}{2}\ln k')=$$

$$=0+4\frac{kk'^2K}{\pi^2}\frac{dK}{dk}+4\frac{kk'K^2}{\pi^2}\frac{dk'}{dk}=4\frac{kk'^2K}{\pi^2}\frac{(E-k'^2K)}{kk'^2}-4\frac{k^2K^2}{\pi^2}=\frac{4K}{\pi^2}(E-k'^2K)-4\frac{k^2K^2}{\pi^2} \tag{H.7}$$

ed essendo che: $k'^2=1-k^2$, segue che: $k'^2K=K-Kk^2$ e allora la (H.7) fornisce:

$4q\frac{d}{dq}\ln\theta_4=\frac{4K}{\pi^2}(E-k'^2K)-4\frac{k^2K^2}{\pi^2}=\frac{4K}{\pi^2}(E-K+Kk^2)-4\frac{k^2K^2}{\pi^2}=\frac{4K}{\pi^2}(E-K)$, ossia:

$4q\frac{d}{dq}\ln\theta_4=\frac{4K}{\pi^2}(E-K)$, che altro non è che la (H.2)! cvd.

Adesso, dalla (H.1) si ottiene:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\dot\theta_4}{\theta_4}}{\theta_3^4} \tag{H.8}$$

e ricordando che da un confronto tra la (F.9) e la (F.10) scaturisce che $\theta_4(q)=\theta_3(-q)$, ossia: $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=\sum_{n=-\infty}^{\infty}(-q)^{n^2}$ e considerando altresì le espressioni trovate in Appendice F per le varie θ_i ((F.11)---(F.14)), si ha:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\frac{d}{dq}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\frac{d}{dq}\sum_{n=-\infty}^{\infty}(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=$$

$$=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}[(-1)^{n^2}(q)^{n^2}]}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}\frac{d}{dq}(q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4q\frac{\sum_{n=-\infty}^{\infty}n^2q^{n^2-1}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2q^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=$$

$$=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}\ \text{(H.9) , ossia:}\ \boxed{\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}}\ . \tag{H.10}$$

Ora, per la $q=e^{-\pi\frac{K'}{K}}=e^{-\pi\sqrt{r}}$ appena ricordata, si ha:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{r}})^4} \tag{H.11}$$

e visto che con r→∞ succede che $e^{-\pi\sqrt{r}}$ va a zero molto velocemente (**da cui l'estrema efficacia della formula di Ramanujan…**), allora:

$$\lim_{r\to\infty}\alpha(r)=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}{\sum_{n=-\infty}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}}}{(\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2\sqrt{r}})^4}=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{[\sum_{n=-\infty,n\neq o}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}]}{[\sum_{n=-\infty,n\neq o}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}]}}{(\sum_{n=-\infty,n\neq o}^{\infty}e^{-\pi\cdot n^2\sqrt{r}}+e^{-\pi\cdot 0^2\sqrt{r}})^4}=$$

$$=\lim_{r\to\infty}\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{[\sum_{n=-\infty,n\neq o}^{\infty}n^2(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+0]}{[\sum_{n=-\infty,n\neq o}^{\infty}(-1)^{n^2}e^{-\pi\cdot n^2\sqrt{r}}+1]}}{(\sum_{n=-\infty,n\neq o}^{\infty}e^{-\pi\cdot n^2\sqrt{r}}+1)^4}\to\frac{\frac{1}{\pi}-\sqrt{\infty(lento)}\cdot 4\frac{[0(veloce)+0]}{[0+1]}}{(0+1)^4}\to\frac{1}{\pi}\quad !!! \tag{H.12}$$

Volendo poi avere una valutazione concreta della **rapidità di convergenza** di α(∞) ad 1/π, ripartiamo dalla (H.10) e ricordiamo che nel limite di r→∞ , si ha che q→0 ed allora:

$$\alpha(r)=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{\sum_{n=-\infty}^{\infty}n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty}(-q)^{n^2}}}{(\sum_{n=-\infty}^{\infty}q^{n^2})^4}=\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{0+2\sum_{n=1}^{\infty}n^2(-q)^{n^2}}{1+2\sum_{n=1}^{\infty}(-q)^{n^2}}}{(1+2\sum_{n=1}^{\infty}q^{n^2})^4}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{2(-q)^1+8(-q)^4+...}{1+2(-q)^1+2(-q)^4+...}}{[1+2q^1+2q^4+...]^4}\cong$$

$$\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{(-2q+8q^4+...)}{1-2q+2q^4+...}}{[1+2q+...]^4}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4\frac{(-2q+...)}{1-2q+...}}{(1+8q+...)}\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4(-2q+...)(1+2q+...)}{(1+8q+...)}\cong$$

$$\cong\frac{\frac{1}{\pi}-\sqrt{r}\,4(-2q-4q^2+...)}{(1+8q+...)}\cong[\frac{1}{\pi}-\sqrt{r}\,8(-q-2q^2+...)](1-8q+...)\cong$$

$$\cong\frac{1}{\pi}(1-8q+...)-\sqrt{r}\,8(-q-2q^2+...)(1-8q+...)\cong\frac{1}{\pi}-\frac{8q}{\pi}-\sqrt{r}\,8(-q+...)(1-8q+...)\cong$$

$$\cong\frac{1}{\pi}-\frac{8q}{\pi}-\sqrt{r}\,8(-q+8q^2+...)\cong\frac{1}{\pi}-\frac{8q}{\pi}+8q\sqrt{r}+...\cong\frac{1}{\pi}+8(\sqrt{r}-\frac{1}{\pi})q$$, da cui:

$$\boxed{\alpha(r)-\frac{1}{\pi}\to 8(\sqrt{r}-\frac{1}{\pi})q=8(\sqrt{r}-\frac{1}{\pi})e^{-\pi\sqrt{r}}\to 0}\quad !!!\ \text{cvd.} \tag{H.13}$$

(tendenza molto veloce, come, conseguentemente, veloce è la convergenza della serie del Pi di Ramanujan, poiché $e^{-\pi\sqrt{r}}$ sopraffà molto velocemente $\sqrt{r}$, per r→∞)

APPENDICE I: Sulla somma di Ramanujan di tutti i naturali (-1/12).

1 – La Funzione Zeta di Riemann – definizione:

$$\boxed{\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}} \tag{1.1}$$

ovviamente, per una questione di convergenza, s>1 (nel caso di s reale) oppure Re(s)>1 in caso di s complesso.

Tuttavia, abbiamo valori di $\zeta(s)$ anche per s<1 e vogliamo qui sottolineare il valore: $\zeta(-1)=-\frac{1}{12}$, che per la Funzione Zeta di Riemann, ricordiamolo, è ufficiale!

La serie (1.1) non convergerebbe per $s\le 1$, ma si può estendere il dominio tramite altre funzioni, così come, ad esempio, la serie geometrica $S(x)=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots$ converge solo per $x\in]-1,+1[$, sebbene, volendo vedere questa serie come lo sviluppo di Taylor di $f(x)=\frac{1}{1-x}$, allora S(x) può vedere la sua definizione estesa a tutto il dominio di f(x), che ha valori ovunque, tranne che in x=1.

E lo stesso succede a $\zeta(s)$.

2 – La parentela con i numeri primi – Formula Prodotto di Eulero:

Ricordiamo che: $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots$; moltiplicando ora entrambi i membri per $\frac{1}{2^s}$, si ha:

$$\frac{1}{2^s}\zeta(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\ldots; \tag{2.1}$$

sottraendo ora una dall'altra, si ottiene:

$$\zeta(s)-\frac{1}{2^s}\zeta(s)=\zeta(s)(1-\frac{1}{2^s})=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{15^s}+\ldots; \tag{2.2}$$

moltiplichiamo ora entrambi i membri per $\frac{1}{3^s}$:

$$\frac{1}{3^s}(1-\frac{1}{2^s})\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\ldots \tag{2.3}$$

e sottraendo ora la (2.3) dalla (2.2), si ottiene:

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{25^s}+\ldots \tag{2.4}$$

Reiterando il metodo, moltiplichiamo entrambi i membri della (2.4) per $\frac{1}{5^s}$, da cui:

$$\frac{1}{5^s}(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{35^s}+\frac{1}{55^s}+\ldots \tag{2.5}$$

e ancora sottraendo la (2.5) dalla (2.4):

$$(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\frac{1}{23^s}+\frac{1}{29^s}+\ldots\quad . \tag{2.6}$$

Già si vede, nella (2.6), che il termine dominante è l'1, al quale si aggiungono termini sempre minori dell'unità; e si nota altresì, che il secondo membro della (2.6) è meno ricco di quello della

(2.4), il quale, a sua volta, ha meno termini pesanti di quello della (2.2), sicché, continuando così, ad un certo punto si giungerà all'unità e basta:

$$.......(1-\frac{1}{13^s})(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1 \;, \tag{2.7}$$

e notiamo che al primo membro della (2.7) compaiono tutti i numeri primi. Ricavando allora $\zeta(s)$ dalla (2.7), si ha:

$$\zeta(s)=\frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})(1-\frac{1}{7^s})(1-\frac{1}{11^s})(1-\frac{1}{13^s}).......}=\prod_p (1-p^{-s})^{-1} \text{ , ossia:}$$

$$\boxed{\zeta(s)=\prod_p (1-p^{-s})^{-1}} \quad \text{(p numero primo)} \tag{2.8}$$

che è appunto la **Formula Prodotto di Eulero**, che imparenta la Z di Riemann con i numeri primi.
E si evince anche che i numeri primi sono infiniti, sebbene la dimostrazione di Euclide del fatto che i numeri primi siano infiniti resta sempre la più affascinante:
il Teorema di Euclide afferma che i numeri primi sono infiniti. Euclide lo dimostrò per assurdo nel 300 a.C., provando che, per quanto grande sia un insieme finito di numeri primi, ne esisterà sempre un altro non incluso in esso. La Dimostrazione è per Assurdo. Il ragionamento logico si articola nei seguenti passaggi: l'ipotesi (che vogliamo smentire): i numeri primi sono in quantità finita e possono essere elencati; si calcola il numero N formato dal prodotto di tutti questi numeri primi e si aggiunge 1; si hanno così due possibilità:
-se N è primo, allora abbiamo trovato un nuovo numero primo che non apparteneva all'elenco iniziale.
-Se N non è primo, ma è composto, allora deve essere divisibile per qualche numero primo. Tuttavia, se dividiamo N per uno qualsiasi dei numeri primi elencati, otterremo sempre il resto di 1, visto che avevamo sommato appunto 1; pertanto, il divisore primo di N deve essere un numero primo nuovo, non presente nella lista. La conclusione è che in entrambi i casi, esiste sempre un nuovo numero primo. L'elenco iniziale non poteva quindi essere completo, dimostrando che i numeri primi sono infiniti.

<u>Finti generatori di numeri primi</u>: consideriamo i primi generati dalla seguente formula: $p_n = n^2 - 79n + 1601$; bene, per $0 \le n \le 79$ i p_n sono tutti primi, ma già per n=80 non più; infatti: $p_{80}=41^2$!!!!

Sempre riguardo la parentela tra Funzione Zeta di Riemann e numeri primi, indichiamo con $\pi(N)$ il numero di numeri primi che si incontrano nel contare fino ad N: per N=1000, si incontrano circa 170 numeri primi ($\pi(1000) \approx 170$); poi: $\pi(10^6) \approx 80.000$, $\pi(10^9) \approx 51 \cdot 10^6$, $\pi(10^{12}) \approx 38 \cdot 10^9$ e così via. Pare dunque (**Teorema dei Numeri Primi**) che: $\pi(N) \approx \frac{N}{\ln N}$.

Inoltre, consideriamo la funzione logaritmo integrale: $\int_0^x \frac{1}{\ln t} dt$ e definiamo: $Li(x) = \int_2^x \frac{1}{\ln t} dt$; si ha:

$Li(x) = \int_2^x \frac{1}{\ln t} dt \approx \pi(x) \cong \frac{x}{\ln x}$ (numero di primi entro x); infatti, integrando per parti:

$Li(x)=\int_2^x[\frac{1}{\ln t}]\cdot\{1\}\cdot dt=\left|[\frac{1}{\ln t}]\{t\}\right|_2^x-\int_2^x[(-1)\frac{1}{\ln^2 t}\frac{1}{t}]\{t\}dt=\frac{x}{\ln x}-\frac{2}{\ln 2}+\int_2^x\frac{1}{\ln^2 t}dt$, ossia, per x non piccolissime: $\int_2^x\frac{1}{\ln t}dt-\cancel{\int_2^x\frac{1}{\ln^2 t}dt}\approx\int_2^x\frac{1}{\ln t}dt\approx Li(x)\approx\frac{x}{\ln x}-\cancel{\frac{2}{\ln 2}}\approx\frac{x}{\ln x}\approx\pi(x)$.

(infatti, l'area sottesa da $\frac{1}{\ln^2 t}$ diventa trascurabile, col crescere di t, rispetto a quella sottesa da $\frac{1}{\ln t}$)

Riprendiamo, poi, la Formula Prodotto di Eulero (2.8): $\zeta(s)=\prod_p(1-p^{-s})^{-1}$ ed eseguiamo il logaritmo di entrambi i membri: $\ln\zeta(s)=-\sum_p(1-\frac{1}{p^s})=-\int_1^\infty\ln(1-\frac{1}{x^s})d\pi(x)$, dopo aver reinterpretato la sommatoria come un integrale e considerato che in un conteggio fino ad un x generico ci sono $\pi(x)$ primi. Integrando ora per parti, si ha:

$$\ln\zeta(s)=-\int_1^\infty\ln(1-\frac{1}{x^s})d\pi(x)=-\ln(1-\frac{1}{x^s})\pi(x)\Big|_1^\infty+\int_1^\infty\pi(x)\cdot d\ln(1-\frac{1}{x^s})=$$

$$=-[\ln(1-\frac{1}{\infty^s})\pi(\infty)-\ln(1-\frac{1}{1^s})\pi(1)]+\int_1^\infty\pi(x)\cdot\frac{sx^{-s-1}}{(1-\frac{1}{x^s})}dx=[0\cdot\infty-(-\infty\cdot 0)]+s\int_1^\infty\frac{\pi(x)}{x(x^s-1)}dx=$$

$$\boxed{=\ln\zeta(s)=s\int_1^\infty\frac{\pi(x)}{x(x^s-1)}dx\,.}$$

Oppure, come alternativa, $\ln\zeta(s)=-\sum_p(1-\frac{1}{p^s})=-\sum_{n=2}^\infty[\pi(n)-\pi(n-1)]\ln(1-\frac{1}{n^s})=$

$=-\sum_{n=2}^\infty\pi(n)[\ln(1-\frac{1}{n^s})-\ln(1-\frac{1}{(n-1)^s})]=\sum_{n=2}^\infty\pi(n)[\int_n^{n+1}\frac{s}{x(x^s-1)}dx]=s\int_2^\infty\frac{\pi(n)}{x(x^s-1)}dx]$, che è sostanzialmente il risultato già ottenuto.

-<u>valori complessi di s</u>:

Se nella Zeta di Riemann $\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}$ consideriamo valori complessi di s, tipo s=a+ib , allora ricordiamo innanzitutto che la funzione ha come zeri anche i complessi coniugati s* di s , ossia le s*=a-ib . E, a questo punto, forniamo alcuni zeri complessi di Zeta:
s_1~(1/2)+i14,135 , s_2~(1/2)+i21,022 , s_3~(1/2)+i25,011 e, naturalmente, anche gli s_i* .
Effettivamente, usando, come esempio, s_1, si ha:

$$\zeta(\frac{1}{2}+i14{,}135)=\sum_{n=1}^\infty\frac{1}{n^s}=\frac{1}{1^{(\frac{1}{2}+i14.135)}}+\frac{1}{2^{(\frac{1}{2}+i14{,}135)}}+\frac{1}{3^{(\frac{1}{2}+i14{,}135)}}+...=\frac{1}{1^{\frac{1}{2}}1^{i14{,}135}}+\frac{1}{2^{\frac{1}{2}}2^{i14{,}135}}+\frac{1}{3^{\frac{1}{2}}3^{i14{,}135}}+...=$$

$$=\frac{1}{e^{i14{,}135\ln 1}}+\frac{1}{2^{\frac{1}{2}}e^{i14{,}135\ln 2}}+\frac{1}{3^{\frac{1}{2}}e^{i14{,}135\ln 3}}+...=1+\frac{1}{1{,}41e^{i9{,}8}}+\frac{1}{1{,}73e^{i15{,}53}}+...=$$

$$=1+\frac{1}{1{,}41(\cos 9{,}8+i\sin 9{,}8)}+\frac{1}{1{,}73(\cos 15{,}53+i\sin 15{,}53)}+...=$$

$$=1+\frac{(-0{,}93+i0{,}37)}{1{,}41(-0{,}93-i0{,}37)(-0{,}93+i0{,}37)}+\frac{(-0{,}98-i0{,}177)}{1{,}73(-0{,}98+i0{,}177)(-0{,}98-i0{,}177)}+...=$$

$$=1+\frac{(-0,93+i0,37)}{1,41\cdot 1,0018}+\frac{(-0,98-i0,177)}{1,73\cdot 0,9917}+\ldots=1+\frac{(-0,93+i0,37)}{1,41}+\frac{(-0,98-i0,177)}{1,72}+\ldots=$$

$=1-0,66+i0,262-0,570-i0,103+\ldots=1-1,23+i0,159\approx 0+i0=\underline{0}$ (tenendo conto delle approssimazioni avute anche nel fornire i valori degli zeri, nonché nel basso numero di termini utilizzati).

E analizzando i tre zeri complessi qui forniti, si ha poi l'Ipotesi di Riemann: tutti gli zeri non banali hanno parte reale pari a 1/2.

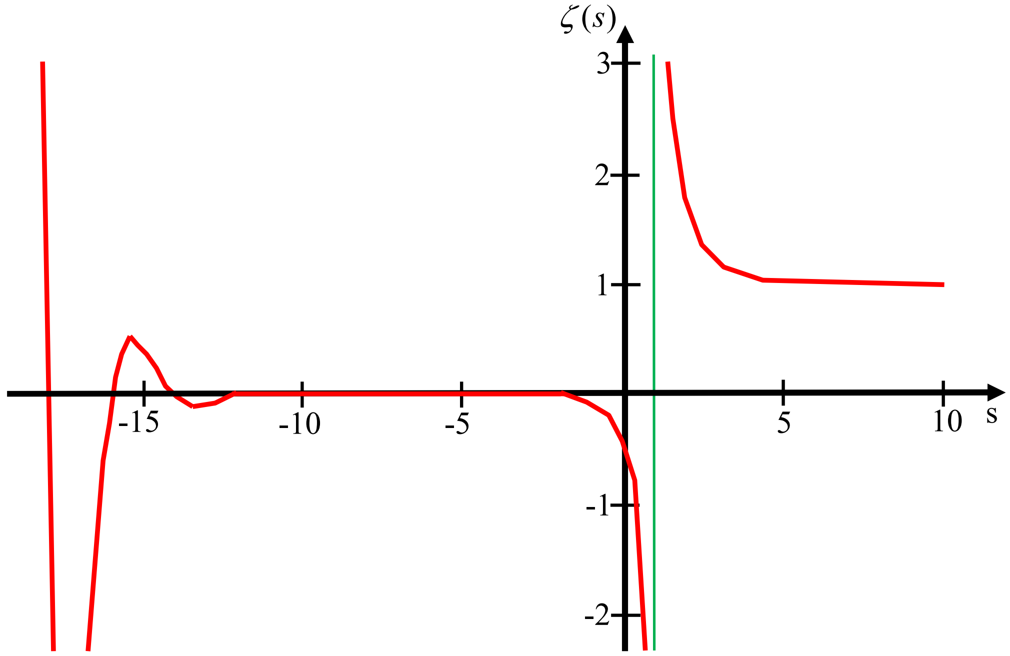

Fig. 2.1: La Funzione Zeta di Riemann $\zeta(s)$ graficata in un ampio range di s.

3 – Estensione del dominio della Funzione Zeta di Riemann:

a) tramite la serie η(s):

consideriamo la serie eta:

$$\eta(s)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\ldots \qquad (3a.1)$$

che converge per s>1, e notiamo facilmente che possiamo vederla nel seguente modo:

$$\eta(s)=(1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\ldots)=(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots)-2(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\ldots)=$$

$$=\zeta(s)-2\frac{1}{2^s}\zeta(s)=\zeta(s)(1-\frac{1}{2^{s-1}})\ \rightarrow$$

$$\boxed{\zeta(s)=\eta(s)(1-\frac{1}{2^{s-1}})^{-1}} \qquad (3a.2)$$

Allora, visto che $\eta(1/2)\cong 0,6$, segue che $\zeta(1/2)=\eta(1/2)(1-\frac{1}{2^{s-1}})^{-1}=0,6(1-\frac{1}{2^{-1/2}})^{-1}\cong -1,46$ ed abbiamo così ottenuto un valore di $\zeta(s)$ per un valore (appena sopra lo zero) di s, cioè (1/2), che è fuori dal range di convergenza della serie (1.1) , ossia s>1.

b) <u>tramite i Numeri di Bernoulli</u>:

con riferimento alla Subappendice A, dove vengono introdotti e forniti i numeri di Bernoulli B_i , si ha:

$$\boxed{\zeta(2n)=\frac{2^{2n-1}}{(2n)!}\left|B_{2n}\right|\pi^{2n}} \quad \text{e} \tag{3b.1}$$

$$\boxed{\zeta(1-n)=(-1)^{n+1}\frac{B_n}{n} \quad .} \tag{3b.2}$$

da cui, ad esempio:

n=2 $\longrightarrow$ $\zeta(1-2)=\zeta(-1)=(-1)^3\frac{B_2}{2}=-\frac{1}{12}$, n=3 $\longrightarrow$ $\zeta(1-3)=\zeta(-2)=(-1)^4\frac{B_3}{3}=0$,

n=4 $\longrightarrow$ $\zeta(1-4)=\zeta(-3)=(-1)^5\frac{B_4}{4}=\frac{1}{120}$, eccetera.

<u>Deduzione</u>:

Sappiamo che $p(x)=\frac{\sin x}{x}$ vale zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, dunque:

$$p(x)=\frac{\sin x}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi}).........=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2}).....=\prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2] \ .$$

Se poniamo $\alpha=\frac{x^2}{\pi^2}$, allora:

$$p(\alpha)=\prod_{n=1}^{\infty}[1-\frac{\alpha}{n^2}]. \tag{3b.3}$$

$$(\ p(0)=\prod_{n=1}^{\infty}[1-\frac{0}{n^2}]=\prod_{n=1}^{\infty}1=1\ )$$

Sappiamo anche, con Taylor, che: $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.....$ e dunque:

$$p(x)=\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+..... \tag{3b.4}$$

Ora, Taylor sulla (3b.3) dà:

$$p(\alpha)=\prod_{n=1}^{\infty}[1-\frac{\alpha}{n^2}]=p(0)+p'(0)\alpha+\frac{p''(0)}{2!}\alpha^2+\frac{p'''(0)}{3!}\alpha^3+... \tag{3b.5}$$

$\frac{d}{d\alpha}\ln p(\alpha)=\frac{1}{p(\alpha)}p'(\alpha)$, così: $\quad p'(\alpha)=p(\alpha)\frac{d}{d\alpha}\ln p(\alpha)=p(\alpha)\frac{d}{d\alpha}[\ln\prod(1-\frac{\alpha}{n^2})]=$

$$=p(\alpha)\frac{d}{d\alpha}[\sum\ln(1-\frac{\alpha}{n^2})]=p(\alpha)\sum\frac{d}{d\alpha}\ln(1-\frac{\alpha}{n^2})=p(\alpha)\sum\frac{-\frac{1}{n^2}}{(1-\frac{\alpha}{n^2})}=p(\alpha)\sum\frac{1}{(\alpha-n^2)} \ .$$

Quando $\alpha=0$, $p'(0)=p(0)\sum\frac{1}{(-n^2)}=-\sum\frac{1}{n^2}$ e stando alle (3b.4) e (3b.5), ciò deve valere $-\frac{x^2}{3!}$ ed anche $p'(0)\alpha=p'(0)\frac{x^2}{\pi^2}$, così: $-\frac{x^2}{\pi^2}\sum\frac{1}{n^2}=-\frac{x^2}{3!}$ e $\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$. Siccome la Funzione Zeta di Riemann è definita nel seguente modo: $\zeta(z)=\sum_{n=1}^\infty\frac{1}{n^z}$, allora: $\boxed{\zeta(2)=\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6}.}$

(e $p'(0)=p(0)\sum\frac{1}{(-n^2)}=-\frac{\pi^2}{6}$)

Ora calcoliamo $p''(\alpha)$: siccome $p'(\alpha)=p(\alpha)\sum\frac{1}{(\alpha-n^2)}$, allora:

$$p''(\alpha)=p'(\alpha)\sum\frac{1}{(\alpha-n^2)}-p(\alpha)\sum\frac{1}{(\alpha-n^2)^2},$$ così:

$$p''(0)=-p'(0)\sum\frac{1}{n^2}-p(0)\sum\frac{1}{n^4}=\frac{\pi^4}{36}-\sum\frac{1}{n^4},$$

ma per le (3b.4) e (3b.5), $\frac{p''(0)}{2!}\alpha^2=\frac{x^4}{5!}$, così: $\frac{1}{2}(\frac{\pi^4}{36}-\sum\frac{1}{n^4})\frac{x^4}{\pi^4}=\frac{x^4}{5!}$ e finalmente:

$\boxed{\zeta(4)=\sum_1^\infty\frac{1}{n^4}=\frac{\pi^4}{90}}$ (e $p''(0)=\frac{\pi^4}{36}-\sum\frac{1}{n^4}=\frac{\pi^4}{36}-\frac{\pi^4}{90}$).

Proseguendo ancora, otteniamo $p'''(\alpha)$ a partire da $p''(\alpha)$:

$$p'''(\alpha)=\frac{d}{d\alpha}p''(\alpha)=\frac{d}{d\alpha}[p'(\alpha)\sum\frac{1}{(\alpha-n^2)}-p(\alpha)\sum\frac{1}{(\alpha-n^2)^2}]=$$

$$=p''(\alpha)\sum\frac{1}{(\alpha-n^2)}-p'(\alpha)\sum\frac{1}{(\alpha-n^2)^2}-p'(\alpha)\sum\frac{1}{(\alpha-n^2)^2}+2p(\alpha)\sum\frac{1}{(\alpha-n^2)^3}=$$

$$=p''(\alpha)\sum\frac{1}{(\alpha-n^2)}-2p'(\alpha)\sum\frac{1}{(\alpha-n^2)^2}+2p(\alpha)\sum\frac{1}{(\alpha-n^2)^3}$$ e, per α=0 :

$$p'''(0)=-p''(0)\sum\frac{1}{n^2}-2p'(0)\sum\frac{1}{n^4}-2p(0)\sum\frac{1}{n^6}=-(\frac{\pi^4}{36}-\frac{\pi^4}{90})\frac{\pi^2}{6}+2\frac{\pi^2}{6}\frac{\pi^4}{90}-2\cdot1\cdot\sum\frac{1}{n^6}=$$

$=-\frac{\pi^2}{6}\frac{\pi^4}{36}+3\frac{\pi^2}{6}\frac{\pi^4}{90}-2\sum\frac{1}{n^6}=\frac{\pi^2}{6}(\frac{\pi^4}{30}-\frac{\pi^4}{36})-2\sum\frac{1}{n^6}=\frac{\pi^6}{1080}-2\sum\frac{1}{n^6}$, ma il solito confronto tra i due precedenti sviluppi di Taylor impone che: $\frac{p'''(0)}{3!}\alpha^3=-\frac{x^6}{7!}$, da cui:

$$\frac{1}{6}(\frac{\pi^6}{1080}-2\sum\frac{1}{n^6})\frac{x^6}{\pi^6}=-\frac{x^6}{7!}\rightarrow\frac{1}{6480}-\frac{1}{3}\frac{1}{\pi^6}\sum\frac{1}{n^6}=-\frac{1}{5040}\rightarrow\sum\frac{1}{n^6}=3\pi^6\frac{11520}{32659200}=\frac{\pi^6}{945},$$

ossia: $\boxed{\zeta(6)=\sum_1^\infty\frac{1}{n^6}=\frac{\pi^6}{945}.}$

Riassumendo: $\zeta(2)=\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$, $\zeta(4)=\sum_1^\infty\frac{1}{n^4}=\frac{\pi^4}{90}$, $\zeta(6)=\sum_1^\infty\frac{1}{n^6}=\frac{\pi^6}{945}$ e ricordando i Numeri di Bernoulli introdotti in Subappendice A, ossia:
$B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$ eccetera, deduciamo che:

$$\boxed{\zeta(2n)=\frac{2^{2n-1}}{(2n)!}|B_{2n}|\pi^{2n}\ .} \qquad (3b.6)$$

Inoltre, ricordando un po' di valori di $\zeta(s)$, ossia questi:

$\zeta(-1)=-\frac{1}{12}$, $\zeta(-2)=0$, $\zeta(-3)=\frac{1}{120}$, $\zeta(0)=-\frac{1}{2}$, $\zeta(\frac{3}{2})=2{,}612$, $\zeta(2)=\frac{\pi^2}{6}=1{,}645$, $\zeta(\frac{5}{2})=1{,}341$, $\zeta(3)=1{,}202$, $\zeta(\frac{7}{2})=1{,}127$, $\zeta(5)=1{,}0369$, $\zeta(7)=1{,}0083$, $\zeta(4)=\frac{\pi^4}{90}=1{,}0823$, $\zeta(6)=\frac{\pi^6}{945}=1{,}0173$, si evince anche la seguente, sempre coinvolgente i Numeri di Bernoulli:

$$\boxed{\zeta(1-n)=(-1)^{n+1}\frac{B_n}{n}\ ;} \qquad (3b.7)$$

infatti, come esempio, prendendo n=1, si ha:

$\zeta(1-1)=\zeta(0)=(-1)^{1+1}\frac{(-1/2)}{1}=-\frac{1}{2}$ (cosa vera) e, come ulteriore esempio, per n=2:

$\zeta(1-2)=\zeta(-1)=(-1)^{2+1}\frac{B_2}{2}=-1\frac{(1/6)}{2}=-\frac{1}{12}$, ossia: $\boxed{\zeta(-1)=-\frac{1}{12}}$, (cosa vera) , ossia di nuovo quel famigerato -1/12 !!!

c) <u>con l'**Equazione Funzionale**</u>:

l'Equazione Funzionale fornisce $\zeta(1-s)$ in termini di $\zeta(s)$ e ci permette di scendere ad s=0 ed anche sotto zero; così, se ad esempio conosciamo già $\zeta(16)$, possiamo ottenere $\zeta(-15)$.

Equazione Funzionale col fattoriale:

$$\zeta(1-s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)(s-1)!\zeta(s)\ . \qquad (3c.1)$$

Si ha che s ed (1-s) sono qui intercambiabili, da cui:

$$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})(-s)!\zeta(1-s)\ . \qquad (3c.2)$$

Equazione Funzionale con la Gamma di Eulero ($\Gamma(s+1)=s!$):

$$\boxed{\zeta(1-s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)\Gamma(s)\zeta(s)} \quad \text{e} \qquad (3c.3)$$

$$\boxed{\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)} \qquad (3c.4)$$

Bene, partiamo dalla definizione della Gamma di Eulero in x': $\Gamma(s)=\int_0^{\infty}e^{-x'}x'^{s-1}dx'$; se in questa poniamo (1/2)s in luogo di s e $n^2\pi x$ in luogo di x' ($x'=n^2\pi x\ \rightarrow\ dx'=n^2\pi dx$), abbiamo:

$$\Gamma(\frac{1}{2}s)=\int_0^\infty e^{-n^2\pi x}(n^2\pi x)^{\frac{1}{2}s-1}(n^2\pi)dx=\int_0^\infty e^{-n^2\pi x}(n^2\pi)^{\frac{1}{2}s-1}x^{\frac{1}{2}s-1}(n^2\pi)dx=(n^2\pi)^{\frac{1}{2}s}\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx,$$

da cui: $\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx=\dfrac{\Gamma(\frac{1}{2}s)}{n^s\pi^{\frac{1}{2}s}}$; applicando ora la sommatoria, si ottiene:

$$\sum_{n=1}^{\infty}\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx=\int_0^\infty x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty}e^{-n^2\pi x}dx=\sum_{n=1}^{\infty}\frac{\Gamma(\frac{1}{2}s)}{n^s\pi^{\frac{1}{2}s}}=\frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}}\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}}\zeta(s)$$, da cui:

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{\frac{1}{2}s}}=\int_0^\infty x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty}e^{-n^2\pi x}dx \quad . \tag{3c.5}$$

Denominando ora $f(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$, segue che, per la (3c.5):

$$\zeta(s)=\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\int_0^\infty x^{\frac{1}{2}s-1}f(x)dx \quad . \tag{3c.6}$$

Ora, qui sotto, in Subappendice D, dimostreremo a breve la seguente:

$$\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty}e^{-n^2\pi\frac{1}{x}} \tag{3c.7}$$

e la stessa può essere così scomposta:

$$\sum_{n=-\infty}^{-1}e^{-n^2\pi x}+e^{-n^2\pi 0}+\sum_{n=1}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}(\sum_{n=-\infty}^{-1}e^{-n^2\pi\frac{1}{x}}+e^{-n^2\pi 0}+\sum_{n=1}^{\infty}e^{-n^2\pi\frac{1}{x}})$$, ossia:

$f(x)+1+f(x)=\frac{1}{\sqrt{x}}[f(\frac{1}{x})+1+f(\frac{1}{x})]$ e quindi: $2f(x)+1=\frac{1}{\sqrt{x}}[2f(\frac{1}{x})+1]$, da cui, ancora:

$f(x)=\frac{1}{2\sqrt{x}}[2f(\frac{1}{x})+1]-\frac{1}{2}$ e così la (3c.6) fornisce:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\int_0^\infty x^{\frac{1}{2}s-1}f(x)dx=\int_0^1 x^{\frac{1}{2}s-1}f(x)dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\int_0^1 x^{\frac{1}{2}s-1}\{\frac{1}{2\sqrt{x}}[2f(\frac{1}{x})+1]-\frac{1}{2}\}dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\int_0^1 x^{\frac{1}{2}s-1}\frac{1}{2\sqrt{x}}dx+\int_0^1 x^{\frac{1}{2}s-1}(-\frac{1}{2})dx+\int_0^1 x^{\frac{1}{2}s-1}\frac{1}{\sqrt{x}}f(\frac{1}{x})dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{2}\int_0^1 x^{\frac{1}{2}s-1-\frac{1}{2}}dx-\frac{1}{s}\left|x^{\frac{1}{2}s-1+1}\right|_0^1+\int_0^1 x^{\frac{1}{2}s-1-\frac{1}{2}}f(\frac{1}{x})dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{2(\frac{1}{2}s-\frac{3}{2}+1)}\left|x^{\frac{1}{2}s-\frac{3}{2}+1}\right|_0^1-\frac{1}{s}+\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=$$

$$=\frac{1}{(s-1)}-\frac{1}{s}+\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx=\frac{1}{s(s-1)}+\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx+\int_1^\infty x^{\frac{1}{2}s-1}f(x)dx \quad . \tag{3c.8}$$

Adesso, nell'integrale $\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}f(\frac{1}{x})dx$ della (3c.8) effettuiamo il seguente cambio di variabile:

$y=\frac{1}{x} \rightarrow \; dy=-\frac{1}{x^2}dx \rightarrow dx=-\frac{1}{y^2}dy$ e x=(0 , 1) dà y=(∞ , 1) → $\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}} f(\frac{1}{x})dx=$

$$=\int_\infty^1 (\frac{1}{y})^{\frac{1}{2}s-\frac{3}{2}} f(y)(-\frac{1}{y^2}dy)=-\int_\infty^1 (y)^{\frac{-1}{2}s+\frac{3}{2}} f(y)y^{-2}dy=-\int_\infty^1 (y)^{\frac{-1}{2}s-\frac{1}{2}} f(y)dy=\int_1^\infty x^{\frac{-1}{2}s-\frac{1}{2}} f(x)dx \qquad (3c.9)$$

e dalla (3c.9) messa dentro la (3c.8) otteniamo:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty x^{\frac{-1}{2}s-\frac{1}{2}} f(x)dx+\int_1^\infty x^{\frac{1}{2}s-1} f(x)dx=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \quad ,$$

ossia: $$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \quad . \qquad (3c.10)$$

Notiamo ora che il secondo membro della (3c.10) resta immutato se si scambia (s) con (1-s), da cui, effettuando tale scambio, la (3c.10) fornisce:

$$\pi^{\frac{-1}{2}+\frac{1}{2}s}\Gamma(\frac{1-s}{2})\zeta(1-s)=\frac{1}{s(s-1)}+\int_1^\infty (x^{\frac{-1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})f(x)dx \; ; \qquad (3c.11)$$

ora, dal confronto tra la (3c.10) e la (3c.11) si ottiene finalmente:

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)=\pi^{\frac{-1}{2}+\frac{1}{2}s}\Gamma(\frac{1-s}{2})\zeta(1-s) \qquad (3c.12)$$

e si verifica facilmente che scambiando s con (1-s) anche nella (3c.12) si ottiene la (3c.12) stessa.

Ricaviamo ora $\zeta(1-s)$ e $\zeta(s)$ dalla (3c.12); si ha:

$$\zeta(1-s)=\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})}\zeta(s) \qquad (3c.13)$$

$$\zeta(s)=\pi^{\frac{-1}{2}+s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2}s)}\zeta(1-s) \qquad (3c.14)$$

e ricordiamo anche la Formula di Riflessione $\frac{2}{\Gamma(s)\Gamma(1-s)}=\frac{2\sin s\pi}{\pi}$ (eq. (C.1) in Subappendice C) e scriviamola prima per $\frac{1-s}{2}$ in luogo di s e poi anche per $\frac{s}{2}$ in luogo di s, ottenendo:

$$\frac{2}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}=\frac{2\sin(\frac{1-s}{2}\pi)}{\pi}$$ (e qui ora moltiplichiamo il 1° membro sopra e sotto per $\Gamma(\frac{1}{2}s)$)

$$\frac{2}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}=\frac{2\sin(\frac{\pi s}{2})}{\pi}$$ (e qui ora moltiplichiamo il 1° membro sopra e sotto per $\Gamma(\frac{1-s}{2})$)

ottenendo:

$$\frac{2\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)}=\frac{2\sin(\frac{1-s}{2}\pi)}{\pi} \rightarrow \boxed{\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1-s}{2})}}=\frac{\sin(\frac{1-s}{2}\pi)}{\pi}\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)$$

$$\frac{2\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})}=\frac{2\sin(\frac{\pi s}{2})}{\pi}\rightarrow\boxed{\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2}s)}}=\frac{\sin(\frac{\pi s}{2})}{\pi}\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})$$

ed inseriamo ora le quantità nei due riquadri blu rispettivamente nelle (3c.13) e (3c.14), ottenendo:

$$\zeta(1-s)=\pi^{\frac{1}{2}-s}\frac{\sin(\frac{1-s}{2}\pi)}{\pi}\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)\zeta(s)=\pi^{\frac{-1}{2}-s}\boxed{\Gamma(\frac{1+s}{2})\Gamma(\frac{1}{2}s)}\sin(\frac{1-s}{2}\pi)\zeta(s) \tag{3c.15}$$

$$\zeta(s)=\pi^{\frac{-1}{2}+s}\frac{\sin(\frac{\pi s}{2})}{\pi}\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})\zeta(1-s)=\pi^{\frac{-3}{2}+s}\boxed{\Gamma(1-\frac{s}{2})\Gamma(\frac{1-s}{2})}\sin(\frac{\pi s}{2})\zeta(1-s) \tag{3c.16}$$

Ricordiamo ora la Formula di Duplicazione di Legendre: $\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$ (eq. (E.1) in Subappendice E) e ponendo in essa, al posto di s, prima $\frac{s}{2}$ e poi $(\frac{1}{2}-\frac{s}{2})$, otteniamo le seguenti:

$$s\rightarrow\frac{s}{2}\qquad\rightarrow\qquad\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})=2^{1-s}\pi^{\frac{1}{2}}\Gamma(s)$$

$$s\rightarrow(\frac{1}{2}-\frac{s}{2})\qquad\rightarrow\qquad\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})=2^{s}\pi^{\frac{1}{2}}\Gamma(1-s)$$

ed usiamo queste due rispettivamente nei riquadri verdi delle (3c.15) e (3c.16), ottenendo:

$$\zeta(1-s)=\pi^{\frac{-1}{2}-s}[2^{1-s}\pi^{\frac{1}{2}}\Gamma(s)]\sin(\frac{1-s}{2}\pi)\zeta(s)=2^{1-s}\pi^{-s}\sin(\frac{1-s}{2}\pi)\Gamma(s)\zeta(s)$$

$$\zeta(s)=\pi^{\frac{-3}{2}+s}[2^{s}\pi^{\frac{1}{2}}\Gamma(1-s)]\sin(\frac{\pi s}{2})\zeta(1-s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

che altro non sono che la (3c.3) e la (3c.4)!!!!!

d) con questa **bizzarra** dimostrazione:

$$\boxed{S=\sum_{n=1}^{\infty}n=1+2+3+4+5+..........=-\frac{1}{12}} \tag{3d.1}$$

(qui S sarebbe la $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ della (1.1) con s forzato a -1)

Dimostrazione:

$S_1=\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+1-1+........=1-(1-1+1-1+........)=1-S_1$, da cui: $S_1=\frac{1}{2}$.

$S_2=\sum_{n=0}^{\infty}(-1)^n(n+1)=1-2+3-4+5-6+........$ e inoltre: $2S_2=S_2+S_2$, ossia:

$$\begin{array}{ll}1-2+3-4+5-6+..............+ & S_2+\\ \quad 1-2+3-4+5-6+........= & S_2=\\ \hline 1-1+1-1+1-1+........ & S_1\end{array}$$

dunque: $2S_2 = S_1 = \frac{1}{2}$, da cui: $S_2 = \frac{1}{4}$. Inoltre: $S - 4S = -3S$ e: $-4S = -4 - 8 - 12 - 16 - .....$, da cui:

$$\begin{array}{l} 1 \boxed{+2} +3 \boxed{+4} +5 \boxed{+6} +..............+ \\ \quad \boxed{-4} \quad\; \boxed{-8} \quad\;\; \boxed{-12} -.............= \\ \hline 1 \boxed{-2} +3 \boxed{-4} +5 \boxed{-6} +......... \end{array} \qquad \begin{array}{c} S+ \\ (-4S)= \\ \hline S_2 \end{array}$$

, ossia: $S - 4S = -3S = S_2 = \frac{1}{4}$, da cui:

$\boxed{S = -\frac{1}{12}}$. ($\zeta(-1) = -\frac{1}{12}$, cosa purtroppo vera per la $\zeta(s)$ di Riemann)

e) <u>con quest'altro ragionamento</u>:

però, succede che tale bizzarro risultato (S=-1/12) del punto d) si ripresenta anche per altra via; infatti,consideriamo la Funzione Gamma di Eulero $\Gamma(s)$:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \tag{3e.1}$$

dove, naturalmente: $\Gamma(1) = 1$; si ha:

$$\Gamma(n+1) = n\Gamma(n); \tag{3e.2}$$

infatti, dopo una integrazione per parti:

$\Gamma(s) = e^{-t} \frac{t^s}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-t} t^s dt = 0 + \frac{1}{s} \Gamma(s+1)$, ed iterando la (3e.2) ($\Gamma(n+1) = n\Gamma(n)$), otteniamo: $\Gamma(n+1) = n!$. Consideriamo ora la Funzione Zeta di Riemann:

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \quad ; \tag{3e.3}$$

possiamo scrivere, dopo aver considerato la Funzione Gamma di Eulero ($\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$), che:

Γ di Eulero scritta in t': $\Gamma(s) = \int_0^\infty e^{-t'} t'^{(s-1)} dt'$ $\rightarrow$ (cambio di variabile): t'=nt $\rightarrow$ dt'=ndt $\rightarrow$

$$\rightarrow \quad \Gamma(s) = \int_0^\infty e^{-t'} t'^{(s-1)} dt' = \int_0^\infty e^{-nt} (nt)^{(s-1)} ndt = n^s \int_0^\infty e^{-nt} t^{(s-1)} dt \quad \rightarrow \quad \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-nt} t^{(s-1)} dt \quad \rightarrow$$

$$\rightarrow \quad \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \frac{1}{\Gamma(s)} \int_0^\infty e^{-nt} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^\infty e^{-nt} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^\infty (e^{-t})^n dt,$$

ma essendo che vale il seguente sviluppo di Taylor: $\frac{x}{1-x} = \sum_{n=1}^\infty x^n$, segue che: $\frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^\infty (e^{-t})^n$ e dunque:

$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^\infty (e^{-t})^n dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1-e^{-t}} dt$; valendo ora anche il seguente sviluppo di Taylor: $e^x = \frac{1}{n!} \sum_{n=0}^\infty x^n$, segue che:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1-e^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1-(1-t+\frac{t^2}{2}-\frac{t^3}{6}+...)} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{t-\frac{t^2}{2}+\frac{t^3}{6}-...} dt =$$

$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}}{t(1-\frac{t}{2}+\frac{t^2}{6}-...)}dt=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\frac{e^{-t}}{1-[(\frac{t}{2}-\frac{t^2}{6})+...]}dt$$ e valendo ora anche il seguente sviluppo di Taylor: $\frac{1}{1-x}=1+x+x^2+x^3+...=\sum_{n=0}^{\infty}x^n$, segue che: ($x\rightarrow(\frac{t}{2}-\frac{t^2}{6})$)

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\frac{e^{-t}}{1-[(\frac{t}{2}-\frac{t^2}{6})+...]}dt=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}[1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}+...)]dt=$$

$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(1+\frac{t}{2}+\frac{t^2}{12}+...)dt=\frac{1}{\Gamma(s)}[\int_0^\infty t^{s-2}e^{-t}dt+\int_0^\infty\frac{t}{2}t^{s-2}e^{-t}dt+\int_0^\infty\frac{t^2}{12}t^{s-2}e^{-t}dt+...]=$$

$$=\frac{1}{\Gamma(s)}[\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\frac{1}{12}\Gamma(s+1)+...] \qquad (3e.4)$$

ed essendo che, per la (3e.2): $\Gamma(s+1)=s\Gamma(s)$ e inoltre anche quest'ultima con $s-1$ al posto di s, che dà: $\Gamma(s)=(s-1)\Gamma(s-1)$, segue che:

$$\zeta(s)=\frac{1}{\Gamma(s)}[\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\frac{1}{12}\Gamma(s+1)+...]=\frac{1}{\Gamma(s)}[\frac{\Gamma(s)}{(s-1)}+\frac{1}{2}\Gamma(s)+\frac{1}{12}s\Gamma(s)+...]=[\frac{1}{(s-1)}+\frac{1}{2}+\frac{1}{12}s+...]$$

ossia: $\zeta(s)=\frac{1}{(s-1)}+\frac{1}{2}+\frac{1}{12}s+...$ e per $s=-1\rightarrow$

$$\zeta(-1)=\sum_{n=1}^{\infty}\frac{1}{n^{-1}}=\sum_{n=1}^{\infty}n=1+2+3+4+...=\frac{1}{(-1-1)}+\frac{1}{2}+\frac{1}{12}(-1)+...=-\frac{1}{12} \quad (*) \qquad (3e.5)$$

ossia:

$$\boxed{\sum_{n=1}^{\infty}n=1+2+3+4+...=-\frac{1}{12}}$$

e ci risiamo!

(*): la (3e.5) è un risultato apparentemente parziale, poiché ci sarebbero altri termini rappresentati dal (**+ …**), ma tutti questi termini che abbiamo tralasciato sono tutti nulli; essi, infatti, nascerebbero in seno alla (3e.4); analizziamo allora, come esempio, i successivi due termini tralasciati:

$$[1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}+...)]\ \rightarrow\ [1+(\frac{t}{2}-\frac{t^2}{6})+(\frac{t^2}{4}\boxed{-\frac{t^3}{6}}+\boxed{\frac{t^4}{36}})]\ \rightarrow$$

$$\rightarrow......\boxed{+\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(-\frac{t^3}{6})dt}\boxed{+\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-t}(\frac{t^4}{36})dt}=-\frac{1}{6}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+1}e^{-t}dt+\frac{1}{36}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+2}e^{-t}dt=$$

$$=\frac{-\Gamma(s+2)}{6\Gamma(s)}+\frac{\Gamma(s+3)}{36\Gamma(s)}=\frac{-(s+1)\Gamma(s+1)}{6\Gamma(s)}+\frac{(s+2)\Gamma(s+2)}{36\Gamma(s)}=\frac{-(s+1)\Gamma(s+1)}{6\Gamma(s)}+\frac{(s+2)(s+1)\Gamma(s+1)}{36\Gamma(s)},$$

che, sempre per $s=-1$, vale 0.

SUBAPPENDICI:

SUBAPPENDICE A: I numeri di Bernoulli.

Consideriamo la seguente funzione:

$$f(x)=\frac{x}{e^x-1} \tag{A.1}$$

il cui limite per x→0 esiste (De L'Hopital) e lo stesso dicasi per le sue derivate. Sviluppandola allora in serie di Taylor intorno a zero, si ottiene:

$$f(x)=\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\ , \tag{A.2}$$

con $B_n=f^{(n)}(0)$ e $B_0=1$, $B_1=-1/2$, $B_2=1/6$ eccetera, che sono i Numeri di Bernoulli. Per avere un metodo di calcolo di tali numeri, notiamo che dalle (A.1) ed (A.2) si ha:

$$x=(e^x-1)\frac{x}{e^x-1}=(e^x-1)\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=(\frac{1}{n!}\sum_{n=0}^{\infty}x^n-1)\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=(x+\frac{x^2}{2!}+\frac{x^3}{3!}+...)(B_0+B_1x+B_2\frac{x^2}{2!}+B_3\frac{x^2}{3!}+...)=$$

$$=x[B_0+(\frac{B_0}{2!}+B_1)x+(\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!})x^2+(\frac{B_0}{4!}+\frac{B_1}{3!}+\frac{B_2}{2!2!}+\frac{B_3}{3!})x^3+(\frac{B_0}{5!}+\frac{B_1}{4!}+\frac{B_2}{3!2!}+\frac{B_3}{2!3!}+\frac{B_4}{4!})x^4+...]$$

e visto che al primo membro c'è solo x , mentre all'ultimo membro vi è $B_0x+...$, allora segue che:

$$\begin{aligned}&B_0=1\ ,\\ &\frac{B_0}{2!}+B_1=0\ \rightarrow\ B_1=-\frac{1}{2}\\ &\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!}=0\ \rightarrow\ B_2=\frac{1}{6}\end{aligned} \tag{A.3}$$

………………

e così via. Inoltre, tutte le (A.3) possono essere così riassunte:

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k=0\,,\qquad (\text{con }\binom{n}{k}=\frac{n!}{(n-k)!k!}) \tag{A.4}$$

Se poi prendiamo, ad esempio, l'ultima delle (A.3), ossia la $\frac{B_0}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!}=0$, noteremo che:

$$\frac{B_2}{2!}=-(\frac{B_0}{3!}+\frac{B_1}{2!})=-(\frac{1}{3!}B_0+\frac{1}{2!}B_1)=-[\frac{1}{3!}\binom{3}{0}B_0+\frac{1}{3!}B_1\binom{3}{1}]=-\frac{1}{3!}\sum_{k=0}^{2-1}\binom{3}{k}B_k\ \text{, ossia:}$$

$$B_2=-\frac{2!}{3!}\sum_{k=0}^{2-1}\binom{3}{k}B_k=-\frac{1}{3}\sum_{k=0}^{2-1}\binom{3}{k}B_k\ \text{ e, in generale:}$$

$$\boxed{B_n=-\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k} \tag{A.5}$$

tramite la quale è possible calcolare in modo ricorsivo e più semplice i vari B_i :
$B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$ eccetera e possiamo anche dire che tutti quelli dispari, a partire da 3, sono nulli.

SUBAPPENDICE B: La funzione Γ (Gamma) di Eulero.

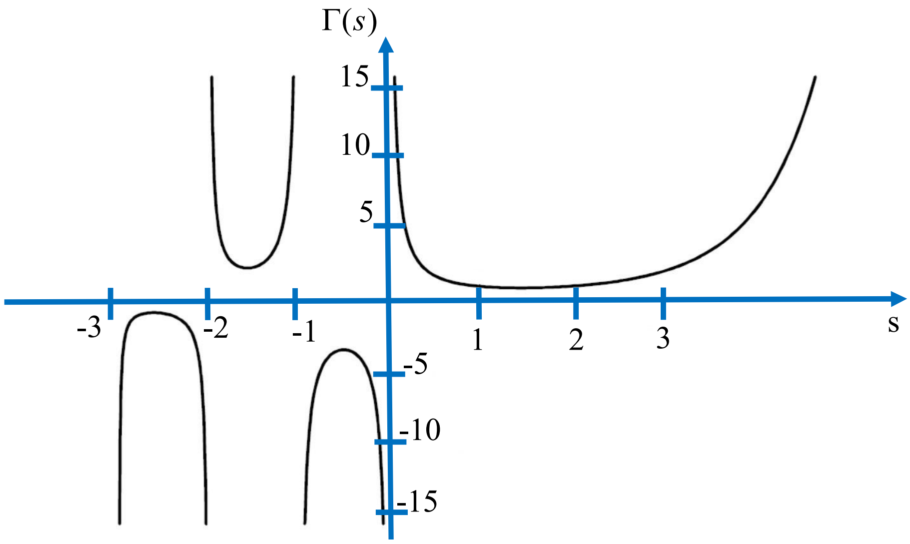

Fig. B.1: La Gamma di Eulero graficata.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{B.1}$$

Naturalmente, $\Gamma(1) = 1$. Poi,

$$\Gamma(n+1) = n\Gamma(n)\,; \tag{B.2}$$

infatti, dopo una integrazione per parti:

$\Gamma(z) = e^{-t} \left. \dfrac{t^z}{z} \right|_0^\infty + \dfrac{1}{z}\int_0^\infty e^{-t} t^z dt = 0 + \dfrac{1}{z}\Gamma(z+1)$ ed iterando la $\Gamma(n+1) = n\Gamma(n)$, otteniamo:

$$\Gamma(n+1) = n!\ . \tag{B.3}$$

INTEGRALE DI GAUSS

Consideriamo i due integrali (uguali tra loro):

$$I = \int_0^\infty e^{-\alpha x^2} dx \qquad\qquad I = \int_0^\infty e^{-\alpha y^2} dy \tag{B.4}$$

Moltiplicandoli tra loro: $I^2 = \int_0^\infty \int_0^\infty e^{-\alpha(x^2+y^2)} dxdy$ e, in coordinate polari:

($x^2 + y^2 = r^2$, $dS = dxdy = rdrd\theta$), si ha:

$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-\alpha r^2} rdrd\theta = \dfrac{\pi}{2}\int_0^\infty e^{-\alpha r^2} rdr = -\dfrac{\pi}{4\alpha}\left| e^{-\alpha r^2} \right|_0^\infty = \dfrac{\pi}{4\alpha}$, perciò:

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\ . \tag{B.5}$$

VALORI PARTICOLARI DELLA FUNZIONE GAMMA DI EULERO

La (B.3) con n=0 fornisce Γ(1)=0!=1, mentre con n=1, fornisce Γ(2)=1!=1.
Inoltre, la (B.1) con z=1/2 diventa:

$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$; adesso, ponendo $t=w^2$, si ha: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \int_0^\infty e^{-w^2} w^{-1} 2w dw = 2\int_0^\infty e^{-w^2} dw \qquad \text{(B.6)}$$

e per la (B.5) con α=1, si ha: $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-w^2} dw = \sqrt{\pi}$.

Infine, per la (B.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Per riassumere: Γ(1)=1, Γ(2)=1, $\Gamma(1/2) = \sqrt{\pi}$, Γ(3/2) = $\frac{\sqrt{\pi}}{2}$. (B.7)

SUBAPPENDICE C: LA FORMULA DI RIFLESSIONE.

Eccola: $\frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi}$. (C.1)

Infatti, sappiamo che $\frac{\sin x}{x}$ vale zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, sicchè:

$\frac{\sin x}{x} = (1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi}).......... = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})..... = \prod_{k=1}^{\infty}[1-(\frac{x}{k\pi})^2]$, e dunque:

$$\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \qquad \text{(C.2)}$$

Inoltre, un fattoriale possiamo anche vederlo così:

$n! = \lim_{k\to\infty} \frac{k!k^n}{(n+1)...(n+k)} \cdot \lim_{k\to\infty} \frac{(k+1)(k+2)...(k+n)}{k^n}$; infatti, il denominatore del primo limite è:

$(n+1)...(n+k) = \frac{(n+k)!}{n!}$, mentre da entrambi i numeratori possiamo raccogliere:

$k!(k+1)...(k+n) = (k+n)!$, così resta solo n! . Poi, vediamo che il secondo limite vale 1:

$\lim_{k\to\infty} \frac{(k+1)(k+2)...(k+n)}{k^n} = 1$, poichè si riduce alla forma $\lim_{k\to\infty} \frac{k^n}{k^n}$. Perciò: $n! = \lim_{k\to\infty} \frac{k!k^n}{(n+1)...(n+k)}$.

Siccome $\Gamma(x+1) = x!$, si ha: $\Gamma(x+1) = x! = \lim_{k\to\infty} \frac{k!k^x}{(x+1)...(x+k)}$ e dopo aver diviso sia numeratore che denominatore per k!:

$$\Gamma(x+1) = \lim_{k\to\infty} \frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \qquad \text{(C.3)}$$

Ora, introduciamo la costante $\gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} - \ln k$, e poniamo:

($\gamma = \lim_{k\to\infty} \gamma_k = \lim_{k\to\infty} (\sum_{n=1}^{k} \frac{1}{k} - \ln k) \approx 0{,}55721$, Costante di Eulero-Mascheroni);

sulla base di ciò, la (C.3) diventa: $\Gamma(x+1)=e^{-\gamma x}\lim_{k\to\infty}\frac{e^x}{(1+x)}\frac{e^{x/2}}{(1+x/2)}....\frac{e^{x/k}}{(1+x/k)}$. (C.4)

Infatti, tutti gli e^x, $e^{x/2}$,…, $e^{x/k}$ si elidono con i termini in $e^{-\gamma x}$ ed il termine $k^x=e^{(\ln k)x}$ pure proviene da $e^{-\gamma x}$. Inoltre, siccome $\Gamma(x+1)=x\Gamma(x)$, la (C.4) diventa:

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}}=xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \quad \text{(C.5)}$$

Ora, poichè $\Gamma(y+1)=y\Gamma(y)$, segue che $\Gamma(y)=\frac{1}{y}\Gamma(y+1)$ e dopo aver sostituito y con –x:

$\Gamma(-x)=-\frac{1}{x}\Gamma(1-x)$ e dunque: $\Gamma(1-x)=-x\Gamma(-x)$. Valutiamo $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x)=[x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}]=[A]\cdot[B]=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}, \quad \text{(C.6)}$$

dove A è la (C.5) invertita e B è la (C.4) con x scambiato con –x. Perciò, la (C.6) è:

$\Gamma(x)\Gamma(1-x)=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, e per la (C.2) (($\frac{\pi x}{\sin \pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) abbiamo:

$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}$ che è proprio la (C.1).

SUBAPPENDICE D: L'EQUAZIONE $\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty}e^{-n^2\pi\frac{1}{x}}$.

Tale equazione ci è servita più sopra (eq. (3c.7)) per la dimostrazione dell'Equazione Funzionale. Bene, adesso, semplificando "molto" le cose, in modo euristico consideriamo una funzione f(t) e prendiamone la Trasformata di Fourier:

$f(p)=\int_{-\infty}^{\infty}f(t)e^{-ipt}dt$ ($p\equiv\omega=2\pi f \;\to\; f\Rightarrow n \;\to\; p=2\pi n$), da cui:

$f(n)=\int_{-\infty}^{\infty}f(t)e^{-2\pi\,\mathrm{int}}dt$ ed applichiamo la sommatoria ad entrambi i membri:

$$\sum_{n=-\infty}^{\infty}f(n)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)e^{-2\pi\,\mathrm{int}}dt \ . \quad \text{(D.1)}$$

Ora, nella (D.1) consideriamo una $f(t)=e^{-\pi\cdot t^2/x}$, da cui, per la (D.1):

$\sum_{n=-\infty}^{\infty}f(n)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)e^{-2\pi\,\mathrm{int}}dt=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\pi\cdot t^2/x}e^{-2\pi\,\mathrm{int}}dt=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt$, ossia:

$$\sum_{n=-\infty}^{\infty}f(n)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt \quad \text{(D.2)}$$

ed essendo $f(t)=e^{-\pi\cdot t^2/x}$, allora, considerando la sessa struttura anche per f(n), ossia: $f(n)=e^{-\pi\cdot n^2/x}$, si avrà dalla (D.2) che:

$$\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2/x}=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt \quad \text{(D.3)}$$

Prendiamo ora l'integrale nella (D.3): $I=\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt$ ed effettuiamo la sostituzione $t=\sqrt{x}u$:

($u=\frac{t}{\sqrt{x}}$, $dt=\sqrt{x}du$ e t=(-∞,+∞) >>> u=(-∞,+∞) →

$I=\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt=\sqrt{x}\int_{-\infty}^{\infty}e^{-\pi\cdot u^2-2\pi in\sqrt{x}u}du=\sqrt{x}\int_{-\infty}^{\infty}e^{-\pi(u+in\sqrt{x})^2-\pi n^2x}du=\sqrt{x}e^{-\pi n^2x}\int_{-\infty}^{\infty}e^{-\pi(u+in\sqrt{x})^2}du$;
adesso, con la sostituzione $v=u+in\sqrt{x}$ (dv=du e gli estremi di integrazione ovviamente non cambiano), si ha: $I=\sqrt{x}e^{-\pi n^2x}\int_{-\infty}^{\infty}e^{-\pi(u+in\sqrt{x})^2}du=\sqrt{x}e^{-\pi n^2x}\int_{-\infty}^{\infty}e^{-\pi v^2}dv=2\sqrt{x}e^{-\pi n^2x}\int_{0}^{\infty}e^{-\pi v^2}dv$, trattandosi di un integrando che è una funzione pari. Adesso, con una seconda sostituzione $r=\pi v^2$ ($dr=2\pi v dv$, $dv=\frac{dr}{2\pi v}=\frac{dr}{2\pi\sqrt{\frac{r}{\pi}}}=\frac{dr}{2\sqrt{\pi r}}$ e stessi estremi), si ha:

$$I=2\sqrt{x}e^{-\pi n^2x}\int_{0}^{\infty}e^{-\pi v^2}dv=2\sqrt{x}e^{-\pi n^2x}\int_{0}^{\infty}e^{-r}\frac{dr}{2\sqrt{\pi r}}=\sqrt{\frac{x}{\pi}}e^{-\pi n^2x}\int_{0}^{\infty}e^{-r}r^{-\frac{1}{2}}dr=\sqrt{\frac{x}{\pi}}e^{-\pi n^2x}\Gamma(\frac{1}{2})=\sqrt{\frac{x}{\pi}}e^{-\pi n^2x}\sqrt{\pi}=$$

$=\sqrt{x}e^{-\pi n^2x}$, e allora la (D.3) fornisce:

$\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2/x}=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\pi\cdot t^2/x)-2\pi\,\mathrm{int}}dt=\sqrt{x}\sum_{n=-\infty}^{\infty}e^{-\pi n^2x}$, da cui $\sum_{n=-\infty}^{\infty}e^{-\pi n^2x}=\frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty}e^{-\pi\cdot n^2/x}$, ossia l'asserto. Essendo $\theta(z)=\sum_{n=-\infty}^{\infty}e^{-n^2z}$ la Funzione Theta di Jacobi, si potrà anche scrivere:

$$\theta(\pi x)=\frac{1}{\sqrt{x}}\theta(\frac{\pi}{x}).$$

SUBAPPENDICE E: La Formula di Duplicazione di Legendre.

$\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$ (Formula di Duplicazione di Legendre) (E.1)

-<u>Prima dimostrazione</u>:

In Subappendice C abbiamo visto che: $\Gamma(x+1)=\lim_{k\to\infty}\frac{k!k^x}{(x+1)...(x+k)}$ e allora se poniamo x=s-1, otteniamo: $\Gamma(s)=\lim_{k\to\infty}\frac{k!k^{(s-1)}}{(s)...(s-1+k)}=\lim_{k\to\infty}\frac{1\cdot 2\cdot....(k-1)kk^{(s-1)}}{(s)...(s-1+k)}=\lim_{k\to\infty}\frac{1\cdot 2\cdot....(k-1)}{(s)(s+1)...(s-1+k)}k^s$. (E.2)

Sfruttando ora il fatto che $\lim_{k\to\infty}\frac{k}{(s+k)}=1$, possiamo aggiungere nella (E.2) k a numeratore ed (s+k) al denominatore, ottenendo:

$\Gamma(s)=\lim_{k\to\infty}\frac{1\cdot 2\cdot....(k-1)k}{(s)(s+1)...(s-1+k)(s+k)}k^s=\lim_{k\to\infty}\frac{k!}{(s)(s+1)...(s+k)}k^s$ (E.3)

Adesso, se nella (E.3) poniamo s→s+(1/2) , otteniamo:

$\Gamma(s+\frac{1}{2})=\lim_{k\to\infty}\frac{k!}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}k^{s+\frac{1}{2}}$ mentre se poniamo s→2s , otteniamo:

$\Gamma(2s)=\lim_{k\to\infty}\frac{k!}{(2s)...(2s+k)}k^{2s}$ ed in quest'ultima poniamo anche k→2k , tanto il limite è per k che va ad infinito, dunque anche 2k ci va e resta tutto proporzionato: $\Gamma(2s)=\lim_{k\to\infty}\frac{(2k)!}{(2s)...(2s+2k)}(2k)^{2s}$;

a questo punto, valutiamo la quantità $\dfrac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}$:

$$\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}=\lim_{k\to\infty}[\frac{k!}{(s)...(s+k)}k^s\frac{k!}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}k^{s+\frac{1}{2}}\frac{(2s+0)...(2s+2k)}{(2k)!(2k)^{2s}}]=$$

$$=\lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}}{(2k)!}\frac{1}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}\frac{2^{2k+1}(s)...(s+k)}{2^{2s}(s)...(s+k)}]=$$

$$=\frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}2^{2k+1}}{(2k)!}\frac{1}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}]=\frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2k^{\frac{1}{2}}2^{2k+1}}{k(2k)!}\frac{k}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}]=$$

$$=\frac{1}{2^{2s}}\lim_{k\to\infty}[\frac{(k!)^2 2^{2k+1}}{(2k)!k^{\frac{1}{2}}}\frac{k}{(s+\frac{1}{2})...(s+\frac{1}{2}+k)}]=\frac{1}{2^{2s}}const$$, ossia:

$$\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}=\frac{1}{2^{2s}}const \quad , \tag{E.4}$$

dove $const=\lim_{k\to\infty}\{\frac{(k!)^2 2^{2k+1}}{(2k)!k^{\frac{1}{2}}}\frac{(k)}{(s+\frac{1}{2})...[s+\frac{1}{2}+(k)]}\}$; per trovare const , poniamo nella (E.4) $s=\frac{1}{2}$:

$\dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(2\frac{1}{2})}=\dfrac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(1)}=\Gamma(\frac{1}{2})=\sqrt{\pi}=\dfrac{1}{2^{2\frac{1}{2}}}const=\dfrac{1}{2}const$, da cui: $const=2\sqrt{\pi}$ e, dunque, la (E.4) fornisce: $\dfrac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}=\dfrac{1}{2^{2s}}2\sqrt{\pi}=2^{1-2s}\sqrt{\pi}$, ossia proprio la (E.1), cvd.

-<u>Seconda dimostrazione</u>:

-Premessa sulla Funzione Beta di Eulero:

eccola: $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ (E.5)
(convergente per m>0 ed n>0)

-Si ha che: $B(m,n)=B(n,m)$; infatti, con la sostituzione x=1-y, si ha:
$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^1(1-y)^{m-1}y^{n-1}dx=\int_0^1 y^{n-1}(1-y)^{m-1}dx=B(n,m)$.

-Si ha inoltre che:
$B(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$; (E.6)
infatti, usando la sostituzione $x=\sin^2\theta$ nella (E.5), si ottiene:

$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}2\sin\theta\cos\theta d\theta=$
$=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$.

-Si ha anche che: $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$; (E.7)

infatti, con la sostituzione $t=x^2$ (e ovviamente $dt=2xdx$) nella funzione Γ:
$\Gamma(m)=\int_0^\infty e^{-t}t^{m-1}dt$, si ottiene: $\Gamma(m)=\int_0^\infty e^{-t}t^{m-1}dt=2\int_0^\infty e^{-x^2}x^{2m-1}dx$ e similmente:
$\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$ e dunque:
$\Gamma(m)\Gamma(n)=4(\int_0^\infty e^{-x^2}x^{2m-1}dx)(\int_0^\infty e^{-y^2}y^{2n-1}dy)=4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dxdy$;
passando ora alle coordinate polari ($x=\rho\cos\alpha$, $y=\rho\sin\alpha$), si ottiene:
($dS=dxdy=(\rho d\alpha)d\rho=\rho d\rho d\alpha$)
$\Gamma(m)\Gamma(n)=4\int_{\alpha=0}^{\pi/2}\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\rho d\alpha=$
$=4(\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho)[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha]$; (E.8)

prendiamo ora il primo integrale: $\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho$; possiamo anche vederlo così:
$\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^\infty\rho^{2(m+n)}\dfrac{1}{\rho}e^{-\rho^2}d\rho$ ed effettuando ora la seguente sostituzione: $\rho^2=g$
(e, di conseguenza: $d\rho=\dfrac{1}{2\rho}dg$), si ha:
$\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\int_{\rho=0}^\infty\rho^{2(m+n)}\dfrac{1}{\rho}e^{-\rho^2}d\rho=\int_{\rho=0}^\infty g^{(m+n)}e^{-g}\dfrac{1}{2\rho^2}dg=\dfrac{1}{2}\int_{\rho=0}^\infty g^{(m+n)}e^{-g}\dfrac{1}{g}dg=$
$=\dfrac{1}{2}\int_{\rho=0}^\infty g^{(m+n)-1}e^{-g}dg=\dfrac{1}{2}\Gamma(m+n)$ ossia, riscrivendola:
$\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho=\dfrac{1}{2}\int_{\rho=0}^\infty g^{(m+n)-1}e^{-g}dg=\dfrac{1}{2}\Gamma(m+n)$ (E.9)

e le (E.9)+(E.6) nella (E.8) ci danno:
$\Gamma(m)\Gamma(n)=4(\int_{\rho=0}^\infty\rho^{2(m+n)-1}e^{-\rho^2}d\rho)[\int_{\alpha=0}^{\pi/2}(\cos^{2m-1}\alpha)(\sin^{2n-1}\alpha)d\alpha]=4[\dfrac{1}{2}\Gamma(m+n)][\dfrac{1}{2}B(m,n)]=$
$=\Gamma(m+n)B(m,n)$, da cui, finalmente: $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

-La (E.7) nella (E.6) fornisce inoltre:
$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\dfrac{1}{2}B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$ (E.10)

(m>0 ed n>0).

Finita questa premessa, scriviamo la Funzione Beta $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ per z_1 e z_2 :

$B(z_1,z_2)=\int_0^1 x^{z_1-1}(1-x)^{z_2-1}dx$ e con $z_1 = z_2$: $B(z,z)=\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)}=\int_0^1 x^{z-1}(1-x)^{z-1}dx$ e con la sostituzione $x=\frac{1+u}{2}$ (e $dx=\frac{1}{2}du$ ed x=0 → u=-1 + x=1 → u=1) si ha:

$B(z,z)=\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)}=\int_0^1 x^{z-1}(1-x)^{z-1}dx=\frac{1}{2}\int_{-1}^1(\frac{1+u}{2})^{z-1}(\frac{1-u}{2})^{z-1}du=\frac{1}{2^{2z-1}}\int_{-1}^1(1-u^2)^{z-1}du$, da cui:

$$2^{2z-1}\Gamma(z)\Gamma(z)=\Gamma(2z)\int_{-1}^1(1-u^2)^{z-1}du=2\Gamma(2z)\int_0^1(1-u^2)^{z-1}du\,. \tag{E.11}$$

(gli estremi di integrazione sono cambiati, avendo un integrando che è una funzione pari)

Ora, effettuando la sostituzione $x=u^2$ nella precedente Beta di Eulero in z_1 , z_2 (e dx=2udu ed x=0 → u=0 + x=1 → u=1) si ha: $B(z_1,z_2)=\int_0^1 x^{z_1-1}(1-x)^{z_2-1}dx=2\int_0^1 uu^{2z_1-2}(1-u^2)^{z_2-1}du$ e ponendo

$z_1=\frac{1}{2}$ e $z_2=z$, si ha: $$B(\frac{1}{2},z)=\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(\frac{1}{2}+z)}=2\int_0^1(1-u^2)^{z-1}du \tag{E.12}$$

ed usando ora la (E.12) nella (E.11), si ottiene:

$2^{2z-1}\Gamma(z)\Gamma(z)=\Gamma(2z)\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(\frac{1}{2}+z)}$, da cui: $\Gamma(z)\Gamma(\frac{1}{2}+z)=2^{-2z+1}\Gamma(2z)\Gamma(\frac{1}{2})$ e ricordando che

$\Gamma(\frac{1}{2})=\sqrt{\pi}$, segue che: $\Gamma(z)\Gamma(\frac{1}{2}+z)=2^{-2z+1}\Gamma(2z)\sqrt{\pi}$, <u>che altro non è che la (E.1), ossia l'asserto</u>.