# The Limits of Geometric Idealization in Michelson–Morley-Type Models: Wavefronts and Extended Reflectors

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**Abstract**

In standard treatments of the Michelson–Morley experiment, the round-trip time is calculated along a central path that identifies reflection with the geometric center of the mirror. This choice is adopted without proof: the reflector is treated as point-like, or the central ray is selected on symmetry grounds, without verifying that the reduction preserves the property being calculated.

This article provides that verification. By parameterizing the reflecting surface with a continuous coordinate, the total round-trip time is calculated as a function of the reflection point. In the geometric time-of-flight model with point-like source and receiver, the physically relevant path is the one that minimizes the total time. In general, the corresponding minimum point coincides neither with the geometric center of the reflector nor with the first point reached by the incident wavefront.

The main result is that, although the minimum-time reflection point is displaced from the geometric center by an amount of order $O(\beta^2)L$, the round-trip time evaluated along the central path differs from the minimum time only at order $O(\beta^4)$, with $\beta = v/c$, whereas the anisotropy predicted by Galilean kinematics already appears at order $O(\beta^2)$.

As a corollary, not all geometrically plausible reductions have the same status. The first-contact construction—which identifies the relevant reflection point with the first point reached by the incident wavefront during the outward propagation alone—produces an error already at order $O(\beta^2)$ and cancels the entire angular dependence. Two geometrically analogous reductions of the same system therefore have opposite methodological status, and the difference can be diagnosed from the perturbative structure of the error.

The immediate motivation for this work comes from an acoustic analogue proposed by Norbert Feist. In that context, the author had previously adopted the first-contact construction as an explanation of an experimentally reported isotropic result. The present article shows that this construction was incorrect and identifies the error: minimizing the outward propagation time alone is not equivalent to minimizing the total round-trip time.

**Keywords:**
Michelson–Morley experiment; controlled idealization; approximation; extended reflectors; path selection; wavefront geometry; perturbative order; acoustic analogues.

## 1. Introduction

The Michelson–Morley experiment [1] occupies a central place in the history of physics and in reflection on the relationship between modeling and theoretical change. In standard textbook treatments [2], the expected effect is obtained by comparing the round-trip times of light along arms oriented differently with respect to the hypothesized motion of the apparatus through a privileged medium. The geometric construction is simple: it reduces propagation to one ray for each arm and displays the second-order anisotropic dependence predicted by Galilean kinematics.

In these treatments, the relevant path is identified with the ray passing through the geometric center of the mirror. The choice is presented as natural—the reflector is treated as point-like, or the symmetry of the problem suggests that path—but it is never demonstrated that this reduction preserves the property being calculated. The question posed by this article is precise: if the point-reflector idealization is abandoned and an extended surface is considered explicitly, does the minimum-time path coincide with the central one? And if not, is the discrepancy negligible at the relevant perturbative order?

To answer this question, the reflector is parameterized by a continuous coordinate $s$, and the total round-trip time $t_{tot}(s)$ is calculated for every possible reflection point. In the geometric time-of-flight model adopted here—point-like source and receiver, high-frequency regime—the physically relevant path for the first arrival is the one that minimizes this function. The minimum point $s_{\min}$ is displaced from the geometric center by an amount of order $O(\beta^2)L$. However, because the time is stationary at the minimum point, the temporal effect of this displacement enters quadratically: the time evaluated at the center differs from the minimum only at order $O(\beta^4)$. Since the Galilean anisotropy appears at second order, the standard choice is controlled at the relevant perturbative level. The usual treatment does not select the exact minimum-time path, but the round-trip time it assigns differs from the exact minimum only at an order higher than that of the effect it is intended to represent.

This result has a consequence that goes beyond the justification of the standard treatment. It shows that whether a geometric reduction is controlled is not an absolute property of the model, but depends on the target effect and on the order at which the error appears. A general criterion emerges: if the effect to be represented appears at order $O(\beta^{\mathrm{n}})$ and the correction introduced by the reduction appears only at $O(\beta^{\mathrm{m}})$ with $m > n$, the simplification is perturbatively controlled with respect to that effect.

The plausibility of this criterion is illustrated by contrast with an alternative construction. The immediate motivation for this work comes from an acoustic analogue proposed by Norbert Feist [3, 4], in which an ultrasonic range finder and a reflector rigidly mounted on a moving apparatus were reported to yield a round-trip time independent of the angle between the source-reflector axis and the direction of motion. In a previous analysis [5], the author proposed explaining this result by identifying the relevant reflection point with the first point reached by the incident wavefront. This construction does indeed produce an exactly isotropic time, but—as the present work shows—for the wrong reason: the first-contact point is determined by minimizing the outward propagation time alone, not the total time. The two criteria do not coincide, and the difference between the first-contact path and the minimum-time path appears already at order $O(\beta^2)$, that is, at the same order as the anisotropy under study.
The isotropy obtained with that construction is an artifact of path selection.

The comparison between the two reductions—the central path with error $O(\beta^4)$ and the first-contact path with error $O(\beta^2)$—shows, in an analytically tractable form, how two geometrically analogous simplifications can have opposite methodological status with respect to the same target property.

The article is organized as follows. Section 2 develops the wavefront model for an extended planar reflector and derives the minimum-time point and the corresponding total time. Section 3 compares the exact result with the first-contact construction and with the central path, and derives the perturbative structure of the respective errors. Section 4 discusses the methodological implications and formulates the perturbative control criterion. The Conclusions assess the scope of the result and the limits of its application to the actual acoustic apparatus.

**2. Wavefront Model for an Extended Planar Reflector**

The model developed in this section is a geometric time-of-flight model in the high-frequency limit. Here, high frequency is meant in the asymptotic sense that the wavelengths carrying the relevant part of the signal are small compared with the characteristic geometric length scales of the system, $\lambda/\ell \ll 1$. In this limit, propagation times can be described to leading order by geometric rays; diffraction, interference, and the amplitudes of the different contributions require a more complete wave-field treatment [6].

The source, which also acts as receiver, is treated as point-like; the reflector is planar, rigidly attached to it, and sufficiently extended to contain the reflection point selected by the minimization problem. The objective is to derive the total round-trip time as a function of a generic point on the reflecting surface, without assuming in advance which point is physically relevant.

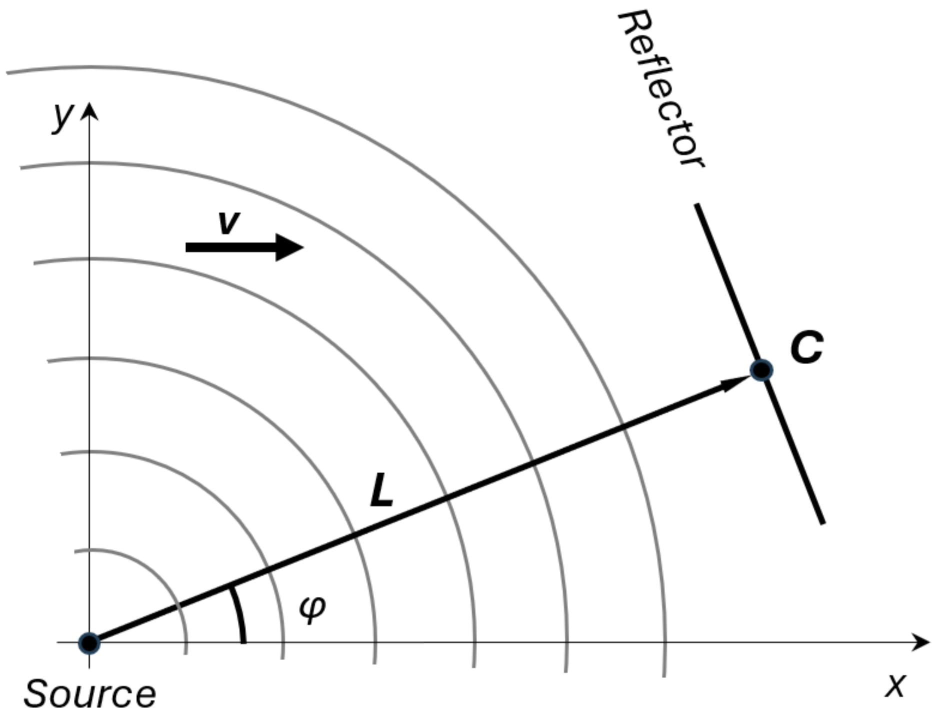


*Fig. 1 Geometry of the source-reflector system*

Consider the reference frame in which the propagation medium is at rest; in this frame wave propagation is isotropic with speed $c$.

At the initial time $t = 0$, the source/receiver is at the origin and emits a spherical wavefront.

The entire source-reflector apparatus moves uniformly with velocity $v$ along the $x$ axis.

The line joining the source to the geometric center ***C*** of the reflector has length ***L*** and forms an angle $\varphi$ with the direction of motion.

The reflector surface is assumed to be planar, perpendicular to this line, and rigidly attached to the source.

For non-collinear orientations, the direction of motion and the source-center axis define a plane of symmetry of the system. In the collinear cases, axial symmetry allows any plane containing the common axis to be chosen.
The point that minimizes the total time lies in this plane; it is therefore sufficient to parameterize the section of the reflector contained in it by a single coordinate s.
The three-dimensional verification, in which the coordinate orthogonal to the plane contributes to the time only through a positive quadratic term, is given in Appendix A.

Let s be a coordinate along the intersection of this plane with the reflector surface, with $s = 0$ at the geometric center ***C***.

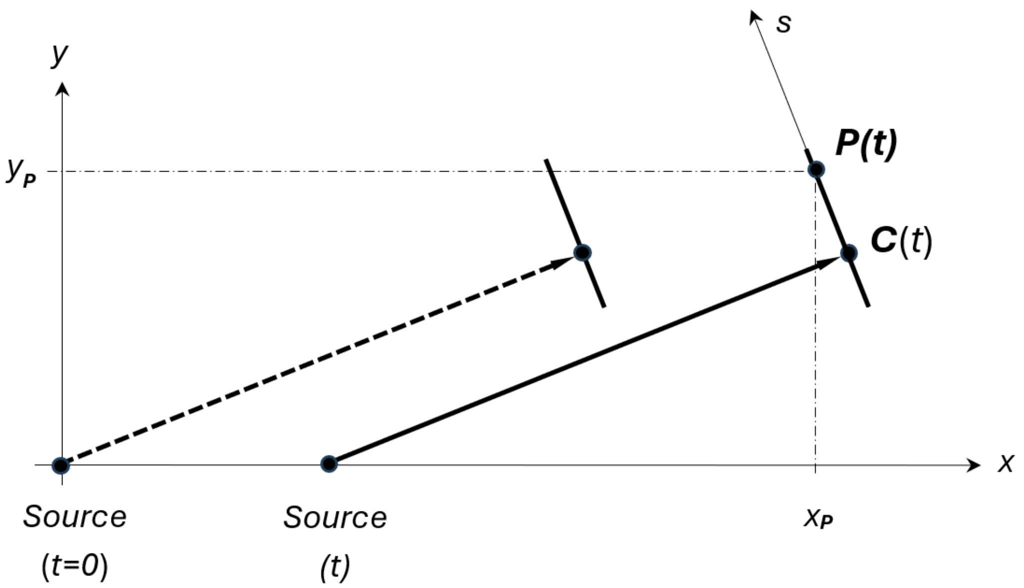


*Fig. 2 The source-reflector system at a generic time t*

A generic point ***P*** on the reflector is therefore described, at time $t$, by the coordinates

$$x_P(s,t) = L\cos(\varphi) + v\,t - s\sin(\varphi)\,, \qquad\qquad y_P(s,t) = L\sin(\varphi) + s\cos(\varphi)$$

The forward propagation time $t_1(s)$ is defined as the time required for the incident spherical wavefront emitted at $t = 0$ to reach the point $P(s, t_1)$.
In the rest frame of the medium, this condition is expressed as:

$$[x_P(s,t_1)]^2 + [y_P(s,t_1)]^2 = (ct_1)^2$$

Substituting the coordinates of ***P*** gives:

$$[L\cos(\varphi) + v\,t_1 - s\sin(\varphi)]^2 + [L\sin(\varphi) + s\cos(\varphi)]^2 = (ct_1)^2$$

Expanding and collecting terms in $t_1$, the equation becomes:

$$(c^2 - v^2){t_1}^2 - 2v[L\cos(\varphi) - s\sin(\varphi)]t_1 - (L^2 + s^2) = 0$$

Selecting the positive root corresponding to a physical propagation time gives:

$$t_1(s) = \frac{v[L\cos(\varphi) - s\sin(\varphi)] + \sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

At time $t_1(s)$, the point $P(s, t_1)$ is reached by the incident wavefront.
In the Huygens construction adopted here, it may be represented as the source of a secondary wave contribution.
Let $t_2(s)$ be the time required for the secondary wave emitted from $P(s, t_1)$ to return to the moving receiver.
At the final arrival time $t_1 + t_2$, the receiver is at the point:

$$R_{rec} = (v(t_1 + t_2), 0)$$

The return-propagation condition is therefore:

$$[v(t_1 + t_2) - x_P(s, t_1)]^2 + [0 - y_P(s, t_1)]^2 = (ct_2)^2$$

Substituting the coordinates of $P(s, t_1)$, the first difference becomes:

$$v(t_1 + t_2) - [L\cos(\varphi) + v\,t_1 - s\sin(\varphi)] = vt_2 - [L\cos(\varphi) - s\sin(\varphi)]$$

In this way, the explicit term $vt_1$ cancels identically. The return equation reduces to

$$\{vt_2 - [L\cos(\varphi) - s\ \sin(\varphi)]\}^2 + [L\sin(\varphi) + s\ \cos(\varphi)]^2 = (ct_2)^2$$

Expanding and collecting terms in $t_2$ gives:

$$(c^2 - v^2){t_2}^2 + 2v[L\cos(\varphi) - s\sin(\varphi)]t_2 - (L^2 + s^2) = 0$$

Selecting the positive root again gives:

$$t_2(s) = \frac{-v[L\cos(\varphi) - s\sin(\varphi)] + \sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

The total round-trip time associated with the point $s$ is:

$$t_{tot}(s) = t_1(s) + t_2(s)$$

Adding the two expressions, the linear terms outside the square root cancel, giving:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}$$

Expanding the radicand and collecting powers of $s$, the expression becomes:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{L^2[c^2 - v^2\sin^2(\varphi)] + s^2[c^2 - v^2\cos^2(\varphi)] - 2Lv^2 s\sin(\varphi)\cos(\varphi)}$$

Within the present model, this equation gives the round-trip time of a signal constrained to reach the reflector at the point specified by the coordinate $s$ and then return to the receiver.
For brevity, define $F(s)$ as the radicand:

$$F(s) = L^2[c^2 - v^2\sin^2(\varphi)] + s^2[c^2 - v^2\cos^2(\varphi)] - 2Lv^2 s\sin(\varphi)\cos(\varphi)$$

The total time can therefore be written as:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{F(s)}$$

Each value of $s$ identifies, within the present model, a possible round-trip path and the corresponding time $t_{tot}(s)$.
For a pulse and a point-like receiver sensitive to the first arrival, the relevant time is the minimum of this family of times over the reflector surface.

In the geometric-propagation limit, the same selection can be expressed through Fermat's principle [6]: the relevant path is stationary with respect to small variations of the reflection point. In the present problem, as will be shown, the stationary point is unique and coincides with the global minimum. The model does not, however, determine the amplitudes of the different contributions, diffraction, or the complete response of a real receiver.

Since the factor $2/(c^2 - v^2)$ is positive for $v < \mathrm{c}$ , and the square root is a strictly monotonic function, minimizing the total time $t_{tot}(s)$ is equivalent to minimizing $F(s)$.

Differentiating $F(s)$ with respect to $s$ gives:

$$\frac{dF}{ds} = 2s[c^2 - v^2\cos^2(\varphi)] - 2Lv^2\sin(\varphi)\cos(\varphi)$$

Imposing the stationarity condition $dF/ds = 0$, one obtains

$$\boxed{s_{\min} = \frac{Lv^2\sin(\varphi)\cos(\varphi)}{c^2 - v^2\cos^2(\varphi)}}$$

The point $s_{\min}$ therefore identifies the first arrival of the reflected signal at the moving receiver.
Substituting $s_{\min}$ into the quadratic polynomial $F(s)$ gives:

$$F_{\min} = F(s_{\min}) = L^2\frac{c^2(c^2 - v^2)}{c^2 - v^2\cos^2(\varphi)}$$

The structure of the minimum can be made even more transparent by rewriting the polynomial in canonical form, obtained by completing the square:

$$F(s) = F_{\min} + [c^2 - v^2\cos^2(\varphi)](\mathrm{s} - s_{\min})^2$$

Since $c^2 - v^2\cos^2(\varphi) > 0$, the second term is non-negative and vanishes only for $\mathrm{s} = s_{\min}$; the latter is therefore the unique global minimum.

The analysis assumes that the reflector is sufficiently extended for $s_{\min}$ to lie on its physical surface. If the theoretical minimum point lay outside the reflector, the physically accessible minimum would instead be located at one of the edges.

Introducing the dimensionless parameter $\beta = v/c$, the minimum point can be written as:

$$s_{\min} = L\frac{\beta^2 \sin(\varphi)\cos(\varphi)}{1-\beta^2\cos^2(\varphi)}$$

For $\beta \ll 1$ one obtains:

$$s_{\min} = L\beta^2 \sin(\varphi)\cos(\varphi) + O(\beta^4)L,$$

and therefore, in order-of-magnitude terms:

$$s_{\min} = O(\beta^2)L.$$

The displacement of the minimum point from the geometric center is thus second order in the relative speed. This result will be useful in the next section for understanding why evaluating the time at the geometric center produces only a fourth-order error.

Substituting the value of $F_{\min}$ into the expression for the total time gives:

$$t_{tot,min} = t_{tot}(s_{\min}) = \frac{2}{c^2 - v^2}\sqrt{F_{\min}}$$

Consequently, the minimum total round-trip time is:

$$\boxed{t_{tot,min} = \frac{2Lc}{\sqrt{c^2 - v^2}\sqrt{c^2 - v^2\cos^2(\varphi)}}}$$

This expression depends explicitly on the angle $\varphi$. It follows that, within the present geometric time-of-flight model, deidealizing the reflector does not eliminate the anisotropy predicted by Galilean kinematics.

The finite extent of the reflector alone therefore cannot explain the isotropy reported by Feist. Interpreting the actual apparatus requires a more complete model, discussed in the Conclusions.

## 3. Comparison with Two Single-Point Geometric Reductions

The extended model developed in Section 2 provides the reference against which two simpler geometric constructions can be evaluated, both based on selecting a single point on the reflector: the first-contact point of the incident wavefront and the geometric center of the surface.

### 3.1 The First-Contact Construction

A first, geometrically intuitive criterion is to identify the relevant reflection point with the first point on the surface reached by the incident wavefront. Denote this point by ***D*** and its coordinate along the reflector by $s_D$.

The path constrained to pass through ***D*** is associated with the total round-trip time:

$$t_{tot}(s_D) = \frac{2Lc}{c^2 - v^2}$$

This expression is exactly independent of the angle $\varphi$. The first-contact construction therefore seems to suggest that the finite extent of the reflector is sufficient to eliminate the dependence of the total time on the orientation of the apparatus.

This conclusion, however, results from an incorrect path selection. The point ***D*** is determined by the first-contact condition during the outward propagation alone; in general, it is not the point that minimizes the total round-trip time to the moving receiver. As shown in Section 2, the relevant point in the present model is instead $s_{\min}$.

The calculation of $t_{tot}(s_D)$ is therefore correct for the path constrained to pass through ***D***, but it does not identify the first reflected signal reaching the point-like receiver. The angular independence obtained with this construction is consequently an artifact of the choice of reflection point.

This construction is the same as that proposed in a previous analysis by the author and subsequently withdrawn after the path-selection error was identified.

### 3.2 The Central Path and the Perturbative Structure of the Error

The second single-point geometric reduction consists of assuming that reflection occurs at the geometric center ***C*** of the reflector, corresponding to $s = 0$. This is the construction used in standard treatments of the Michelson–Morley problem.

Evaluating the expression derived in Section 2 at the center gives:

$$t_{tot}(0) = \frac{2L\sqrt{c^2 - v^2\sin^2(\varphi)}}{c^2 - v^2}$$

For every non-collinear orientation, that is, for $\varphi \neq 0, \pi$, one has $\sqrt{c^2 - v^2\sin^2(\varphi)} < \mathrm{c}$.
Consequently, the total time associated with the geometric center is smaller than the total time associated with the first-contact point: $t_{tot}(0) < t_{tot}(s_D)$.
Since $s_{\min}$ minimizes the total time over the reflector surface, one also has: $t_{tot,min} \leq t_{tot}(0)$.
Therefore, for $\sin(\varphi) \neq 0$:

$$t_{tot,min} \leq t_{tot}(0) < t_{tot}(s_D)$$

Equality between $t_{tot,min}$ and $t_{tot}(0)$ occurs when the minimum point coincides with the center, namely when: $\sin(\varphi)\cos(\varphi) = 0$.

The reason why the central path approximates the minimum-time path so well can be understood directly from the canonical form of $F(s)$ derived in Section 2:

$$F(s) = F_{\min} + [c^2 - v^2\cos^2(\varphi)](s - s_{\min})^2$$

Setting $s = 0$ gives:

$$F(0) - F_{\min} = [c^2 - v^2\cos^2(\varphi)]\, {s_{\min}}^2$$

Since:

$$s_{\min} = O(\beta^2)L$$

the difference between $F(0)$ and $F_{\min}$ is fourth order:

$$\frac{F(0) - F_{\min}}{F_{\min}} = O(\beta^4)$$

The exact reflection point is therefore displaced from the center by a second-order amount; however, because the time is stationary at the minimum point, the temporal error produced by this displacement enters quadratically and appears only at fourth order.
To determine this correction explicitly, rewrite the two times in terms of the dimensionless parameter $\beta = v/c$.

For the central path:

$$t_{tot}(0) = \frac{2L}{c}\frac{\sqrt{1 - \beta^2\sin^2(\varphi)}}{1 - \beta^2}$$

For the minimum-time path:

$$t_{tot,min} = \frac{2L}{\mathrm{c}}\frac{1}{\sqrt{1 - \beta^2}\sqrt{1 - \beta^2\cos^2(\varphi)}}$$

For $\beta \ll 1$, the expansion of the time associated with the geometric center is:

$$t_{tot}(0) = \frac{2L}{c}\left\{1 + \frac{1}{2}\beta^2[1 + \cos^2(\varphi)] + \frac{1}{8}\beta^4[3 + 6\cos^2(\varphi) - \cos^4(\varphi)] + O(\beta^6)\right\}$$

The expansion of the minimum time is instead:

$$t_{tot,min} = \frac{2L}{c}\left\{1 + \frac{1}{2}\beta^2[1 + \cos^2(\varphi)] + \frac{1}{8}\beta^4[3 + 2\cos^2(\varphi) + 3\cos^4(\varphi)] + O(\beta^6)\right\}$$

The two results agree through second order. Subtracting them gives:

$$t_{tot}(0) - t_{tot,min} = \frac{L}{c}\beta^4 \sin^2(\varphi)\cos^2(\varphi) + O\left(\frac{L}{c}\beta^6\right)$$

The angular coefficient is bounded by: $0 \leq \sin^2(\varphi)\cos^2(\varphi) \leq 1/4$.

The correction vanishes when the minimum point coincides with the center and, in the fundamental interval $0 \leq \varphi \leq \pi/2$, reaches its maximum value for $\varphi = \pi/4$.

The central path therefore does not yield the exact time of the extended model, but it fully reproduces its angular dependence through order $O(\beta^2)$, at which the relevant anisotropic effect appears.

Comparison with the first-contact construction makes the different structure of the two errors evident:

$$t_{tot}(s_D) - t_{tot,min} = \frac{L}{c}\beta^2 \sin^2(\varphi) + O\left(\frac{L}{c}\beta^4\right)$$

whereas:

$$t_{tot}(0) - t_{tot,min} = \frac{L}{c}\beta^4 \sin^2(\varphi)\cos^2(\varphi) + O\left(\frac{L}{c}\beta^6\right)$$

The first-contact construction therefore changes the result at the same order as the effect under study and cancels its angular dependence. Evaluation at the geometric center, by contrast, introduces only a higher-order correction. The methodological significance of this difference will be examined in Section 4.

**4. Perturbative Control of Geometric Reductions**

Section 3 produced two quantitative results that can be considered together:

$$t_{tot}(0) - t_{tot,min} = \frac{L}{c}\beta^4 \sin^2(\varphi)\cos^2(\varphi) + O\left(\frac{L}{c}\beta^6\right)$$

$$t_{tot}(s_D) - t_{tot,min} = \frac{L}{c}\beta^2 \sin^2(\varphi) + O\left(\frac{L}{c}\beta^4\right)$$

The Galilean anisotropy that both reductions are intended to represent appears at order $O(\beta^2)$. The comparison of orders is therefore immediate: the central path introduces an error two perturbative orders higher than the target effect; the first-contact path introduces an error at the same order and cancels its angular dependence.

From this asymmetry a general criterion emerges. Let $O(\beta^{\mathrm{n}})$ be the order at which the target property appears, and $O(\beta^{\mathrm{m}})$ the order at which the correction introduced by removing the simplification appears.
When $m > n$, the error is perturbatively separated from the effect under study: the simplification is controlled with respect to that property, provided that the correction coefficients remain regular. By contrast, when $m \leq n$, the error enters before the effect or at the same order and may change its coefficient, alter its dependence on the relevant variables, or suppress it entirely.

In the present case the central path satisfies $m = 4 > n = 2$, and the angular coefficient of the correction is bounded by $0 \leq \sin^2(\varphi)\cos^2(\varphi) \leq 1/4$.
No anomalous amplification compensates for the suppression in powers of $\beta$. The criterion is satisfied. By contrast, the first-contact path satisfies $m = n = 2$: the criterion is not satisfied, and the calculation shows the consequences explicitly—the exact isotropy of the time $t_{tot}(s_D)$ is an artifact of path selection, not a property of the system.

It is worth noting that the central path admits two distinct readings: it can be interpreted as the exact result for an idealized system with a point-like reflector placed at the geometric center, or as an approximation to the minimum in the extended system, obtained by evaluating $t_{tot}(s)$ at $\mathrm{s} = 0$ rather than at $s_{\min}$.
The two readings correspond to different representational relations between model and target system. They do not, however, alter the quantitative diagnosis: in both cases the discrepancy with respect to the extended model appears at $O(\beta^4)$. The distinction between idealization and approximation—discussed in [7, 8, 9]—is therefore irrelevant for perturbative control; what matters is the structure of the error, not how the simplification is classified.

Finally, the relative nature of the criterion is worth making explicit. The central path is controlled for the study of second-order anisotropy, but it would not be controlled in an analysis concerned with fourth-order corrections: in that case the neglected term would belong to the effect to be determined. Controllability is not an absolute property of a simplification, but a relation among the simplification, the target property, and the required level of accuracy.

**5. Conclusions**

The present work has addressed a question that standard treatments of the Michelson–Morley experiment do not ask explicitly: is the reduction to the central path—identifying reflection with the geometric center of the mirror—a legitimate choice, and in what sense?

The answer is yes, but it requires demonstration. Parameterizing the reflector by a continuous coordinate and calculating the total round-trip time for every possible reflection point shows that the minimum-time point is displaced from the geometric center by an amount of order $O(\beta^2)L$. Because the time is stationary at the minimum point, the temporal effect of this displacement enters quadratically: the time evaluated at the center differs from the minimum only at order $O(\beta^4)$. The Galilean anisotropy appears at second order.
The standard treatment does not select the exact minimum-time path of the extended geometric model; however, the round-trip time it assigns differs from the minimum only at an order higher than that of the property it is intended to represent. The choice is therefore controlled at the relevant perturbative level.

This result has a precise methodological consequence, which can be formulated as a general criterion: a geometric reduction is perturbatively controlled with respect to a target property $O(\beta^{\mathrm{n}})$ when the correction introduced by removing it appears at $O(\beta^{\mathrm{m}})$ with $m > n$ and the correction coefficients remain regular. By contrast, when $m \leq n$, the error is not perturbatively separated from the effect under study and may modify or suppress it.

The first-contact construction illustrates the case $m = n$ in an analytically tractable form.
Identifying the relevant reflection point with the first point reached by the incident wavefront means minimizing the outward propagation time alone, not the total time. The two criteria produce distinct points, and the difference between the first-contact path and the minimum-time path appears already at order $O(\beta^2)$.
The time associated with that construction is exactly isotropic, but the isotropy is an artifact: it depends on which path is selected, not on a property of the system. This construction had been proposed by the author in a previous analysis [5] as a geometric explanation of the result reported by Feist; the present work identifies the error and withdraws that explanation.

With regard to Feist's acoustic analogue, the conclusion is limited but clear. In the geometric time-of-flight model adopted here—high frequency, point-like source and receiver, first arrival—the finite extent of the reflector does not eliminate the Galilean anisotropy. An adequate explanation of the reported experimental result would require modeling the extent and directivity of the transducers, the threshold and temporal response of the receiver, and the actual criterion by which the apparatus identifies the return signal. These developments lie outside the geometric problem solved here.

The main result, however, remains internal to the model and does not depend on the reliability of Feist's apparatus: the standard Michelson–Morley treatment makes a tacit geometric choice that can be tested, and the test confirms that choice. Deidealizing the reflector does not weaken that treatment; on the contrary, it establishes its robustness at the relevant perturbative order.

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**Appendix A. Detailed Derivations**

This appendix collects the algebraic checks referred to in Sections 2–4. In particular, it shows that the three-dimensional minimization can be reduced to the plane defined by the direction of motion and the source-center axis, verifies in compact form the expressions for the forward and return times, derives the location and value of the minimum by completing the square, derives the first-contact point, and presents the perturbative expansions used to compare the two geometric reductions.
All calculations are carried out in the inertial reference frame in which the propagation medium is at rest.
In this frame the wave propagates isotropically with speed $c$, while the source-reflector apparatus moves with velocity $v$ along the $x$ axis. We assume $v < c$.

## A.1 Three-Dimensional Reduction to the Plane of Symmetry

In the main text the reflector is parameterized by a single coordinate $s$ lying in the plane defined by the direction of motion and the source-center axis. This subsection shows that this reduction is not an additional assumption: within the geometric model considered, the three-dimensional minimum of the total time necessarily lies in that plane.

For non-collinear orientations, the direction of the apparatus velocity and the source-center axis define a plane $\mathbf{\Pi}$. In the collinear cases, the symmetry is axial and any plane containing the common axis may be chosen.

Introduce two orthogonal coordinates on the planar surface of the reflector:
- $s$, along the intersection between the reflector and the plane defined by the direction of motion and the source-center axis;
- $u$, along the direction within the surface orthogonal to that plane.

The geometric center ***C*** of the reflector corresponds to $s = 0$ and $u = 0$.
Choose the Cartesian axes so that the plane $\mathbf{\Pi}$ coincides with the plane $(x, y)$, while the direction associated with $u$ coincides with the $z$ axis.
In the reference frame in which the propagation medium is at rest, the source is at the origin at time $t = 0$, while the entire apparatus moves with velocity $v$ along the $x$ axis.
A generic point ***P*** on the reflector therefore has coordinates:

$$x_P(s, u, t) = L\cos(\varphi) + v\,t - s\sin(\varphi)$$

$$y_P(s, u, t) = L\sin(\varphi) + s\cos(\varphi)$$

$$z_P(s, u, t) = u$$

For brevity, define:

$$A(s) = L\cos(\varphi) - s\sin(\varphi)$$

$$B(s) = L\sin(\varphi) + s\cos(\varphi)$$

The coordinates become:

$$x_P(s, u, t) = A(s) + v\,t$$

$$y_P(s, u, t) = B(s)$$

$$z_P(s, u, t) = u$$

The two quantities $A(s)$ and $B(s)$ satisfy the identity

$$A^2(s) + B^2(s) = [L\cos(\varphi) - s\sin(\varphi)]^2 + [L\sin(\varphi) + s\cos(\varphi)]^2$$

Expanding the two squares:

$$\begin{aligned} A^2(s) + B^2(s) = {} & [L^2\cos^2(\varphi) - 2Ls\sin(\varphi)\cos(\varphi) + s^2\sin^2(\varphi)] \\ & + [L^2\sin^2(\varphi) + 2Ls\sin(\varphi)\cos(\varphi) + s^2\cos^2(\varphi)] \end{aligned}$$

The cross terms cancel:

$$A^2(s) + B^2(s) = L^2[\cos^2(\varphi) + \sin^2(\varphi)] + s^2[\sin^2(\varphi) + \cos^2(\varphi)]$$

Since $\sin^2(\varphi) + \cos^2(\varphi) = 1$, one obtains:

$$A^2(s) + B^2(s) = L^2 + s^2$$

Let $t_1(s,u)$ be the time required for the spherical wavefront emitted by the source at time $t = 0$ to reach the point P on the reflector.
In the rest frame of the medium, the forward-propagation condition is:

$$[x_P(s,u,t_1)]^2 + [y_P(s,u,t_1)]^2 + [z_P(s,u,t_1)]^2 = (ct_1)^2$$

Substituting the coordinates:

$$[A(s) + v\,t_1]^2 + B^2(s) + u^2 = c^2{t_1}^2$$

It follows:

$$A^2(s) + 2vA(s)t_1 + v^2{t_1}^2 + B^2(s) + u^2 = c^2{t_1}^2$$

Using:

$$A^2(s) + B^2(s) = L^2 + s^2$$

one obtains:

$$L^2 + s^2 + 2vA(s)t_1 + v^2{t_1}^2 + u^2 = c^2{t_1}^2$$

Ordering the terms by powers of $t_1$:

$$(c^2 - v^2){t_1}^2 - 2\,vA(s)t_1 - (L^2 + s^2 + u^2) = 0$$

The positive root is:

$$t_1(s,u) = \frac{vA(s) + \sqrt{v^2A^2(s) + (c^2 - v^2)(L^2 + s^2 + u^2)}}{c^2 - v^2}$$

The detailed solution of the quadratic equation and the choice of the physical root will be shown in A.2 for the planar case $u = 0$. For the purposes of the present subsection, the dependence on the transverse coordinate $u$ is of primary interest.

At time $t_1$, the reflection event is located at:

$$\boldsymbol{P}(s,u,t_1) = (A(s) + v\,t_1, B(s), u)$$

Let $t_2(s,u)$ be the time required for the secondary wave contribution generated at that point to reach the receiver.
At the final time $t_1 + t_2$, the receiver, which moves with the apparatus along the $x$ axis, is at:

$$\boldsymbol{R}_{rec} = (v(t_1 + t_2), 0, 0)$$

The difference between the receiver coordinate $x$ and that of the reflection event is:

$$v(t_1 + t_2) - [A(s) + v\, t_1] = vt_1 + vt_2 - A(s) - v\, t_1 = vt_2 - A(s)$$

The return-propagation condition is therefore:

$$[vt_2 - A(s)]^2 + B^2(s) + u^2 = (c\, t_2)^2$$

Expanding the square:

$$[vt_2 - A(s)]^2 = v^2 {t_2}^2 - 2vA(s)t_2 + A^2(s)$$

It follows:

$$v^2 {t_2}^2 - 2vA(s)t_2 + A^2(s) + B^2(s) + u^2 = c^2 {t_2}^2$$

Using:

$$A^2(s) + B^2(s) = L^2 + s^2$$

one obtains:

$$v^2 {t_2}^2 - 2vA(s)t_2 + L^2 + s^2 + u^2 = c^2 {t_2}^2$$

Rearranging:

$$(c^2 - v^2){t_2}^2 + 2vA(s)t_2 - (L^2 + s^2 + u^2) = 0$$

The positive root is:

$$t_2(s,u) = \frac{-vA(s) + \sqrt{v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2 + u^2)}}{c^2 - v^2}$$

The total time associated with the surface point $(s, u)$ is:

$$t_{tot}(s,u) = t_1(s,u) + t_2(s,u)$$

Substituting the two expressions, the terms $vA(s)$ and $-vA(s)$ cancel:

$$t_{tot}(s,u) = \frac{2}{c^2 - v^2} \sqrt{v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2 + u^2)}$$

Now separate the term containing $u$:

$$v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2 + u^2) = v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2) + (c^2 - v^2)u^2$$

Define:

$$F(s) = v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)$$

The total time therefore takes the form:

$$t_{tot}(s,u) = \frac{2}{c^2 - v^2} \sqrt{F(s) + (c^2 - v^2)u^2}$$

For each fixed value of s, define the radicand as:

$$G(s,u) = F(s) + (c^2 - v^2)u^2$$

The derivative with respect to the transverse coordinate is:

$$\frac{\partial G(s,u)}{\partial u} = 2(c^2 - v^2)u$$

The stationarity condition:

$$\frac{\partial G(s,u)}{\partial u} = 0$$

gives:

$$u = 0.$$

The second derivative is:

$$\frac{\partial^2 G(s,u)}{\partial u^2} = 2(c^2 - v^2) > 0$$

The point $u = 0$ is therefore the unique minimum with respect to the transverse coordinate. Equivalently:

$$t_{tot}(s,u) \geq t_{tot}(s,0)$$

with equality only for:

$$u = 0$$

The three-dimensional minimum of the total time therefore lies in the plane $\mathbf{\Pi}$ defined by the direction of motion and the source-center axis.

The minimization may therefore be reduced, without loss of generality within the model considered, to the planar function:

$$t_{tot}(s) \equiv t_{tot}(s,0)$$

The detailed derivation of the forward and return times in the plane will be developed in subsection A.2.

## A.2 Derivation of the Forward and Return Times in the Plane

Subsection A.1 showed that the three-dimensional minimum of the total time lies in the plane defined by the direction of motion and the source-center axis. We may therefore set $u = 0$ and describe every relevant point of the reflector by the single coordinate $s$.

Recall the definitions:

$$A(s) = L\cos(\varphi) - s\sin(\varphi)$$

$$B(s) = L\sin(\varphi) + s\cos(\varphi)$$

The coordinates of the reflector point ***P***, at a generic time $t$, are:

$$x_P(s,t) = A(s) + v\,t$$

$$y_P(s,t) = B(s)$$

From the previous subsection one also has:

$$A^2(s) + B^2(s) = L^2 + s^2$$

### A.2.1 Forward-Propagation Time

Let $t_1(s)$ be the time taken by the spherical wavefront, emitted by the source at the origin at time $t = 0$, to reach the reflector point specified by the coordinate $s$.
In the rest frame of the medium, the wavefront has radius $ct_1$. The point reached at time $t_1$ has coordinates:

$$x_P(s,t_1) = A(s) + v\,t_1$$

$$y_P(s,t_1) = B(s)$$

The propagation condition is:

$$[x_P(s,t_1)]^2 + [y_P(s,t_1)]^2 = c^2{t_1}^2$$

Substituting the coordinates:

$$[A(s) + v\,t_1]^2 + B^2(s) = c^2{t_1}^2$$

It follows:

$$A^2(s) + 2v\,A(s)t_1 + v^2{t_1}^2 + B^2(s) = c^2{t_1}^2$$

Using:

$$A^2(s) + B^2(s) = L^2 + s^2$$

One obtains:

$$L^2 + s^2 + 2v\,A(s)t_1 + v^2{t_1}^2 = c^2{t_1}^2$$

Rearranging the terms:

$$(c^2 - v^2){t_1}^2 - 2v\,A(s)t_1 - (L^2 + s^2) = 0$$

Writing $A(s)$ explicitly:

$$(c^2 - v^2){t_1}^2 - 2v\,[L\cos(\varphi) - s\sin(\varphi)]t_1 - (L^2 + s^2) = 0$$

The equation has the general form: $\quad a_1{t_1}^2 + b_1t_1 + d_1 = 0$

with: $\quad a_1 = c^2 - v^2 \qquad b_1 = -2vA(s) \qquad d_1 = -(L^2 + s^2)$

The discriminant is: $\quad \Delta_1 = {b_1}^2 - 4\,a_1d_1$

Hence:

$$\Delta_1 = {b_1}^2 - 4\, a_1 d_1 = 4\, v^2 A^2(s) + 4\,(c^2 - v^2)(L^2 + s^2)$$

$$\Delta_1 = 4[\, v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)]$$

The quadratic formula gives:

$$t_1 = \frac{-b_1 \pm \sqrt{\Delta_1}}{2\, a_1}$$

Since:

$$\sqrt{\Delta_1} = 2\sqrt{\, v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)}$$

substituting:

$$t_1 = \frac{vA(s) \pm \sqrt{\, v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

Since $v < c$, the denominator is positive. Moreover, $(c^2 - v^2)(L^2 + s^2) > 0$, and therefore:

$$\sqrt{\, v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)} > |vA(s)|$$

The root with the minus sign therefore has a negative numerator and corresponds to a nonphysical time. The positive root is:

$$t_1(s) = \frac{vA(s) + \sqrt{\, v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

Writing $A(s)$ explicitly:

$$t_1(s) = \frac{v[L\cos(\varphi) - s\sin(\varphi)] + \sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

**A.2.2 Return-Propagation Time**

At time $t_1(s)$, the incident wavefront reaches the reflector point specified by the coordinate $s$.
The reflection event has coordinates:

$$\boldsymbol{P}(s, t_1) = \big(A(s) + v\, t_1, B(s)\big)$$

Let $t_2(s)$ be the time required for the signal generated at this event to reach the receiver.
At the final time $t_1 + t_2$, the receiver is at the point:

$$R_{rec} = (v(t_1 + t_2), 0)$$

The vector from the reflection event to the final position of the receiver has components:

$$\Delta x = v(t_1 + t_2) - [A(s) + v\, t_1] = vt_2 - A(s)$$

$$\Delta y = 0 - B(s) = -B(s)$$

During the time $t_2$, the signal travels a distance $c\, t_2$. The return-propagation condition is therefore:

$$\Delta x^2 + \Delta y^2 = (ct_2)^2$$

Substituting:

$$[vt_2 - A(s)]^2 + B^2(s) = c^2 {t_2}^2$$

It follows:

$$v^2{t_2}^2 - 2vA(s)t_2 + A^2(s) + B^2(s) = c^2{t_2}^2$$

Using:

$$A^2(s) + B^2(s) = L^2 + s^2$$

one obtains:

$$v^2{t_2}^2 - 2vA(s)t_2 + L^2 + s^2 = c^2{t_2}^2$$

hence:

$$(c^2 - v^2){t_2}^2 + 2vA(s)t_2 - (L^2 + s^2) = 0$$

The equation has the form:

$$a_2{t_2}^2 + b_2t_2 + d_2 = 0$$

with:

$$a_2 = c^2 - v^2$$

$$b_2 = 2vA(s)$$

$$d_2 = -(L^2 + s^2)$$

The discriminant is:

$$\Delta_2 = {b_2}^2 - 4\,a_2d_2 = 4\,v^2A^2(s) + 4\,(c^2 - v^2)(L^2 + s^2)$$

$$\Delta_2 = 4[\,v^2A^2(s) + (c^2 - v^2)(L^2 + s^2)]$$

The discriminant coincides with that of the forward equation: $\Delta_2 = \Delta_1$.
The quadratic formula gives:

$$t_2 = \frac{-2\,vA(s) \pm \sqrt{\Delta_2}}{2\,(c^2 - v^2)}$$

Since:

$$\sqrt{\Delta_2} = 2\sqrt{\,v^2A^2(s) + (c^2 - v^2)(L^2 + s^2)}$$

the two roots are:

$$t_2 = \frac{-vA(s) \pm \sqrt{\,v^2A^2(s) + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

The root with the minus sign has a numerator that is always negative and must therefore be discarded. By contrast, the root with the plus sign has a positive numerator and gives the physically admissible time:

$$t_2(s) = \frac{-vA(s) + \sqrt{\,v^2A^2(s) + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

Writing $A(s)$ explicitly:

$$t_2(s) = \frac{-v[L\cos(\varphi) - s\sin(\varphi)] + \sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}}{c^2 - v^2}$$

**A.2.3 Total Time Associated with the Point s**

The total round-trip time associated with the path that meets the reflector at the point specified by $s$ is:

$$t_{tot}(s) = t_1(s) + t_2(s)$$

Adding the expressions for $t_1(s)$ and $t_2(s)$ obtained above, the linear terms cancel while the two radicals add, yielding:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2)}$$

Writing $A(s)$ explicitly:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)}$$

Define $F(s)$ as the radicand:

$$F(s) = v^2[L\cos(\varphi) - s\sin(\varphi)]^2 + (c^2 - v^2)(L^2 + s^2)$$

Expanding the square:

$$F(s) = v^2[L^2\cos^2(\varphi) - 2Ls\sin(\varphi)\cos(\varphi) + s^2\sin^2(\varphi)] + (c^2 - v^2)(L^2 + s^2)$$

$$F(s) = L^2v^2\cos^2(\varphi) - 2Lv^2s\sin(\varphi)\cos(\varphi) + s^2v^2\sin^2(\varphi) + L^2c^2 + s^2c^2 - L^2v^2 - v^2s^2$$

Collecting terms in powers of $s$ gives

$$F(s) = L^2v^2\cos^2(\varphi) + L^2c^2 - L^2v^2 + s^2c^2 + s^2v^2\sin^2(\varphi) - v^2s^2 - 2Lv^2s\sin(\varphi)\cos(\varphi)$$

$$F(s) = L^2[v^2\cos^2(\varphi) + c^2 - v^2] + s^2[c^2 + v^2\sin^2(\varphi) - v^2] - 2Lv^2s\sin(\varphi)\cos(\varphi)$$

$$F(s) = L^2\{v^2[\cos^2(\varphi) - 1] + c^2\} + s^2\{c^2 + v^2[\sin^2(\varphi) - 1]\} - 2Lv^2s\sin(\varphi)\cos(\varphi)$$

Using: $1 - \cos^2(\varphi) = \sin^2(\varphi)$

and: $1 - \sin^2(\varphi) = \cos^2(\varphi)$

one obtains the final form:

$$F(s) = L^2[c^2 - v^2\sin^2(\varphi)] + s^2[c^2 - v^2\cos^2(\varphi)] - 2Lv^2s\sin(\varphi)\cos(\varphi)$$

and the total time can therefore be written as:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{F(s)}$$

This expression is exact within the geometric model considered. Each value of $s$ identifies a path constrained to meet the reflector at a specific point; determining the first-arrival path now requires minimizing $F(s)$, which is carried out in A.3.

## A.3 Minimization of the Total Time

From subsection A.2, the total time associated with the reflector point specified by the coordinate $s$ is:

$$t_{tot}(s) = \frac{2}{c^2 - v^2}\sqrt{F(s)}$$

where:

$$F(s) = L^2[c^2 - v^2\sin^2(\varphi)] + s^2[c^2 - v^2\cos^2(\varphi)] - 2Lv^2 s\sin(\varphi)\cos(\varphi)$$

Since $v < c$ is assumed, the factor preceding the square root is positive. Moreover, the square root is a strictly increasing function of its argument. Consequently, the values of $s$ that minimize $t_{tot}(s)$ coincide with those that minimize $F(s)$.
The problem can therefore be reduced to minimizing the quadratic polynomial $F(s)$.
To make the structure of the polynomial more transparent, define:

$$a = c^2 - v^2\cos^2(\varphi)$$

$$b = Lv^2\sin(\varphi)\cos(\varphi)$$

$$d = L^2[c^2 - v^2\sin^2(\varphi)]$$

With these definitions:

$$F(s) = as^2 - 2bs + d$$

Since:

$$0 \leq \cos^2(\varphi) \leq 1$$

one has:

$$v^2\cos^2(\varphi) \leq v^2 < c^2$$

and therefore:

$$a = c^2 - v^2\cos^2(\varphi) > 0$$

The parabola represented by $F(s)$ opens upward and has a unique global minimum.

### A.3.1 Determination of the Minimum Point

Differentiating $F(s)$ with respect to $s$, since $a$, $b$, and $d$ do not depend on $s$:

$$\frac{dF}{ds} = \frac{d}{ds}(as^2 - 2bs + d) = 2as - 2b$$

The stationarity condition $dF/ds = 0$ gives:

$$2as_{\min} - 2b = 0$$

hence:

$$s_{\min} = \frac{b}{\mathrm{a}}$$

Substituting the expressions for $a$ and $b$:

$$s_{\min} = \frac{Lv^2\sin(\varphi)\cos(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

The second derivative of $F(s)$ is:

$$\frac{d^2F}{ds^2} = 2a = 2[c^2 - v^2\cos^2(\varphi)]$$

Since the term in parentheses is positive, one has $d^2F/ds^2 > 0$.
The stationary point $s_{\min}$ is therefore a local minimum. Since $F(s)$ is a quadratic polynomial with a positive leading coefficient, this minimum is also unique and global.

Introducing the dimensionless parameter $\beta = v/c$, the expression for $s_{\min}$ can be written as:

$$s_{\min} = \frac{Lv^2 \sin(\varphi)\cos(\varphi)}{c^2 - v^2\cos^2(\varphi)} = \frac{L\beta^2 c^2 \sin(\varphi)\cos(\varphi)}{c^2 - \beta^2 c^2 \cos^2(\varphi)}$$

Hence:

$$\boxed{s_{\min} = L\frac{\beta^2 \sin(\varphi)\cos(\varphi)}{1 - \beta^2\cos^2(\varphi)}}$$

For $\beta \ll 1$:

$$\frac{1}{1 - \beta^2\cos^2(\varphi)} = 1 + \beta^2\cos^2(\varphi) + \beta^4\cos^4(\varphi) + O(\beta^6)$$

Therefore:

$$s_{\min} = L\beta^2 \sin(\varphi)\cos(\varphi)\,[1 + \beta^2\cos^2(\varphi) + O(\beta^4)]$$

$$s_{\min} = L\beta^2 \sin(\varphi)\cos(\varphi) + L\beta^4 \sin(\varphi)\cos^3(\varphi) + O(\beta^6)L$$

To leading nonzero order: $s_{\min} = L\beta^2 \sin(\varphi)\cos(\varphi) + O(\beta^4)L.$

In order-of-magnitude terms: $s_{\min} = O(\beta^2)L.$

The displacement of the minimum-time point from the geometric center is therefore second order in $\beta$.

**A.3.2 Completing the Square**

Start from the form:

$$F(s) = as^2 - 2bs + d$$

$$F(s) = a\left[s^2 - 2\frac{b}{a}s\right] + d$$

$$F(s) = a\left[\left(\mathrm{s} - \frac{b}{a}\right)^2 - \frac{b^2}{a^2}\right] + d$$

$$F(s) = a\left(\mathrm{s} - \frac{b}{a}\right)^2 - \frac{b^2}{a} + d$$

Since:

$$s_{\min} = \frac{b}{\mathrm{a}}$$

Hence:

$$F_{\min} \equiv F(s_{\min}) = d - \frac{b^2}{a}$$

$F(s)$ can then be written in canonical form:

$$F(s) = F_{\min} + a(\mathrm{s} - s_{\min})^2$$

This form makes the minimum immediately apparent.

Indeed: $c^2 - v^2\cos^2(\varphi) > 0$

and: $(\mathrm{s} - s_{\min})^2 \geq 0.$

Therefore: $F(s) \geq F_{\min}$, with equality only for $\mathrm{s} = s_{\min}$.

### A.3.3 Explicit Calculation of $F_{min}$

From the definition:

$$F_{\min} = d - \frac{b^2}{a}$$

Substituting:

$$d = L^2[c^2 - v^2\sin^2(\varphi)]$$

$$b^2 = L^2v^4\sin^2(\varphi)\cos^2(\varphi)$$

$$a = c^2 - v^2\cos^2(\varphi)$$

one obtains:

$$F_{\min} = L^2[c^2 - v^2\sin^2(\varphi)] - \frac{L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

It follows:

$$F_{\min} = \frac{L^2[c^2 - v^2\sin^2(\varphi)][c^2 - v^2\cos^2(\varphi)] - L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

Expanding the numerator:

$$\begin{aligned}[c^2 - v^2\sin^2(\varphi)][c^2 - v^2\cos^2(\varphi)] &= c^4 - c^2v^2\cos^2(\varphi) - c^2v^2\sin^2(\varphi) + v^4\sin^2(\varphi)\cos^2(\varphi) = \\ &= c^4 - c^2v^2[\sin^2(\varphi) + \cos^2(\varphi)] + v^4\sin^2(\varphi)\cos^2(\varphi) = c^4 - c^2v^2 + v^4\sin^2(\varphi)\cos^2(\varphi)\end{aligned}$$

Therefore:

$$F_{min} = \frac{L^2[c^4 - c^2v^2 + v^4\sin^2(\varphi)\cos^2(\varphi)] - L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

The two fourth-degree terms in $v$ cancel:

$$\boxed{F_{min} = \frac{L^2c^2[c^2 - v^2]}{c^2 - v^2\cos^2(\varphi)}}$$

The complete canonical form of $F(s)$ is therefore:

$$\boxed{F(s) = \frac{L^2c^2[c^2 - v^2]}{c^2 - v^2\cos^2(\varphi)} + [c^2 - v^2\cos^2(\varphi)](\mathrm{s} - s_{\min})^2}$$

**A.3.4 Minimum Total Time**
At the point $\mathrm{s} = s_{\min}$:

$$t_{tot,min} = \frac{2}{c^2 - v^2}\sqrt{F_{\min}}$$

Substituting $F_{\min}$:

$$t_{tot,min} = \frac{2}{c^2 - v^2}\sqrt{\frac{L^2c^2[c^2 - v^2]}{c^2 - v^2\cos^2(\varphi)}}$$

It follows:

$$\boxed{t_{tot,min} = \frac{2Lc}{\sqrt{c^2 - v^2}\sqrt{c^2 - v^2\cos^2(\varphi)}}}$$

This is the expression used in the main text.
This expression retains an explicit dependence on the angle $\varphi$.

**A.3.5 Comparison Between the Geometric Center and the Minimum**
The canonical form makes it possible to compare directly the value of $F(s)$ at the geometric center, specified by $\mathrm{s} = 0$, with the minimum value.
Setting $\mathrm{s} = 0$:

$$F(0) = F_{min} + [c^2 - v^2\cos^2(\varphi)](0 - s_{\min})^2$$

Hence:

$$F(0) - F_{min} = [c^2 - v^2\cos^2(\varphi)]s^2{}_{\min}$$

Since:

$$s_{\min} = \frac{Lv^2\sin(\varphi)\cos(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

Then:

$$F(0) - F_{min} = [c^2 - v^2\cos^2(\varphi)]\frac{L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{[c^2 - v^2\cos^2(\varphi)]^2}$$

$$F(0) - F_{min} = \frac{L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

Dividing by:

$$F_{min} = \frac{L^2c^2[c^2 - v^2]}{c^2 - v^2\cos^2(\varphi)}$$

One obtains:

$$\frac{F(0) - F_{min}}{F_{min}} = \frac{L^2v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2 - v^2\cos^2(\varphi)}\frac{c^2 - v^2\cos^2(\varphi)}{L^2c^2[c^2 - v^2]} = \frac{v^4\sin^2(\varphi)\cos^2(\varphi)}{c^2[c^2 - v^2]}$$

Introducing $\beta$:

$$\frac{F(0) - F_{min}}{F_{min}} = \frac{\beta^4c^4\sin^2(\varphi)\cos^2(\varphi)}{c^4[1 - \beta^2]} = \frac{\beta^4\sin^2(\varphi)\cos^2(\varphi)}{1 - \beta^2}$$

For $\beta \ll 1$:

$$\frac{1}{1 - \beta^2} = 1 + \beta^2 + O(\beta^4)$$

therefore:

$$\frac{F(0) - F_{min}}{F_{min}} = \beta^4 \sin^2(\varphi)\cos^2(\varphi) + O(\beta^6)$$

In perturbative-order terms:

$$\frac{F(0) - F_{min}}{F_{min}} = O(\beta^4)$$

## A.4 Minimization of the Forward Time and the First-Contact Construction

The first-contact point ***D*** is the point on the reflector first reached by the incident wavefront.
It is therefore determined by minimizing only the forward time $t_1(s)$, not by minimizing the total round-trip time $t_{tot}(s)$.
Denote by $s_D$ the coordinate of the point ***D*** and by $t_D$ the corresponding first-contact time:

$$t_D = t_1(s_D)$$

The reflector is assumed to be sufficiently extended to contain the point $s_D$.

### A.4.1 Implicit Definition and Derivative of the Forward Time

The forward-propagation condition can be written as:

$$H(s, t_1) = 0$$

where:

$$H(s, t_1) = [L\cos(\varphi) + v\,t_1 - s\sin(\varphi)]^2 + [L\sin(\varphi) + s\cos(\varphi)]^2 - c^2{t_1}^2$$

This relation must locally define $t_1$ as a differentiable function of $s$.
We therefore calculate the partial derivative of $H$ with respect to $t_1$:

$$\frac{\partial H}{\partial t_1} = 2v[L\cos(\varphi) + v\,t_1 - s\sin(\varphi)] - 2c^2 t_1$$

Using:

$$A(s) = L\cos(\varphi) - s\sin(\varphi)$$

One obtains

$$\frac{\partial H}{\partial t_1} = 2vA(s) - 2(c^2 - v^2)t_1$$

From the physical root, calculated in A.2:

$$t_1(s) = \frac{vA(s) + \sqrt{F(s)}}{c^2 - v^2}$$

it follows:

$$(c^2 - v^2)t_1(s) - vA(s) = \sqrt{F(s)}$$

Therefore, on the physical root:

$$\frac{\partial H}{\partial t_1} = -2\sqrt{F(s)}$$

Since:

$$F(s) = v^2 A^2(s) + (c^2 - v^2)(L^2 + s^2) > 0$$

one has:

$$\frac{\partial H}{\partial t_1} < 0$$

for every value of $s$.
The implicit function theorem therefore guarantees that the relation:

$$H(s, t_1) = 0$$

defines a differentiable function:

$$t_1 = t_1(s)$$

Differentiating the identity:

$$H\big(s, t_1(s)\big) = 0$$

with respect to $s$:

$$\frac{\partial H}{\partial s} + \frac{\partial H}{\partial t_1}\frac{dt_1}{ds} = 0$$

Since $\partial H/\partial t_1 \neq 0$, we can solve for the derivative:

$$\frac{dt_1}{ds} = -\frac{\partial H/\partial s}{\partial H/\partial t_1}$$

We calculate the partial derivative of $H$ with respect to $s$:

$$\frac{\partial H}{\partial s} = -2\sin(\varphi)[L\cos(\varphi) + v\,t_1 - s\sin(\varphi)] + 2\cos(\varphi)\,[L\sin(\varphi) + s\cos(\varphi)]$$

$$\frac{\partial H}{\partial s} = -2L\cos(\varphi)\sin(\varphi) - 2v\,t_1\sin(\varphi) + 2s\sin^2(\varphi) + 2L\sin(\varphi)\cos(\varphi) + 2\,s\,cos^2(\varphi)$$

$$\frac{\partial H}{\partial s} = -2v\,t_1\sin(\varphi) + 2s\sin^2(\varphi) + \mathrm{s}\,cos^2(\varphi)$$

$$\frac{\partial H}{\partial s} = 2[\mathrm{s} - v\,t_1\sin(\varphi)]$$

Substituting $\partial H/\partial s$ and $\partial H/\partial t_1$ into the implicit derivative:

$$\frac{dt_1}{ds} = -\frac{\partial H/\partial s}{\partial H/\partial t_1} = -\frac{2[\mathrm{s} - v\,t_1\sin(\varphi)]}{2\sqrt{F(s)}} = -\frac{\mathrm{s} - v\,t_1\sin(\varphi)}{\sqrt{F(s)}}$$

Since the denominator is strictly positive, the stationarity condition:

$$\frac{dt_1}{ds} = 0$$

is equivalent to:

$$\mathrm{s} - v\,t_1\sin(\varphi) = 0$$

At the point ***D***:

$$s_D = v\,t_{\mathrm{D}}\sin(\varphi)$$

### A.4.2 Determination of $t_D$ and $s_D$

To determine $t_{\mathrm{D}}$, substitute $s_D = v\, t_{\mathrm{D}} \sin(\varphi)$ into the coordinates of the reflector point reached at time $t_{\mathrm{D}}$.

The coordinate $x$ is:

$$x_D = L\cos(\varphi) + v\, t_{\mathrm{D}} - s_D \sin(\varphi)$$

Substituting $s_D$:

$$x_D = L\cos(\varphi) + v\, t_{\mathrm{D}} - v\, t_{\mathrm{D}} \sin^2(\varphi)$$

$$x_D = L\cos(\varphi) + v\, t_{\mathrm{D}} cos^2(\varphi)$$

$$x_D = \cos(\varphi)\,[L + v\, t_{\mathrm{D}} \cos(\varphi)]$$

The coordinate $y$ is:

$$y_D = L\sin(\varphi) + s_D \cos(\varphi)$$

Substituting $s_D$:

$$y_D = L\sin(\varphi) + v\, t_{\mathrm{D}} \sin(\varphi)\cos(\varphi)$$

$$y_D = \sin(\varphi)\,[L + v\, t_{\mathrm{D}} \cos(\varphi)]$$

The distance of the point ***D*** from the origin therefore satisfies:

$${x_D}^2 + {y_D}^2 = cos^2(\varphi)[L + v\, t_{\mathrm{D}} \cos(\varphi)]^2 + \sin^2(\varphi)[L + v\, t_{\mathrm{D}} \cos(\varphi)]^2$$

$${x_D}^2 + {y_D}^2 = [cos^2(\varphi) + \sin^2(\varphi)][L + v\, t_{\mathrm{D}} \cos(\varphi)]^2$$

$${x_D}^2 + {y_D}^2 = [L + v\, t_{\mathrm{D}} \cos(\varphi)]^2$$

Since ***D*** belongs to the incident wavefront at time $t_{\mathrm{D}}$, one also has:

$${x_D}^2 + {y_D}^2 = c^2 {t_{\mathrm{D}}}^2$$

It follows:

$$c^2 {t_{\mathrm{D}}}^2 = [L + v\, t_{\mathrm{D}} \cos(\varphi)]^2$$

$$c\, t_{\mathrm{D}} = L + v\, t_{\mathrm{D}} \cos(\varphi)$$

$$t_{\mathrm{D}}[c - v\ \cos(\varphi)] = L$$

Since $v < c$ :

$$c - v\ \cos(\varphi) \geq c - v > 0$$

Therefore:

$$\boxed{t_{\mathrm{D}} = \frac{L}{c - v\ \cos(\varphi)}}$$

Substituting into $s_D = v\, t_{\mathrm{D}} \sin(\varphi)$:

$$\boxed{s_D = \frac{Lv\sin(\varphi)}{c - v\ \cos(\varphi)}}$$

**A.4.3 Verification of the Global Minimum**
From subsection A.2, the forward-propagation time is:

$$t_1(s) = \frac{vA(s) + \sqrt{F(s)}}{c^2 - v^2}$$

where:

$$A(s) = L\cos(\varphi) - s\sin(\varphi)$$

and:

$$F(s) = as^2 - 2bs + d$$

with:

$$a = c^2 - v^2\cos^2(\varphi)$$

$$b = Lv^2\sin(\varphi)\cos(\varphi)$$

$$d = L^2[c^2 - v^2\sin^2(\varphi)]$$

Since $\boldsymbol{A(s)}$ is a linear function of $\boldsymbol{s}$, its second derivative is zero. Therefore:

$$\frac{d^2 t_1}{ds^2} = \frac{1}{c^2 - v^2}\frac{d^2}{ds^2}\sqrt{F(s)}$$

Calculate the derivatives of $F(s)$:

$$\frac{dF}{ds} = \frac{d}{ds}(as^2 - 2bs + d) = 2as - 2b = 2(as - b)$$

$$\frac{d^2F}{ds^2} = 2a$$

and the second derivative of the square root:

$$\frac{d}{ds}\sqrt{F(s)} = \frac{\frac{dF}{ds}}{2\sqrt{F(s)}}$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{d}{ds}\left(\frac{d}{ds}\sqrt{F(s)}\right) = \frac{d}{ds}\left(\frac{\frac{dF}{ds}}{2\sqrt{F(s)}}\right)$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{\frac{d^2F}{ds^2}2\sqrt{F(s)} - \frac{dF}{ds}\frac{\frac{dF}{ds}}{\sqrt{F(s)}}}{\left(2\sqrt{F(s)}\right)^2}$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{1}{2\sqrt{F(s)}}\frac{d^2F(s)}{ds^2} - \frac{\left(\frac{dF(s)}{ds}\right)^2}{4\left(F(s)\right)^{3/2}}$$

Substituting:

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{a}{\sqrt{F(s)}} - \frac{(as-b)^2}{\left(F(s)\right)^{3/2}}$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{aF(s) - (as - b)^2}{[F(s)]^{3/2}}$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{a(as^2 - 2bs + d) - (a^2s^2 - 2\mathrm{a}bs + b^2)}{[F(s)]^{3/2}}$$

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{\mathrm{a}d - b^2}{[F(s)]^{3/2}}$$

Calculate $\mathrm{a}d - b^2$:

$$\mathrm{a}d - b^2 = [c^2 - v^2\cos^2(\varphi)]\, L^2[c^2 - v^2\sin^2(\varphi)] - [Lv^2 \sin(\varphi)\cos(\varphi)]^2$$

simplifying:

$$\mathrm{a}d - b^2 = L^2c^2(c^2 - v^2)$$

it follows:

$$\frac{d^2}{ds^2}\sqrt{F(s)} = \frac{L^2c^2(c^2 - v^2)}{[F(s)]^{3/2}}$$

Substituting into the expression for $\frac{d^2t_1}{ds^2}$:

$$\frac{d^2t_1}{ds^2} = \frac{1}{c^2 - v^2}\frac{L^2c^2(c^2 - v^2)}{[F(s)]^{3/2}} = \frac{L^2c^2}{[F(s)]^{3/2}}$$

Since:

$$F(s) = v^2A^2(s) + (c^2 - v^2)(L^2 + s^2) > 0$$

For every $s$, one has:

$$\frac{d^2t_1}{ds^2} = \frac{L^2c^2}{[F(s)]^{3/2}} > 0$$

The function $t_1(s)$ is therefore strictly convex. It can have at most one stationary point, and every stationary point is necessarily the unique global minimum.

Since the stationary point was derived in the preceding subsections:

$$s_D = \frac{Lv\sin(\varphi)}{c - v\ \cos(\varphi)}$$

one concludes that:

$$t_1(s_D) = \min_s t_1(s)$$

The point ***D*** is therefore the first point reached by the incident wavefront.

**A.4.4 Total Time of the Path Through *D***

The total time associated with the path constrained to meet the reflector at the point ***D*** is:

$$t_{tot}(s_D) = \frac{2}{c^2 - v^2}\sqrt{F(s_D)}$$

To obtain $\boldsymbol{F(s_D)}$, use the expression for the forward time derived in A.2:

$$t_1(s) = \frac{vA(s) + \sqrt{F(s)}}{c^2 - v^2}$$

where:

$$A(s) = L\cos(\varphi) - s\sin(\varphi)$$

At the point ***D***:

$$t_\mathrm{D} = \frac{vA(s_D) + \sqrt{F(s_D)}}{c^2 - v^2}$$

It follows:

$$(c^2 - v^2)t_\mathrm{D} = vA(s_D) + \sqrt{F(s_D)}$$

It follows:

$$\sqrt{F(s_D)} = (c^2 - v^2)t_\mathrm{D} - vA(s_D)$$

From the relation:

$$s_D = v\, t_\mathrm{D}\sin(\varphi)$$

one has:

$$A(s_D) = L\cos(\varphi) - s_D\sin(\varphi) = L\cos(\varphi) - v\, t_\mathrm{D}\sin^2(\varphi)$$

Consequently:

$$\sqrt{F(s_D)} = (c^2 - v^2)t_\mathrm{D} - vL\cos(\varphi) + v^2 t_\mathrm{D}\sin^2(\varphi)$$

$$\boldsymbol{\sqrt{F(s_D)} = t_\mathrm{D}\left[c^2 - v^2 + v^2\sin^2(\varphi)\right] - vL\cos(\varphi)}$$

$$\boldsymbol{\sqrt{F(s_D)} = t_\mathrm{D}\left[c^2 - v^2\cos^2(\varphi)\right] - vL\cos(\varphi)}$$

From the relation:

$$t_\mathrm{D} = \frac{L}{c - v\cos(\varphi)}$$

it follows:

$$L = t_\mathrm{D}[c - v\cos(\varphi)]$$

Substituting:

$$\sqrt{F(s_D)} = t_\mathrm{D}[c^2 - v^2\cos^2(\varphi)] - vt_\mathrm{D}[c - v\cos(\varphi)]\cos(\varphi)$$

$$\sqrt{F(s_D)} = t_\mathrm{D}[c^2 - v^2\cos^2(\varphi) - v\,c\cos(\varphi) + v^2\cos^2(\varphi)]$$

$$\sqrt{F(s_D)} = t_\mathrm{D}[c^2 - v\,c\cos(\varphi)]$$

$$\sqrt{F(s_D)} = c\, t_\mathrm{D}[c - v\cos(\varphi)]$$

Since:

$$t_\mathrm{D}[c - v\cos(\varphi)] = L$$

One obtains:

$$\sqrt{F(s_D)} = Lc$$

and therefore:

$$F(s_D) = L^2c^2$$

The total time of the path through the first-contact point is therefore:

$$\boxed{t_{tot}(s_D) = \frac{2Lc}{c^2 - v^2}}$$

This expression is exactly independent of the angle $\varphi$. The isotropy, however, concerns the path constrained to pass through ***D***, not the path that minimizes the total round-trip time.

**A.4.5 Comparison with the Point of Minimum Total Time**

From subsection A.3:

$$s_{\min} = \frac{Lv^2 \sin(\varphi)\cos(\varphi)}{c^2 - v^2\cos^2(\varphi)}$$

Factoring the denominator:

$$s_{\min} = \frac{Lv^2 \sin(\varphi)\cos(\varphi)}{[c - v\ \cos(\varphi)][c + v\ \cos(\varphi)]}$$

Since:

$$s_D = \frac{Lv\sin(\varphi)}{c - v\ \cos(\varphi)}$$

the exact relation follows:

$$\boxed{s_{\min} = s_D \frac{v\ \cos(\varphi)}{c + v\ \cos(\varphi)}}$$

The two points are therefore generally distinct.
In the interval:

$$0 < \varphi < \frac{\pi}{2}$$

one has:

$$0 < \frac{v\ \cos(\varphi)}{c + v\ \cos(\varphi)} < 1$$

and therefore:

$$0 < s_{\min} < s_D$$

The first-contact point minimizes only the forward time; whereas the point $s_{\min}$ minimizes the total round-trip time. The perturbative difference between the times associated with the two paths will be derived in A.5.

### A.5 Perturbative Expansions and Comparison Between the Paths

Introduce the dimensionless parameter $\beta = v/c$, with $\beta \ll 1$.

The expansions are obtained from the binomial expansion:

$$(1-x)^{\alpha} = 1 - \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + O(x^3)$$

Since $x = O(\beta^2)$ in the expressions considered, the terms are retained through order $O(\beta^4)$.

#### A.5.1 Central Path and Minimum-Time Path

For the central path, passing through the geometric center C and corresponding to $s = 0$, the total round-trip time is:

$$t_{tot}(0) = \frac{2L\sqrt{c^2 - v^2\sin^2(\varphi)}}{c^2 - v^2}$$

In terms of $\beta$:

$$t_{tot}(0) = \frac{2L}{c}\frac{\sqrt{1-\beta^2\sin^2(\varphi)}}{1-\beta^2}$$

Using:

$$\frac{1}{1-\beta^2} = 1 + \beta^2 + \beta^4 + O(\beta^6)$$

and:

$$\sqrt{1-\beta^2\sin^2(\varphi)} = 1 - \frac{1}{2}\beta^2\sin^2(\varphi) - \frac{1}{8}\beta^4\sin^4(\varphi) + O(\beta^6)$$

Multiplying the two expressions:

$$\frac{\sqrt{1-\beta^2\sin^2(\varphi)}}{1-\beta^2} = 1 + \beta^2\left[1 - \frac{1}{2}\sin^2(\varphi)\right] + \beta^4\left[1 - \frac{1}{2}\sin^2(\varphi) - \frac{1}{8}\sin^4(\varphi)\right] + O(\beta^6)$$

Since:

$$1 - \frac{1}{2}\sin^2(\varphi) = \frac{1}{2}[1 + \cos^2(\varphi)]$$

and:

$$1 - \frac{1}{2}\sin^2(\varphi) - \frac{1}{8}\sin^4(\varphi) = \frac{1}{8}[3 + 6\cos^2(\varphi) - \cos^4(\varphi)]$$

one obtains:

$$t_{tot}(0) = \frac{2L}{c}\left\{1 + \frac{1}{2}\beta^2[1+\cos^2(\varphi)] + \frac{1}{8}\beta^4[3 + 6\cos^2(\varphi) - \cos^4(\varphi)] + O(\beta^6)\right\}$$

For the minimum-time path, A.3 gives:

$$t_{tot,min} = \frac{2L\mathrm{c}}{\sqrt{c^2 - v^2}\sqrt{c^2 - v^2\cos^2(\varphi)}}$$

In terms of $\beta$:

$$t_{tot,min} = \frac{2L}{\mathrm{c}} \frac{1}{\sqrt{1 - \beta^2}\sqrt{1 - \beta^2\cos^2(\varphi)}}$$

Using:

$$(1 - \beta^2)^{-1/2} = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + O(\beta^6)$$

and:

$$\left(1 - \beta^2\cos^2(\varphi)\right)^{-1/2} = 1 + \frac{1}{2}\beta^2\cos^2(\varphi) + \frac{3}{8}\beta^4\cos^4(\varphi) + O(\beta^6)$$

Multiplying the two expansions gives:

$$\frac{1}{\sqrt{1 - \beta^2}\sqrt{1 - \beta^2\cos^2(\varphi)}} = 1 + \frac{1}{2}\beta^2[1 + \cos^2(\varphi)] + \beta^4\left[\frac{3}{8} + \frac{1}{4}\cos^2(\varphi) + \frac{3}{8}\cos^4(\varphi)\right] + O(\beta^6)$$

$$\frac{1}{\sqrt{1 - \beta^2}\sqrt{1 - \beta^2\cos^2(\varphi)}} = 1 + \frac{1}{2}\beta^2[1 + \cos^2(\varphi)] + \frac{1}{8}\beta^4[3 + 2\cos^2(\varphi) + 3\cos^4(\varphi)] + O(\beta^6)$$

Therefore:

$$t_{tot,min} = \frac{2L}{c}\left\{1 + \frac{1}{2}\beta^2[1 + \cos^2(\varphi)] + \frac{1}{8}\beta^4[3 + 2\cos^2(\varphi) + 3\cos^4(\varphi)] + O(\beta^6)\right\}$$

**A.5.2 Difference Between the Central Path and the Minimum**

Comparing $t_{tot}(0)$ and $t_{tot,min}$ shows that the zeroth-order and second-order terms coincide. The first difference appears at fourth order:

$$t_{tot}(0) - t_{tot,min} \approx \frac{2L}{c}\left\{\frac{1}{8}\beta^4[6\cos^2(\varphi) - \cos^4(\varphi)] - \frac{1}{8}\beta^4[2\cos^2(\varphi) + 3\cos^4(\varphi)]\right\}$$

$$t_{tot}(0) - t_{tot,min} \approx \frac{2L}{c}\left\{\frac{1}{8}\beta^4[4\cos^2(\varphi) - 4\cos^4(\varphi)]\right\}$$

$$t_{tot}(0) - t_{tot,min} \approx \frac{L}{c}\frac{1}{4}\beta^4\{4\cos^2(\varphi)[1 - \cos^2(\varphi)]\}$$

Hence

$$t_{tot}(0) - t_{tot,min} = \frac{L}{c}\beta^4\sin^2(\varphi)\cos^2(\varphi) + O\left(\frac{L}{c}\beta^6\right)$$

The central path therefore agrees with the minimum time through order $O(\beta^2)$ and differs from it only at fourth order.

### A.5.3 First-Contact Path

From subsection A.4:

$$t_{tot}(s_D) = \frac{2Lc}{c^2 - v^2}$$

In terms of $\beta$:

$$t_{tot}(s_D) = \frac{2L}{c}\frac{1}{1-\beta^2}$$

Therefore:

$$t_{tot}(s_D) = \frac{2L}{c}[1 + \beta^2 + \beta^4 + O(\beta^6)]$$

The expression is independent of the angle $\varphi$.
It follows:

$$t_{tot}(s_D) - t_{tot,min} = \frac{L}{c}\beta^2 \sin^2(\varphi) + O\left(\frac{L}{c}\beta^4\right)$$

### A.5.4 Final Comparison

$$t_{tot}(0) - t_{tot,min} = \frac{L}{c}\beta^4 \sin^2(\varphi)\cos^2(\varphi) + O\left(\frac{L}{c}\beta^6\right)$$

$$t_{tot}(s_D) - t_{tot,min} = \frac{L}{c}\beta^2 \sin^2(\varphi) + O\left(\frac{L}{c}\beta^4\right)$$

The central path introduces only a fourth-order correction and preserves the angular dependence present at second order.
By contrast, the first-contact path differs from the minimum already at order $O(\beta^2)$, that is, at the same order as the anisotropic effect, and produces a time exactly independent of the angle.