Cosmological Special Theory of Relativity: Lorentz Transformation with Scale Factor and Form-Invariance of the Robertson-Walker Metric

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Abstract

We present a formulation of Cosmological Special Theory of Relativity (CSTR) in which the standard Lorentz transformation is extended to incorporate the expansion of the universe via the Robertson-Walker scale factor. By defining physical coordinates $X^i = a(t_0)x^i$ where x^i are comoving coordinates and $a(t_0)$ is the scale factor evaluated at the cosmological time t_0 of the observer, we show that the flat Robertson-Walker line element $ds^2 = -c^2dt^2 + a(t_0)^2\delta_{ij}dx^i dx^j$ is form-invariant under a modified Lorentz transformation. The transformation $ct = \gamma(ct' + (v_0/c)X')$, $X = \gamma(X' + v_0t')$, $Y = Y'$, $Z = Z'$ with $\gamma = (1 - v_0^2/c^2)^{-1/2}$ preserves ds^2. In the limit $a(t_0) = a(t_0') = 1$, the transformation reduces exactly to the Lorentz transformation of special relativity. We discuss the postulates, local nature, physical interpretation as peculiar velocity, relativity of scale factor measurement, and implications for Maxwell and Klein-Gordon equations in expanding background. The framework provides a kinematical bridge between special relativity and the cosmological principle without invoking full general relativity dynamics.

Keywords: Cosmological Special Relativity, Lorentz transformation, Robertson-Walker metric, scale factor, cosmological principle, peculiar velocity

1 Introduction

Special relativity (SR) is founded on the invariance of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ under Lorentz transformations. Its domain is strictly inertial frames in a non-expanding, gravity-free spacetime. Cosmology, on the other hand, is governed by the Robertson-Walker (RW) metric which explicitly contains a time-dependent scale factor $a(t)$ encoding the expansion of space.

A natural question arises: Is there a special relativistic kinematics that is compatible with the RW metric form? That is, can we define a transformation between inertial observers located at different cosmological times such that each observer measures the line element in the RW form with its own scale factor?

The Cosmological Special Theory of Relativity (CSTR), proposed and developed by Yi between 2019 and 2025 (Yi, 2025; Carmeli, 2002), attempts to answer this. Unlike Carmeli's CSTR which introduces the Hubble constant as a second invariant, the present formulation, which we call Yi-CSTR, directly inserts $a(t_0)$ evaluated at a fixed cosmological instant into the Lorentz transformation.

In standard cosmology textbooks, it is well known that the RW metric is locally Minkowskian when expressed in physical coordinates $\vec{R} = a(t)\vec{x}$. However, the form-invariance under Lorentz-type boosts between cosmological inertial frames is rarely written explicitly as a global transformation law.

In this paper we systematize the transformation proposed as

$$ct = \gamma\left(ct' + \frac{v_0}{c}x'a(t'_0)\right) \tag{1}$$

$$xa(t_0) = \gamma\left(x'a(t'_0) + v_0 t'\right) \tag{2}$$

$$ya(t_0) = y'a(t'_0) \tag{3}$$

$$za(t_0) = z'a(t'_0) \tag{4}$$

and prove

$$ds^2 = -c^2dt^2 + a(t_0)^2[dx^2 + dy^2 + dz^2] = -c^2dt'^2 + a(t'_0)^2[dx'^2 + dy'^2 + dz'^2]. \tag{5}$$

We clarify assumptions, derivation, physical meaning, limits, and extensions to field equations.

2 Robertson-Walker Metric and Cosmological Inertial Frames

For spatial flatness $k = 0$, the RW metric is

$$ds^2 = -c^2dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{6}$$

where t is cosmic time, x^i are comoving coordinates, $a(t)$ is dimensionless scale factor (or with length dimension depending on normalization).

At a given epoch t_0 fixed for an observer, we define the instantaneous physical coordinate system:

$$X(t_0) \equiv a(t_0)x, \quad Y(t_0) \equiv a(t_0)y, \quad Z(t_0) \equiv a(t_0)z. \tag{7}$$

Then $ds^2 = -c^2dt^2 + dX^2 + dY^2 + dZ^2 + O(x\,da)$. Near the origin $x \approx 0$ or for an infinitesimal interval where $da = \dot{a}dt$ contributes at second order, the xda term is negligible. This is the precise statement of the equivalence principle: RW is locally Minkowskian.

We define a *cosmological inertial frame* $S(t_0)$ as a frame where at instant t_0, free particles with zero peculiar velocity remain at constant comoving coordinate x, and local physics obeys SR in terms of X^i.

Similarly, a second observer at t'_0 defines $S'(t'_0)$ with $X' = a(t'_0)x'$.

The cosmological principle demands that both S and S' write the metric in the RW form, possibly with different scale factor values $a(t_0)$ and $a(t'_0)$.

3 Postulates of Cosmological Special Relativity

We propose two postulates extending Einstein's:

P1. Principle of cosmological relativity: The laws of physics, expressed in terms of physical coordinates $X^i = a(t_0)x^i$, take the same form in all cosmological inertial frames.

P2. Invariance of c and form-invariance of ds^2: The speed of light c is invariant in physical coordinates, and the RW line element in the instantaneous form (5) is invariant under transformation between cosmological inertial frames.

Note P2 reduces to Einstein's second postulate when $a = 1$.

4 Derivation of Modified Lorentz Transformation

Consider boost along x-direction with peculiar velocity v_0. Peculiar velocity is defined as $v_{pec} = a(t)\dot{x}$, distinct from Hubble flow $v_H = \dot{a}x$.

We demand linear transformation between (ct, X) and (ct', X') preserving $-(ct)^2+X^2$:

Standard Lorentz group in 1+1 dimension gives uniquely

$$ct = \gamma(ct' + \beta X') \tag{8}$$

$$X = \gamma(X' + \beta ct') \tag{9}$$

$$Y = Y' \tag{10}$$

$$Z = Z', \tag{11}$$

where $\beta = v_0/c$, $\gamma = (1-\beta^2)^{-1/2}$.

Rewriting $X = a(t_0)x$, $X' = a(t_0')x'$ we obtain exactly (1)-(4):

$$\boxed{\begin{aligned} ct &= \gamma\left(ct' + \frac{v_0}{c}x'a(t_0')\right) \\ xa(t_0) &= \gamma\left(x'a(t_0') + v_0t'\right) \\ ya(t_0) &= y'a(t_0') \\ za(t_0) &= z'a(t_0') \end{aligned}} \tag{12}$$

Inverse transformation follows from orthogonality of Lorentz group:

$$ct' = \gamma\left(ct - \frac{v_0}{c}xa(t_0)\right) \tag{13}$$

$$x'a(t_0') = \gamma\left(xa(t_0) - v_0t\right). \tag{14}$$

When $a(t_0) = a(t_0') = a_0 = const$, this is isomorphic to standard Lorentz transformation.

5 Proof of Form-Invariance of Robertson-Walker Metric

Consider differential form at fixed t_0, t_0' so $da(t_0) = da(t_0') = 0$ for the transformation:

$dX = a(t_0)dx$, $dX' = a(t_0')dx'$ etc.

Then

$$d(ct) = \gamma\left(d(ct') + \beta dX'\right) \tag{15}$$

$$dX = \gamma\left(dX' + \beta d(ct')\right) \tag{16}$$

$$dY = dY', \quad dZ = dZ'. \tag{17}$$

Compute:

$$-d(ct)^2 + dX^2 = -\gamma^2(d(ct') + \beta dX')^2 + \gamma^2(dX' + \beta d(ct'))^2 \tag{18}$$

$$= -\gamma^2\left[d(ct')^2 + 2\beta d(ct')dX' + \beta^2 dX'^2\right] \tag{19}$$

$$+\gamma^2\left[dX'^2 + 2\beta dX'd(ct') + \beta^2 d(ct')^2\right] \tag{20}$$

$$= \gamma^2(1-\beta^2)\left[-d(ct')^2 + dX'^2\right] \tag{21}$$

$$= -d(ct')^2 + dX'^2, \tag{22}$$

since $\gamma^2(1-\beta^2) = 1$.

Adding $dY^2 + dZ^2 = dY'^2 + dZ'^2$ yields

$$-c^2dt^2 + a^2(t_0)(dx^2 + dy^2 + dz^2) = -c^2dt'^2 + a^2(t_0')(dx'^2 + dy'^2 + dz'^2), \tag{23}$$

which is (5). Q.E.D.

If we retain da terms, $dX = adx + x\dot{a}dt$. Near origin $|x| \ll c/H$, $x\dot{a}dt = xHadt$ is suppressed by $Hx/c \ll 1$. Thus invariance holds locally within Hubble radius, consistent with SR as local theory.

6 Limiting Cases

(i) Standard SR limit: Set $a(t_0) = a(t'_0) = 1$. Then (1)-(4) become ordinary Lorentz transformation. The metric reduces to Minkowski.

(ii) Galilean-Cosmological limit: $v_0 \ll c$, $\gamma \approx 1$,

$$t \approx t', \tag{24}$$

$$xa(t_0) \approx x'a(t'_0) + v_0 t', \tag{25}$$

$$ya(t_0) \approx y'a(t'_0), \tag{26}$$

$$za(t_0) \approx z'a(t'_0), \tag{27}$$

i.e., Galilean transformation for physical distances.

(iii) Comoving limit: $v_0 = 0$, $\gamma = 1$, then $ct = ct'$, $X = X'$, which implies $xa(t_0) = x'a(t'_0)$. This expresses cosmological principle: physical distance to a comoving object scales with a.

7 Physical Interpretation

7.1 Peculiar Velocity vs Hubble Flow

v_0 is not the recession velocity HD. It is the relative peculiar velocity of two cosmological inertial frames. Hubble law remains $v_H = (\dot{a}/a)X$ and is separate. The total velocity of a distant galaxy seen from S is $v_{tot} = v_H + v_{pec}$ where v_{pec} transforms via Einstein velocity addition with factor γ.

7.2 Relativity of Scale Factor

Equation (2) implies $a(t_0)$ and $a(t'_0)$ are related via transformation if x, x', t, t' are specified. Observers in relative motion do not agree on simultaneous scale factor value, analogous to relativity of simultaneity. For events simultaneous in S' ($dt' = 0$), $dt = \gamma\beta dX'/c \neq 0$, so t_0 and t'_0 differ. Thus expansion history is observer-dependent at second order $O(v_0 HX/c^2)$.

7.3 Cosmological Time Dilation

From (1), $dt = \gamma(dt' + (v_0/c^2)dX')$. For a clock at rest in S' ($dX' = 0$), $dt = \gamma dt'$, standard time dilation, but physical time interval is measured by cosmic time.

8 Extension to Field Equations

Following the same replacement $\partial_X = a^{-1}(t_0)\partial_x$, the operators become:

$$\partial_\mu^{(cos)} = \left(\frac{1}{c}\partial_t, \frac{1}{a(t_0)}\nabla_x\right). \tag{28}$$

Then Maxwell equations in $S(t_0)$ take form:

$$\frac{1}{a(t_0)}\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{29}$$

$$\frac{1}{a(t_0)}\nabla \times \vec{E} = -\partial_t \vec{B}, \tag{30}$$

$$\frac{1}{a(t_0)}\nabla \cdot \vec{B} = 0, \tag{31}$$

$$\frac{1}{a(t_0)}\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0\partial_t \vec{E}. \tag{32}$$

These are form-invariant under (1)-(4) provided $\vec{E}, \vec{B}, \rho, \vec{J}$ transform as in SR with X coordinates. Similarly, Klein-Gordon equation:

$$\left[-\frac{1}{c^2}\partial_t^2 + \frac{1}{a^2(t_0)}\nabla_x^2 - \frac{m^2c^2}{\hbar^2}\right]\phi = 0, \tag{33}$$

is invariant.

This shows CSTR provides a consistent kinematical framework for quantum fields in expanding background at instant t_0, without full curved spacetime QFT.

9 Cosmological Implications and Discussion

- **Local Lorentz Invariance of RW:** It is textbook knowledge that RW is locally Lorentz invariant. The present transformation makes it explicit: at each t_0, tangent space has Lorentz symmetry for physical coordinates. This is not new physics but a useful parametrization.

- **Horizon and Causality:** Since c is invariant in physical coordinates, light cone $dX = cdt$ is preserved. However $X = a(t_0)x$, so comoving horizon $\int cdt/a(t)$ is unchanged. The transformation does not by itself solve horizon problem; dynamics of $a(t)$ still required.

- **Comparison with Carmeli CSTR:** Carmeli introduced $\tau = H_0^{-1}$ as second invariant. Yi-CSTR keeps only c invariant but makes metric form-invariant with scale factor. Both reduce to SR locally.

- **Validity Domain:** As noted, $dX = adx + xda$ full differential yields extra terms $\sim HX$. Invariance holds for $X \ll c/H$ (sub-Hubble) and for time intervals $\Delta t \ll H^{-1}$. For cosmological scales $X \sim c/H$, GR corrections appear.

- **Experimental Test:** Peculiar velocity addition law predicts $v'_{pec} = (v_{pec} - v_0)/(1 - v_{pec}v_0/c^2)$ independent of a, testable via redshift-space distortions. Any a-dependence in transverse Doppler effect would be second-order.

10 Conclusion

We have formulated and proved that the modified Lorentz transformation $ct = \gamma(ct' + (v_0/c)x'a(t'_0))$, $xa(t_0) = \gamma(x'a(t'_0) + v_0 t')$, $ya(t_0) = y'a(t'_0)$, $za(t_0) = z'a(t'_0)$ preserves the form of the flat Robertson-Walker metric. In the limit $a \to 1$ it reduces to standard Lorentz transformation, hence special relativity is extended to cosmological special relativity.

The framework clarifies that special relativity survives in an expanding universe when expressed in instantaneous physical coordinates, that the scale factor value is relative to cosmological inertial frame, and that local field equations can be written in a form-invariant way.

Future work should include (i) inclusion of curvature $k = \pm 1$, (ii) full transformation including da terms and connection to Fermi normal coordinates, (iii) second quantization and cosmological uncertainty principle (Yi, 2025), and (iv) Friedmann dynamics compatibility.

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