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SPHERICAL AND CYLINDRICAL COORDINATES FROM THE TENSORS OF THE RELATIVITY

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Abstract: in physics the use of the spherical or cylindrical coordinates is often the only way to develop a subject at an acceptable level of difficulty, as such a use allows us to benefit from the simplifications coming from the existing symmetry (spherical or cylindrical, indeed). Here is a treatise with tensors and also a classical one, without tensors.

1-PREAMBLE ON COORDINATE SYSTEMS

-CARTESIAN COORDINATES:

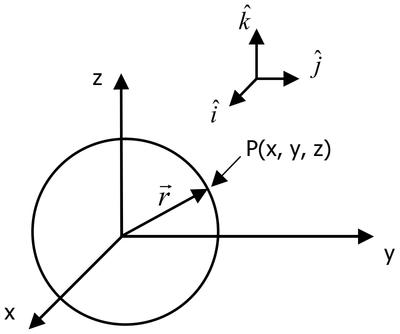


Fig. 1.1

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad , \tag{1.1}$$

$$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k} \quad , \tag{1.2}$$

V is a scalar and $\vec{F}$ is a vector. The divergence gives a scalar, while the gradient and the curl give a vector.

$$gradV = \nabla V = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}) \tag{1.3}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) \tag{1.4}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{j}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{1.5}$$

$$\Delta V = \nabla^2 V = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})V \tag{1.6}$$

-SPHERICAL COORDINATES:

We go from (x,y,z) to (ρ,θ,φ). φ spans from 0 to 2π, while θ spans between 0 and π.

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\theta)\hat{\theta} + (\rho\sin\theta d\varphi)\hat{\varphi} \tag{1.7}$$

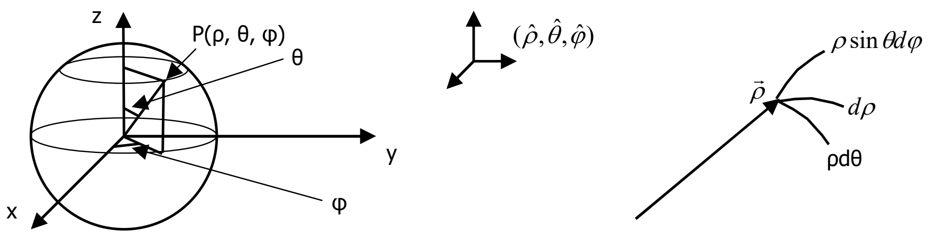


Fig. 1.2

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \tag{1.8}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}) \tag{1.9}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho^2 \sin\theta}[\frac{\partial}{\partial \rho}(\rho^2 \sin\theta F_\rho) + \frac{\partial}{\partial \theta}(\rho \sin\theta F_\theta) + \frac{\partial}{\partial \varphi}(\rho F_\varphi)] \tag{1.10}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \rho \sin\theta \hat{\varphi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_\rho & \rho F_\theta & \rho \sin\theta F_\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho^2 \sin\theta}[\frac{\partial(\rho \sin\theta F_\varphi)}{\partial \theta} - \frac{\partial(\rho F_\theta)}{\partial \varphi}]\hat{\rho} + \frac{1}{\rho \sin\theta}[\frac{\partial F_\rho}{\partial \varphi} - \frac{\partial(\rho \sin\theta F_\varphi)}{\partial \rho}]\hat{\theta} + \frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}]\hat{\varphi} \tag{1.11}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} \tag{1.12}$$

-CYLINDRICAL COORDINATES:

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$\varphi \in [0, 2\pi)$

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\varphi)\hat{\varphi} + (dz)\hat{z} \tag{1.13}$$

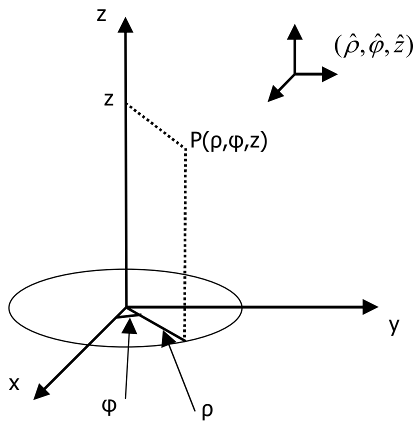


Fig. 1.3

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z}) \tag{1.14}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \varphi}(F_\varphi) + \frac{\partial}{\partial z}(\rho F_z)] \tag{1.15}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} = \frac{1}{\rho}[\frac{\partial(F_z)}{\partial \varphi} - \frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho} + [\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}]\hat{\varphi} + \frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}]\hat{\varphi} \tag{1.16}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.17}$$

2-TENSOR CALCULUS

-**derivative of a 4-vector**:

we have a 4-vector, but, in general, also just a vector: $\vec{V} = V^{\mu}\vec{\tau}_{\mu}$ (V^{μ} are the components, while the $\vec{\tau}_{\mu}$ are the versors and the summation over μ is understood, due to the Einstein's Convention, that is: $\vec{V} = V^{\mu}\vec{\tau}_{\mu} = \sum_{\mu} V^{\mu}\vec{\tau}_{\mu} = V^{x}\vec{\tau}_{x} + V^{y}\vec{\tau}_{y} + V^{z}\vec{\tau}_{z} = V^{x}\hat{x} + V^{y}\hat{y} + V^{z}\hat{z} \equiv V^{r}\hat{r} + V^{\theta}\hat{\theta} + V^{\varphi}\hat{\varphi}$).

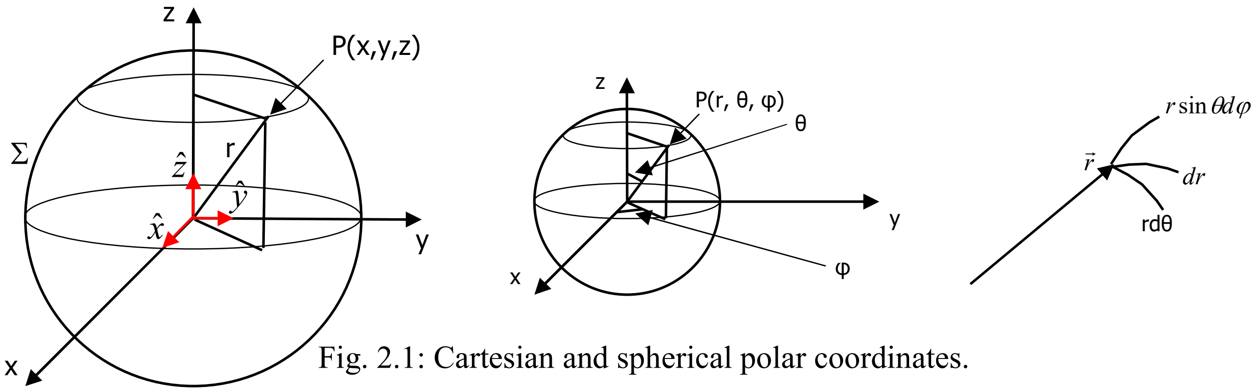


Fig. 2.1: Cartesian and spherical polar coordinates.

Now, by naming, more conveniently, the versors in the following way: $(\hat{x}, \hat{y}, \hat{z}) = (\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{e}_x, \hat{e}_y, \hat{e}_z)$ for the Cartesians and $(\hat{r}, \hat{\theta}, \hat{\varphi}) = (\hat{e}_r, \hat{e}_{\theta}, \hat{e}_{\varphi})$ for the sphericals, and also the components in this way: $(V^x, V^y, V^z) = (V^1, V^2, V^3)$ for the Cartesians and $(V^r, V^{\theta}, V^{\varphi})$ for the sphericals (which are $[dr, rd\theta, (r\sin\theta)d\varphi]$ in case we refer to the position vector $d\vec{P}$), we have:

-**for the Cartesians**:

$\vec{V} = V^{x}\hat{x} + V^{y}\hat{y} + V^{z}\hat{z} = (\hat{e}_1 V^1 + \hat{e}_2 V^2 + \hat{e}_3 V^3) = \sum_{i=1}^{3} \hat{e}_i V^i = \hat{e}_i V^i$, that is:

$$\vec{V} = \hat{e}_i V^i \quad . \tag{2.1}$$

Then, if the vector is the de position vector $d\vec{P}$, then we have:

$d\vec{P} = dx\hat{x} + dy\hat{y} + dz\hat{z} = dr\hat{r} + rd\theta\hat{\theta} + d\varphi(r\sin\theta)\hat{\varphi} = (\hat{e}_1 dx^1 + \hat{e}_2 dx^2 + \hat{e}_3 dx^3) = \sum_{i=1}^{3} \hat{e}_i dx^i = \hat{e}_i dx^i$, so:

$d\vec{P} = \hat{e}_i dx^i$. Now, we make the derivative of the vector (2.1) ($\vec{V} = \hat{e}_{\mu} V^{\mu}$) with respect to a coordinate x^{ν}:

$$\frac{d\vec{V}}{dx^{\nu}} = \frac{d(\hat{e}_{\mu}V^{\mu})}{dx^{\nu}} = \frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu} + V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}; \tag{2.2}$$

the indices change from one term to another (μ→σ), as they change over all the values (there is an understood summation over repeated indices) and they do not necessarily have to be simultaneously the same in both terms.

In case of (flat) Cartesian coordinates, the derivative is still $\frac{d\vec{V}}{dx^{\nu}} = \frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu}$, as, being this about Euclidean geometry, then the versors $(\hat{x}, \hat{y}, \hat{z})$, that are the $\hat{e}_i$, are fixed and so their derivatives are null ($\frac{d\hat{e}_{\sigma}}{dx^{\nu}} = 0$). A different situation is that of a non Euclidean geometry.

Now, we multiply the right side of the (2.2) by the unit quantity $\hat{e}^{\mu}\hat{e}_{\mu}$, where $\hat{e}^{\mu}$ is the contravariant tensor, which obviously exists, so that its product with the covariant one ($\hat{e}_{\mu}$) yields, by definition, the unit, indeed:

$$\frac{d\vec{V}}{dx^{\nu}}=\frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu}+V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}=(\frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu}+V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}})\hat{e}^{\mu}\hat{e}_{\mu}=\frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu}\hat{e}^{\mu}\hat{e}_{\mu}+V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}\hat{e}^{\mu}\hat{e}_{\mu}=[\frac{dV^{\mu}}{dx^{\nu}}+V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}\hat{e}^{\mu}]\hat{e}_{\mu}$$

well, the quantity between the square brackets is the covariant derivative and it's a tensor:

$$V^{\mu}_{;\nu}=\frac{dV^{\mu}}{dx^{\nu}}+V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}\hat{e}^{\mu}=\frac{dV^{\mu}}{dx^{\nu}}+V^{\sigma}\Gamma^{\mu}_{\sigma\nu} \qquad \text{(covariant derivative)} \qquad (2.3)$$

where the $\Gamma^{\mu}_{\sigma\nu}=\frac{d\hat{e}_{\sigma}}{dx^{\nu}}\hat{e}^{\mu}$ (2.4)

are the Christoffel symbols (affine connection).

In case of a vector which is expressed on the reciprocal base ($\vec{V}=V_{\mu}\vec{\tau}^{\mu}$), or better: $\vec{V}=\hat{e}^{\mu}V_{\mu}$, we have: $\frac{d\vec{V}}{dx^{\nu}}=\frac{d(\hat{e}^{\mu}V_{\mu})}{dx^{\nu}}=\frac{dV_{\mu}}{dx^{\nu}}\hat{e}^{\mu}+V_{\sigma}\frac{d\hat{e}^{\sigma}}{dx^{\nu}}$; now, we multiply the right side by the unit quantity $\hat{e}_{\mu}\hat{e}^{\mu}$ where $\hat{e}_{\mu}$ is the covariant versor, which obviously exists, so that its product with the contravariant one ($\hat{e}^{\mu}$) yields, by definition, the unit, indeed:

$$\frac{d\vec{V}}{dx^{\nu}}=\frac{dV_{\mu}}{dx^{\nu}}\hat{e}^{\mu}+V_{\sigma}\frac{d\hat{e}^{\sigma}}{dx^{\nu}}=(\frac{dV_{\mu}}{dx^{\nu}}\hat{e}^{\mu}+V_{\sigma}\frac{d\hat{e}^{\sigma}}{dx^{\nu}})\hat{e}_{\mu}\hat{e}^{\mu}=\frac{dV_{\mu}}{dx^{\nu}}\hat{e}^{\mu}\hat{e}_{\mu}\hat{e}^{\mu}+V_{\sigma}\frac{d\hat{e}^{\sigma}}{dx^{\nu}}\hat{e}_{\mu}\hat{e}^{\mu}=[\frac{dV_{\mu}}{dx^{\nu}}+V_{\sigma}\frac{d\hat{e}^{\sigma}}{dx^{\nu}}\hat{e}_{\mu}]\hat{e}^{\mu},$$

but: $\frac{\partial}{\partial x^{\nu}}(\hat{e}^{\sigma}\hat{e}_{\mu})=\frac{\partial}{\partial x^{\nu}}\delta^{\sigma}_{\mu}=0=\frac{\partial\hat{e}^{\sigma}}{\partial x^{\nu}}\hat{e}_{\mu}+\hat{e}^{\sigma}\frac{\partial\hat{e}_{\mu}}{\partial x^{\nu}}$, so: $\frac{\partial\hat{e}^{\sigma}}{\partial x^{\nu}}\hat{e}_{\mu}=-\hat{e}^{\sigma}\frac{\partial\hat{e}_{\mu}}{\partial x^{\nu}}$, and then:

$$\frac{d\vec{V}}{dx^{\nu}}=[\frac{dV_{\mu}}{dx^{\nu}}-V_{\sigma}\hat{e}^{\sigma}\frac{\partial\hat{e}_{\mu}}{\partial x^{\nu}}]\hat{e}^{\mu}=[\frac{dV_{\mu}}{dx^{\nu}}-V_{\sigma}\Gamma^{\sigma}_{\mu\nu}]\hat{e}^{\mu}=[V_{\mu;\nu}]\hat{e}^{\mu}$$, therefore:

$$V_{\mu;\nu}=\frac{dV_{\mu}}{dx^{\nu}}-V_{\sigma}\Gamma^{\sigma}_{\mu\nu} \ .$$

So we can state that: $\Gamma^{k}_{ij}=0 \ \longrightarrow$ no curvature.

-<u>derivative of a tensor</u>:

we have that: $T^{\mu\sigma}_{\lambda;\rho}=\frac{\partial}{\partial x^{\rho}}T^{\mu\sigma}_{\lambda}+\Gamma^{\mu}_{\rho\nu}T^{\nu\sigma}_{\lambda}+\Gamma^{\sigma}_{\rho\nu}T^{\mu\sigma}_{\lambda}-\Gamma^{k}_{\lambda\rho}T^{\mu\sigma}_{k}$ (2.5)

in fact ($T^{\mu\sigma}_{\lambda}$ are the components, without "versors" and moreover $\hat{e}_{\mu}\hat{e}^{\mu}=\delta^{\mu}_{\mu}=1$):

T(tensor)= $T^{\mu\sigma}_{\lambda}\hat{e}_{\mu}\hat{e}_{\sigma}\hat{e}^{\lambda}$, so:

$$\frac{\partial}{\partial x^{\rho}}(T^{\mu\sigma}_{\lambda}\hat{e}_{\mu}\hat{e}_{\sigma}\hat{e}^{\lambda})=(\frac{\partial}{\partial x^{\rho}}T^{\mu\sigma}_{\lambda})(\hat{e}_{\mu}\hat{e}_{\sigma}\hat{e}^{\lambda})+T^{\nu\sigma}_{\lambda}(\frac{\partial}{\partial x^{\rho}}\hat{e}_{\nu})\hat{e}_{\sigma}\hat{e}^{\lambda}(\hat{e}_{\mu}\hat{e}^{\mu})+T^{\mu\nu}_{\lambda}(\frac{\partial}{\partial x^{\rho}}\hat{e}_{\nu})\hat{e}_{\mu}\hat{e}^{\lambda}(\hat{e}_{\sigma}\hat{e}^{\sigma})+$$

$$-T^{\mu\sigma}_{k}(\frac{\partial}{\partial x^{\rho}}\hat{e}_{\mu})\hat{e}^{k}\hat{e}_{\sigma}(\hat{e}_{\lambda}\hat{e}^{\lambda})=\frac{\partial}{\partial x^{\rho}}T^{\mu\sigma}_{\lambda}+\Gamma^{\mu}_{\rho\nu}T^{\nu\sigma}_{\lambda}+\Gamma^{\sigma}_{\rho\nu}T^{\mu\sigma}_{\lambda}-\Gamma^{k}_{\lambda\rho}T^{\mu\sigma}_{k}=T^{\mu\sigma}_{\lambda;\rho} \quad \text{qed.}$$

(as : $\frac{\partial}{\partial x^{\nu}}\hat{e}^{\mu}\hat{e}_{k}=\frac{\partial}{\partial x^{\nu}}\delta^{\mu}_{k}=0=\frac{\partial\hat{e}^{\mu}}{\partial x^{\nu}}\hat{e}_{k}+\hat{e}^{\mu}\frac{\partial\hat{e}_{k}}{\partial x^{\nu}}$, from which: $\frac{\partial\hat{e}^{\mu}}{\partial x^{\nu}}\hat{e}_{k}=-\hat{e}^{\mu}\frac{\partial\hat{e}_{k}}{\partial x^{\nu}}$)

If we go back to the position vector $d\vec{P}=\hat{e}_{i}dx^{i}$, whose components are ($dx^{1},dx^{2},dx^{3},(dx^{4})$) in the curvilinear base ($\hat{e}_{1},\hat{e}_{2},\hat{e}_{3},(\hat{e}_{4})$), then we said there exists a reciprocal set of four ($\hat{e}^{1},\hat{e}^{2},\hat{e}^{3},(\hat{e}^{4})$) so that, by definition: $\hat{e}_{i}\cdot\hat{e}^{j}=\delta^{j}_{i}$. Now, if we consider another base ($\hat{e}'_{1},\hat{e}'_{2},\hat{e}'_{3},(\hat{e}'_{4})$), then, being $d\vec{P}$ still the same, we will have: $d\vec{P}=\hat{e}'_{l}\,dx'^{l}$ and

$$\hat{e}_{i}dx^{i}=\hat{e}'_{l}\,dx'^{l} \ , \qquad (2.6)$$

where the coordinates go from dx^{i} to dx'^{l} .

Now, after multiplying both sides of the (2.6) by $\hat{e}^i$, we have: $dx^i = dx'^l\, \hat{e}'_l \cdot \hat{e}^i$, but we also notice that, due to the partial derivatives, we have the following obvious relation: $dx^i = \frac{\partial x^i}{\partial x'^l} dx'^l$, from which: $\frac{\partial x^i}{\partial x'^l} = \hat{e}'_l \cdot \hat{e}^i$, that is: $\hat{e}'_l = \hat{e}_i \frac{\partial x^i}{\partial x'^l}$ (2.7)

and the (2.7) is the equation for the change of base.

-<u>law of transformation for the components of a 4-vector</u>:

let $\vec{V} = V^i \hat{e}_i$ be a generic vector expressed through the curvilinear coordinates in the base $\hat{e}_i$; conversely, in the base $\hat{e}'_l$ we have: $\vec{V} = V'^l\, \hat{e}'_l$ and for the (2.7):

$$\vec{V} = V'^l\, \hat{e}'_l = V'^l \frac{\partial x^i}{\partial x'^l} \hat{e}_i = V^i \hat{e}_i \text{ , where:}$$

$$V^i = V'^l \frac{\partial x^i}{\partial x'^l} \tag{2.8}$$

which is the equation for the transformation of the components of a 4-vector after a base change. Of course its inverse expression is: $V'^i = V^l \frac{\partial x'^i}{\partial x^l}$

-<u>law of transformation for the components of a 4-tensor</u>:

on General Relativity we say that we can get a tensor T with rank n when we multiply the components of n vectors. So, if we have two vectors V and S: (where we are simultaneously reminding how their components transform)

$$V'^\mu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu} \quad \text{e} \quad S'^\lambda = S^\sigma \frac{\partial x'^\lambda}{\partial x^\sigma} \quad >>>>$$

$$T'^{\mu\lambda} = V'^\mu S'^\lambda = V^\nu \frac{\partial x'^\mu}{\partial x^\nu} S^\sigma \frac{\partial x'^\lambda}{\partial x^\sigma} = V^\nu S^\sigma \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} = \boxed{T^{\nu\sigma} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} = T'^{\mu\lambda}} \tag{2.9}$$

and this is how the components of a rank 2 tensor transform. Then, you can proceed similarly for higher rank tensors.

Well, now, from Fig. 2.1 we notice that for a sphere (so we are into three dimensions) the square magnitude dP^2 of the position vector is (as $d\vec{P} = \hat{e}_i dx^i$):

$$dP^2 = (\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = (\hat{e}_\mu \hat{e}_\nu) dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \text{ ,} \tag{2.10}$$

and through the jump to the spherical polar coordinates: $\mu,\nu \to r,\theta,\varphi$

$$dP^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{rr} dx^r dx^r + g_{\theta\theta} dx^\theta dx^\theta + g_{\varphi\varphi} dx^\varphi dx^\varphi + \cancel{2g_{\theta\varphi} dx^\theta dx^\varphi} + \cancel{2g_{r\theta} dx^r dx^\theta} + \cancel{2g_{r\varphi} dx^r dx^\varphi} =$$

$= g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$ (a simple use of the Pythagorean Theorem on a sphere) and here we see that we have no mixed terms (such as $d\theta d\varphi$), then the metric tensor g is diagonal, then all the mixed terms are null ($g_{\theta\varphi} = g_{\varphi\theta} = g_{r\theta} = g_{\theta r} = g_{r\varphi} = g_{\varphi r} = 0$) and so, being that:

$$g_{\mu\nu}g^{\mu\nu}=1 \rightarrow g_{\mu\nu}=\frac{1}{g^{\mu\nu}} \rightarrow g_{rr}=1 \text{ (and } g^{rr}=\frac{1}{g_{rr}}=1), \; g_{\theta\theta}=r^2 \text{ (and } g^{\theta\theta}=\frac{1}{r^2}),$$

$$g_{\varphi\varphi}=r^2\sin^2\theta \text{ (and } g^{\varphi\varphi}=\frac{1}{r^2\sin^2\theta}) \text{ and} \qquad (2.11)$$

$$g_{\theta\varphi}=g_{\varphi\theta}=g^{\theta\varphi}=g^{\varphi\theta}=g_{r\theta}=g_{\theta r}=g^{r\theta}=g^{\theta r}=g_{r\varphi}=g_{\varphi r}=g^{r\varphi}=g^{\varphi r}=0.$$

and being that $g_{\mu\nu}=\hat{e}_\mu\hat{e}_\nu$ and $g^{\mu\nu}=\hat{e}^\mu\hat{e}^\nu$, it follows that:

$$\hat{e}_x=1 \;,\; \hat{e}_y=1 \;,\; \hat{e}_z=1 \;,\; \hat{e}^x=1 \;,\; \hat{e}^y=1 \;,\; \hat{e}^z=1 \;; \qquad (2.12)$$

$$\hat{e}_r=1 \;,\; \hat{e}_\theta=r \;,\; \hat{e}_\varphi=r\sin\theta \;,\; \hat{e}^r=1 \;,\; \hat{e}^\theta=\frac{1}{r} \;,\; \hat{e}^\varphi=\frac{1}{r\sin\theta} \;. \qquad (2.13)$$

Now, let's go back to the (2.4), that is: $\Gamma^\mu_{\sigma\nu}=\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu$ and to the (2.10), that is: $dP^2=g_{\mu\nu}dx^\mu dx^\nu$; well, such a (2.10) has a general validity for a curved space, while for a fully flat space (of Minkowski), it gets like this: $dP^2=\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$, where $\eta_{\alpha\beta}=(1,1,1)$, as in a flat space we have:
$dP^2=\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta=(d\xi^1)^2+(d\xi^2)^2+(d\xi^3)^2=1\cdot(dx)^2+1\cdot(dy)^2+1\cdot(dz)^2=dx^2+dy^2+dz^2$.
Now, by expressing a generic versor $\hat{e}_i$ (curvilinear) as a function of the versor of the flat geometry $(\hat{i},\hat{j},\hat{k})$, then we obviously have: $\hat{e}_i=\frac{\partial\xi^1}{\partial x^i}\hat{i}+\frac{\partial\xi^2}{\partial x^i}\hat{j}+\frac{\partial\xi^3}{\partial x^i}\hat{k}=(\sum_\alpha)\eta_\alpha\frac{\partial\xi^\alpha}{\partial x^i}=\eta_\alpha\frac{\partial\xi^\alpha}{\partial x^i}$ and we will also have a dual form for the reciprocal set of three $\hat{e}^j$ (so that, by def. of reciprocity: $\hat{e}^j\hat{e}_j=\delta^j_i$): $\hat{e}^j=(\sum_\alpha)\eta^\alpha\frac{\partial x^j}{\partial\xi^\alpha}=\eta^\alpha\frac{\partial x^j}{\partial\xi^\alpha}$ and so the coefficient $\Gamma^\mu_{\nu\sigma}$ $(=\Gamma^\mu_{\sigma\nu})$ of the (2.4) can be written also in the following way: $\Gamma^\mu_{\nu\sigma}=\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu=\frac{d}{dx^\nu}(\eta_\alpha\frac{\partial\xi^\alpha}{\partial x^\sigma})\eta^\alpha\frac{\partial x^\mu}{\partial\xi^\alpha}=\eta_\alpha\eta^\alpha\frac{\partial^2\xi^\alpha}{\partial x^\sigma\partial x^\nu}\frac{\partial x^\mu}{\partial\xi^\alpha}$, and the last one can be also written in a simpler way, withouth the unit coefficients η:

$$\Gamma^\mu_{\nu\sigma}=\frac{\partial^2\xi^\alpha}{\partial x^\sigma\partial x^\nu}\frac{\partial x^\mu}{\partial\xi^\alpha} \;. \qquad (2.14)$$

-**<u>the relation between</u>** $g_{\mu\nu}$ **<u>and</u>** $\Gamma^\lambda_{\mu\nu}$:

by equalling the metric tensor (2.10) of the curvilinear case to that for the case of a flat space, we have: $dP^2=g_{\mu\nu}dx^\mu dx^\nu=\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$, from which: $g_{\mu\nu}=\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}$; now, we use, on the last one, this $\frac{\partial}{\partial x^\lambda}$: $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}=\frac{\partial^2\xi^\alpha}{\partial x^\lambda\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}+\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu}\eta_{\alpha\beta}$; now, after reminding that, due to the (2.14): $\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}=\Gamma^\lambda_{\mu\nu}\frac{\partial\xi^\alpha}{\partial x^\lambda}$, it follows that:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda}=\Gamma^\rho_{\lambda\mu}\frac{\partial\xi^\alpha}{\partial x^\rho}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}+\Gamma^\rho_{\lambda\nu}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\alpha}{\partial x^\rho}\eta_{\alpha\beta}=\Gamma^\rho_{\lambda\mu}g_{\rho\nu}+\Gamma^\rho_{\lambda\nu}g_{\rho\mu} \qquad (2.15)$$

Similarly, then we also calculate $\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}$ and $\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}$, from which: $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}=2g_{k\nu}\Gamma^{k}_{\lambda\mu}$;
now, after defining a reciprocal matrix (or tensor) $g^{\nu\sigma}$ so that: $g^{\nu\sigma}g_{k\nu}=\delta^{\sigma}_{k}$, then we will have:

$$\Gamma^{\sigma}_{\lambda\mu}=\Gamma^{\sigma}_{\mu\lambda}=\frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right). \tag{2.16}$$

Now, in the (2.16) we put σ=μ , from which: $\quad (g_{AB}=g_{BA}$ and $g^{AB}=g^{BA})$

$$\Gamma^{\mu}_{\mu\lambda}=\frac{1}{2}g^{\nu\mu}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right)=\frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}+\frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}\right)=\frac{1}{2}g^{\mu\nu}\frac{\partial g_{\nu\mu}}{\partial x^{\lambda}}=\frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}}. \tag{2.17}$$

Now we notice that: $\frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}}=\frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^{\lambda}}=\frac{1}{2}\frac{\partial}{\partial x^{\lambda}}\ln g$; (2.18)

where g is the determinant of the matrix $g_{\rho\mu}$; in fact, $g_{\rho\mu}=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$ and $g^{\rho\mu}=\begin{pmatrix} g^{11} & g^{12}\\ g^{21} & g^{22}\end{pmatrix}$ (if we consider the simple case of a 2x2 matrix)
and according to the formulas for the inversion of a matrix, we have:

$g^{11}=g_{22}/g$, $g^{12}=g^{21}=-g_{12}/g=-g_{21}/g$,

$g^{22}=g_{11}/g$; $|g_{\rho\mu}|=\begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}=g_{11}g_{22}-g_{12}g_{21}=g_{11}g_{22}-(g_{12})^{2}$,

$\frac{\partial g}{\partial x^{\lambda}}=\frac{\partial}{\partial x^{\lambda}}[g_{11}g_{22}-(g_{12})^{2}]=\frac{\partial g_{11}}{\partial x^{\lambda}}g_{22}+\frac{\partial g_{22}}{\partial x^{\lambda}}g_{11}-2g_{12}\frac{\partial g_{12}}{\partial x^{\lambda}}=g[g^{11}\frac{\partial g_{11}}{\partial x^{\lambda}}+g^{22}\frac{\partial g_{22}}{\partial x^{\lambda}}+2g^{12}\frac{\partial g_{12}}{\partial x^{\lambda}}]=$

$=gg^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}}$; so, according to the (2.18), the (2.17) becomes:

$\Gamma^{\mu}_{\mu\lambda}=\frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}}=\frac{1}{2}\frac{\partial}{\partial x^{\lambda}}\ln g=\frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^{\lambda}}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\lambda}}\sqrt{g}$. The involvement of the determinant g will turn out to be useful as when we let the classical expressions stand (without g), then we would have to calculate directly several Christoffel's symbols, which, conversely, by the expressions with g, can be no longer calculated, as they turn to be buried.

Now, we consider the (2.3): $V^{\mu}_{;\nu}=\frac{dV^{\mu}}{dx^{\nu}}+V^{\sigma}\Gamma^{\mu}_{\sigma\nu}$ and we inspect it on the specific case of μ=ν :

$V^{\mu}_{;\mu}=\frac{dV^{\mu}}{dx^{\mu}}+V^{\sigma}\Gamma^{\mu}_{\sigma\mu}$ $\quad(\Gamma^{\mu}_{\sigma\mu}=\Gamma^{\mu}_{\mu\sigma})$ and being σ a silent/dummy index, then we can change its name (λ , as an example), so:

$$V^{\mu}_{;\mu}=\frac{\partial V^{\mu}}{\partial x^{\mu}}+\Gamma^{\mu}_{\mu\lambda}V^{\lambda} \tag{2.19}$$

which, due to the (2.18), becomes:

$$V^{\mu}_{;\mu}=\frac{\partial V^{\mu}}{\partial x^{\mu}}+\Gamma^{\mu}_{\mu\lambda}V^{\lambda}=\frac{\partial V^{\mu}}{\partial x^{\mu}}+\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\lambda}}\sqrt{g}\right)V^{\lambda}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\lambda}}(\sqrt{g}V^{\lambda}). \tag{2.20}$$

At last, out of completeness, regarding the derivative of a rank 2 tensor:

$T^{\mu\nu}_{;\lambda}=\frac{\partial}{\partial x^{\lambda}}T^{\mu\nu}+\Gamma^{\mu}_{\lambda k}T^{k\nu}+\Gamma^{\nu}_{\lambda k}T^{\mu k}$, (generic) and with λ=μ , we have: (2.21)

$$T^{\mu\nu}_{;\mu}=\frac{\partial}{\partial x^{\mu}}T^{\mu\nu}+\Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu}+\Gamma^{\mu}_{\mu\lambda}T^{\mu\lambda}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}T^{\mu\nu})+\Gamma^{\mu}_{\mu\lambda}T^{\mu\lambda}\quad. \tag{2.22}$$

-**<u>the Geodesic Equation</u>**:

A free falling parachutist does not have any floor over which his body can rest; therefore, he does not detect any gravitational acceleration and he feels as if floating in the vacuum. He realizes he is falling only if he looks at the moving objects around. Therefore, for a free falling particle, there is a reference system in which $\vec{a}=0$, that is:

$$\frac{\partial^2\xi^\alpha}{\partial\tau^2}=0 \quad \text{(free falling in Euclidean coordinates)} \tag{2.23}$$

with $d\tau^2=-\eta_{ik}d\xi^i d\xi^k$. But we can see the (2.23) in the following way:

$$\frac{\partial^2\xi^\alpha}{\partial\tau^2}=0=\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right)=\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2}+\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

If we multiply both sides by $\frac{\partial x^\lambda}{\partial\xi^\alpha}$ and consider that: $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha}=\delta^\lambda_\mu$, we'll have:

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \quad \text{(free falling in curvilinear coordinates)} \tag{2.24}$$

(equation of the geodesic, where geodesic, on the Earth, is the shortest path between two places)

with $\Gamma^\lambda_{\mu\nu}=\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\lambda}{\partial\xi^\alpha}$ indeed.

Now we go on with the reasoning and in order to find the equation of the transformation of the affine connection, we write again the equation of the affine connection in a different reference frame: $\Gamma'^\lambda_{\mu\nu}=\frac{\partial^2\xi^\alpha}{\partial x'^\mu\partial x'^\nu}\frac{\partial x'^\lambda}{\partial\xi^\alpha}$, from which:

$$\Gamma'^\lambda_{\mu\nu}=\frac{\partial x'^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x'^\mu\partial x'^\nu}=\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial\xi^\alpha}{\partial x^\sigma}\right)=\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial\xi^\alpha}\left[\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial^2\xi^\alpha}{\partial x^\tau\partial x^\sigma}+\frac{\partial^2x^\sigma}{\partial x'^\mu\partial x'^\nu}\frac{\partial\xi^\alpha}{\partial x^\sigma}\right]=$$

$$=\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}+\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2x^\rho}{\partial x'^\mu\partial x'^\nu}\ ; \tag{2.25}$$

now, we differentiate $\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu}=\delta^\lambda_\nu$: $\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2x^\rho}{\partial x'^\mu\partial x'^\nu}=-\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2x'^\lambda}{\partial x^\rho\partial x^\sigma}$, from which:

$$\Gamma'^\lambda_{\mu\nu}=\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}-\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2x'^\lambda}{\partial x^\rho\partial x^\sigma}\ , \tag{2.26}$$

which is the equation of the transformation of the affine connection, indeed; we see that it doesn't transform as a tensor, like the (2.8) $V^i=V'^l\frac{\partial x^i}{\partial x'^l}$, so it is not exactly a tensor. By the way, regarding the (2.8), or better its inverse: $V'^\mu=V^\nu\frac{\partial x'^\mu}{\partial x^\nu}$, we have, by differentiating with respect to x'^λ :

$$\frac{\partial V'^\mu}{\partial x'^\lambda}=\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial V^\nu}{\partial x^\rho}+\frac{\partial^2x'^\mu}{\partial x^\nu\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu; \tag{2.27}$$

now, we legitimately write again the (2.8) with the new indices: $V'^\kappa=V^\eta\frac{\partial x'^\kappa}{\partial x^\eta}$ and we multiply both sides by $\Gamma'^\mu_{\lambda\kappa}$: $\Gamma'^\mu_{\lambda\kappa}V'^\kappa=\Gamma'^\mu_{\lambda\kappa}V^\eta\frac{\partial x'^\kappa}{\partial x^\eta}$, but the (2.26) with its indices consistently readapted to provide $\Gamma'^\mu_{\lambda\kappa}$ indeed, provides:

$$\Gamma'^\mu_{\lambda\kappa}V'^\kappa=\left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\kappa}\Gamma^\nu_{\rho\sigma}-\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\kappa}\frac{\partial^2x'^\mu}{\partial x^\rho\partial x^\sigma}\right)V^\eta\frac{\partial x'^\kappa}{\partial x^\eta}=\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma}V^\sigma-\frac{\partial^2x'^\mu}{\partial x^\rho\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\lambda}V^\sigma \tag{2.28}$$

Now, by summing side to side the (2.27) and the (2.28), we get:
$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}}+\Gamma'^{\mu}_{\lambda\kappa}V'^{\kappa}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\lambda}}(\frac{\partial V^{\nu}}{\partial x^{\rho}}+\Gamma^{\nu}_{\rho\sigma}V^{\sigma})$ which tells us that the covariant derivative (2.3) is a tensor, indeed, as it transforms as a tensor.
Out of completeness, we repeat everything about the contravariant derivative: by starting from

$$V'_{\mu}=V_{\rho}\frac{\partial x^{\rho}}{\partial x'^{\mu}} \qquad (2.29)$$

and by differentiating with respect to x'^{ν} : $\frac{\partial V'_{\mu}}{\partial x'^{\nu}}=\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial V_{\rho}}{\partial x^{\sigma}}+\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}}V_{\rho}$; (2.30)

now, the (2.29) in λ yields: $V'_{\lambda}=V_{\kappa}\frac{\partial x^{\kappa}}{\partial x'^{\lambda}}$ and by multiplying side to side by $\Gamma'^{\lambda}_{\mu\nu}$:

$\Gamma'^{\lambda}_{\mu\nu}V'_{\lambda}=\Gamma'^{\lambda}_{\mu\nu}V_{\kappa}\frac{\partial x^{\kappa}}{\partial x'^{\lambda}}$, but $\Gamma'^{\lambda}_{\mu\nu}$ yields, due to the (2.25) with its indices readapted for the case:

$$\Gamma'^{\lambda}_{\mu\nu}V'_{\lambda}=(\frac{\partial x'^{\lambda}}{\partial x^{\tau}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\tau}_{\rho\sigma}+\frac{\partial x'^{\lambda}}{\partial x^{\tau}}\frac{\partial^2 x^{\tau}}{\partial x'^{\mu}\partial x'^{\nu}})V_{\kappa}\frac{\partial x^{\kappa}}{\partial x'^{\lambda}}=\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\kappa}_{\rho\sigma}V_{\kappa}+\frac{\partial^2 x^{\kappa}}{\partial x'^{\mu}\partial x'^{\nu}}V_{\kappa}; \qquad (2.31)$$

Now, by subtracting the (2.31) from the (2.30), we have:
$\frac{\partial V'_{\mu}}{\partial x'^{\nu}}-\Gamma'^{\lambda}_{\mu\nu}V'_{\lambda}=\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}(\frac{\partial V_{\rho}}{\partial x^{\sigma}}-\Gamma^{\kappa}_{\rho\sigma}V_{\kappa})$ q.e.d.
See also Appendix 1 for further details on tensors.

3-CHANGE OF COORDINATES

We remind that because of the (2.10), that is: $dP^2=(\hat{e}_{\mu}dx^{\mu})(\hat{e}_{\nu}dx^{\nu})=(\hat{e}_{\mu}\hat{e}_{\nu})dx^{\mu}dx^{\nu}=g_{\mu\nu}dx^{\mu}dx^{\nu}$, we have that: $(\hat{e}_i\hat{e}_i)=g_{ii}=(\hat{e}_i)^2$, from which:
$\hat{e}_i=\sqrt{g_{ii}}$ and by the same token, regarding the reciprocal set of three:
$dP^2=(\hat{e}^{\mu}dx_{\mu})(\hat{e}^{\nu}dx_{\nu})=(\hat{e}^{\mu}\hat{e}^{\nu})dx_{\mu}dx_{\nu}=g^{\mu\nu}dx_{\mu}dx_{\nu}\;\rightarrow\;(\hat{e}^i\hat{e}^i)=g^{ii}=(\hat{e}^i)^2$, from which:

$$\hat{e}^i=\sqrt{g^{ii}} \qquad (3.1)$$

while, due to the definition itself of vector $\vec{V}$, we have:

$$\vec{V}=V_i\hat{e}^i=V_i\sqrt{g^{ii}}\,\hat{i}\,(=V^i\hat{e}_i=V^i\sqrt{g_{ii}}\,\hat{i}) , \qquad (3.2)$$

-<u>the GRADIENT with tensors</u>:

let $V=V(x^i)=V(x^1,x^2,\ldots,x^n)$ be a scalar; $V=V(x,y,z)$ in the Cartesian case, and $V=V(\rho,\theta,\varphi)$ in the spherical one; so:

$$gradV=\nabla V=\hat{e}^i\frac{\partial V}{\partial x^i}=\hat{e}^i(\nabla V)_i(=\hat{e}^1\frac{\partial V}{\partial x^1}+\hat{e}^2\frac{\partial V}{\partial x^2}+\ldots+\hat{e}^n\frac{\partial V}{\partial x^n})(\equiv\hat{x}\frac{\partial V}{\partial x}+\hat{y}\frac{\partial V}{\partial y}+\hat{z}\frac{\partial V}{\partial z}\equiv)$$

$$(\equiv\hat{e}^r\frac{\partial V}{\partial r}+\hat{e}^{\theta}\frac{\partial V}{\partial\theta}+\hat{e}^{\varphi}\frac{\partial V}{\partial\varphi}=)\text{ (for the (3.2)) }(=\sqrt{g^{rr}}\frac{\partial V}{\partial r}\hat{r}+\sqrt{g^{\theta\theta}}\frac{\partial V}{\partial\theta}\hat{\theta}+\sqrt{g^{\varphi\varphi}}\frac{\partial V}{\partial\varphi}\hat{\varphi}=$$

$=\sqrt{1}\frac{\partial V}{\partial r}\hat{r}+\sqrt{\frac{1}{r^2}}\frac{\partial V}{\partial\theta}\hat{\theta}+\sqrt{\frac{1}{r^2\sin^2\theta}}\frac{\partial V}{\partial\varphi}\hat{\varphi}=\frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}\hat{\varphi})$ and so we know that the gradient operator (on a scalar, of course) provides a vector and we have also obtained the expressions for the Cartesian gradient and for the spherical one:

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}}$$ (that is the (1.3))

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}}$$ (that is the (1.9))

At last, if we consider the cylindrical case, then with reference to Fig. 1.3, we have:
($d\vec{P} = \hat{e}_i dx^i$) , $dP^2 = (\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = (\hat{e}_\mu \hat{e}_\nu)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu$ and by passing to the cylindricals: $\mu,\nu \to r,\varphi,z$:

$$dP^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{rr}dx^r dx^r + g_{\varphi\varphi}dx^\varphi dx^\varphi + g_{zz}dx^z dx^z + \cancel{2g_{\varphi z}dx^\varphi dx^z} + \cancel{2g_{r\varphi}dx^r dx^\varphi} + \cancel{2g_{rz}dx^r dx^z} =$$

$= g_{rr}dr^2 + g_{\varphi\varphi}d\varphi^2 + g_{zz}dz^2 = dr^2 + r^2 d\varphi^2 + dz^2$, from which:

$$\boxed{g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \ , \ g_{zz} = 1 \quad (\sqrt{g} = \sqrt{1\cdot r^2 \cdot 1} = r) \quad \text{and} \quad g^{rr} = 1 \ , \ g^{\varphi\varphi} = \frac{1}{r^2} \ , \ g^{zz} = 1 \ .}$$

Therefore:

$$gradV = \nabla V = \hat{e}^i \frac{\partial V}{\partial x^i} = \hat{e}^i(\nabla V)_i \ (= \hat{e}^1\frac{\partial V}{\partial x^1} + \hat{e}^2\frac{\partial V}{\partial x^2} + \ldots + \hat{e}^n\frac{\partial V}{\partial x^n}) \ (\equiv \hat{e}^r\frac{\partial V}{\partial r} + \hat{e}^\varphi\frac{\partial V}{\partial \varphi} + \hat{e}^z\frac{\partial V}{\partial z} =)$$

(for the (3.2)) $(= \sqrt{g^{rr}}\frac{\partial V}{\partial r}\hat{r} + \sqrt{g^{\varphi\varphi}}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \sqrt{g^{zz}}\frac{\partial V}{\partial z}\hat{z} =$

$= \sqrt{1}\frac{\partial V}{\partial r}\hat{r} + \sqrt{\frac{1}{r^2}}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \sqrt{1}\frac{\partial V}{\partial z}\hat{z} = \boxed{\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z} = \nabla V \ )}$, that is exactly the (1.14)!!!

-**<u>the DIVERGENCE with tensors</u>**:

now, we start from the (2.20): $V^\mu_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}(\sqrt{g}V^\lambda)$, where the double λ on the right side, due to the Einstein Summation Convention, has the task of forcing the summation, therefore it can easily become a μ, it too, from which:

$$V^\mu_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}(\sqrt{g}V^\mu) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^1) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^2) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^3) \qquad (3.3)$$

and we also remind, about the Cartesian coordinates, that, as already shown previously:

$$dP^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 = 1\cdot(dx)^2 + 1\cdot(dy)^2 + 1\cdot(dz)^2 = dx^2 + dy^2 + dz^2$$

from which: $|g_{\mu\nu}| = |\eta_{\mu\nu}| = \begin{vmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zx} & \eta_{zy} & \eta_{zz} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & 0 & 0 \\ 0 & \eta_{yy} & 0 \\ 0 & 0 & \eta_{zz} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = g$,

while regarding the case of the spherical coordinates, for the (2.11), we have:

$$|g_{\mu\nu}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} g_{rr} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta = g$$

and then the (3.3) yields:

-**<u>for the Cartesian ones</u>**: $V^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^1)+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^2)+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^3)=$

$=\frac{1}{\sqrt{1}}\frac{\partial}{\partial x}(\sqrt{1}V^x)+\frac{1}{\sqrt{1}}\frac{\partial}{\partial y}(\sqrt{1}V^y)+\frac{1}{\sqrt{1}}\frac{\partial}{\partial z}(\sqrt{1}V^z)=\boxed{V^{\mu}_{;\mu}=\frac{\partial}{\partial x}V^x+\frac{\partial}{\partial y}V^y+\frac{\partial}{\partial z}V^z=\vec{\nabla}\cdot\vec{V}\,,}$ that is the classical Cartesian divergence already provided through the (1.4)!

-**<u>for the spherical ones</u>**: $V^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^{1-sph})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^{2-sph})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^{3-sph})=$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})]; \qquad (3.4)$$

now, for the (3.2), we have:

$$\vec{V}=V_1\hat{e}^r+V_2\hat{e}^\theta+V_3\hat{e}^\varphi=V_i\hat{e}^i=V_i\sqrt{g^{ii}}\hat{i}=(V_r\sqrt{g^{rr}})\hat{r}+(V_\theta\sqrt{g^{\theta\theta}})\hat{\theta}+(V_\varphi\sqrt{g^{\varphi\varphi}})\hat{\varphi}=V^{1-sph}\hat{r}+V^{2-sph}\hat{\theta}+V^{3-sph}\hat{\varphi}$$

from which we have, for the (3.4):

$$V^{\mu}_{;\mu}=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r\sqrt{g^{rr}})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V_\theta\sqrt{g^{\theta\theta}})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V_\varphi\sqrt{g^{\varphi\varphi}})]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r\cdot 1)+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V_\theta\frac{1}{r})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V_\varphi\frac{1}{r\sin\theta})]=$$

$=\boxed{\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r)+\frac{\partial}{\partial\theta}(r\sin\theta\cdot V_\theta)+\frac{\partial}{\partial\varphi}(r\cdot V_\varphi)]=div\cdot\vec{V}=\vec{\nabla}\cdot\vec{V}\,,}$ which is exactly the (1.10) (spherical divergence)!

-**<u>for the cylindrical ones</u>**: $V^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^{1-cyl})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^{2-cyl})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^{3-cyl})=$

$$=\frac{1}{r}\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{1}{r}\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{1}{r}\frac{\partial}{\partial z}(rV^{3-cyl})=\frac{1}{r}[\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{\partial}{\partial z}(rV^{3-cyl})];$$

now, for the (3.2), we have:

$$\vec{V}=V_1\hat{e}^r+V_2\hat{e}^\varphi+V_3\hat{e}^z=V_i\hat{e}^i=V_i\sqrt{g^{ii}}\hat{i}=(V_r\sqrt{g^{rr}})\hat{r}+(V_\varphi\sqrt{g^{\varphi\varphi}})\hat{\varphi}+(V_z\sqrt{g^{zz}})\hat{z}=V^{1-cyl}\hat{r}+V^{2-cyl}\hat{\varphi}+V^{3-cyl}\hat{z}$$

from which we have:

$$V^{\mu}_{;\mu}=\frac{1}{r}[\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{\partial}{\partial z}(rV^{3-cyl})]=\frac{1}{r}[\frac{\partial}{\partial r}(rV_r\sqrt{g^{rr}})+\frac{\partial}{\partial\varphi}(rV_\varphi\sqrt{g^{\varphi\varphi}})+\frac{\partial}{\partial z}(rV_z\sqrt{g^{zz}})]=$$

$$=\frac{1}{r}[\frac{\partial}{\partial r}(rV_r\cdot 1)+\frac{\partial}{\partial\varphi}(rV_\varphi\frac{1}{r})+\frac{\partial}{\partial z}(rV_z\cdot 1)]=\boxed{\frac{1}{r}[\frac{\partial}{\partial r}(rV_r)+\frac{\partial}{\partial\varphi}(V_\varphi)+\frac{\partial}{\partial z}(rV_z)]=div\cdot\vec{V}=\vec{\nabla}\cdot\vec{V}\,,}$$

which is exactly the (1.15) (cylindrical divergence)!

-**LAPLACIAN with tensors**:

let $\mathrm{V}=\mathrm{V}(\mathrm{x}^{\mathrm{i}})=\mathrm{V}(\mathrm{x}^1,\mathrm{x}^2,\ldots,\mathrm{x}^{\mathrm{n}})$ be a scalar; $\mathrm{V}=\mathrm{V}(\mathrm{x},\mathrm{y},\mathrm{z})$ in the Cartesian case and $\mathrm{V}=\mathrm{V}(\rho,\theta,\varphi)$ in the spherical one; then, we know that:

$div(gradV)=\nabla\cdot(\nabla V)=\nabla^2 V=\Delta V$, which is the scalar laplacian (a scalar) of V.

But let's remind now the (3.3) for the divergence of a vector $\vec{S}$: $S^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}S^{\mu})=\vec{\nabla}\cdot\vec{S}$ and we lower the index to S^{μ} through (as per use and definition) the metric tensor: $S^{\mu}=g^{\eta\mu}S_{\eta}$, from which:

$$S^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}S^{\mu})=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}g^{\eta\mu}S_{\eta})=\vec{\nabla}\cdot\vec{S} \tag{3.5}$$

and now, in the (3.5) we will insert, as the S_{η}, the vector gradient (a vector) $(\nabla V)_{\eta}$ we found before:

$$S^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}g^{\eta\mu}S_{\eta})=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\nabla V)_{\eta}]=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})]=\nabla^2 V\ ; \tag{3.6}$$

Now, after reminding the matrices under the (3.3), that are:

$$|g_{\mu\nu}|=|\eta_{\mu\nu}|=\begin{vmatrix}\eta_{11}&\eta_{12}&\eta_{13}\\\eta_{21}&\eta_{22}&\eta_{23}\\\eta_{31}&\eta_{32}&\eta_{33}\end{vmatrix}=\begin{vmatrix}\eta_{xx}&\eta_{xy}&\eta_{xz}\\\eta_{yx}&\eta_{yy}&\eta_{yz}\\\eta_{zx}&\eta_{zy}&\eta_{zz}\end{vmatrix}=\begin{vmatrix}\eta_{xx}&0&0\\0&\eta_{yy}&0\\0&0&\eta_{zz}\end{vmatrix}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1=g$$ for the Cartesians,

and for the sphericals:

$$|g_{\mu\nu}|=\begin{vmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{vmatrix}=\begin{vmatrix}g_{rr}&g_{r\theta}&g_{r\varphi}\\g_{\theta r}&g_{\theta\theta}&g_{\theta\varphi}\\g_{\varphi r}&g_{\varphi\theta}&g_{\varphi\varphi}\end{vmatrix}=\begin{vmatrix}g_{rr}&0&0\\0&g_{\theta\theta}&0\\0&0&g_{\varphi\varphi}\end{vmatrix}=\begin{vmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2\theta\end{vmatrix}=r^4\sin^2\theta=g$$, then it follows, for the (3.6), that:

-**for the Cartesians**: $\nabla^2 V=\Delta V=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})]=\frac{1}{\sqrt{1}}\frac{\partial}{\partial x^{\mu}}[\sqrt{1}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})]=\frac{\partial}{\partial x^{\mu}}[g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})]=$

$$=\frac{\partial}{\partial x^{\mu}}[g^{1\mu}(\frac{\partial V}{\partial x^1})+g^{2\mu}(\frac{\partial V}{\partial x^2})+g^{3\mu}(\frac{\partial V}{\partial x^3})]=\frac{\partial}{\partial x^{\mu}}[g^{1\mu}(\frac{\partial V}{\partial x})+g^{2\mu}(\frac{\partial V}{\partial y})+g^{3\mu}(\frac{\partial V}{\partial z})]=$$

$$=\frac{\partial}{\partial x^1}[g^{11}(\frac{\partial V}{\partial x})+g^{21}\cancel{(\frac{\partial V}{\partial y})}+g^{31}\cancel{(\frac{\partial V}{\partial z})}]+\frac{\partial}{\partial x^2}[g^{12}\cancel{(\frac{\partial V}{\partial x})}+g^{22}(\frac{\partial V}{\partial y})+g^{32}\cancel{(\frac{\partial V}{\partial z})}]+$$

$$+\frac{\partial}{\partial x^3}[g^{13}\cancel{(\frac{\partial V}{\partial x})}+g^{23}\cancel{(\frac{\partial V}{\partial y})}+g^{33}(\frac{\partial V}{\partial z})]=\frac{\partial}{\partial x}[g^{11}(\frac{\partial V}{\partial x})]+\frac{\partial}{\partial y}[g^{22}(\frac{\partial V}{\partial y})]+\frac{\partial}{\partial z}[g^{33}(\frac{\partial V}{\partial z})]=$$

$$=g^{11}(\frac{\partial^2 V}{\partial x^2})+g^{22}(\frac{\partial^2 V}{\partial y^2})+g^{33}(\frac{\partial^2 V}{\partial z^2})=\boxed{(\frac{\partial^2 V}{\partial x^2})+(\frac{\partial^2 V}{\partial y^2})+(\frac{\partial^2 V}{\partial z^2})=\Delta V}$$!!!! (that is the (1.6))

-**for the sphericals**:

$$\nabla^2 V=\Delta V=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x^\mu}[\sqrt{r^4\sin^2\theta}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\mu}[r^2\sin\theta\cdot g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\mu}[r^2\sin\theta\cdot g^{1\mu}(\frac{\partial V}{\partial x^1})+r^2\sin\theta\cdot g^{2\mu}(\frac{\partial V}{\partial x^2})+r^2\sin\theta\cdot g^{3\mu}(\frac{\partial V}{\partial x^3})]=$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\mu}[r^2\sin\theta\cdot g^{r\mu}(\frac{\partial V}{\partial r})+r^2\sin\theta\cdot g^{\theta\mu}(\frac{\partial V}{\partial \theta})+r^2\sin\theta\cdot g^{\varphi\mu}(\frac{\partial V}{\partial \varphi})]=$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^r}[r^2\sin\theta\cdot g^{rr}(\frac{\partial V}{\partial r})+\cancel{r^2\sin\theta\cdot g^{\theta r}(\frac{\partial V}{\partial \theta})}+\cancel{r^2\sin\theta\cdot g^{\varphi r}(\frac{\partial V}{\partial \varphi})}]+$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\theta}[\cancel{r^2\sin\theta\cdot g^{r\theta}(\frac{\partial V}{\partial r})}+r^2\sin\theta\cdot g^{\theta\theta}(\frac{\partial V}{\partial \theta})+\cancel{r^2\sin\theta\cdot g^{\varphi\theta}(\frac{\partial V}{\partial \varphi})}]+$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\varphi}[\cancel{r^2\sin\theta\cdot g^{r\varphi}(\frac{\partial V}{\partial r})}+\cancel{r^2\sin\theta\cdot g^{\theta\varphi}(\frac{\partial V}{\partial \theta})}+r^2\sin\theta\cdot g^{\varphi\varphi}(\frac{\partial V}{\partial \varphi})]=$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}[r^2\sin\theta\cdot g^{rr}(\frac{\partial V}{\partial r})]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}[r^2\sin\theta\cdot g^{\theta\theta}(\frac{\partial V}{\partial \theta})]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \varphi}[r^2\sin\theta\cdot g^{\varphi\varphi}(\frac{\partial V}{\partial \varphi})]=$$

$$=\frac{\sin\theta}{r^2\sin\theta}\frac{\partial}{\partial r}[r^2\cdot 1\cdot(\frac{\partial V}{\partial r})]+\frac{r^2}{r^2\sin\theta}\frac{\partial}{\partial \theta}[\sin\theta\cdot\frac{1}{r^2}(\frac{\partial V}{\partial \theta})]+\frac{r^2\sin\theta}{r^2\sin\theta}\frac{\partial}{\partial \varphi}[\frac{1}{r^2\sin^2\theta}(\frac{\partial V}{\partial \varphi})]=$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\frac{\partial V}{\partial r})]+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}[\sin\theta\cdot\frac{1}{r^2}(\frac{\partial V}{\partial \theta})]+\frac{\partial}{\partial \varphi}[\frac{1}{r^2\sin^2\theta}(\frac{\partial V}{\partial \varphi})]=$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\frac{\partial V}{\partial r})]+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}[\sin\theta(\frac{\partial V}{\partial \theta})]+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \varphi}[(\frac{\partial V}{\partial \varphi})]=$$

$$=(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial V}{\partial r}2r)+[\frac{1}{r^2}\frac{\sin\theta}{\sin\theta}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2}\frac{1}{\sin\theta}(\frac{\partial V}{\partial \theta})\cos\theta]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}=$$

$$\boxed{=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}=\Delta V}$$ (that is the (1.12))!!!

-**for the cylindricals**:

$$\nabla^2 V=\Delta V=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{\sqrt{r^2}}\frac{\partial}{\partial x^\mu}[\sqrt{r^2}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{1\mu}(\frac{\partial V}{\partial x^1})+rg^{2\mu}(\frac{\partial V}{\partial x^2})+rg^{3\mu}(\frac{\partial V}{\partial x^3})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{r\mu}(\frac{\partial V}{\partial r})+rg^{\varphi\mu}(\frac{\partial V}{\partial \varphi})+rg^{z\mu}(\frac{\partial V}{\partial z})]=\frac{1}{r}\frac{\partial}{\partial x^r}[rg^{rr}(\frac{\partial V}{\partial r})+\cancel{rg^{\varphi r}(\frac{\partial V}{\partial \varphi})}+\cancel{rg^{zr}(\frac{\partial V}{\partial z})}]+$$

$$+\frac{1}{r}\frac{\partial}{\partial x^\varphi}[\cancel{rg^{r\varphi}(\frac{\partial V}{\partial r})}+rg^{\varphi\varphi}(\frac{\partial V}{\partial \varphi})+\cancel{rg^{z\varphi}(\frac{\partial V}{\partial z})}]+\frac{1}{r}\frac{\partial}{\partial x^z}[\cancel{rg^{rz}(\frac{\partial V}{\partial r})}+\cancel{rg^{\varphi z}(\frac{\partial V}{\partial \varphi})}+rg^{zz}(\frac{\partial V}{\partial z})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial r}[rg^{rr}(\frac{\partial V}{\partial r})]+\frac{1}{r}\frac{\partial}{\partial \varphi}[rg^{\varphi\varphi}(\frac{\partial V}{\partial \varphi})]+\frac{1}{r}\frac{\partial}{\partial z}[rg^{zz}(\frac{\partial V}{\partial z})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial r}[r\cdot 1\cdot(\frac{\partial V}{\partial r})]+\frac{1}{r}\frac{\partial}{\partial \varphi}[r\frac{1}{r^2}(\frac{\partial V}{\partial \varphi})]+\frac{1}{r}\frac{\partial}{\partial z}[r\cdot 1\cdot(\frac{\partial V}{\partial z})]=$$

$$\boxed{= \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = \nabla^2 V}$$, (that is the (1.17))!!!

-**curl**: $(curl\vec{V})^i = (\vec{\nabla}\times\vec{V})^i = \varepsilon^{ijk}\frac{\partial V_k}{\partial x^j}$

ε^{ijk} Levi-Civita Emisymmetric Tensor, which changes its sign every time two contiguous indices are swapped; its value is +1 if ijk=123, 231, 321 , then it is -1 if ijk=321, 132, 213 and is 0 if two indices have the same value. The same can be said for ε_{ijk} .

We previously proved the formula (2.9) for the transformation of the components of a tensor: $T'^{\mu\lambda} = T^{\nu\sigma}\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\sigma}}$ and so, in order to pass from the Cartesian emisymmetric tensor "ε" to the spherical one "e", we have:

$$e'^{ijk} = \varepsilon^{lmn}\frac{\partial x'^{i}}{\partial x^{l}}\frac{\partial x'^{j}}{\partial x^{m}}\frac{\partial x'^{k}}{\partial x^{n}} \quad ; \tag{3.7}$$

and, in the opposite case: $\varepsilon^{lmn} = e'^{ijk}\frac{\partial x^{l}}{\partial x'^{i}}\frac{\partial x^{m}}{\partial x'^{j}}\frac{\partial x^{n}}{\partial x'^{k}}$;

then, if, as usual, through the metric tensor g we raise and lower indices of the emisymmetric tensor, then we have, as an example, the mixed emisymmetric:

$$e'^{i}_{\ jk} = \varepsilon^{l}_{\ mn}\frac{\partial x'^{i}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{j}}\frac{\partial x^{n}}{\partial x'^{k}} \tag{3.8}$$

where the second and third partial derivatives (in m and n) are inverted, as there m and n are low indices, while l (el) stayed high. But let's analyse those three partial derivatives:

$$\frac{\partial x'^{i}}{\partial x^{l}}$$
$$\frac{\partial x^{m}}{\partial x'^{j}}$$
$$\frac{\partial x^{n}}{\partial x'^{k}}$$

in considering the (2.7), that is $\hat{e}'_l = \hat{e}_i\frac{\partial x^i}{\partial x'^l}$, which is here rewritten which the names of the indices swapped: $\hat{e}'_i = \hat{e}_l\frac{\partial x^l}{\partial x'^i}$; then, for those three partial derivatives we have, by considering the sequence of Cartesian values 1,2,3 and spherical r,θ,φ in the (3.8), as there is a sequence of different letters (l,m,n)/(i,j,k), then we, too, displace them in a sequence, indeed (1,2,3)(=(x,y,z)) / (r,θ,φ), so:

$$\frac{\partial x'^{i}}{\partial x^{l}} \equiv \hat{e}_l\hat{e}'^{i} = \sqrt{g_{ll}}\sqrt{g'^{ii}} = \sqrt{g_{11}}\sqrt{g'^{rr}} = \sqrt{g_{xx}}\sqrt{g'^{rr}} = 1\cdot 1 = 1$$

$$\frac{\partial x^{m}}{\partial x'^{j}} = \hat{e}'_j\,\hat{e}^{m} = \sqrt{g'_{jj}}\sqrt{g^{mm}} = \sqrt{g_{22}}\sqrt{g'^{\theta\theta}} = \sqrt{g_{yy}}\sqrt{g'^{\theta\theta}} = 1\cdot\frac{1}{r} = \frac{1}{r}$$

$$\frac{\partial x^{n}}{\partial x'^{k}} = \hat{e}'_k\,\hat{e}^{n} = \sqrt{g'_{kk}}\sqrt{g^{nn}} = \sqrt{g_{33}}\sqrt{g'^{\varphi\varphi}} = \sqrt{g_{zz}}\sqrt{g'^{\varphi\varphi}} = 1\cdot\frac{1}{r\sin\theta} = \frac{1}{r\sin\theta}$$

and so the transformation equation (3.8) yields:

$$e'^{i}_{\ jk} = \varepsilon^{l}_{\ mn}\frac{\partial x'^{i}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{j}}\frac{\partial x^{n}}{\partial x'^{k}} = \varepsilon^{l}_{\ mn}(1\cdot\frac{1}{r}\frac{1}{r\sin\theta}) = \sqrt{g}\,\varepsilon^{l}_{\ mn} \quad . \tag{3.9}$$

Then, of course we also have that: $(curl\vec{V})_i = (\vec{\nabla}\times\vec{V})_i = \varepsilon_{ijk}\dfrac{\partial V^k}{\partial x_j}$.

-**<u>case of the Cartesian coordinates</u>**:

$$(curl\vec{V})^i = (\vec{\nabla}\times\vec{V})^i = \varepsilon^{ijk}\frac{\partial V_k}{\partial x^j} = \varepsilon^{ij1}\frac{\partial V_1}{\partial x^j} + \varepsilon^{ij2}\frac{\partial V_2}{\partial x^j} + \varepsilon^{ij3}\frac{\partial V_3}{\partial x^j} =$$

$$= (\varepsilon^{i11}\frac{\partial V_1}{\partial x^1} + \varepsilon^{i12}\frac{\partial V_2}{\partial x^1} + \varepsilon^{i13}\frac{\partial V_3}{\partial x^1}) + (\varepsilon^{i21}\frac{\partial V_1}{\partial x^2} + \varepsilon^{i22}\frac{\partial V_2}{\partial x^2} + \varepsilon^{i23}\frac{\partial V_3}{\partial x^2}) + (\varepsilon^{i31}\frac{\partial V_1}{\partial x^3} + \varepsilon^{i32}\frac{\partial V_2}{\partial x^3} + \varepsilon^{i33}\frac{\partial V_3}{\partial x^3}) =$$

$$= (\varepsilon^{i11}\frac{\partial V_x}{\partial x} + \varepsilon^{i12}\frac{\partial V_y}{\partial x} + \varepsilon^{i13}\frac{\partial V_z}{\partial x}) + (\varepsilon^{i21}\frac{\partial V_x}{\partial y} + \varepsilon^{i22}\frac{\partial V_y}{\partial y} + \varepsilon^{i23}\frac{\partial V_z}{\partial y}) + (\varepsilon^{i31}\frac{\partial V_x}{\partial z} + \varepsilon^{i32}\frac{\partial V_y}{\partial z} + \varepsilon^{i33}\frac{\partial V_z}{\partial z})$$

so:

$$(curl\vec{V})^1 = (\vec{\nabla}\times\vec{V})^1 = (curl\vec{V})^x = (\vec{\nabla}\times\vec{V})^x = (\varepsilon^{111}\frac{\partial V_x}{\partial x} + \varepsilon^{112}\frac{\partial V_y}{\partial x} + \varepsilon^{113}\frac{\partial V_z}{\partial x}) + (\varepsilon^{121}\frac{\partial V_x}{\partial y} + \varepsilon^{122}\frac{\partial V_y}{\partial y} + \varepsilon^{123}\frac{\partial V_z}{\partial y}) +$$

$$+ (\varepsilon^{131}\frac{\partial V_x}{\partial z} + \varepsilon^{132}\frac{\partial V_y}{\partial z} + \varepsilon^{133}\frac{\partial V_z}{\partial z}) = (0\frac{\partial V_x}{\partial x} + 0\frac{\partial V_y}{\partial x} + 0\frac{\partial V_z}{\partial x}) + (0\frac{\partial V_x}{\partial y} + 0\frac{\partial V_y}{\partial y} + \varepsilon^{123}\frac{\partial V_z}{\partial y}) +$$

$$+ (0\frac{\partial V_x}{\partial z} + \varepsilon^{132}\frac{\partial V_y}{\partial z} + 0\frac{\partial V_z}{\partial z}) = \varepsilon^{123}\frac{\partial V_z}{\partial y} + \varepsilon^{132}\frac{\partial V_y}{\partial z} = (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})$$ and similarly:

$$(curl\vec{V})^2 = (\vec{\nabla}\times\vec{V})^2 = (curl\vec{V})^y = (\vec{\nabla}\times\vec{V})^y = (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})$$, and:

$$(curl\vec{V})^3 = (\vec{\nabla}\times\vec{V})^3 = (curl\vec{V})^z = (\vec{\nabla}\times\vec{V})^z = (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})$$; in total:

$$\boxed{curl\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}) + \hat{j}(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}) + \hat{k}(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})}$$

, which is exactly the (1.5) !!!

-**<u>case of the spherical coordinates</u>**:

$$(curl\vec{V})^i = (\vec{\nabla}\times\vec{V})^i = e^i_{jk}\frac{\partial V^k}{\partial x_j} =$$ (for the (3.9))

$$= \sqrt{g}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j} = \frac{1}{r^2\sin\theta}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j} = \frac{1}{r^2\sin\theta}(\varepsilon^i_{j1}\frac{\partial V^1}{\partial x_j} + \varepsilon^i_{j2}\frac{\partial V^2}{\partial x_j} + \varepsilon^i_{j3}\frac{\partial V^3}{\partial x_j}) =$$

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^i_{11}\frac{\partial V^1}{\partial x_1} + \varepsilon^i_{12}\frac{\partial V^2}{\partial x_1} + \varepsilon^i_{13}\frac{\partial V^3}{\partial x_1}) + (\varepsilon^i_{21}\frac{\partial V^1}{\partial x_2} + \varepsilon^i_{22}\frac{\partial V^2}{\partial x_2} + \varepsilon^i_{23}\frac{\partial V^3}{\partial x_2}) + (\varepsilon^i_{31}\frac{\partial V^1}{\partial x_3} + \varepsilon^i_{32}\frac{\partial V^2}{\partial x_3} + \varepsilon^i_{33}\frac{\partial V^3}{\partial x_3})] =$$

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^i_{11}\frac{\partial V^r}{\partial x_1} + \varepsilon^i_{12}\frac{\partial V^\theta}{\partial x_1} + \varepsilon^i_{13}\frac{\partial V^\varphi}{\partial x_1}) + (\varepsilon^i_{21}\frac{\partial V^r}{\partial x_2} + \varepsilon^i_{22}\frac{\partial V^\theta}{\partial x_2} + \varepsilon^i_{23}\frac{\partial V^\varphi}{\partial x_2}) + (\varepsilon^i_{31}\frac{\partial V^r}{\partial x_3} + \varepsilon^i_{32}\frac{\partial V^\theta}{\partial x_3} + \varepsilon^i_{33}\frac{\partial V^\varphi}{\partial x_3})]$$

so: $(curl\vec{V})^1 = (\vec{\nabla}\times\vec{V})^1 = (curl\vec{V})^r = (\vec{\nabla}\times\vec{V})^r =$ (also for the complementary of the (3.2))

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^1_{11}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial r} + \varepsilon^1_{12}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r} + \varepsilon^1_{13}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r})] +$$

$$+ \frac{1}{r^2\sin\theta}[\varepsilon^1_{21}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta} + \varepsilon^1_{22}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\theta} + \varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}] +$$

$$+\frac{1}{r^2\sin\theta}[\varepsilon^1_{31}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}+\varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}+\varepsilon^1_{33}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\varphi}]=$$

$$=\frac{1}{r^2\sin\theta}[(0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial r}+0\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r})]+$$

$$+\frac{1}{r^2\sin\theta}[0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta}+0\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\theta}+\varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}]+$$

$$+\frac{1}{r^2\sin\theta}[0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}+\varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\varphi}]=$$

$$=\frac{1}{r^2\sin\theta}[\varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}+\varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]=\frac{1}{r^2\sin\theta}[1\cdot\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}+(-1)\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}-\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]$$ and similarly:

$$(curl\vec{V})^2=(\vec{\nabla}\times\vec{V})^2=(curl\vec{V})^\theta=(\vec{\nabla}\times\vec{V})^\theta=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r}]$$

$$(curl\vec{V})^3=(\vec{\nabla}\times\vec{V})^3=(curl\vec{V})^\varphi=(\vec{\nabla}\times\vec{V})^\varphi=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta}]$$

and in total: (and once again, for the complementary of the (2.34))

$$curl\vec{V}=\vec{\nabla}\times\vec{V}=\hat{e}_r(\vec{\nabla}\times\vec{V})^r+\hat{e}_\theta(\vec{\nabla}\times\vec{V})^\theta+\hat{e}_\varphi(\vec{\nabla}\times\vec{V})^\varphi=$$

$$=\frac{1}{r^2\sin\theta}[\sqrt{g_{rr}}(\vec{\nabla}\times\vec{V})^r\hat{r}+\sqrt{g_{\theta\theta}}(\vec{\nabla}\times\vec{V})^\theta\hat{\theta}+\sqrt{g_{\varphi\varphi}}(\vec{\nabla}\times\vec{V})^\varphi\hat{\varphi}]=$$

$$=\frac{\sqrt{g_{rr}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}-\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]\hat{r}+\frac{\sqrt{g_{\theta\theta}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r}]\hat{\theta}+$$

$$+\frac{\sqrt{g_{\varphi\varphi}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta}]\hat{\varphi};$$ now, by reminding that: $\sqrt{g^{rr}}=1$, $\sqrt{g^{\theta\theta}}=\frac{1}{r}$ and $\sqrt{g^{\varphi\varphi}}=\frac{1}{r\sin\theta}$, but also: $\sqrt{g_{rr}}=1$, $\sqrt{g_{\theta\theta}}=r$ and $\sqrt{g_{\varphi\varphi}}=r\sin\theta$, then it follows that:

$$curl\vec{V}=\frac{1}{r^2\sin\theta}[\frac{\partial(r\sin\theta V^\varphi)}{\partial\theta}-\frac{\partial(rV^\theta)}{\partial\varphi}]\hat{r}+\frac{r}{r^2\sin\theta}[\frac{\partial(1\cdot V^r)}{\partial\varphi}-\frac{\partial(r\sin\theta V^\varphi)}{\partial r}]\hat{\theta}+$$

$$+\frac{r\sin\theta}{r^2\sin\theta}[\frac{\partial(rV^\theta)}{\partial r}-\frac{\partial(1\cdot V^r)}{\partial\theta}]\hat{\varphi}=$$

$$\boxed{=\vec{\nabla}\times\vec{V}=\frac{1}{r^2\sin\theta}[\frac{\partial(r\sin\theta V^\varphi)}{\partial\theta}-\frac{\partial(rV^\theta)}{\partial\varphi}]\hat{r}+\frac{1}{r\sin\theta}[\frac{\partial V^r}{\partial\varphi}-\frac{\partial(r\sin\theta V^\varphi)}{\partial r}]\hat{\theta}+\frac{1}{r}[\frac{\partial(rV^\theta)}{\partial r}-\frac{\partial V^r}{\partial\theta}]\hat{\varphi}}$$

that is exactly the (1.11)!!!

-**<u>case of the cylindrical coordinates</u>**:

$$(curl\vec{V})^i=(\vec{\nabla}\times\vec{V})^i=e^i_{jk}\frac{\partial V^k}{\partial x_j}=\qquad \text{(for the (3.9))}$$

$$=\sqrt{g}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j}=\frac{1}{r}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j}=\frac{1}{r}(\varepsilon^i_{j1}\frac{\partial V^1}{\partial x_j}+\varepsilon^i_{j2}\frac{\partial V^2}{\partial x_j}+\varepsilon^i_{j3}\frac{\partial V^3}{\partial x_j})=$$

$$=\frac{1}{r}[(\varepsilon_{11}^{i}\frac{\partial V^{1}}{\partial x_1}+\varepsilon_{12}^{i}\frac{\partial V^{2}}{\partial x_1}+\varepsilon_{13}^{i}\frac{\partial V^{3}}{\partial x_1})+(\varepsilon_{21}^{i}\frac{\partial V^{1}}{\partial x_2}+\varepsilon_{22}^{i}\frac{\partial V^{2}}{\partial x_2}+\varepsilon_{23}^{i}\frac{\partial V^{3}}{\partial x_2})+(\varepsilon_{31}^{i}\frac{\partial V^{1}}{\partial x_3}+\varepsilon_{32}^{i}\frac{\partial V^{2}}{\partial x_3}+\varepsilon_{33}^{i}\frac{\partial V^{3}}{\partial x_3})]=$$

$$=\frac{1}{r}[(\varepsilon_{11}^{i}\frac{\partial V^{r}}{\partial x_1}+\varepsilon_{12}^{i}\frac{\partial V^{\theta}}{\partial x_1}+\varepsilon_{13}^{i}\frac{\partial V^{\varphi}}{\partial x_1})+(\varepsilon_{21}^{i}\frac{\partial V^{r}}{\partial x_2}+\varepsilon_{22}^{i}\frac{\partial V^{\theta}}{\partial x_2}+\varepsilon_{23}^{i}\frac{\partial V^{\varphi}}{\partial x_2})+(\varepsilon_{31}^{i}\frac{\partial V^{r}}{\partial x_3}+\varepsilon_{32}^{i}\frac{\partial V^{\theta}}{\partial x_3}+\varepsilon_{33}^{i}\frac{\partial V^{\varphi}}{\partial x_3})]$$

so: $(curl\vec{V})^1=(\vec{\nabla}\times\vec{V})^1=(curl\vec{V})^r=(\vec{\nabla}\times\vec{V})^r=$ (also for the complementary of the (3.2))

$$=\frac{1}{r}[(\varepsilon_{11}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial r}+\varepsilon_{12}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}+\varepsilon_{13}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r})]+\frac{1}{r}[\varepsilon_{21}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}+\varepsilon_{22}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial\varphi}+\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}]+$$

$$+\frac{1}{r}[\varepsilon_{31}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}+\varepsilon_{33}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial z}]=\frac{1}{r}[(0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial r}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}+0\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r})]+$$

$$+\frac{1}{r}[0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial\varphi}+\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}]+\frac{1}{r}[0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}+0\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial z}]=$$

$$=\frac{1}{r}[\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]=\frac{1}{r}[1\cdot\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}+(-1)\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]=\frac{1}{r}[\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]$$

and, similarly:

$$(curl\vec{V})^2=(\vec{\nabla}\times\vec{V})^2=(curl\vec{V})^{\varphi}=(\vec{\nabla}\times\vec{V})^{\varphi}=\frac{1}{r}[\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}-\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r}]$$

$$(curl\vec{V})^3=(\vec{\nabla}\times\vec{V})^3=(curl\vec{V})^{z}=(\vec{\nabla}\times\vec{V})^{z}=\frac{1}{r}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}]$$

and in total: (and once again, for the complementary of the (2.34))

$$curl\vec{V}=\vec{\nabla}\times\vec{V}=\hat{e}_r(\vec{\nabla}\times\vec{V})^r+\hat{e}_{\varphi}(\vec{\nabla}\times\vec{V})^{\varphi}+\hat{e}_z(\vec{\nabla}\times\vec{V})^z=$$

$$=\frac{1}{r}[\sqrt{g_{rr}}(\vec{\nabla}\times\vec{V})^r\hat{r}+\sqrt{g_{\varphi\varphi}}(\vec{\nabla}\times\vec{V})^{\varphi}\hat{\varphi}+\sqrt{g_{zz}}(\vec{\nabla}\times\vec{V})^z\hat{z}]=$$

$$=\frac{\sqrt{g_{rr}}}{r}[\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]\hat{r}+\frac{\sqrt{g_{\varphi\varphi}}}{r}[\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}-\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r}]\hat{\varphi}+$$

$+\frac{\sqrt{g_{zz}}}{r}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}]\hat{z}$; now, by reminding that: $\sqrt{g^{rr}}=1$, $\sqrt{g^{\varphi\varphi}}=\frac{1}{r}$ and $\sqrt{g^{zz}}=1$,

but also: $\sqrt{g_{rr}}=1$, $\sqrt{g_{\varphi\varphi}}=r$ and $\sqrt{g_{zz}}=1$, it follows that:

$$curl\vec{V}=\frac{1}{r}[\frac{\partial V^{z}}{\partial\varphi}-\frac{\partial(rV^{\varphi})}{\partial z}]\hat{r}+\frac{r}{r}[\frac{\partial V^{r}}{\partial z}-\frac{\partial V^{z}}{\partial r}]\hat{\varphi}+\frac{1}{r}[\frac{\partial(rV^{\varphi})}{\partial r}-\frac{\partial V^{r}}{\partial\varphi}]\hat{z}=$$

$$\boxed{=\frac{1}{r}[\frac{\partial V^{z}}{\partial\varphi}-\frac{\partial(rV^{\varphi})}{\partial z}]\hat{r}+[\frac{\partial V^{r}}{\partial z}-\frac{\partial V^{z}}{\partial r}]\hat{\varphi}+\frac{1}{r}[\frac{\partial(rV^{\varphi})}{\partial r}-\frac{\partial V^{r}}{\partial\varphi}]\hat{z}=\vec{\nabla}\times\vec{V}}$$

that is the dual of the (1.16)!!!

See also Appendix 2 for further details on change of coordinates.

APPENDIX 1: FURTHER DETAILS ON TENSOR CALCULUS.

We will mainly analyse the case of the rotating motion in polar coordinates, which is a subject that contains the curvature, but we will do that with tensors. And then we will consider also the sphere. Here is a list of the main tensor equations of the general relativity:

-the Metric Tensor: $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$

-the geodesic equation: $\dfrac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = 0$ (free fall, in curvilinear coordinates, over x^λ)

-the Christoffel Symbols: $\Gamma^\lambda_{\mu\nu} = \dfrac{1}{2} g^{\sigma\lambda} \left(\dfrac{\partial g_{\nu\sigma}}{\partial x^\mu} + \dfrac{\partial g_{\mu\sigma}}{\partial x^\nu} - \dfrac{\partial g_{\nu\mu}}{\partial x^\sigma}\right)$ $\quad (\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu})$.

-the Riemann-Christoffel curvature tensor:

$$R^\lambda_{\mu\nu k} = (\Gamma^\lambda_{\mu\nu})_k - (\Gamma^\lambda_{\mu k})_\nu + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\nu\eta} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\nu\eta}$$

-the Ricci tensor $R_{\mu k} = \dfrac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^k} - \dfrac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\lambda\eta}$ $\quad (R_{\mu k} = R_{k\mu})$, $R_{\mu k} = g^{\lambda\nu} R_{\lambda\mu\nu k}$

-the scalar curvature R (of Ricci): $R = g^{\mu k} R_{\mu k}$

-the Einstein's Equations: $R_{\mu\nu} - \dfrac{1}{2} g_{\mu\nu} R = -\dfrac{8\pi G}{c^4} T_{\mu\nu}$

(We remind that, from a context to another, the names of the indices can change, as they span over a range of 4 values, like the number of the coordinates, but the consistency among them must be preserved, as if, for example, a μ becomes a ν, then it must do so, that is ν, everywhere into the equation…)

Rotating motion

Now we get into two dimensions and represent in Fig. 1 both Cartesian x,y and polar r,θ axes, then we imagine a point P, an end of the vector $\vec{P}$, which rotates counterclockwise and we have also identified the versor $\hat{r}$ and the tangent versor $\hat{t}$.

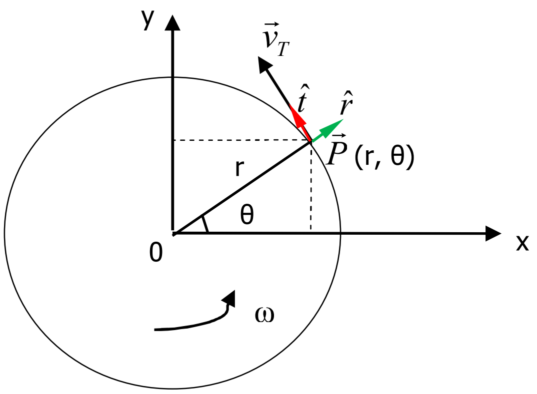


Fig. 1: Circumference.

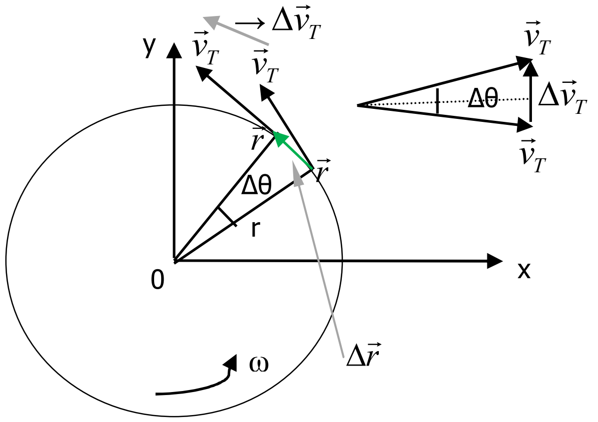


Fig. 2: Calculation of the centripetal acceleration.

Now we preliminarily remind the following facts:

-angular velocity ω: the round angle is 360° (2π) and in case of rotation, by definition, the time taken to run a cycle (a round) is the period T. It follows that for the angular velocity:

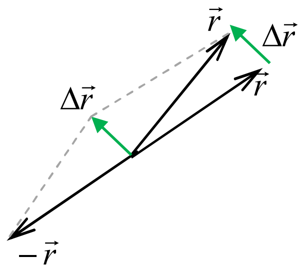
$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \frac{2\pi}{T} \; .$$

-about the peripheral velocity v_P, we know that if it's numerically constant, it informs us about the rapidity by which the circumference (the entire cycle) is run (in the time T, which is the period):
$v_p = \frac{2\pi r}{T} = \frac{2\pi}{T} r = \boxed{\omega r = v_p = v_T}$. (with vectors: $\vec{v}_P = \vec{v}_T = \vec{\omega} \times \vec{r}$) , so such a vector product orients like the tangent versor $\hat{t}$. In fact, $\vec{v}_T$ is orthogonal to $\vec{r}$, as we can see in Fig. 2, where the little green vector $\Delta\vec{r}$ is orthogonal to $\vec{r}$, indeed; more specifically, we have that:

$\vec{v}_T = \frac{\Delta\vec{r}}{\Delta t} = \omega r \hat{t}$; in fact, with reference to Fig. 2, and with Δ which becomes "d":

$$\vec{v}_T = \frac{\Delta\vec{r}}{\Delta t} \to \frac{d\vec{r}}{dt} = \frac{2r\sin(d\theta/2)}{dt}\hat{t} = 2r\frac{\sin(d\theta/2)}{dt(d\theta/2)}(d\theta/2)\hat{t} = r\frac{\sin(d\theta/2)}{(d\theta/2)}(d\theta/dt)\hat{t} = \omega r\frac{\sin(d\theta/2)}{(d\theta/2)}\hat{t} \to \vec{\omega}\times\vec{r}$$

as: $\lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{(\Delta\theta/2)} = 1 \equiv \frac{\sin(d\theta/2)}{(d\theta/2)}$. Therefore, by the derivative of $\vec{r}$, that is with $\frac{d\vec{r}}{dt}$, we got $\vec{v}_T$ which is not oriented like $\vec{r}$, but like $\hat{t}$, which can be obtained by the counterclockwise rotation of $\hat{r}$ about 90°; so: $\frac{d}{dt} \to$ direction at 90° counterclockwise.



Conversely, if the point P moves with any kind of motion, then:

$$\frac{d\vec{P}}{dt} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cdot\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = v_r\hat{r} + r(\vec{\omega}\times\hat{r}) = \vec{v}_r + \omega r\hat{t} = \vec{v}_r + \vec{v}_T\,; \qquad (1)$$

all this because $\frac{d\hat{r}}{dt} = (\vec{\omega}\times\hat{r})$, as $\vec{\omega}$ gets out of the paper sheet and by vectorially multiplying with $\hat{r}$, it gives $\hat{t}$ indeed, that, as we predicted before, is in the counterclockwise rotation about 90° of $\hat{r}$ indeed.

-At last, the rotation determines a centripetal acceleration, as $\vec{v}_T$ can be constant in its modulus, but it's not about its direction:

$\vec{a}_C = \frac{\Delta\vec{v}_T}{\Delta t} = -\omega^2 R\hat{r}$; in fact, with reference to Fig. 2 (the vector triangle in the top right), and with the Δ becoming "d":

$$\vec{a}_C = \frac{\Delta\vec{v}_T}{\Delta t} = \frac{2v_T\sin(d\theta/2)}{dt}(-\hat{r}) = 2v_T\frac{\sin(d\theta/2)}{dt(d\theta/2)}(d\theta/2)(-\hat{r}) = v_T\frac{\sin(d\theta/2)}{(d\theta/2)}(d\theta/dt)(-\hat{r}) \to -\vec{\omega}\times\vec{v}_T =$$

$$= -\vec{\omega}\times(\vec{\omega}\times\vec{r}) \quad (a_C = -\omega^2 r), \text{ as: } \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{(\Delta\theta/2)} = 1 \equiv \frac{\sin(d\theta/2)}{(d\theta/2)}.$$

Now, the (1) tells us that: $\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cdot\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = v_r\hat{r} + r(\vec{\omega}\times\hat{r}) = v_r\hat{r} + (\omega r)\hat{t} = \vec{v}_r + \vec{v}_T =$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$, as $\dot{r}$ is $\frac{dr}{dt}$, while $\hat{t}$ is indeed the same as the tangential direction (counterclockwise) $\hat{\theta}$ of the polar coordinates and, at last, of course: $\omega = \frac{d\theta}{dt} = \dot{\theta}$. By deriving further:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta}) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}(\dot{\theta}\hat{\theta}) + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} =$$

$= \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - \dot{\theta}^2 r\hat{r} = \hat{r}(\ddot{r} - \dot{\theta}^2 r) + \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta})$, that is:

$$\vec{a} = \hat{r}(\ddot{r} - \dot{\theta}^2 r) + \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) \qquad (2)$$

(as $\frac{d\hat{r}}{dt}$, as we proved before, yields $\omega|\hat{r}|\hat{t} = \omega 1\hat{\theta} = \dot{\theta}\hat{\theta}$, that is ω x the versor obtained by rotating $\hat{r}$ about 90° counterclockwise, that is $\hat{t}$, that is again $\hat{\theta}$, while $\frac{d\hat{\theta}}{dt}$ yields $\omega|\hat{\theta}|(-\hat{r}) = -\omega 1\hat{r} = -\dot{\theta}\hat{r}$) and so on. The $(2\dot{r}\dot{\theta})\hat{\theta}$ is the **Coriolis** component (acceleration).
The (2) has a tangential component (//) and a radial/centripetal one ($\perp$), so:

$$\boxed{\vec{a} = \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) = \vec{a}_{//} + \vec{a}_{\perp}} \qquad (= a^{\lambda} = a^{\lambda}_{//} + a^{\lambda}_{\perp}) \qquad (3)$$

Now, in case of a rotating motion on a circle, where a point rotates by keeping always at the same distance r and with a fixed angular speed $\omega = \dot{\theta} = const$, it's $\dot{r} = \ddot{r} = \ddot{\theta} = 0$, so, due to the (3):
$\boxed{\vec{a}} = \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) = \hat{\theta}(2\cdot 0\cdot 0 + r) + \hat{r}(0 - \dot{\theta}^2 r) = -\dot{\theta}^2 r = \boxed{-\omega^2 r}$ (a result we already got).

<u>The geodesic equation</u>

A free falling parachutist does not have any floor over which his body can rest; therefore, he does not detect any gravitational acceleration and he feels as if floating in the vacuum. He realizes he is falling only if he looks at the moving objects around. Therefore, for a free falling particle, there is a reference system in which $\vec{a} = 0$, that is:
$\vec{a} = \frac{\partial^2 \xi^{\alpha}}{\partial \tau^2} = 0$ (free falling in Euclidean coordinates)
Of course, if the space is curved (such as that of the polar coordinates), then the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ will show up, to show the curvature and the free fall (that is the motion along a geodesic) will be: $\vec{a} = \frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0$ (free fall in curvilinear coordinates)
(equation of the geodesic, where geodesic, on the Earth, is the shortest path between two places)

In order to carry out calculations which lead us to the (3) through the tensor calculus (the geodesic equation), we need all the values of the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ and in order to get them, that are:
$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu})$, we need to know the metric $g^{\mu\nu}$ first, so:

-<u>metric</u>: now, we remind again the polar coordinates of Fig. 1, from which, due to the Pythagorean Theorem: $d\tau^2 = -(\hat{e}_{\mu}dx^{\mu})(\hat{e}_{\nu}dx^{\nu}) = -(\hat{e}_{\mu}\hat{e}_{\nu})dx^{\mu}dx^{\nu} = -g_{\mu\nu}dx^{\mu}dx^{\nu}$, $\mu,\nu \to r,\theta$
$d\tau^2 = -g_{\mu\nu}dx^{\mu}dx^{\nu} = -g_{rr}dx^{r}dx^{r} - g_{\theta\theta}dx^{\theta}dx^{\theta} - 2g_{r\theta}dx^{r}dx^{\theta} = -g_{rr}dr^2 - g_{\theta\theta}d\theta^2 - \cancel{2g_{r\theta}drd\theta} =$
$= -(dr^2 + r^2 d\theta^2)$ and here we see that we have no cross terms $drd\theta$, so the metric tensor g is diagonal, that is the cross terms are null $(g_{r\theta} = g_{\theta r} = 0)$ and so, again, as $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu} \to$

$$\boxed{(g_{\mu\nu}g^{\mu\nu} \equiv "1") \to g_{\mu\nu} \equiv \frac{1}{g^{\mu\nu}},\ g_{rr} = 1\ (\text{and } g^{rr} = 1^{-1} = 1) \to g_{\theta\theta} = r^2\ (\text{and } g^{\theta\theta} = \frac{1}{r^2}),\ \text{and } g_{r\theta} = g_{\theta r} = g^{r\theta} = g^{\theta r} = 0.}$$

-<u>Christoffel symbols</u>: we know from the general relativity and from the previous pages that:

$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu})$, from which:

<u>case 1 (λ=r)</u>: $\Gamma^{r}_{\mu\nu} = \frac{1}{2} g^{\sigma r} (\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{r}_{\mu\nu} = \Gamma^{r}_{\nu\mu})$ and so:

(μ=ν=θ): $\boxed{\Gamma^{r}_{\theta\theta}} = \frac{1}{2} g^{\sigma r} (\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} + \frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} - \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) = (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{rr} \frac{\partial g_{\theta r}}{\partial \theta} + \frac{1}{2} g^{\theta r} \frac{\partial g_{\theta\theta}}{\partial \theta}) + (\frac{1}{2} g^{rr} \frac{\partial g_{\theta r}}{\partial \theta} + \frac{1}{2} g^{\theta r} \frac{\partial g_{\theta\theta}}{\partial \theta}) - (\frac{1}{2} g^{rr} \frac{\partial g_{\theta\theta}}{\partial x^{r}} + \frac{1}{2} g^{\theta r} \frac{\partial g_{\theta\theta}}{\partial x^{\theta}}) =$

$= (\frac{1}{2} 1 \cdot 0 + \frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta}) + (\frac{1}{2} g^{rr} \frac{\partial 0}{\partial \theta} + \frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta}) - (\frac{1}{2} g^{rr} \frac{\partial g_{\theta\theta}}{\partial r} + \frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta}) = -\frac{1}{2} g^{rr} \frac{\partial g_{\theta\theta}}{\partial r} = -\frac{1}{2} (1) \frac{\partial r^{2}}{\partial r} = \boxed{-r}$

(μ=ν=r): $\boxed{\Gamma^{r}_{rr}} = \frac{1}{2} g^{\sigma r} (\frac{\partial g_{r\sigma}}{\partial x^{r}} + \frac{\partial g_{r\sigma}}{\partial x^{r}} - \frac{\partial g_{rr}}{\partial x^{\sigma}}) = (\frac{1}{2} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial r}) + (\frac{1}{2} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial r}) - (\frac{1}{2} g^{\sigma r} \frac{\partial g_{rr}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial r}) + (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial r}) - (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{rr}}{\partial x^{\sigma}}) = (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} + \frac{1}{2} g^{\theta r} \frac{\partial g_{r\theta}}{\partial r}) + (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} + \frac{1}{2} g^{\theta r} \frac{\partial g_{r\theta}}{\partial r}) -$

$- (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial x^{r}} + \frac{1}{2} g^{\theta r} \frac{\partial g_{rr}}{\partial x^{\theta}}) = (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} + \frac{1}{2} 0 \frac{\partial 0}{\partial r}) + (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} + \frac{1}{2} 0 \frac{\partial 0}{\partial r}) - (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} + \frac{1}{2} 0 \frac{\partial g_{rr}}{\partial \theta}) =$

$= \frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} = \frac{1}{2} 1 \frac{\partial 1}{\partial r} = \boxed{0\,.}$

(μ=r and ν=θ or vice versa): $\boxed{\Gamma^{r}_{r\theta}} = \Gamma^{r}_{\theta r} = \frac{1}{2} g^{\sigma r} (\frac{\partial g_{\theta\sigma}}{\partial x^{r}} + \frac{\partial g_{r\sigma}}{\partial x^{\theta}} - \frac{\partial g_{\theta r}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial r}) + (\frac{1}{2} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta r}}{\partial x^{\sigma}}) = (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial r}) + (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial \theta}) - (\frac{1}{2} \sum_{\sigma} g^{\sigma r} \frac{\partial g_{\theta r}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial r} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta\sigma}}{\partial r}) + (\frac{1}{2} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial \theta} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{r\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta r}}{\partial x^{\sigma}} + \frac{1}{2} g^{\sigma r} \frac{\partial g_{\theta r}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{rr} \frac{\partial g_{\theta r}}{\partial r} + \frac{1}{2} g^{\theta r} \frac{\partial g_{\theta\theta}}{\partial r}) + (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial \theta} + \frac{1}{2} g^{\theta r} \frac{\partial g_{r\theta}}{\partial \theta}) - (\frac{1}{2} g^{rr} \frac{\partial g_{\theta r}}{\partial x^{r}} + \frac{1}{2} g^{\theta r} \frac{\partial g_{\theta r}}{\partial x^{\theta}}) =$

$= (\frac{1}{2} g^{rr} \frac{\partial 0}{\partial r} + \frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial r}) + (\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial \theta} + \frac{1}{2} 0 \frac{\partial 0}{\partial \theta}) - (\frac{1}{2} g^{rr} \frac{\partial 0}{\partial r} + \frac{1}{2} 0 \frac{\partial 0}{\partial \theta}) = \frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial \theta} = \frac{1}{2} 1 \frac{\partial 1}{\partial \theta} = \boxed{0\ .}$

And again (λ=θ):

<u>case 2(λ=θ)</u>: $\Gamma^{\theta}_{\mu\nu} = \frac{1}{2} g^{\sigma\theta} (\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{\theta}_{\mu\nu} = \Gamma^{\theta}_{\nu\mu})$ and so:

(μ=ν =θ): $\boxed{\Gamma^{\theta}_{\theta\theta}} = \frac{1}{2} g^{\sigma\theta} (\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} + \frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} - \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) = (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} \sum_{\sigma} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} \sum_{\sigma} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} \sum_{\sigma} g^{\sigma\theta} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) + (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\sigma}}{\partial \theta}) - (\frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}} + \frac{1}{2} g^{\sigma\theta} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}) =$

$= (\frac{1}{2} g^{r\theta} \frac{\partial g_{\theta r}}{\partial \theta} + \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial \theta}) + (\frac{1}{2} g^{r\theta} \frac{\partial g_{\theta r}}{\partial \theta} + \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial \theta}) - (\frac{1}{2} g^{r\theta} \frac{\partial g_{\theta\theta}}{\partial x^{r}} + \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial x^{\theta}}) =$

$$=(\frac{1}{2}0\frac{\partial 0}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})+(\frac{1}{2}0\frac{\partial 0}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})-(\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}=\frac{1}{2}(\frac{1}{r^2})\frac{\partial r^2}{\partial\theta}=\boxed{0}$$

(μ=ν =r): $\boxed{\Gamma^{\theta}_{rr}}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{r\sigma}}{\partial x^r}+\frac{\partial g_{r\sigma}}{\partial x^r}-\frac{\partial g_{rr}}{\partial x^\sigma})=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{rr}}{\partial x^\sigma})=$

$$=(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{rr}}{\partial x^\sigma})=(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial r})+(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial r})-$$

$$-(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial x^r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{rr}}{\partial x^\theta})=(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial r})+(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial r})-(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 1}{\partial\theta})=\boxed{0}.$$

(μ=r and ν=θ or vice versa): $\boxed{\Gamma^{\theta}_{r\theta}}=\Gamma^{\theta}_{\theta r}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\theta\sigma}}{\partial x^r}+\frac{\partial g_{r\sigma}}{\partial x^\theta}-\frac{\partial g_{\theta r}}{\partial x^\sigma})=$

$$=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta r}}{\partial x^\sigma})=(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{\theta r}}{\partial x^\sigma})=$$

$$=(\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta 1}}{\partial r}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta 2}}{\partial r})+(\frac{1}{2}g^{1\theta}\frac{\partial g_{r1}}{\partial\theta}+\frac{1}{2}g^{2\theta}\frac{\partial g_{r2}}{\partial\theta})-(\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta r}}{\partial x^1}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta r}}{\partial x^2})=$$

$$=(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial\theta})-(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial x^r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta r}}{\partial x^\theta})=$$

$$=(\frac{1}{2}0\frac{\partial 0}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}0\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial\theta})-(\frac{1}{2}0\frac{\partial g_{\theta r}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial\theta})=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r}=\frac{1}{2}\frac{1}{r^2}\frac{\partial r^2}{\partial r}=\boxed{\frac{1}{r}}.$$

By summing things up:

$$\boxed{\Gamma^{r}_{\theta\theta}=-r\ ,\ \Gamma^{r}_{rr}=0\ ,\ \Gamma^{r}_{r\theta}=\Gamma^{r}_{\theta r}=0\ ,\ \Gamma^{\theta}_{\theta\theta}=0\ ,\ \Gamma^{\theta}_{rr}=0\ ,\ \Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=\frac{1}{r}\ .}$$

-<u>The Geodesic equation</u>: $\vec{a}=\frac{d^2x^\lambda}{d\tau^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$; (4)

we have: <u>(λ=r)</u>: $a^r=\frac{d^2x^r}{d\tau^2}+\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0\ \to$ (dτ→dt because dτ=dt/γ=t, as v<<c)

$$\to\ \frac{d^2r}{dt^2}+\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0=\frac{d^2r}{dt^2}+\sum_{\mu,\nu}\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=$$

$$=\frac{d^2r}{dt^2}+\sum_\mu[\Gamma^{r}_{\mu 1}\frac{dx^\mu}{dt}\frac{dx^1}{dt}+\Gamma^{r}_{\mu 2}\frac{dx^\mu}{dt}\frac{dx^2}{dt}]=\frac{d^2r}{dt^2}+\sum_\mu[\Gamma^{r}_{\mu r}\frac{dx^\mu}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{\mu\theta}\frac{dx^\mu}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{1r}\frac{dx^1}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{2r}\frac{dx^2}{dt}\frac{dx^r}{dt}]+[\Gamma^{r}_{1\theta}\frac{dx^1}{dt}\frac{dx^\theta}{dt}+\Gamma^{r}_{2\theta}\frac{dx^2}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}\frac{dx^r}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{\theta r}\frac{dx^\theta}{dt}\frac{dx^r}{dt}]+[\Gamma^{r}_{r\theta}\frac{dx^r}{dt}\frac{dx^\theta}{dt}+\Gamma^{r}_{\theta\theta}\frac{dx^\theta}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}\frac{dr}{dt}\frac{dr}{dt}+\Gamma^{r}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{r}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{r}_{\theta\theta}\frac{d\theta}{dt}\frac{d\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}(\frac{dr}{dt})^2+\Gamma^{r}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{r}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{r}_{\theta\theta}(\frac{d\theta}{dt})^2]=$$

$$=\frac{d^2r}{dt^2}+[0(\frac{dr}{dt})^2+0\frac{d\theta}{dt}\frac{dr}{dt}]+[0\frac{dr}{dt}\frac{d\theta}{dt}+(-r)(\frac{d\theta}{dt})^2]=\frac{d^2r}{dt^2}-r(\frac{d\theta}{dt})^2=\boxed{\ddot{r}-r\dot{\theta}^2=a^r=0.}\qquad(5)$$

<u>(λ=θ)</u>: $a^\theta=\frac{d^2x^\theta}{d\tau^2}+\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0\ \to$

$$\to\ \frac{d^2\theta}{dt^2}+\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0=\frac{d^2\theta}{dt^2}+\sum_{\mu,\nu}\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=\frac{d^2\theta}{dt^2}+\sum_\mu[\Gamma^{\theta}_{\mu 1}\frac{dx^\mu}{dt}\frac{dx^1}{dt}+\Gamma^{\theta}_{\mu 2}\frac{dx^\mu}{dt}\frac{dx^2}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+\sum_{\mu}[\Gamma^{\theta}_{\mu r}\frac{dx^{\mu}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{\mu\theta}\frac{dx^{\mu}}{dt}\frac{dx^{\theta}}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{1r}\frac{dx^{1}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{2r}\frac{dx^{2}}{dt}\frac{dx^{r}}{dt}]+[\Gamma^{\theta}_{1\theta}\frac{dx^{1}}{dt}\frac{dx^{\theta}}{dt}+\Gamma^{\theta}_{2\theta}\frac{dx^{2}}{dt}\frac{dx^{\theta}}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}\frac{dx^{r}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{\theta r}\frac{dx^{\theta}}{dt}\frac{dx^{r}}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dx^{r}}{dt}\frac{dx^{\theta}}{dt}+\Gamma^{\theta}_{\theta\theta}\frac{dx^{\theta}}{dt}\frac{dx^{\theta}}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}\frac{dr}{dt}\frac{dr}{dt}+\Gamma^{\theta}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{\theta}_{\theta\theta}\frac{d\theta}{dt}\frac{d\theta}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}(\frac{dr}{dt})^2+\Gamma^{\theta}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{\theta}_{\theta\theta}(\frac{d\theta}{dt})^2]=$$

$$=\frac{d^2\theta}{dt^2}+[0(\frac{dr}{dt})^2+\frac{1}{r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt}+0(\frac{d\theta}{dt})^2]=\frac{d^2\theta}{dt^2}+\frac{2}{r}\frac{d\theta}{dt}\frac{dr}{dt}=\ddot{\theta}+\frac{2}{r}\dot{\theta}\dot{r}=$$

$\boxed{=\frac{1}{r}(r\ddot{\theta}+2\dot{\theta}\dot{r})=a^{\theta}=0}$ (the $(2\dot{r}\dot{\theta})\hat{\theta}$ is the **Coriolis** component (acceleration). (6)

So, from the (4), (5) and (6) we have that:

$\boxed{\vec{a}=\vec{a}_{//}+\vec{a}_{\perp}=a^{\lambda}=a^{r}+a^{\theta}=\hat{r}(\ddot{r}-\dot{\theta}^2 r)+\frac{\hat{\theta}'}{r}(2\dot{r}\dot{\theta}+r\ddot{\theta})=\hat{r}(\ddot{r}-\dot{\theta}^2 r)+\hat{\theta}(2\dot{r}\dot{\theta}+r\ddot{\theta})}$, **that is exactly the (3)!**

($\vec{a}=0$ is required only if we claim the free fall)

($\hat{\theta}'=r\hat{\theta}$ as, in such a tensor context, all the versors must have the same measuring unit, out of consistency; therefore, if $[\hat{r}]=metre$, then the same must be for $\hat{\theta}'$, so $\hat{\theta}$ will get multiplied by r)

Let us recall the components we just obtained as the accelerations of the motion:
$a^{r}=(\ddot{r}-\dot{\theta}^2 r)\hat{r}$ and $a^{\theta}=(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\theta}$. Now, by putting ourselves in a gravitational field, we have that the force F=ma is radial, so:

$F^{r}=ma^{r}=m(\ddot{r}-\dot{\theta}^2 r)\hat{r}=-G\frac{Mm}{r^2}\hat{r}$ and

$F^{\theta}=ma^{\theta}=m(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\theta}=0$

The F^{θ} can be seen as: $(2\dot{r}\dot{\theta}+r\ddot{\theta})=0=\frac{d}{dt}(r^2\dot{\theta})$, because: $\frac{d}{dt}(r^2\dot{\theta})=(2r\dot{r})\dot{\theta}+r^2\ddot{\theta}=0$, from which: $(2\dot{r})\dot{\theta}+r\ddot{\theta}=0$, but if $\frac{d}{dt}(r^2\dot{\theta})=0\rightarrow$

$\rightarrow \boxed{r^2\dot{\theta}=const}$,

which is nothing but the expression for the **Second Kepler's Law**:

II-*A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

In fact, with reference to Fig. 3, we have for the small area A swept by r and the relevant areolar velocity A':

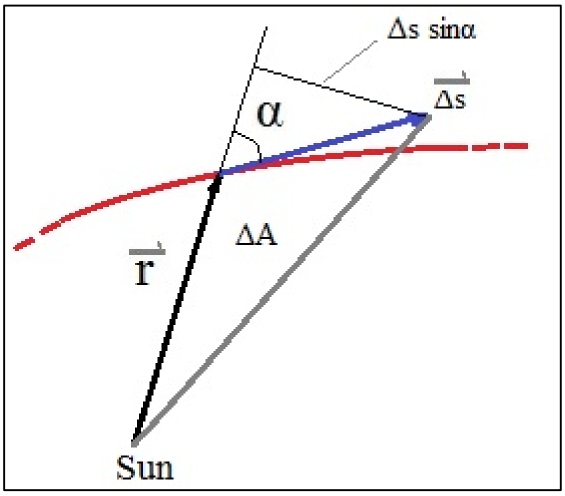


Fig. 3: On the areolar velocity.

$$\Delta A = \frac{1}{2} r \cdot \Delta s \sin\alpha = \frac{1}{2}(\vec{r} \times \Delta\vec{s}) \text{ and } \dot{A} = \frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}(\vec{r} \times \frac{\Delta\vec{s}}{\Delta t}) = \frac{1}{2}(\vec{r} \times \vec{v}) = \frac{1}{2}(rv\sin\alpha)(\hat{\theta}) =$$

$$= \frac{1}{2} r v^{\theta}(\hat{\theta}) = \frac{1}{2} r\omega r(\hat{\theta}) = \frac{1}{2} r^2 \dot{\theta}(\hat{\theta})$$, which is constant, due to the previous eq. $r^2\dot{\theta} = const$:

$$\boxed{\left|\frac{dA}{dt}\right| = \frac{1}{2} r^2 \dot{\theta} = const} \quad .$$

At last, let us evaluate the Ricci Tensor and the curvature scalar (of Ricci) R; with reference to Fig. 1, we have:

$$R_{\mu k} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^k} - \frac{\partial \Gamma^{\lambda}_{\mu k}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\lambda\eta} \quad (R_{\mu k} = R_{k\mu})$$

(μ=k=r): $\boxed{R_{rr}} = \frac{\partial \Gamma^{\lambda}_{r\lambda}}{\partial x^r} - \frac{\partial \Gamma^{\lambda}_{rr}}{\partial x^{\lambda}} + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta} = (\frac{\partial \Gamma^1_{r1}}{\partial r} + \frac{\partial \Gamma^2_{r2}}{\partial r}) - (\frac{\partial \Gamma^1_{rr}}{\partial x^1} + \frac{\partial \Gamma^2_{rr}}{\partial x^2}) + \sum_{\lambda,\eta} \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \sum_{\lambda,\eta} \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta} =$

$$= (\frac{\partial \Gamma^r_{rr}}{\partial r} + \frac{\partial \Gamma^{\theta}_{r\theta}}{\partial r}) - (\frac{\partial \Gamma^r_{rr}}{\partial r} + \frac{\partial \Gamma^{\theta}_{rr}}{\partial \theta}) + \sum_{\lambda} [\Gamma^1_{r\lambda}\Gamma^{\lambda}_{r1} + \Gamma^2_{r\lambda}\Gamma^{\lambda}_{r2}] - \sum_{\lambda} [\Gamma^1_{rr}\Gamma^{\lambda}_{\lambda1} + \Gamma^2_{rr}\Gamma^{\lambda}_{\lambda2}] =$$

$$= (\frac{\partial 0}{\partial r} + \frac{\partial (1/r)}{\partial r}) - (\frac{\partial 0}{\partial r} + \frac{\partial 0}{\partial \theta}) + \sum_{\lambda} [\Gamma^r_{r\lambda}\Gamma^{\lambda}_{rr} + \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}] - \sum_{\lambda} [\Gamma^r_{rr}\Gamma^{\lambda}_{\lambda r} + \Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}] =$$

$$= \frac{\partial (1/r)}{\partial r} + \sum_{\lambda} [0\Gamma^{\lambda}_{rr} + \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}] - \sum_{\lambda} [0\Gamma^{\lambda}_{\lambda r} + \Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}] = -\frac{1}{r^2} + \sum_{\lambda} \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta} - \sum_{\lambda} \Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta} =$$

$$= -\frac{1}{r^2} + [\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta} + \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}] - [\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta} + \Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}] = -\frac{1}{r^2} + [\Gamma^{\theta}_{r1}\Gamma^1_{r\theta} + \Gamma^{\theta}_{r2}\Gamma^2_{r\theta}] - [\Gamma^{\theta}_{rr}\Gamma^1_{1\theta} + \Gamma^{\theta}_{rr}\Gamma^2_{2\theta}] =$$

$$= -\frac{1}{r^2} + [\Gamma^{\theta}_{rr}\Gamma^r_{r\theta} + \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{r\theta}] - [\Gamma^{\theta}_{rr}\Gamma^r_{r\theta} + \Gamma^{\theta}_{rr}\Gamma^{\theta}_{\theta\theta}] = -\frac{1}{r^2} + [0\Gamma^r_{r\theta} + \frac{1}{r}\frac{1}{r}] - [0\Gamma^r_{r\theta} + 0\Gamma^{\theta}_{\theta\theta}] = \boxed{0\,.}$$

(μ=k=θ) $\boxed{R_{\theta\theta}} = \frac{\partial \Gamma^{\lambda}_{\theta\lambda}}{\partial x^{\theta}} - \frac{\partial \Gamma^{\lambda}_{\theta\theta}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta} - \Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta} = (\frac{\partial \Gamma^1_{\theta1}}{\partial \theta} + \frac{\partial \Gamma^2_{\theta2}}{\partial \theta}) - (\frac{\partial \Gamma^1_{\theta\theta}}{\partial x^1} + \frac{\partial \Gamma^2_{\theta\theta}}{\partial x^2}) + \sum_{\lambda,\eta} \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta} - \sum_{\lambda,\eta} \Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta} =$

$$= (\frac{\partial \Gamma^r_{\theta r}}{\partial \theta} + \frac{\partial \Gamma^{\theta}_{\theta\theta}}{\partial \theta}) - (\frac{\partial \Gamma^r_{\theta\theta}}{\partial x^r} + \frac{\partial \Gamma^{\theta}_{\theta\theta}}{\partial x^{\theta}}) + \sum_{\lambda} [\Gamma^1_{\theta\lambda}\Gamma^{\lambda}_{\theta1} + \Gamma^2_{\theta\lambda}\Gamma^{\lambda}_{\theta2}] - \sum_{\lambda} [\Gamma^1_{\theta\theta}\Gamma^{\lambda}_{\lambda1} + \Gamma^2_{\theta\theta}\Gamma^{\lambda}_{\lambda2}] =$$

$$= (\frac{\partial 0}{\partial \theta} + \frac{\partial 0}{\partial \theta}) - (\frac{\partial (-r)}{\partial r} + \frac{\partial 0}{\partial \theta}) + \sum_{\lambda} [\Gamma^r_{\theta\lambda}\Gamma^{\lambda}_{\theta r} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta}] - \sum_{\lambda} [\Gamma^r_{\theta\theta}\Gamma^{\lambda}_{\lambda r} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\theta}] =$$

$$= 1 + \sum_{\lambda} [\Gamma^r_{\theta\lambda}\Gamma^{\lambda}_{\theta r} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta}] - \sum_{\lambda} [(-r)\Gamma^{\lambda}_{\lambda r} + 0\Gamma^{\lambda}_{\lambda\theta}] =$$

$$= 1 + \sum_{\lambda} [\Gamma^r_{\theta\lambda}\Gamma^{\lambda}_{\theta r} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta}] + \sum_{\lambda} r\Gamma^{\lambda}_{\lambda r} = 1 + [\Gamma^r_{\theta1}\Gamma^1_{\theta r} + \Gamma^r_{\theta2}\Gamma^2_{\theta r}] + [\Gamma^{\theta}_{\theta1}\Gamma^1_{\theta\theta} + \Gamma^{\theta}_{\theta2}\Gamma^2_{\theta\theta}] + [r\Gamma^1_{1r} + r\Gamma^2_{2r}] =$$

$$= 1+[\Gamma^r_{\theta r}\Gamma^r_{\theta r}+\Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}]+[\Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta}+\Gamma^\theta_{\theta\theta}\Gamma^\theta_{\theta\theta}]+[r\Gamma^r_{rr}+r\Gamma^\theta_{\theta r}] = 1+[0\cdot 0+(-r)\frac{1}{r}]+[\frac{1}{r}(-r)+0\cdot 0]+[r\cdot 0+r\frac{1}{r}] =$$

$$= 1+[0\cdot 0-1]+[-1+0\cdot 0]+[0+1]=\boxed{0\ .}$$

(μ=r and k=θ or vice versa): $\boxed{R_{r\theta}=R_{\theta r}} = \frac{\partial\Gamma^\lambda_{r\lambda}}{\partial x^\theta}-\frac{\partial\Gamma^\lambda_{r\theta}}{\partial x^\lambda}+\Gamma^\eta_{r\lambda}\Gamma^\lambda_{\theta\eta}-\Gamma^\eta_{r\theta}\Gamma^\lambda_{\lambda\eta} =$

$$= \frac{\partial\Gamma^1_{r1}}{\partial\theta}+\frac{\partial\Gamma^2_{r2}}{\partial\theta}-\frac{\partial\Gamma^1_{r\theta}}{\partial x^1}-\frac{\partial\Gamma^2_{r\theta}}{\partial x^2}+\sum_{\lambda,\eta}\Gamma^\eta_{r\lambda}\Gamma^\lambda_{\theta\eta}-\sum_{\lambda,\eta}\Gamma^\eta_{r\theta}\Gamma^\lambda_{\lambda\eta} = \frac{\partial\Gamma^r_{rr}}{\partial\theta}+\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}-\frac{\partial\Gamma^r_{r\theta}}{\partial x^r}-\frac{\partial\Gamma^\theta_{r\theta}}{\partial x^\theta}+$$

$$+\sum_\lambda [\Gamma^1_{r\lambda}\Gamma^\lambda_{\theta 1}+\Gamma^2_{r\lambda}\Gamma^\lambda_{\theta 2}]-\sum_\lambda [\Gamma^1_{r\theta}\Gamma^\lambda_{\lambda 1}+\Gamma^2_{r\theta}\Gamma^\lambda_{\lambda 2}] = \frac{\partial\Gamma^r_{rr}}{\partial\theta}+\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}-\frac{\partial\Gamma^r_{r\theta}}{\partial r}-\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}+$$

$$+\sum_\lambda [\Gamma^r_{r\lambda}\Gamma^\lambda_{\theta r}+\Gamma^\theta_{r\lambda}\Gamma^\lambda_{\theta\theta}]-\sum_\lambda [\Gamma^r_{r\theta}\Gamma^\lambda_{\lambda r}+\Gamma^\theta_{r\theta}\Gamma^\lambda_{\lambda\theta}] = \frac{\partial 0}{\partial\theta}+\frac{\partial(1/r)}{\partial\theta}-\frac{\partial 0}{\partial r}-\frac{\partial(1/r)}{\partial\theta}+$$

$$+\sum_\lambda [\Gamma^r_{r\lambda}\Gamma^\lambda_{\theta r}+\Gamma^\theta_{r\lambda}\Gamma^\lambda_{\theta\theta}]-\sum_\lambda [0\Gamma^\lambda_{\lambda r}+\frac{1}{r}\Gamma^\lambda_{\lambda\theta}] = 0+0-0-0+[\Gamma^r_{r1}\Gamma^1_{\theta r}+\Gamma^r_{r2}\Gamma^2_{\theta r}]+[\Gamma^\theta_{r1}\Gamma^1_{\theta\theta}+\Gamma^\theta_{r2}\Gamma^2_{\theta\theta}]-[\frac{1}{r}\Gamma^1_{1\theta}+\frac{1}{r}\Gamma^2_{2\theta}] =$$

$$= [\Gamma^r_{rr}\Gamma^r_{\theta r}+\Gamma^r_{r\theta}\Gamma^\theta_{\theta r}]+[\Gamma^\theta_{rr}\Gamma^r_{\theta\theta}+\Gamma^\theta_{r\theta}\Gamma^\theta_{\theta\theta}]-[\frac{1}{r}\Gamma^r_{r\theta}+\frac{1}{r}\Gamma^\theta_{\theta\theta}] = [0\cdot 0+0\frac{1}{r}]+[0\Gamma^r_{\theta\theta}+\Gamma^\theta_{r\theta}0]-[\frac{1}{r}0+\frac{1}{r}0]=\boxed{0\ .}$$

By summing things up: $\boxed{R_{rr}=0\ ,\ \ R_{\theta\theta}=0\ \ ,\ \ R_{r\theta}=R_{\theta r}=0\ .}$

Finally, for what the curvature scalar R is concerned:

$$\boxed{R} = g^{\mu k}R_{\mu k} = g^{rr}R_{rr}+g^{\theta\theta}R_{\theta\theta} = 1\cdot 0+\frac{1}{r^2}0=\boxed{0}\ .$$

We just got R=0 because, as a matter of facts, the polar plane (r,θ) we based on, is, in the end, plane/flat indeed, as it stays on the paper sheet.

-<u>Conversely, by considering the surface of a sphere and still keeping into two dimensions (θ,φ; r=const), we have</u>:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

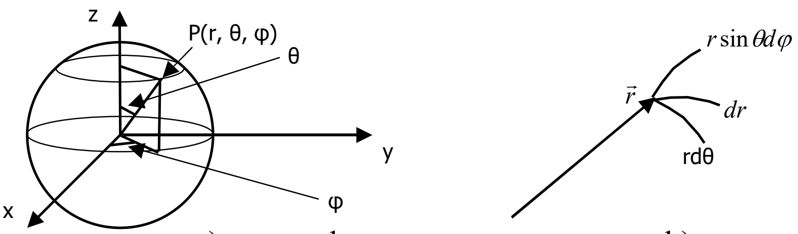


Fig. 4 a) and b)

φ spans between 0 and 2π, while θ spans between 0 and π. Moreover, of course:

$$\begin{cases} r^2 = x^2+y^2+z^2 \\ \varphi = arctg\,\frac{y}{x} \\ \theta = \arccos\frac{z}{r} \end{cases}$$

Well, now, from the Fig. 4 b), we understand that over the surface of the sphere, which is the essence of a sphere (so we are dealing with two dimensions) the square modulus $d\tau^2$ of a vector is: $(d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu\ ,\ \ \mu,\nu \to \theta,\varphi)$

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = -g_{\theta\theta}dx^\theta dx^\theta - g_{\varphi\varphi}dx^\varphi dx^\varphi - 2g_{\theta\varphi}dx^\theta dx^\varphi = -g_{\theta\theta}d\theta^2 - g_{\varphi\varphi}d\varphi^2 - 2g_{\theta\varphi}\cancel{d\theta}d\varphi =$$

$$= -r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2$$ (that's a simple use of the Pythagorean Theorem over a spherical surface)

and here we see that we have no cross terms $d\theta d\varphi$, from which we have that the metric tensor g is diagonal, so the cross terms are null ($g_{\theta\varphi} = g_{\varphi\theta} = 0$) and moreover, being that $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu} \rightarrow$ ($g_{\mu\nu}g^{\mu\nu} \equiv "1"$) $\rightarrow$ $g_{\mu\nu} \equiv \dfrac{1}{g^{\mu\nu}} \rightarrow$

$$\boxed{\rightarrow g_{\theta\theta} = r^2 \text{ (and } g^{\theta\theta} = \frac{1}{r^2}\text{)}, \; g_{\varphi\varphi} = r^2 \sin^2\theta \text{ (and } g^{\varphi\varphi} = \frac{1}{r^2 \sin^2\theta}\text{) and } \; g_{\theta\varphi} = g_{\varphi\theta} = g^{\theta\varphi} = g^{\varphi\theta} = 0 .}$$

Let us explore the variability over λ; for $\lambda = \varphi$ into the expression for the Christoffel Symbols (where φ is one of the two coordinates θ, φ , about the specific case of a sphere), we have:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right) = \Gamma^{\varphi}_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right) \tag{7}$$

Now, with $\mu, \nu \rightarrow \theta, \varphi$ in the (7), it follows that:

$$\Gamma^{\varphi}_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right) = \Gamma^{\varphi}_{\theta\varphi} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\varphi\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \varphi} - \frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}}\right) = \Gamma^{\varphi}_{\varphi\theta} =$$

$= [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\varphi\sigma}}{\partial \theta}] + [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \varphi}] - [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just the variable index σ , as a matter of fact, is repeated):

$$\boxed{\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta}} = [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\varphi 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\varphi 2}}{\partial \theta}] + [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \varphi} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \varphi}] - [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^1} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^2}] =$$

$$= [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\varphi\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\varphi\varphi}}{\partial \theta}] + [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \varphi}] - [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^{\theta}} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^{\varphi}}] =$$

$$= [\frac{1}{2} 0 \cdot 0 + \frac{1/2}{(r^2 \sin^2\theta)} \frac{\partial (r^2 \sin^2\theta)}{\partial \theta}] + [\frac{1}{2} 0 \frac{\partial (\cancel{r^2})}{\cancel{\partial} \varphi} + \frac{1}{2} g^{\varphi\varphi} 0] - [\frac{1}{2} 0 \cdot 0 + \frac{1}{2} g^{\varphi\varphi} 0] = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} 2r^2 \sin\theta \cos\theta =$$

$$\boxed{= \cot\theta .}$$

Conversely, if in the (7) we set $\mu, \nu \rightarrow \theta, \theta$, then it follows that:

$$\Gamma^{\varphi}_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right) = \Gamma^{\varphi}_{\theta\theta} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}\right) =$$

$= [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \theta}] + [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \theta}] - [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just σ , as a matter of fact, is repeated):

$$\boxed{\Gamma^{\varphi}_{\theta\theta}} = [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \theta}] + [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \theta}] - [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta\theta}}{\partial x^1} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta\theta}}{\partial x^2}] =$$

$$= [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \theta}] + [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \theta}] - [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi}] =$$

$$= [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} 0] + [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} 0] - [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi}] = -\frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi} =$$

$$= -\frac{1}{2} g^{\varphi\varphi} \frac{\partial (r^2)}{\partial \varphi} = \boxed{0 \; .}$$

At last, if in the (7) we set $\mu, \nu \rightarrow \varphi, \varphi$, then it follows that:

$$\Gamma^{\varphi}_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right) = \Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\varphi\sigma}}{\partial \varphi} + \frac{\partial g_{\varphi\sigma}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}}\right) =$$

$=[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]+[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]-[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just σ, as a matter of fact, is repeated):

$$\boxed{\Gamma^{\varphi}_{\varphi\varphi}}=[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi 1}}{\partial\varphi}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi 2}}{\partial\varphi}]+[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi 1}}{\partial\varphi}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi 2}}{\partial\varphi}]-[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{1}}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{2}}]=$$

$$=[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]+[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=$$

$$=[\frac{1}{2}0\cdot 0+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]+[\frac{1}{2}0\cdot 0+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]-[\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}=$$

$$=\frac{1}{2}\frac{1}{r^{2}\sin^{2}\theta}\frac{\partial(r^{2}\sin^{2}\theta)}{\partial\varphi}=\frac{1}{2}\frac{1}{r^{2}\sin^{2}\theta}0=\boxed{0\,.}$$

Now we set into the expression for $\Gamma^{\lambda}_{\mu\nu}$ $\lambda=\theta$ (where θ is one of our two coordinates θ,φ , on the specific case of a sphere); then, we have:

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\sigma\lambda}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}) \tag{8}$$

Now, with $\mu,\nu\rightarrow\theta,\varphi$ in the (8), it follows that:

$$\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\theta\varphi}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\varphi\sigma}}{\partial\theta}+\frac{\partial g_{\theta\sigma}}{\partial\varphi}-\frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\varphi\theta}=$$

$=[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\sigma}}{\partial\theta}]+[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\varphi}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just σ, as a matter of fact, is repeated):

$$\boxed{\Gamma^{\theta}_{\theta\varphi}=\Gamma^{\theta}_{\varphi\theta}}=[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi 1}}{\partial\theta}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi 2}}{\partial\theta}]+[\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\varphi}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\varphi}]-[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{1}}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{2}}]=$$

$$=[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta}]+[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\theta}}{\partial\varphi}]=$$

$$=[\frac{1}{2}g^{\theta\theta}0+\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\theta}]+[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}0\cdot 0]-[\frac{1}{2}g^{\theta\theta}0+\frac{1}{2}0\cdot 0]=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}=\frac{1}{2}\frac{1}{r^{2}}\frac{\partial r^{2}}{\partial\varphi}=\boxed{0\,.}$$

Conversely, if in the (8) we set $\mu,\nu\rightarrow\theta,\theta$, then it follows that:

$$\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\theta\theta}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{\partial g_{\theta\sigma}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$$

$=[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]+[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}]=[g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just σ, as a matter of fact, is repeated):

$$\boxed{\Gamma^{\theta}_{\theta\theta}}=[g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\theta}+g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\theta}]-[\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta\theta}}{\partial x^{1}}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta\theta}}{\partial x^{2}}]=[g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\theta}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}]=$$

$$=[g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+0\cdot 0]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial\varphi}]=-\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}=-\frac{1}{2}\frac{1}{r^{2}}\frac{\partial(r^{2})}{\partial\theta}\boxed{=0}\,.$$

At last, if in the (8) we set $\mu,\nu\rightarrow\varphi,\varphi$, then it follows that:

$$\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\varphi\varphi}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\varphi\sigma}}{\partial\varphi}+\frac{\partial g_{\theta\sigma}}{\partial\varphi}-\frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}})=$$

$=[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]+[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\varphi}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}}]$ and being in use the Einstein Convention on the summations on repeated indices in a term, then we have (as just σ, as a matter of fact, is repeated):

$$\boxed{\Gamma^{\theta}_{\varphi\varphi}}=[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi 1}}{\partial\varphi}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi 2}}{\partial\varphi}]+[\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\varphi}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\varphi}]-[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^{1}}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^{2}}]=$$

$$=[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]+[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=$$

$$=[\frac{1}{2}g^{\theta\theta}0+\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}00]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=$$

$$=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}-\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta}=\frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2)}{\partial\varphi}-\frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2\sin^2\theta)}{\partial\theta}=0-\frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2\sin^2\theta)}{\partial\theta}=-\sin\theta\cos\theta=$$

$$=\boxed{-\frac{1}{2}\sin 2\theta\,.}$$

By summing things up, we got, about the metric tensor g:

$$g_{\theta\theta}=r^2\ ,\ g^{\theta\theta}=\frac{1}{r^2}\quad,\quad g_{\varphi\varphi}=r^2\sin^2\theta\ ,\ g^{\varphi\varphi}=\frac{1}{r^2\sin^2\theta}\ ,\ g_{\theta\varphi}=g_{\varphi\theta}=g^{\theta\varphi}=g^{\varphi\theta}=0\ ,$$

while for what the Christoffel Symbols Γ are concerned, we got:

$$\Gamma^{\varphi}_{\theta\varphi}=\Gamma^{\varphi}_{\varphi\theta}=\cot\theta\quad,\quad\Gamma^{\theta}_{\varphi\varphi}=-\frac{1}{2}\sin 2\theta\quad,\quad\Gamma^{\theta}_{\theta\varphi}=\Gamma^{\theta}_{\varphi\theta}=\Gamma^{\varphi}_{\varphi\varphi}=\Gamma^{\varphi}_{\theta\theta}=\Gamma^{\theta}_{\theta\theta}=0\ .$$

At this point, we evaluate the Ricci Tensor $R_{\mu k}=\frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{k}}-\frac{\partial\Gamma^{\lambda}_{\mu k}}{\partial x^{\lambda}}+\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta}-\Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\lambda\eta}$:

$$\boxed{R_{\theta\theta}}=\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta}-\frac{\partial\Gamma^{\lambda}_{\theta\theta}}{\partial x^{\lambda}}+\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}-\Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta}=\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta}-\frac{\partial 0}{\partial x^{\lambda}}+\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}-0\Gamma^{\lambda}_{\lambda\eta}=[\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta}]+[\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}]=$$

($\Gamma^{\lambda}_{\theta\theta}=\Gamma^{\eta}_{\theta\theta}=0$ as $\Gamma^{?}_{\theta\theta}$ is always zero)

$$=[\sum_{\lambda}\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta}]+[\sum_{\lambda,\eta}\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}]=[\sum_{\lambda}\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta}]+\sum_{\lambda}\sum_{\eta}\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}=[\frac{\partial\Gamma^{1}_{\theta 1}}{\partial\theta}+\frac{\partial\Gamma^{2}_{\theta 2}}{\partial\theta}]+[\sum_{\eta}\Gamma^{\eta}_{\theta 1}\Gamma^{1}_{\theta\eta}+\sum_{\eta}\Gamma^{\eta}_{\theta 2}\Gamma^{2}_{\theta\eta}]=$$

$$=[\frac{\partial\Gamma^{\theta}_{\theta\theta}}{\partial\theta}+\frac{\partial\Gamma^{\varphi}_{\theta\varphi}}{\partial\theta}]+[\sum_{\eta}\Gamma^{\eta}_{\theta\theta}\Gamma^{\theta 1}_{\theta\eta}+\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\theta\eta}]=[0+\frac{\partial\cot\theta}{\partial\theta}]+[\sum_{\eta}0\Gamma^{\theta 1}_{\theta\eta}+\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\theta\eta}]=$$

$$=-\frac{1}{\sin^2\theta}+\Gamma^{1}_{\theta\varphi}\Gamma^{\varphi}_{\theta 1}+\Gamma^{2}_{\theta\varphi}\Gamma^{\varphi}_{\theta 2}=-\frac{1}{\sin^2\theta}+\Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\theta\theta}+\Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\theta\varphi}=-\frac{1}{\sin^2\theta}+\Gamma^{\theta}_{\theta\varphi}0+(\Gamma^{\varphi}_{\theta\varphi})^2=-\frac{1}{\sin^2\theta}+\cot^2\theta=$$

$$=-\frac{1}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{\cos^2\theta-1}{\sin^2\theta}=\boxed{-1\,.}$$

$$\boxed{R_{\varphi\varphi}}=\frac{\partial\Gamma^{\lambda}_{\varphi\lambda}}{\partial x^{\varphi}}-\frac{\partial\Gamma^{\lambda}_{\varphi\varphi}}{\partial x^{\lambda}}+\Gamma^{\eta}_{\varphi\lambda}\Gamma^{\lambda}_{\varphi\eta}-\Gamma^{\eta}_{\varphi\varphi}\Gamma^{\lambda}_{\lambda\eta}=\frac{\partial\Gamma^{1}_{\varphi 1}}{\partial\varphi}+\frac{\partial\Gamma^{2}_{\varphi 2}}{\partial\varphi}-\frac{\partial\Gamma^{1}_{\varphi\varphi}}{\partial x^{1}}-\frac{\partial\Gamma^{2}_{\varphi\varphi}}{\partial x^{2}}+\sum_{\lambda}\sum_{\eta}\Gamma^{\eta}_{\varphi\lambda}\Gamma^{\lambda}_{\varphi\eta}-\sum_{\lambda}\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{\lambda}_{\lambda\eta}=$$

$$=[\frac{\partial\Gamma^{\theta}_{\varphi\theta}}{\partial\varphi}+\frac{\partial\Gamma^{\varphi}_{\varphi\varphi}}{\partial\varphi}-\frac{\partial\Gamma^{\theta}_{\varphi\varphi}}{\partial\theta}-\frac{\partial\Gamma^{\varphi}_{\varphi\varphi}}{\partial\varphi}]+[\sum_{\eta}\Gamma^{\eta}_{\varphi 1}\Gamma^{1}_{\varphi\eta}+\sum_{\eta}\Gamma^{\eta}_{\varphi 2}\Gamma^{2}_{\varphi\eta}]-[\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{1}_{1\eta}+\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{2}_{2\eta}]=$$

$$=[0+0-\frac{\partial\Gamma^{\theta}_{\varphi\varphi}}{\partial\theta}-0]+[\sum_{\eta}\Gamma^{\eta}_{\varphi\theta}\Gamma^{\theta}_{\varphi\eta}+\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\eta}]-[\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{\theta}_{\theta\eta}+\sum_{\eta}\Gamma^{\eta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\eta}]=$$

$$=-\frac{\partial(-\frac{1}{2}\sin 2\theta)}{\partial\theta}+[(\Gamma^{1}_{\varphi\theta}\Gamma^{\theta}_{\varphi 1}+\Gamma^{2}_{\varphi\theta}\Gamma^{\theta}_{\varphi 2})+(\Gamma^{1}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 1}+\Gamma^{2}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 2})]-[(\Gamma^{1}_{\varphi\varphi}\Gamma^{\theta}_{\theta 1}+\Gamma^{2}_{\varphi\varphi}\Gamma^{\theta}_{\theta 2})+(\Gamma^{1}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 1}+\Gamma^{2}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 2})]=$$

$$= \cos 2\theta + [(\Gamma^{\theta}_{\varphi\theta}\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi})] - [(\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\theta}_{\theta\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi})] =$$

$$= \cos 2\theta + [(0\cdot 0 + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + 0\cdot 0)] - [(\Gamma^{\theta}_{\varphi\varphi}0 + 0\Gamma^{\theta}_{\theta\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + 0\cdot 0)] = \cos 2\theta + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi} =$$

$$= \cos 2\theta - \frac{1}{2}\cot\theta \sin 2\theta = (2\cos^2\theta - 1) - \frac{1}{2}\cot\theta(2\sin\theta\cos\theta) = 2\cos^2\theta - 1 - \cos^2\theta = \boxed{-\sin^2\theta\ .}$$

$$\boxed{R_{\theta\varphi}} = R_{\varphi\theta} = \frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial x^{\varphi}} - \frac{\partial\Gamma^{\lambda}_{\theta\varphi}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\varphi\eta} - \Gamma^{\eta}_{\theta\varphi}\Gamma^{\lambda}_{\lambda\eta} = \frac{\partial\Gamma^{1}_{\theta 1}}{\partial\varphi} + \frac{\partial\Gamma^{2}_{\theta 2}}{\partial\varphi} - \frac{\partial\Gamma^{1}_{\theta\varphi}}{\partial x^{1}} - \frac{\partial\Gamma^{2}_{\theta\varphi}}{\partial x^{2}} + \sum_{\lambda}\sum_{\eta}\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\varphi\eta} - \sum_{\lambda}\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\lambda}_{\lambda\eta} =$$

$$= \frac{\partial\Gamma^{\theta}_{\theta\theta}}{\partial\varphi} + \frac{\partial\Gamma^{\varphi}_{\theta\varphi}}{\partial\varphi} - \frac{\partial\Gamma^{\theta}_{\theta\varphi}}{\partial\theta} - \frac{\partial\Gamma^{\varphi}_{\theta\varphi}}{\partial\varphi} + [\sum_{\eta}\Gamma^{\eta}_{\theta 1}\Gamma^{1}_{\varphi\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta 2}\Gamma^{2}_{\varphi\eta}] - [\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{1}_{1\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{2}_{2\eta}] =$$

$$= 0 + \frac{\partial\cot\theta}{\partial\varphi} - 0 - \frac{\partial\cot\theta}{\partial\varphi} + [\sum_{\eta}\Gamma^{\eta}_{\theta\theta}\Gamma^{\theta}_{\varphi\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] - [\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\theta}_{\theta\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] =$$

$$= 0 + [\sum_{\eta}\Gamma^{\eta}_{\theta\theta}\Gamma^{\theta}_{\varphi\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] - [\sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\theta}_{\theta\eta} + \sum_{\eta}\Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] =$$

$$= [\Gamma^{1}_{\theta\theta}\Gamma^{\theta}_{\varphi 1} + \Gamma^{2}_{\theta\theta}\Gamma^{\theta}_{\varphi 2} + \Gamma^{1}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 2}] - [\Gamma^{1}_{\theta\varphi}\Gamma^{\theta}_{\theta 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\theta}_{\theta 2} + \Gamma^{1}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 2}] =$$

$$= [\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\varphi\varphi} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\varphi}] - [\Gamma^{\theta}_{\theta\varphi}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\theta\varphi} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\varphi}] =$$

$$= [0\cdot 0 + 0\cdot 0 + 0\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}0] - [0\cdot 0 + \Gamma^{\varphi}_{\theta\varphi}0 + 0\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}0] = \boxed{0\ .}$$

By summing things up again: $R_{\theta\theta} = -1$, $R_{\varphi\varphi} = -\sin^2\theta$ and $R_{\theta\varphi} = R_{\varphi\theta} = 0$.

Finally, we can calculate the scalar R, by contracting the Ricci tensor with the metric tensor g:

$$\boxed{R} = g^{\mu k}R_{\mu k} = g^{1k}R_{1k} + g^{2k}R_{2k} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} = g^{\theta\theta}R_{\theta\theta} + g^{\theta\varphi}R_{\theta\varphi} + g^{\varphi\theta}R_{\varphi\theta} + g^{\varphi\varphi}R_{\varphi\varphi} =$$

$$= \frac{1}{r^2}(-1) + 0\cdot 0 + 0\cdot 0 + (\frac{1}{r^2\sin^2\theta})(-\sin^2\theta) = -\frac{1}{r^2} - \frac{\sin^2\theta}{r^2\sin^2\theta} = \boxed{-\frac{2}{r^2}}$$

; therefore, the curvature R decreases as long as r is increased.

PARALLEL TRANSPORT AND RIEMANN TENSOR

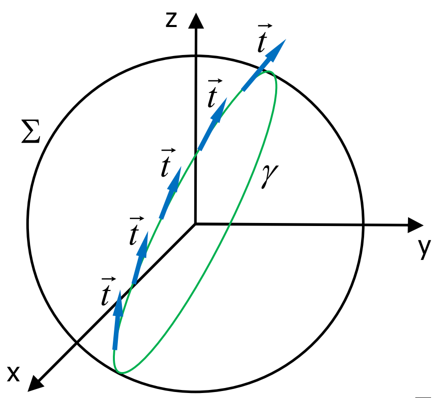


Fig. 5: parallel transport.

Let us consider a small ant in one dimension, which lives on the curve γ (which lies on the surface Σ of the sphere); well, for such a little ant, which does not know anything of the other dimensions, going straight ahead means moving like the tangent vector $\vec{t}$ does; in fact, in every position of that curve, the little ant always keeps the same position details, that are those of the tangent, also because its only versor of its local/personal reference frame is really $\vec{t}$; therefore, so doing, it transports itself parallely. A different situation is that of a tridimensional observer who is integer with the system x,y,z, according to which the small ant, while moving, continuously changes direction.

If the small ant turns around completely, so returning to the start point, then the vector $\vec{t}$ as well, in turning around, will find itself as it was at the start point. And if the path γ was a line, then along such a path we would see $\vec{t}$ having positions which would really be parallel one another.
Then, if we want to have the derivative of a vector along a generic direction $\vec{u}$ (whose components are the u^ν), it's enough to project the derivative $V^\mu_{;\nu}$ along $\vec{u}$, indeed:

$$\nabla_{\vec{u}}(\vec{V}) = u^\nu (V^\mu_{;\nu}\hat{e}_\mu) = u^\nu (\frac{dV^\mu}{dx^\nu} + V^\sigma \Gamma^\mu_{\sigma\nu})\hat{e}_\mu$$

and so, for what has been said so far, the parallel transport of a vector $\vec{V}$ along a direction $\vec{u}$ implies that $V^\mu_{;\nu} = 0 \rightarrow \nabla_{\vec{u}}(\vec{V}) = 0$.
In a curved space (non Euclidean), the parallel transport of a vector on a closed infinitesimal loop, will not bring that vector to the start point and leaving it exactly in a position perfectly parallel to the vector at the start; on this purpose, we imagine to move a vector $\vec{V} = V^1\hat{e}_1 + V^2\hat{e}_2$ along an infinitesimal closed loop $\delta\gamma$, starting from A, like in Fig. 6:

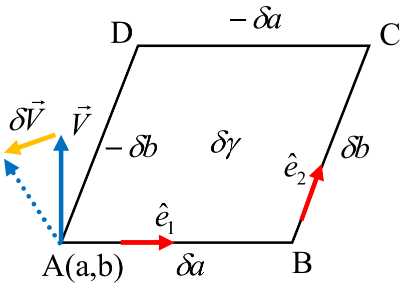


Fig. 6: infinitesimal closed loop.

If at the return to point A the vector $\vec{V}$ appears rotated, then we will be able to say that the surface in δγ is curved: the first step is the parallel transport, from A(a,b) to B, of $\vec{V}$, along $\hat{e}_1$ and parallel transport, according to what has been said so far, means that: $\nabla_{\hat{e}_1}(\vec{V}) = 0$, so:

$$\nabla_{\hat{e}_1}(\vec{V}) = const(V^\mu_{;1}\hat{e}_\mu) = const(\frac{dV^\mu}{dx^1} + V^\sigma\Gamma^\mu_{\sigma 1})\hat{e}_\mu = 0 \rightarrow$$

$$\rightarrow \frac{dV^\mu}{dx^1} = -V^\sigma\Gamma^\mu_{\sigma 1} \text{ , from which: } dV^\mu = -V^\sigma\Gamma^\mu_{\sigma 1}dx^1 \rightarrow$$

$$\rightarrow \int_A^B dV^\mu = V^\mu(B) - V^\mu(A) = -\int_A^B V^\sigma\Gamma^\mu_{\sigma 1}dx^1 \text{ ; so, in B, } V^\mu \text{ will be:}$$

$$V^\mu(B) = V^\mu(A) + (\int_A^B dx^1 \frac{\partial V^\mu}{\partial x^1})_{with\_x^2=b} = V^\mu(A) - \int_a^{a+\delta a} dx^1 V^\sigma\Gamma^\mu_{\sigma 1} \text{ ;} \qquad (9)$$

(in fact, the integral is in dx^1, between A and B, and x^1 is a in A and a+δa in B)

Moreover, about the jump from B to C along $\hat{e}_2$, we have: $(\frac{dV^\mu}{dx^2} + V^\sigma\Gamma^\mu_{\sigma 2}) = 0$, as, by moving (this time) along $\hat{e}_2$, the space coordinate of derivation, that is of variation, will be obviously x^2, so: $dV^\mu = -[V^\sigma\Gamma^\mu_{\sigma 2}]dx^2$, but that's not all, as here we do not have to neglect the fact that on the path BC on which we are working, we have that $x^1 = b + \delta b$ and so the quantity between the square brackets needs to be updated, through Taylor, along x^1 indeed (along δa), that is:

$$[V^\sigma\Gamma^\mu_{\sigma 2}] \rightarrow [V^\sigma\Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma\Gamma^\mu_{\sigma 2})\delta a] \text{ and so, as well as for the previous case:}$$

$$V^{\mu}(C) = V^{\mu}(B) - (\int_B^C V^{\sigma}\Gamma^{\mu}_{\sigma 2} dx^2)_{with\_x^1=b+\delta b} = V^{\mu}(B) - \int_b^{b+\delta b} [V^{\sigma}\Gamma^{\mu}_{\sigma 2} + \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2 \ ; \qquad (10)$$

(in fact, the integral is in dx^2, between B and C, and x^2 is b in B and b+δb in C)

Similarly, about the jump C→D and also by reminding that here we are moving along $-\hat{e}_1$:

$$V^{\mu}(D) = V^{\mu}(C) + (\int_C^D dx^1 \frac{\partial V^{\mu}}{\partial x^1})_{x^2=b+\delta b} = V^{\mu}(C) - (\int_{a+\delta a}^{a} dx^1 V^{\sigma}\Gamma^{\mu}_{\sigma 1})_{x^2=b+\delta b} =$$

(still according to Taylor, along x^2 (along δb), as in such a path, x^2 is not just b, but rather b+δb →)

$$= V^{\mu}(C) - \int_{a+\delta a}^{a} [V^{\sigma}\Gamma^{\mu}_{\sigma 1} + \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1 \qquad (11)$$

and at last, on jump D→A and also reminding that here we are moving along $-\hat{e}_2$:

$$V^{\mu}(A_{END}) = V^{\mu}(D) + (\int_D^{A_{END}} dx^2 \frac{\partial V^{\mu}}{\partial x^2})_{x^1=a} = V^{\mu}(D) - \int_{b+\delta b}^{b} dx^2 V^{\sigma}\Gamma^{\mu}_{\sigma 2} \ . \qquad (12)$$

(and no Taylor expansion here, along x^1, as on this path x^1 is always a and not a+δa)

Therefore, regarding the alleged variation $\delta\vec{V}$: $\delta\vec{V} = V^{\mu}(A_{END}) - V^{\mu}(A_{START})$ and $V^{\mu}(A_{END})$ comes from the (12), while $V^{\mu}(A_{START})$ is deduced by the (9), that is: $V^{\mu}(A) = V^{\mu}(B) + \int_a^{a+\delta a} V^{\sigma}\Gamma^{\mu}_{\sigma 1} dx^1 \ \rightarrow$

→ $\delta\vec{V} = V^{\mu}(D) - \int_{b+\delta b}^{b} V^{\sigma}\Gamma^{\mu}_{\sigma 2} dx^2 - V^{\mu}(B) - \int_a^{a+\delta a} V^{\sigma}\Gamma^{\mu}_{\sigma 1} dx^1$ and using the (10) and (11) respectively to have $V^{\mu}(B)$ and $V^{\mu}(D)$:

$$V^{\mu}(B) = V^{\mu}(C) + \int_b^{b+\delta b} [V^{\sigma}\Gamma^{\mu}_{\sigma 2} + \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2 \quad \text{(deduced by the (10))}$$

$$V^{\mu}(D) = V^{\mu}(C) - \int_{a+\delta a}^{a} [V^{\sigma}\Gamma^{\mu}_{\sigma 1} + \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1 \quad \text{(the (11), unchanged)}$$

from which, by substitution, we have:

$$\delta\vec{V} = \{V^{\mu}(D)\} - \int_{b+\delta b}^{b} V^{\sigma}\Gamma^{\mu}_{\sigma 2} dx^2 - \{V^{\mu}(B)\} - \int_a^{a+\delta a} V^{\sigma}\Gamma^{\mu}_{\sigma 1} dx^1 = \{\cancel{V^{\mu}(C)} - \int_{a+\delta a}^{a} [V^{\sigma}\Gamma^{\mu}_{\sigma 1} + \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1\} -$$

$$- \int_{b+\delta b}^{b} V^{\sigma}\Gamma^{\mu}_{\sigma 2} dx^2 - \{\cancel{V^{\mu}(C)} - \int_b^{b+\delta b} [V^{\sigma}\Gamma^{\mu}_{\sigma 2} + \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2\} - \int_a^{a+\delta a} V^{\sigma}\Gamma^{\mu}_{\sigma 1} dx^1 =$$

$$= -\int_{a+\delta a}^{a} [V^{\sigma}\Gamma^{\mu}_{\sigma 1} + \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1 - \int_{b+\delta b}^{b} V^{\sigma}\Gamma^{\mu}_{\sigma 2} dx^2 - \int_b^{b+\delta b} [V^{\sigma}\Gamma^{\mu}_{\sigma 2} + \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2 - \int_a^{a+\delta a} V^{\sigma}\Gamma^{\mu}_{\sigma 1} dx^1 =$$

$$= \int_a^{a+\delta a} [\cancel{V^{\sigma}\Gamma^{\mu}_{\sigma 1}} + \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1 - \int_a^{a+\delta a} \cancel{V^{\sigma}\Gamma^{\mu}_{\sigma 1}} dx^1 + \int_b^{b+\delta b} \cancel{V^{\sigma}\Gamma^{\mu}_{\sigma 2}} dx^2 - \int_b^{b+\delta b} [\cancel{V^{\sigma}\Gamma^{\mu}_{\sigma 2}} + \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2 =$$

$$= \int_a^{a+\delta a} [\frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b] dx^1 - \int_b^{b+\delta b} [\frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a] dx^2 = [\frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta b]\cdot x^1 \Big|_{x^1=a}^{a+\delta a} - [\frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a]\cdot x^2 \Big|_{x^2=b}^{b+\delta b} =$$

$$= \frac{\partial}{\partial x^2}(V^{\sigma}\Gamma^{\mu}_{\sigma 1})\delta a\delta b - \frac{\partial}{\partial x^1}(V^{\sigma}\Gamma^{\mu}_{\sigma 2})\delta a\delta b = \delta a\delta b[\frac{\partial\Gamma^{\mu}_{\sigma 1}}{\partial x^2}V^{\sigma} + \frac{\partial V^{\eta}}{\partial x^2}\Gamma^{\lambda}_{\eta 1} - \frac{\partial\Gamma^{\mu}_{\sigma 2}}{\partial x^1}V^{\sigma} - \frac{\partial V^{\eta}}{\partial x^1}\Gamma^{\lambda}_{\eta 2}]$$

(in carrying out the derivatives of products of functions, right above here, we notice that new indices have appeared and this is typical of tensor calculus, in order to show that among terms we do not require that there must be a simultaneous equality among the values of the indices themselves) and having previously proved (after a proper readaptation of the indices) that:

$\frac{dV^{\eta}}{dx^1} = -V^{\sigma}\Gamma^{\eta}_{\sigma 1}$ and $\frac{dV^{\eta}}{dx^2} = -V^{\sigma}\Gamma^{\eta}_{\sigma 2}$, it follows that:

$$\delta\vec{V} = \delta a\,\delta b[\frac{\partial\Gamma^{\mu}_{\sigma 1}}{\partial x^2}V^{\sigma} - \frac{\partial\Gamma^{\mu}_{\sigma 2}}{\partial x^1}V^{\sigma} + \frac{\partial V^{\eta}}{\partial x^2}\Gamma^{\mu}_{\eta 1} - \frac{\partial V^{\eta}}{\partial x^1}\Gamma^{\mu}_{\eta 2}] = \delta a\,\delta b[\frac{\partial\Gamma^{\mu}_{\sigma 1}}{\partial x^2}V^{\sigma} - \frac{\partial\Gamma^{\mu}_{\sigma 2}}{\partial x^1}V^{\sigma} - V^{\sigma}\Gamma^{\eta}_{\sigma 2}\Gamma^{\mu}_{\eta 1} + V^{\sigma}\Gamma^{\eta}_{\sigma 1}\Gamma^{\mu}_{\eta 2}] =$$

$$= \delta a\,\delta b[\frac{\partial\Gamma^{\mu}_{\sigma 1}}{\partial x^2} - \frac{\partial\Gamma^{\mu}_{\sigma 2}}{\partial x^1} - \Gamma^{\eta}_{\sigma 2}\Gamma^{\mu}_{\eta 1} + \Gamma^{\eta}_{\sigma 1}\Gamma^{\mu}_{\eta 2}]V^{\sigma}\,.$$

Now, we consider the part between brackets: $R^{\mu}_{\sigma 21} = [\frac{\partial\Gamma^{\mu}_{\sigma 1}}{\partial x^2} - \frac{\partial\Gamma^{\mu}_{\sigma 2}}{\partial x^1} - \Gamma^{\eta}_{\sigma 2}\Gamma^{\mu}_{\eta 1} + \Gamma^{\eta}_{\sigma 1}\Gamma^{\mu}_{\eta 2}]$ and, out of consistency, we confer letters (as indices) also to 1 and 2:

$$\boxed{R^{\mu}_{\sigma\lambda\nu} = [\frac{\partial\Gamma^{\mu}_{\sigma\nu}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\mu}_{\sigma\lambda}}{\partial x^{\nu}} - \Gamma^{\eta}_{\sigma\lambda}\Gamma^{\mu}_{\eta\nu} + \Gamma^{\eta}_{\sigma\nu}\Gamma^{\mu}_{\eta\lambda}]\ ;} \qquad (13)$$

well, we just got the <u>Riemann-Christoffel Curvature Tensor</u> which, as a matter of fact, when is null, it implies that $\delta\vec{V} = 0$, that is, the vector $\vec{V}$ has been left unchanged along its parallel transport, so the space is flat (so, no curvature), that is, as we just said, that's the case when $R^{\mu}_{\sigma\lambda\nu} = 0$.

As a matter of fact, in the case in Fig. 6, that is a (flat) parallelogram, vector $\vec{V}$, when parallel transported, easily returns exactly to the same place, so determining a $d\vec{V} = 0$ and so a curvature $R^{\mu}_{\sigma\lambda\nu} = 0$; all that would be different in case of a surface which resembles a spherical cap, rather than a parallelogram.

APPENDIX 2: THE CLASSICAL (NON TENSORIAL) CALCULATIONS FOR THE CHANGE OF COORDINATES.

1-FORMULAS **2**-DIRECT DEDUCTION OF THE GRADIENT ONLY **3**-COMPLETE DEDUCTION OF GRAD, DIV, CURL AND LAPLACIAN **4**-AN ALTERNATIVE DEDUCTION **5**-THE IMPORTANCE OF THE SPHERICAL AND CYLINDRICAL COORDINATES SUBAPPENDIXES

1-FORMULAS

-CARTESIAN COORDINATES:

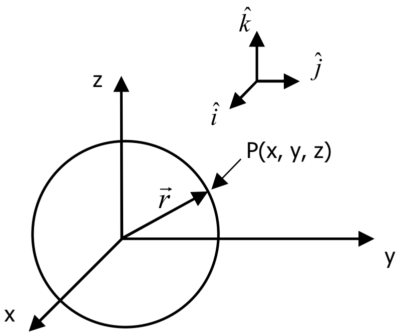


Fig. 1.1

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad , \tag{1.1}$$

$$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k} \quad , \tag{1.2}$$

V is a scalar and $\vec{F}$ is a vector. The divergence gives a scalar, while the gradient and the curl give a vector.

$$gradV = \nabla V = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}) \tag{1.3}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) \tag{1.4}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{j}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{1.5}$$

$$\Delta V = \nabla^2 V = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})V \tag{1.6}$$

-SPHERICAL COORDINATES:

We go from (x,y,z) to (ρ, θ, φ). φ spans from 0 to 2π, while θ spans between 0 and π.

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\theta)\hat{\theta} + (\rho \sin\theta d\varphi)\hat{\varphi} \tag{1.7}$$

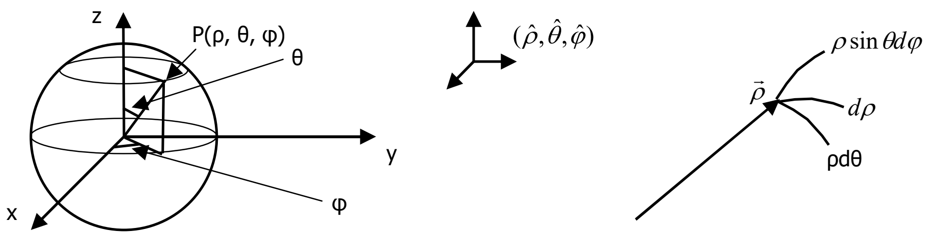


Fig. 1.2

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \tag{1.8}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}) \tag{1.9}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho^2 \sin\theta}[\frac{\partial}{\partial \rho}(\rho^2 \sin\theta F_\rho) + \frac{\partial}{\partial \theta}(\rho \sin\theta F_\theta) + \frac{\partial}{\partial \varphi}(\rho F_\varphi)] \tag{1.10}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \rho\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_\rho & \rho F_\theta & \rho \sin\theta F_\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho^2 \sin\theta}[\frac{\partial(\rho\sin\theta F_\varphi)}{\partial \theta} - \frac{\partial(\rho F_\theta)}{\partial \varphi}]\hat{\rho} + \frac{1}{\rho \sin\theta}[\frac{\partial F_\rho}{\partial \varphi} - \frac{\partial(\rho \sin\theta F_\varphi)}{\partial \rho}]\hat{\theta} + \frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}]\hat{\varphi} \tag{1.11}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} \tag{1.12}$$

-CYLINDRICAL COORDINATES:

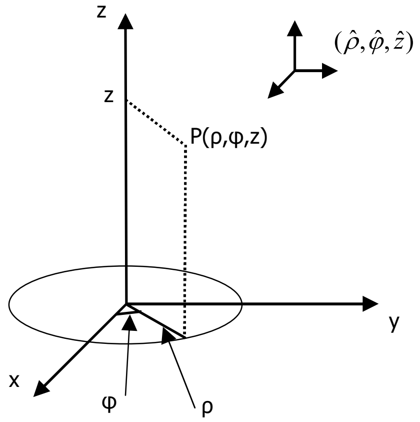


Fig. 1.3

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$\varphi \in [0, 2\pi)$

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\varphi)\hat{\varphi} + (dz)\hat{z} \tag{1.13}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z}) \tag{1.14}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \varphi}(F_\varphi) + \frac{\partial}{\partial z}(\rho F_z)] \tag{1.15}$$

$$curl\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} = \frac{1}{\rho}[\frac{\partial(F_z)}{\partial \varphi} - \frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho} + [\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}]\hat{\varphi} + \frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}]\hat{\varphi} \tag{1.16}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.17}$$

2- DIRECT DEDUCTION OF THE GRADIENT ONLY

(an example of a direct method, but very little effective, so that we will give it up soon)

We know that:

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \tag{2.1}$$

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 & (2.2) \\ \varphi = arctg\frac{y}{x} & (2.3) \\ \theta = \arccos\frac{z}{\rho} & (2.4) \end{cases}$$

and so: $\frac{\partial}{\partial x}=\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$ (2.5)

$$\frac{\partial}{\partial y}=\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi} \quad (2.6) \qquad \frac{\partial}{\partial z}=\frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi} \quad (2.7)$$

Now we differentiate the (2.2): $\rho d\rho = xdx + ydy + zdz$ (2.8)

On the contrary, by differentiating the (2.3): $d\varphi=-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$ (2.9)

(generically: $\partial\varphi=\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy+\cancel{\frac{\partial\varphi}{\partial z}}dz$) and after differentiating the (2.4):

$$-\sin\theta d\theta=-\frac{zx}{\rho^3}dx-\frac{zy}{\rho^3}dy+(\frac{1}{\rho}-\frac{z^2}{\rho^3})dz\,. \quad (2.10)$$

(the $-\frac{1}{\sin\theta}$, later moved to the left side, would be the derivative of the arccos)

Now, we take into account the (2.1) and the (2.8), (2.9), and (2.10), so that:

$$\begin{cases}\frac{\partial\rho}{\partial x}=\sin\theta\cos\varphi & (2.11)\\ \frac{\partial\rho}{\partial y}=\sin\theta\sin\varphi & (2.12)\\ \frac{\partial\rho}{\partial z}=\cos\theta & (2.13)\end{cases} \qquad \begin{cases}\frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi}{\rho\sin\theta} & (2.14)\\ \frac{\partial\varphi}{\partial y}=\frac{\cos\varphi}{\rho\sin\theta} & (2.15)\\ \frac{\partial\varphi}{\partial z}=0 & (2.16)\end{cases}$$

$$\begin{cases}\frac{\partial\theta}{\partial x}=\frac{\cos\theta\cos\varphi}{\rho} & (2.17)\\ \frac{\partial\theta}{\partial y}=\frac{\cos\theta\sin\varphi}{\rho} & (2.18)\\ \frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{\rho} & (2.19)\end{cases}$$

In fact, about the first system of equations, we can say that the (2.11) is (2.8) where dy=dz=0 and by considering that in order to get x we have to multiply ρ by $\sin\theta$ first, to project it on the xy plane, so getting the projection ρ_p and then such a projection has to be multiplied by $\cos\varphi$ to get x indeed. In other words, (2.8) with dy=dz=0 tells us that $\frac{d\rho}{dx}=\frac{x}{\rho}$, which is really $(\sin\theta\cos\varphi)$, according to the projection reasoning just carried out. About the second system, (2.14) would be (2.9) with dy=0 and once again upon the above projection reasoning.

At last, about the third system, (2.17) would be (2.10) with dy=dz=0 and ($z=\rho\cos\theta$) and ($x=\rho\sin\theta\cos\varphi$).

Finally, let's remind gradient and laplacian in spherical coordinates:

By definition, we have: $gradV=\nabla V=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$; if now we use (2.5), (2.6) and (2.7) to express $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ and then we gather all terms with $\frac{\partial}{\partial\rho}$, we get:

$(\nabla)_\rho = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\hat{i} + \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}\hat{j} + \frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho}\hat{k} = \sin\theta\cos\varphi\frac{\partial}{\partial\rho}\hat{i} + \sin\theta\sin\varphi\frac{\partial}{\partial\rho}\hat{j} + \cos\theta\frac{\partial}{\partial\rho}\hat{k} =$

$= \frac{\partial}{\partial\rho}(\hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta) = \hat{\rho}\frac{\partial}{\partial\rho}$. Similarly, $(\nabla)_\theta = \frac{1}{\rho}\frac{\partial}{\partial\theta}\hat{\theta}$ and

$(\nabla)_\varphi = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$, from which, to sum up, we have the components of the gradient in spherical polar coordinates: $(\nabla)_\rho = \hat{\rho}\frac{\partial}{\partial\rho}$, $(\nabla)_\theta = \frac{1}{\rho}\frac{\partial}{\partial\theta}\hat{\theta}$ e $(\nabla)_\varphi = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$.

In order to achieve the scalar laplacian, its definition in Cartesian coordinates is:

$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and in order to calculate every one of those terms, we will first derivate again (2.5), (2.6) and (2.7) and use the results in the above formula for the laplacian, so obtaining:

$\Delta = \frac{1}{\rho^2}(\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$. But we are not going to continue with this way, as it is not convenient; on the contrary, we will use more general ways which follow.

3- COMPLETE DEDUCTION OF GRAD, DIV, CURL AND LAPLACIAN

We preliminarily remind ourselves that in Cartesian coordinates:

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}$; then, if we want to jump from the Cartesian coordinates (x,y,z)-($\hat{i},\hat{j},\hat{k}$) to another curvilinear and generic coordinate system (u,v,w)-($\hat{I},\hat{J},\hat{K}$), we will have relevant equations: x=f(u,v,w), y=g(u,v,w), z=h(u,v,w). Now:

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u}du + \frac{\partial\vec{r}}{\partial v}dv + \frac{\partial\vec{r}}{\partial w}dw \; ; \qquad (3.1)$$

moreover, $\frac{\partial\vec{r}}{\partial u}, \frac{\partial\vec{r}}{\partial v}$ and $\frac{\partial\vec{r}}{\partial w}$ do not necessarily have unitary modulus, so, if we express them as a function of versors: $\frac{\partial\vec{r}}{\partial u} = \left|\frac{\partial\vec{r}}{\partial u}\right|\hat{I} = h_u\hat{I}$, $\frac{\partial\vec{r}}{\partial v} = \left|\frac{\partial\vec{r}}{\partial v}\right|\hat{J} = h_v\hat{J}$ and $\frac{\partial\vec{r}}{\partial w} = \left|\frac{\partial\vec{r}}{\partial w}\right|\hat{K} = h_w\hat{K}$, so:

$$d\vec{r} = h_u du\hat{I} + h_v dv\hat{J} + h_w dw\hat{K} \qquad (3.2)$$

and also: $d\vec{r}\cdot d\vec{r} = ds^2 = h_u^2du^2 + h_v^2dv^2 + h_w^2dw^2$ (3.3)

and about the infinitesimal volume (which is the surface of a parallelogram by a height), we have:

$$dV_{ol} = (h_v dv\hat{J} \times h_w dw\hat{K})\cdot(h_u du\hat{I}) = h_u h_v h_w du dv dw \; . \qquad (3.4)$$

-<u>Gradient</u>: in the new reference frame ($\hat{I},\hat{J},\hat{K}$) it will have this expression:

$$\nabla V = (\nabla V)_u\hat{I} + (\nabla V)_v\hat{J} + (\nabla V)_w\hat{K} \; , \qquad (3.5)$$

with all the various $((\nabla V)_u, (\nabla V)_v, (\nabla V)_w)$ to be found.

Now, as well as with the Cartesian coordinates we have:

$\nabla V\cdot d\vec{r} = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k})[(dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}] = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$,

also here, in the new reference frame, we have:

$dV = \nabla V\cdot d\vec{r} = [(\nabla V)_u\hat{I} + (\nabla V)_v\hat{J} + (\nabla V)_w\hat{K}]\cdot(h_u du\hat{I} + h_v dv\hat{J} + h_w dw\hat{K}) =$

$$= (\nabla V)_u h_u du + (\nabla V)_v h_v dv + (\nabla V)_w h_w dw \; , \qquad (3.6)$$

but being, because of the rules of the differential calculus, also:

$$dV = \frac{\partial V}{\partial u} du + \frac{\partial V}{\partial v} dv + \frac{\partial V}{\partial w} dw, \tag{3.7}$$

then, from the (3.6) and (3.7) we get: $(\nabla V)_u = \frac{1}{h_u}\frac{\partial V}{\partial u}$, $(\nabla V)_v = \frac{1}{h_v}\frac{\partial V}{\partial v}$, $(\nabla V)_w = \frac{1}{h_w}\frac{\partial V}{\partial w}$

and then, for the (3.5), we get our gradient in generic coordinates:

$$\boxed{\nabla V = \frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I} + \frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J} + \frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K}} \tag{3.8}$$

-<u>Divergence</u>:

$$\vec{F} = F_u\hat{I} + F_v\hat{J} + F_w\hat{K} = F_u(\hat{J}\times\hat{K}) + F_v(\hat{K}\times\hat{I}) + F_w(\hat{I}\times\hat{J}) \ , \tag{3.9}$$

but the (3.8) used over u (a small u) yields:

$\nabla u = \frac{1}{h_u}\frac{\partial u}{\partial u}\hat{I} + \frac{1}{h_v}\frac{\partial u}{\partial v}\hat{J} + \frac{1}{h_w}\frac{\partial u}{\partial w}\hat{K} = \frac{1}{h_u}\frac{\partial u}{\partial u}\hat{I} + 0 + 0 = \frac{1}{h_u}\hat{I}$ and similarly: $\nabla v = \frac{1}{h_v}\hat{J}$, $\nabla w = \frac{1}{h_w}\hat{K}$,

which is all summarized here: $\nabla u = \frac{1}{h_u}\hat{I}$, $\nabla v = \frac{1}{h_v}\hat{J}$ and $\nabla w = \frac{1}{h_w}\hat{K}$ (3.10)

and thanks to that, the (3.8) can be rewritten like that:

$$\nabla V = \nabla u\frac{\partial V}{\partial u} + \nabla v\frac{\partial V}{\partial v} + \nabla w\frac{\partial V}{\partial w} \tag{3.11}$$

Now, we take back the (3.9) and have:

$$\vec{F} = F_u(\hat{J}\times\hat{K}) + F_v(\hat{K}\times\hat{I}) + F_w(\hat{I}\times\hat{J}) = F_u h_v h_w(\nabla v\times\nabla w) + F_v h_w h_u(\nabla w\times\nabla u) + F_w h_u h_v(\nabla u\times\nabla v) \tag{3.12}$$

and by applying the divergence to the (3.12), we get:

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w) + F_v h_w h_u(\nabla w\times\nabla u) + F_w h_u h_v(\nabla u\times\nabla v)] = \vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)] +$$
$$+\vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)] + \vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)] \tag{3.13}$$

Now, we figure out the single components of the (3.13):

$$\begin{gathered}\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]\\ \vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)]\\ \vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)]\end{gathered} \tag{3.14}$$

we have from the Subappendix 1 that the following vector identity holds:

$\vec{\nabla}\cdot(V\vec{A}) = \nabla V\cdot\vec{A} + V(\vec{\nabla}\cdot\vec{A})$, so:

$$\begin{gathered}\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)] = \nabla(F_u h_v h_w)\cdot(\nabla v\times\nabla w) + (F_u h_v h_w)\vec{\nabla}\cdot(\nabla v\times\nabla w)\\ \vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)] = \nabla(F_v h_w h_u)\cdot(\nabla w\times\nabla u) + (F_v h_w h_u)\vec{\nabla}\cdot(\nabla w\times\nabla u)\\ \vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)] = \nabla(F_w h_u h_v)\cdot(\nabla u\times\nabla v) + (F_w h_u h_v)\vec{\nabla}\cdot(\nabla u\times\nabla v)\end{gathered} \tag{3.15}$$

and still from the Subappendix 1 we have that the following vector identity holds as well: $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$ and in our case $\vec{A}$ and $\vec{B}$ are the various ∇u , ∇v and ∇w , which are gradients, let's not forget that, and the curl of a gradient is zero, so, in our specific case, $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = 0$, therefore:

$$\begin{gathered}\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)] = \nabla(F_u h_v h_w)\cdot(\nabla v\times\nabla w)\\ \vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)] = \nabla(F_v h_w h_u)\cdot(\nabla w\times\nabla u)\\ \vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)] = \nabla(F_w h_u h_v)\cdot(\nabla u\times\nabla v)\end{gathered} \tag{3.16}$$

As an example, we figure out the term $\nabla(F_u h_v h_w)$ in the first equation of the (3.16), by using the (3.11):

$$\nabla(F_u h_v h_w) = \nabla u \frac{\partial(F_u h_v h_w)}{\partial u} + \nabla v \frac{\partial(F_u h_v h_w)}{\partial v} + \nabla w \frac{\partial(F_u h_v h_w)}{\partial w}, \tag{3.17}$$

from which, still the first equation of the (3.16) becomes:

$$\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] = (\nabla v \times \nabla w) \cdot \nabla(F_u h_v h_w) =$$

$= (\nabla v \times \nabla w) \cdot [\nabla u \frac{\partial(F_u h_v h_w)}{\partial u} + \nabla v \frac{\partial(F_u h_v h_w)}{\partial v} + \nabla w \frac{\partial(F_u h_v h_w)}{\partial w}] = (\nabla v \times \nabla w) \cdot [\nabla u \frac{\partial(F_u h_v h_w)}{\partial u}]$, where the only surviving term of the (3.17) has been $[\nabla u \frac{\partial(F_u h_v h_w)}{\partial u}]$, as the only three-nabla product which doesn't become zero is the first one, indeed: $(\nabla v \times \nabla w) \cdot \nabla u \neq 0$, while the other two are zero: $(\nabla v \times \nabla w) \cdot \nabla v = (\nabla v \times \nabla w) \cdot \nabla w = 0$, as they end up into scalar products among orthogonal vectors, as, due to the (3.10), all the $(\nabla v, \nabla w, \nabla u)$ match the $(\hat{I}, \hat{J}, \hat{K})$.

Therfore, the (3.16) become:

$$\begin{aligned}
&\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] = (\nabla v \times \nabla w) \cdot [\nabla u \frac{\partial(F_u h_v h_w)}{\partial u}] \qquad \text{(and similarly)} \\
&\vec{\nabla} \cdot [F_v h_w h_u (\nabla w \times \nabla u)] = (\nabla w \times \nabla u) \cdot [\nabla v \frac{\partial(F_v h_w h_u)}{\partial v}] \\
&\vec{\nabla} \cdot [F_w h_u h_v (\nabla u \times \nabla v)] = (\nabla u \times \nabla v) \cdot [\nabla w \frac{\partial(F_w h_u h_v)}{\partial w}]
\end{aligned} \tag{3.18}$$

Now, let's develop the first equation of the (3.18):

$$\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] = (\nabla v \times \nabla w) \cdot [\nabla u \frac{\partial(F_u h_v h_w)}{\partial u}] = [(\frac{1}{h_v}\hat{J}) \times (\frac{1}{h_w}\hat{K})] \cdot [\frac{1}{h_u}\hat{I} \frac{\partial(F_u h_v h_w)}{\partial u}] =$$

$= \frac{1}{h_u h_v h_w}[(\hat{J} \times \hat{K}) \cdot \hat{I}] \frac{\partial(F_u h_v h_w)}{\partial u} = \frac{1}{h_u h_v h_w}[\hat{I} \cdot \hat{I}] \frac{\partial(F_u h_v h_w)}{\partial u} = \frac{1}{h_u h_v h_w} \frac{\partial(F_u h_v h_w)}{\partial u}$, so the (3.18) become:

$$\begin{aligned}
&\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] = \frac{1}{h_u h_v h_w} \frac{\partial(F_u h_v h_w)}{\partial u} \qquad \text{(and similarly)} \\
&\vec{\nabla} \cdot [F_v h_w h_u (\nabla w \times \nabla u)] = \frac{1}{h_u h_v h_w} \frac{\partial(F_v h_w h_u)}{\partial v} \\
&\vec{\nabla} \cdot [F_w h_u h_v (\nabla u \times \nabla v)] = \frac{1}{h_u h_v h_w} \frac{\partial(F_w h_u h_v)}{\partial w}
\end{aligned} \tag{3.19}$$

In the end, the (3.13) finally yields:

$$\boxed{\vec{\nabla} \cdot \vec{F} = \frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_w h_u)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial w}]} \tag{3.20}$$

-<u>Curl</u>:

$$\vec{F} = F_u\hat{I} + F_v\hat{J} + F_w\hat{K} = h_u F_u \nabla u + h_v F_v \nabla v + h_w F_w \nabla w$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (h_u F_u \nabla u + h_v F_v \nabla v + h_w F_w \nabla w) = \vec{\nabla} \times [(h_u F_u)\nabla u] + \vec{\nabla} \times [(h_v F_v)\nabla v] + \vec{\nabla} \times [(h_w F_w)\nabla w] \tag{3.21}$$

Let's remind ourselves of the vector identity reported in the Subappendix 1:

$\vec{\nabla} \times (V\vec{B}) = V(\vec{\nabla} \times \vec{B}) + \nabla V \times \vec{B}$ and if in it the vector $\vec{B}$ comes out of a gradient of another scalar A ($\vec{B} = \nabla A$), then:

$$\vec{\nabla} \times (V\nabla A) = V(\vec{\nabla} \times \nabla A) + \nabla V \times \nabla A \ . \tag{3.22}$$

Now, let's write the three components of the (3.21):

$$\vec{\nabla}\times[(h_uF_u)\nabla u]$$
$$\vec{\nabla}\times[(h_vF_v)\nabla v] \quad (3.23)$$
$$\vec{\nabla}\times[(h_wF_w)\nabla w]$$

and let's develop them according to the (3.22):

$$\vec{\nabla}\times[(h_uF_u)\nabla u]=\nabla(h_uF_u)\times\nabla u+(h_uF_u)\vec{\nabla}\times\nabla u=\nabla(h_uF_u)\times\nabla u$$
$$\vec{\nabla}\times[(h_vF_v)\nabla v]=\nabla(h_vF_{vu})\times\nabla v+(h_vF_v)\vec{\nabla}\times\nabla v=\nabla(h_vF_v)\times\nabla v \quad (3.24)$$
$$\vec{\nabla}\times[(h_wF_w)\nabla w]=\nabla(h_wF_w)\times\nabla w+(h_wF_w)\vec{\nabla}\times\nabla w=\nabla(h_wF_w)\times\nabla w$$

(in fact, all the latter terms are zero, as they are curls of gradients)

but because of the (3.11), that is ($\nabla V=\nabla u\frac{\partial V}{\partial u}+\nabla v\frac{\partial V}{\partial v}+\nabla w\frac{\partial V}{\partial w}$), it follows that, for instance, the first equation of the (3.24) becomes: $\nabla(h_uF_u)\times\nabla u=[\frac{\partial(h_uF_u)}{\partial u}\nabla u+\frac{\partial(h_uF_u)}{\partial v}\nabla v+\frac{\partial(h_uF_u)}{\partial w}\nabla w]\times\nabla u=$

$$=0+\frac{\partial(h_uF_u)}{\partial v}(\nabla v\times\nabla u)+\frac{\partial(h_uF_u)}{\partial w}(\nabla w\times\nabla u)=\frac{\partial(h_uF_u)}{\partial v}(\frac{\hat{J}}{h_v}\times\frac{\hat{I}}{h_u})+\frac{\partial(h_uF_u)}{\partial w}(\frac{\hat{K}}{h_w}\times\frac{\hat{I}}{h_u})=$$
$$=\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}(\hat{J}\times\hat{I})+\frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}(\hat{K}\times\hat{I})=-\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}\hat{K}+\frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}\hat{J}$$

and the same happens about the other two equations (3.24):

$$\nabla(h_vF_v)\times\nabla v=-\frac{1}{h_vh_w}\frac{\partial(h_vF_v)}{\partial w}\hat{I}+\frac{1}{h_uh_v}\frac{\partial(h_vF_v)}{\partial u}\hat{K}$$
$$\nabla(h_wF_w)\times\nabla w=-\frac{1}{h_uh_w}\frac{\partial(h_wF_w)}{\partial u}\hat{J}+\frac{1}{h_vh_w}\frac{\partial(h_wF_w)}{\partial v}\hat{I}$$

so, all the (3.24) become:

$$\vec{\nabla}\times[(h_uF_u)\nabla u]=-\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}\hat{K}+\frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}\hat{J}$$
$$\vec{\nabla}\times[(h_vF_v)\nabla v]=-\frac{1}{h_vh_w}\frac{\partial(h_vF_v)}{\partial w}\hat{I}+\frac{1}{h_uh_v}\frac{\partial(h_vF_v)}{\partial u}\hat{K} \quad (3.25)$$
$$\vec{\nabla}\times[(h_wF_w)\nabla w]=-\frac{1}{h_uh_w}\frac{\partial(h_wF_w)}{\partial u}\hat{J}+\frac{1}{h_vh_w}\frac{\partial(h_wF_w)}{\partial v}\hat{I}$$

((((Note: regarding the coming out of the "-" signs, they appear when we invert the natural order:
1,2,3,1,2,3,1,2,3,……..
x,y,z,x,y,z,x,y,z,……..
u,v,w,u,v,w,u,v,w,……
$\hat{\imath},\hat{\jmath},\hat{k},\hat{\imath},\hat{\jmath},\hat{k},\hat{\imath},\hat{\jmath},\hat{k}$,........
$\hat{I},\hat{J},\hat{K},\hat{I},\hat{J},\hat{K},\hat{I},\hat{J},\hat{K}$,........
That is, if we go from 1 to 2, or from 2 to 3, or from 3 to 1 and so on, then we have the "+", while if we go, as an example, from 1 to 3, so skipping the 2, or from 2 to 1, so skipping the 3, or from 3 to 2, so skipping 1, then we receive a "-".))))

Now, let's go back to the issue: thanks to the (3.25), the (3.21) yields:

$$\vec{\nabla}\times\vec{F}=[-\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}\hat{K}+\frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}\hat{J}]+[-\frac{1}{h_vh_w}\frac{\partial(h_vF_v)}{\partial w}\hat{I}+\frac{1}{h_uh_v}\frac{\partial(h_vF_v)}{\partial u}\hat{K}]+$$
$$+[-\frac{1}{h_uh_w}\frac{\partial(h_wF_w)}{\partial u}\hat{J}+\frac{1}{h_vh_w}\frac{\partial(h_wF_w)}{\partial v}\hat{I}]=\frac{1}{h_vh_w}[\frac{\partial(h_wF_w)}{\partial v}-\frac{\partial(h_vF_v)}{\partial w}]\hat{I}+\frac{1}{h_uh_w}[\frac{\partial(h_uF_u)}{\partial w}-\frac{\partial(h_wF_w)}{\partial u}]\hat{J}+$$

$$+\frac{1}{h_u h_v}[\frac{\partial(h_v F_v)}{\partial u}-\frac{\partial(h_u F_u)}{\partial v}]\hat{K}=\boxed{\frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}=\vec{\nabla}\times\vec{F}} \quad (3.26)$$

-<u>Laplacian</u>:

By using the (3.8): $(\nabla V=\frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I}+\frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J}+\frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K})$ into the (3.20):

$(\vec{\nabla}\cdot\vec{F}=\frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u}+\frac{\partial(F_v h_w h_u)}{\partial v}+\frac{\partial(F_w h_u h_v)}{\partial w}])$, we get the Laplacian:

$$\boxed{\vec{\nabla}\cdot\nabla V=\nabla^2 V=\Delta V=\frac{1}{h_u h_v h_w}[\frac{\partial}{\partial u}(\frac{h_v h_w}{h_u}\frac{\partial V}{\partial u})+\frac{\partial}{\partial v}(\frac{h_u h_w}{h_v}\frac{\partial V}{\partial v})+\frac{\partial}{\partial w}(\frac{h_u h_v}{h_w}\frac{\partial V}{\partial w})]} \quad (3.27)$$

-**<u>Summary</u>:**

Let's collect all the expressions we got for all our operators:

$$\nabla V=\frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I}+\frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J}+\frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K}$$

$$\vec{\nabla}\cdot\vec{F}=\frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u}+\frac{\partial(F_v h_w h_u)}{\partial v}+\frac{\partial(F_w h_u h_v)}{\partial w}]$$

$$\vec{\nabla}\times\vec{F}=\frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

$$\nabla^2 V=\frac{1}{h_u h_v h_w}[\frac{\partial}{\partial u}(\frac{h_v h_w}{h_u}\frac{\partial V}{\partial u})+\frac{\partial}{\partial v}(\frac{h_u h_w}{h_v}\frac{\partial V}{\partial v})+\frac{\partial}{\partial w}(\frac{h_u h_v}{h_w}\frac{\partial V}{\partial w})]$$

$$d\vec{r}=h_u du\hat{I}+h_v dv\hat{J}+h_w dw\hat{K}$$

-<u>case of the Cartesian coordinates</u>: about them, we know that: $d\vec{r}=(dx)\hat{i}+(dy)\hat{j}+(dz)\hat{k}$, so we see that: (u,v,w)→(x,y,z) , $(\hat{I},\hat{J},\hat{K})\rightarrow(\hat{i},\hat{j},\hat{k})$ and $(h_u h_v h_w)=(h_x h_y h_z)=(1,1,1)$ and with all those three substitutions into the above equations, we really get the (1.3), (1.4), (1.5) and (1.6):

$$gradV=\nabla V=(\frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k})$$

$$div\cdot\vec{F}=\vec{\nabla}\cdot\vec{F}=(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z})$$

$$curl\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}=\hat{i}(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z})+\hat{j}(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x})+\hat{k}(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y})$$

$$\nabla^2 V=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})V$$

-<u>case of the spherical coordinates</u>: about them, we know that: $d\vec{r} = (d\rho)\hat{\rho} + (\rho d\theta)\hat{\theta} + (\rho \sin\theta d\varphi)\hat{\varphi}$, so we see that: (u,v,w)→(ρ,θ,φ) , $(\hat{I},\hat{J},\hat{K}) \to (\hat{\rho},\hat{\theta},\hat{\varphi})$ and $(h_u h_v h_w) = (h_\rho h_\theta h_\varphi) = (1,\rho,\rho\sin\theta)$ and with such three substitutions into the above generic equations, we get exactly the (1.9), (1.10), (1.11) and (1.12):

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{\rho\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi})$$

$$div\cdot\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{1}{\rho^2\sin\theta}[\frac{\partial}{\partial\rho}(\rho^2\sin\theta F_\rho) + \frac{\partial}{\partial\theta}(\rho\sin\theta F_\theta) + \frac{\partial}{\partial\varphi}(\rho F_\varphi)]$$

$$curl\vec{F} = \vec{\nabla}\times\vec{F} = \frac{1}{\rho^2\sin\theta}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \rho\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ F_\rho & \rho F_\theta & \rho\sin\theta F_\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho^2\sin\theta}[\frac{\partial(\rho\sin\theta F_\varphi)}{\partial\theta} - \frac{\partial(\rho F_\theta)}{\partial\varphi}]\hat{\rho} + \frac{1}{\rho\sin\theta}[\frac{\partial F_\rho}{\partial\varphi} - \frac{\partial(\rho\sin\theta F_\varphi)}{\partial\rho}]\hat{\theta} + \frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial\rho} - \frac{\partial F_\rho}{\partial\theta}]\hat{\varphi}$$

$$\nabla^2 V = \frac{1}{\rho^2\sin\theta}[\frac{\partial}{\partial\rho}(\rho^2\sin\theta\frac{\partial V}{\partial\rho}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{\partial}{\partial\varphi}(\frac{1}{\sin\theta}\frac{\partial V}{\partial\varphi})] =$$

$$= \frac{\partial^2 V}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial\theta} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

-<u>case of the cylindrical coordinates</u>: about them, we know that: $d\vec{r} = (d\rho)\hat{\rho} + (\rho d\varphi)\hat{\varphi} + (dz)\hat{z}$, so we see that: (u,v,w)→(ρ,φ,z) , $(\hat{I},\hat{J},\hat{K}) \to (\hat{\rho},\hat{\varphi},\hat{z})$ and $(h_u h_v h_w) = (h_\rho h_\varphi h_z) = (1,\rho,1)$ and with such three substitutions into the above generic equations, we actually get the (1.14), (1.15), (1.16) and (1.17):

$$gradV = \nabla V = (\frac{\partial V}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial\varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z})$$

$$div\cdot\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{1}{\rho}[\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{\partial}{\partial\varphi}(F_\varphi) + \frac{\partial}{\partial z}(\rho F_z)]$$

$$curl\vec{F} = \vec{\nabla}\times\vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} = \frac{1}{\rho}[\frac{\partial(F_z)}{\partial\varphi} - \frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho} + [\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}]\hat{\varphi} + \frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial\rho} - \frac{\partial F_\rho}{\partial\varphi}]\hat{\varphi}$$

$$\nabla^2 V = \frac{1}{\rho}[\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{\partial}{\partial\varphi}(\frac{1}{\rho}\frac{\partial V}{\partial\varphi}) + \frac{\partial}{\partial z}(\rho\frac{\partial V}{\partial z}) =$$

$$= \frac{\partial^2 V}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

4- AN ALTERNATIVE DEDUCTION

-Calculation of the divergence based on the Divergence Theorem:

First of all, let's figure out the fluxes of the components of the vector $\vec{F} = F_u\hat{I} + F_v\hat{J} + F_w\hat{K}$ (given under its curvilinear frame $(u,v,w)--(\hat{I},\hat{J},\hat{K})$) through the relevant surfaces:

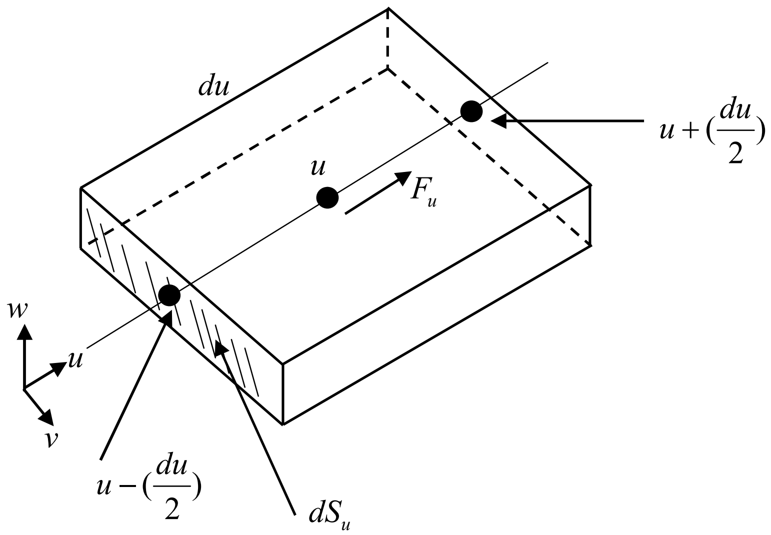


Fig. 4.1.

With reference to Fig. 4.1, we figure out the flux $d\Phi_u$ of F_u through dS_u:

$dS_u = h_v dv h_w dw$ (see the (3.2))

$$d\Phi_u = F_u dS_u\Big|_{in\_[u+\frac{du}{2}],v,w} - F_u dS_u\Big|_{in\_[u-\frac{du}{2}],v,w} = F_u h_v h_w dvdw\Big|_{in\_[u+\frac{du}{2}],v,w} - F_u h_v h_w dvdw\Big|_{in\_[u-\frac{du}{2}],v,w} =$$

(by multiplying up and down by du)=

$$= d\Phi_u = \frac{\partial(F_u h_v h_w)}{\partial u} dudvdw$$

and similar expressions for the other two components, that is, in the end:

$$\boxed{\begin{aligned} d\Phi_u &= \frac{\partial(F_u h_v h_w)}{\partial u} dudvdw \\ d\Phi_v &= \frac{\partial(F_v h_u h_w)}{\partial v} dudvdw \\ d\Phi_w &= \frac{\partial(F_w h_u h_v)}{\partial w} dudvdw \end{aligned}} \qquad (4.1)$$

Now, for the Divergence Theorem shown in the Subappendix 2, we have:

$$\int_V div\vec{F}\cdot dV = \int_S \vec{F}\cdot d\vec{S} \qquad (d\vec{S} = \hat{n}dS)$$

If now we consider the mean value of $div\vec{F}$, that is $\overline{div\vec{F}}$, then we can write:

$\int_V div\vec{F}\cdot dV = \overline{div\vec{F}}\int_V dV = \overline{div\vec{F}}\Delta V = \int_S \vec{F}\cdot d\vec{S}$, so: $\overline{div\vec{F}} = \dfrac{\int_S \vec{F}\cdot d\vec{S}}{\Delta V}$; if now we calculate the limit for $\Delta V \to 0$, then $\overline{div\vec{F}} \to div\vec{F}$, so:

$$div\vec{F} = \lim_{\Delta V\to 0}\frac{\int_S \vec{F}\cdot d\vec{S}}{\Delta V} \qquad (4.2)$$

and also: $div\vec{F} = \lim_{\Delta V\to 0}\dfrac{1}{\Delta V}\lim_{\Delta V\to 0}\int_S \vec{F}\cdot d\vec{S} \equiv \dfrac{1}{dV}\lim_{\Delta V\to 0}\int_S \vec{F}\cdot d\vec{S} \equiv \dfrac{1}{dV}(\vec{F}\cdot d\vec{S}) \equiv \dfrac{1}{dV}d\Phi$ (4.3)

as if the volume shrinks to an infinitesimal value, then the summation (of infinitesimal values) represented by the integral is just a single infinitesimal value.
And also reminding that, because of the (3.4), we have: $dV = h_u h_v h_w du dv dw$.

Then: $div\vec{F} = \frac{1}{dV} d\Phi = \frac{1}{dV}(d\Phi_u + d\Phi_v + d\Phi_w) = \frac{1}{h_u h_v h_w du dv dw}[\frac{\partial(F_u h_v h_w)}{\partial u} du dv dw +$
$+ \frac{\partial(F_v h_u h_w)}{\partial v} du dv dw + \frac{\partial(F_w h_u h_v)}{\partial w} du dv dw] = \frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_u h_w)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial w}]$,
or, after rewriting it again:

$$\boxed{div\vec{F} = \frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_u h_w)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial w}]} \tag{4.4}$$

which is exactly the (3.20) we already got.

-Calculation of the curl, based on the Curl Theorem:

As we show in the Subappendix 2, the following Curl Theorem holds:

$\oint_l \vec{F} \cdot d\vec{l} = \int_S curl\vec{F} \cdot d\vec{S}$ and if we carry out a reasoning similar to that which brought us to the (4.2), we can say that:

$$curl\vec{F} = \lim_{\Delta S \to 0} \frac{\oint_l \vec{F} \cdot d\vec{l}}{\Delta S} \tag{4.5}$$

not only; as well as we got to the (4.3), here too we can say that:

$$curl\vec{F} = \lim_{\Delta S \to 0} \frac{\oint_l \vec{F} \cdot d\vec{l}}{\Delta S} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \lim_{\Delta S \to 0} \oint_l \vec{F} \cdot d\vec{l} \equiv \frac{1}{dS} \overrightarrow{(\vec{F} \cdot d\vec{l})} \tag{4.6}$$

as if the surface ΔS (outlined by $\Delta\vec{l}$) shrinks to dS (outlined by $d\vec{l}$), then the summation represented by the integral (of infinitesimal values) is just one single infinitesimal value.
So, we figure out the quantity $(\vec{F} \cdot d\vec{l})$ of the vector $\vec{F} = F_u \hat{I} + F_v \hat{J} + F_w \hat{K}$ (given in its curvilinear frame $(u, v, w) - -(\hat{I}, \hat{J}, \hat{K})$) through the relevant closed infinitesimal path.

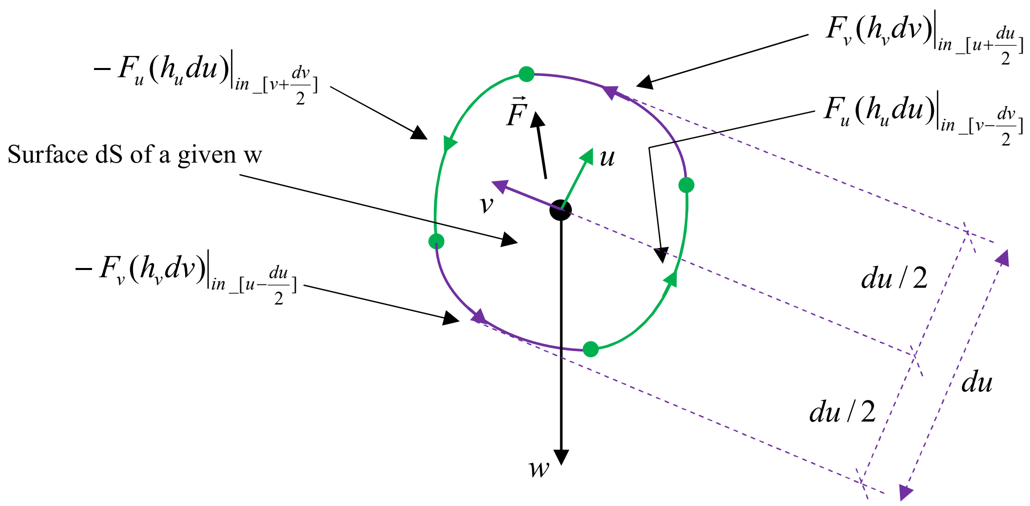


Fig. 4.2.

Because of the (3.2), we have that $d\vec{r} = (h_u du)\hat{I} + (h_v dv)\hat{J} + (h_w dw)\hat{K}$, so all the paths are $(h_u du), (h_v dv), (h_w dw)$, while all the surface contributions are, of course:

$(h_u h_v dudv),(h_u h_w dudw),(h_v h_w dvdw)$.

With reference to Fig. 4.2, we figure out over a given w the contribution $(\vec{F}\cdot d\vec{l})_w$:

$$(\vec{F}\cdot d\vec{l})_w = [F_v(h_v dv)\big|_{in\_[u+\frac{du}{2}]} - F_v(h_v dv)\big|_{in\_[u-\frac{du}{2}]}] - [F_u(h_u du)\big|_{in\_[v+\frac{dv}{2}]} - F_u(h_u du)\big|_{in\_[v-\frac{dv}{2}]}] =$$

$$= [\partial(F_v h_v)dv] - [\partial(F_u h_u)du] = \partial(F_v h_v)dv - \partial(F_u h_u)du \ .$$

And similar calculations for the other two axes, so, in the end:

$$\begin{aligned}(\vec{F}\cdot d\vec{l})_u &= \partial(F_w h_w)dw - \partial(F_v h_v)dv\\ (\vec{F}\cdot d\vec{l})_v &= \partial(F_u h_u)du - \partial(F_w h_w)dw\\ (\vec{F}\cdot d\vec{l})_w &= \partial(F_v h_v)dv - \partial(F_u h_u)du\end{aligned} \qquad (4.7)$$

and so, due to the (4.6), we have:

$$curl\vec{F} = \frac{1}{dS}\overrightarrow{(\vec{F}\cdot d\vec{l})} = \frac{1}{h_v h_w dvdw}(\vec{F}\cdot d\vec{l})_u\hat{I} + \frac{1}{h_u h_w dudw}(\vec{F}\cdot d\vec{l})_v\hat{J} + \frac{1}{h_u h_v dudv}(\vec{F}\cdot d\vec{l})_w\hat{K} =$$

$$= \frac{1}{h_v h_w dvdw}[\partial(F_w h_w)dw - \partial(F_v h_v)dv]\hat{I} + \frac{1}{h_u h_w dudw}[\partial(F_u h_u)du - \partial(F_w h_w)dw]\hat{J} +$$

$$+ \frac{1}{h_u h_v dudv}[\partial(F_v h_v)dv - \partial(F_u h_u)du]\hat{K} =$$

$$= \frac{1}{h_v h_w}[\frac{\partial(h_w F_w)}{\partial v} - \frac{\partial(h_v F_v)}{\partial w}]\hat{I} + \frac{1}{h_u h_w}[\frac{\partial(h_u F_u)}{\partial w} - \frac{\partial(h_w F_w)}{\partial u}]\hat{J} + \frac{1}{h_u h_v dudv}[\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v}]\hat{K} =$$

$$= \frac{1}{h_u h_v h_w}[\frac{\partial(h_w F_w)}{\partial v} - \frac{\partial(h_v F_v)}{\partial w}]h_u\hat{I} + \frac{1}{h_u h_v h_w}[\frac{\partial(h_u F_u)}{\partial w} - \frac{\partial(h_w F_w)}{\partial u}]h_v\hat{J} + \frac{1}{h_u h_v h_w}[\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v}]h_w\hat{K} =$$

$$\boxed{= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K}\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix} = curl\vec{F} = \vec{\nabla}\times\vec{F}}$$

which is exactly the (3.26) we already got.

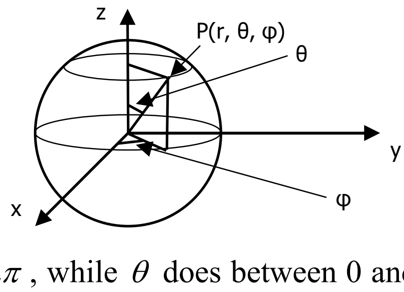
5- THE IMPORTANCE OF THE SPHERICAL AND CYLINDRICAL COORDINATES

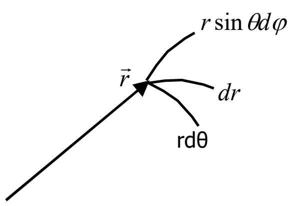
-A-SPHERICAL COORDINATES

General view

Through the spherical polar coordinates (r,θ,φ), calculations will be much easier, opposite to the Cartesian ones x,y,z.

$$\begin{cases} x = r\sin\theta\cos\varphi\\ y = r\sin\theta\sin\varphi\\ z = r\cos\theta\end{cases}$$





(φ spans between 0 and 2π , while θ does between 0 and π). In fact, as an example, on the polar plane the equation of the circle with its center in the origin is $r = R$ (no matter how θ and φ are), while in Cartesian coordinates we have to start from the implicit equation $x^2 + y^2 = R^2$, from which, for the first quadrant xy, we have:

$y = \sqrt{R^2 - x^2}$ (much more difficult) (*)

and in order to calculate the surface of the circle, in polar coordinates we consider the thin crown as thick as dr and at a distance r from the center, whose surface is $dA = 2\pi r \cdot dr$ and by integrating between 0 and R, we get:

$A = 2\pi \int_0^R r \cdot dr = \pi R^2$, while with the Cartesian coordinates, we should integrate (*) between 0 and R to have a quarter of A (area below the curve), from which:

$A = 4\int_0^R \sqrt{R^2 - x^2}\,dx = 4\left|\frac{x\sqrt{R^2 - x^2}}{2} + \frac{R^2}{2}\arcsin\frac{x}{R}\right|_0^R = \pi R^2$, (which is a bit more difficult job…).

The angular momentum in the atom

We know that $\vec{p} = -i\hbar\vec{\nabla}$ and $E = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\Delta$. Now, about the angular momentum L=mvr we know that if a mass point m orbits at r distance from a center point and does it by speed v, we have: (p=mv):

$\vec{L} = \vec{r} \times \vec{p}$, $L = r \times p = -i\hbar r \times \nabla$ (vector product) and as (obviously): $-i\nabla = (-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z})$,

then: $L_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$, $L_y = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$, $L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ and

$L^2 = \vec{L}\vec{L} = L_x^2 + L_y^2 + L_z^2$. (A.1)

By taking into account the above expression for L_z and considering (A.1) and (2.11)……(2.19), we get: $L_z = -i\hbar[r\sin\theta\cos\varphi(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}) - r\sin\theta\sin\varphi(\sin\theta\cos\varphi\frac{\partial}{\partial r} +$

$+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi})]$, which reduces a lot by mutual elimination of terms, so yielding:

$L_z = -i\hbar\frac{\partial}{\partial\varphi}$. Similarly, we get the following:

$L_x = i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi})$, $L_y = -i\hbar(\cos\varphi\frac{\partial}{\partial\theta} - ctg\theta\sin\varphi\frac{\partial}{\partial\varphi})$ and finally, by (A.1), we calculate: $L^2 = -\hbar^2(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}) = -\hbar^2(\frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$ (A.2)

The last equality is due to the obvious development: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} = \frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta}$.

To reach (A.2), we just show, as an example, the calculation for L_x^2 and we will carry out all products one by one, without using ready-to-use formulas on square a+b and so on, as here we are dealing with operators:

$L_x^2 = i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}) \cdot i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}) = -\hbar^2(\sin\varphi\frac{\partial}{\partial\theta}\sin\varphi\frac{\partial}{\partial\theta} +$

$+\sin\varphi\frac{\partial}{\partial\theta}ctg\theta\cos\varphi\frac{\partial}{\partial\varphi} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}) =$

$= -\hbar^2[\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cos\varphi(1 + ctg^2\theta)\frac{\partial}{\partial\varphi} + ctg\theta\cos^2\varphi\frac{\partial}{\partial\theta} +$

$$-\sin\varphi\cos\varphi ctg^2\theta\frac{\partial}{\partial\varphi}+\cos^2\varphi ctg^2\theta\frac{\partial^2}{\partial\varphi^2}]$$

and a similar expression for L_y^2 and a short one for L_z^2. Later, by summing up all the terms, many of them will join or cancel, so leading us to (A.2) indeed.
Please note that $[\varphi, L_z] = i\hbar$.

About the kinetic energy operator T, we know that $\vec{p} = -i\hbar\vec{\nabla}$ and $T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\Delta$, but we also showed that: $\Delta = \frac{1}{r^2}(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$ (**Laplacian in spherical coordinates**) and a comparison between this equation and (A.2) tells us that:

$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$, from which:

$$p^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{L^2}{r^2} = p_r^2+\frac{L^2}{r^2}, \tag{A.3}$$

after naming $p_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = -\hbar^2(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r})$, and after having developed the derivative of a product: $\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$.

Schrodinger's Equation for the atom

We are obviously in a field of forces with a central symmetry generated by the nucleus, so:

$V(\vec{r}) = V(r)$, $H = T + V = \frac{p^2}{2m}+V(r)$, $p^2 = p_r^2+\frac{L^2}{r^2}$ and, according to Schrodinger,

$H\Psi(\vec{r}) = E\Psi(\vec{r})$, that is:

$\frac{1}{2m}(p_r^2+\frac{L^2}{r^2})\Psi + V(r)\Psi = E\Psi$, or:

$$r^2[p_r^2+2m(V(r)-E)]\Psi(\vec{r}) = -L^2\Psi(\vec{r}), \tag{A.4}$$

where (let's remind it):

$$L^2 = -\hbar^2(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}) = -\hbar^2(\frac{\partial^2}{\partial\theta^2}+ctg\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}).$$

Let's find solutions for (A.4) with separated variables: $\Psi(\vec{r}) = \Phi(r)Y(\theta,\varphi)$; by introducing it into (A.4) and after naming by $R = r^2[p_r^2+2m(V(r)-E)]$ the f(r) factor of (A.4) itself, we have:

$R\Phi(r)Y(\theta,\varphi) = -L^2\Phi(r)Y(\theta,\varphi)$, from which:

$$R\Phi(r) = -\Phi(r)\frac{L^2Y(\theta,\varphi)}{Y(\theta,\varphi)} \tag{A.5}$$

and we can name: $\lambda = \frac{L^2Y(\theta,\varphi)}{Y(\theta,\varphi)}$, which is clearly constant with respect to r, as both Y and L^2 don't depend on r. We have:

$$L^2Y(\theta,\varphi) = \lambda Y(\theta,\varphi), \tag{A.6}$$

which is the eigenvalue equation for L^2. As already stated, the $Y(\theta,\varphi)$ will give us infos on the shapes of orbitals, while the other function in r, that is $\Phi(r)$, which will be expressed by us as well,

will tell us just how high is the probability to find the electron in the orbital, when we get far away from the center or when we get closer to the center along r, without changing θ and φ.
Well, after making (A.6) explicit, we get:

$$\frac{\partial^2 Y(\theta,\varphi)}{\partial\theta^2}+ctg\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}+\frac{\lambda}{\hbar^2}Y(\theta,\varphi)=0$$

Out of convenience, let's define $\lambda=\hbar^2\nu(\nu+1)$, where ν is constant, as well as λ, of course. This will be more clear later on. From all this:

$$\frac{\partial^2 Y(\theta,\varphi)}{\partial\theta^2}+ctg\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}+\nu(\nu+1)Y(\theta,\varphi)=0. \quad \text{(A.7)}$$

As well as before, here we look for solutions with separated variables: $Y(\theta,\varphi)=A(\theta)B(\varphi)$; after inserting this into the (A.7):

$A''B+ctg\theta A'B+\frac{1}{\sin^2\theta}AB''+\nu(\nu+1)AB=0$. Let's multiply both sides by $\frac{\sin^2\theta}{AB}$:

$$\sin^2\theta\frac{A''}{A}+\sin\theta\cos\theta\frac{A'}{A}+\nu(\nu+1)\sin^2\theta+\frac{B''}{B}=0. \quad \text{(A.8)}$$

Because of a dimensional matter, we immediately notice that it must be $\frac{B''}{B}=const$, as $\frac{B''}{B}$ is the only item to depend on just φ, while all the other terms depend on just θ. Therefore, after having noticed that Y is periodical with respect to φ, we can write that: $\frac{B''}{B}=-m^2$ (where $B(\varphi)=e^{im\varphi}$ and m=0,±1, ±2,…). By the way, as $L_z=-i\hbar\partial/\partial\varphi$ and applying L_z to $B(\varphi)$, we have: $L_zB(\varphi)=-i\hbar\partial/\partial\varphi\, e^{im\varphi}=\hbar me^{im\varphi}=\hbar mB(\varphi)$, which is an eigenvalue equation for L_z, from which we have the reference to L_z of the quantum number m. The value found for B''/B, if put into the (A.8), yields: $\sin^2\theta A''+\sin\theta\cos\theta A'+[\nu(\nu+1)\sin^2\theta-m^2]A=0$. If now we put $\cos\theta=x$, (and so: $\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}$, $\frac{d^2}{d\theta^2}=-\cos\theta\frac{d}{dx}+\sin^2\theta\frac{d^2}{dx^2}$), we have:

$\sin^2\theta(\sin^2\theta\frac{d^2A}{dx^2}-\cos\theta\frac{dA}{dx})-\sin^2\theta\cos\theta\frac{dA}{dx}+[\nu(\nu+1)\sin^2\theta-m^2]A=0$; finally, by dividing both sides by $\sin^2\theta=1-x^2$, we get:

$$(1-x^2)\frac{d^2A}{dx^2}-2x\frac{dA}{dx}+[\nu(\nu+1)-\frac{m^2}{(1-x^2)}]A=0 \quad \text{(A.9)}$$

known as Legendre Differential Equation; it has three singularities (fuchsian ones) in $x_0=\pm1$ (and at infinite). Those on $x_0=\pm1$ are due to the "dangerous" tending to zero of the denominator of $\frac{1}{(1-x^2)}$ (about the point at the infinite, we will not care much).
Let's write the (A.9) as follows:

$$\frac{d}{dx}[(1-x^2)\frac{dA}{dx}]+[\nu(\nu+1)-\frac{m^2}{(1-x^2)}]A=0 \quad \text{(A.10)}$$

and let's start by considering the easiest case m=0:

$$\frac{d}{dx}[(1-x^2)\frac{dA}{dx}]+\nu(\nu+1)A=0 \quad \text{(A.11)}$$

We look for solutions as a series of powers, like:
$A(x)=x^s\sum_{n=0}^{+\infty}a_nx^n$ and we use that into the (A.11):

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}\sum_n a_n x^{n+s}]+\nu(\nu+1)\sum_n a_n x^{n+s}=0$$
$$\frac{d}{dx}[(1-x^2)\sum_n a_n(n+s)x^{n+s-1}]+\nu(\nu+1)\sum_n a_n x^{n+s}=0$$
$$\frac{d}{dx}[\sum_n a_n(n+s)x^{n+s-1}-\sum_n a_n(n+s)x^{n+s+1}]+\nu(\nu+1)\sum_n a_n x^{n+s}=0$$
$$\sum_n a_n(n+s)(n+s-1)x^{n+s-2}-\sum_n a_n(n+s+1)(n+s)x^{n+s}+\nu(\nu+1)\sum_n a_n x^{n+s}=0$$
Now, into the first summation, we rename n by (n-2):
$$\sum_n a_{n+2}(n+s+2)(n+s+1)x^{n+s}-\sum_n a_n(n+s+1)(n+s)x^{n+s}+\nu(\nu+1)\sum_n a_n x^{n+s}=0$$
$$\sum_n [a_{n+2}(n+s+2)(n+s+1)-a_n(n+s+1)(n+s)+\nu(\nu+1)a_n]x^{n+s}=0$$
$$a_{n+2}(n+s+2)(n+s+1)-a_n[(n+s+1)(n+s)-\nu(\nu+1)]=0$$
$$a_{n+2}=a_n\frac{(n+s+1)(n+s)-\nu(\nu+1)}{(n+s+2)(n+s+1)}$$
Function A is limited in x=1 (that is $\theta=0$), so we sooner or later must have: $(n+s+1)(n+s)-\nu(\nu+1)=0$ in order not to have a divergence, with all coefficients a_n not equal to zero, and as n and s are integer, such must be ν, which will be now called l , and so: $\lambda=\hbar^2 l(l+1)$.
Then, as +m and –m play a role of symmetry, we will have: $-l\le m\le l$, or $|m|\le l$.
Moreover, by a direct use of it, we realize that a solution of the (A.11) is the following:
$$A_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \qquad \text{(A.12)}$$
as well as a solution for the (A.10) is:
$$A_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{|m|/2}\frac{d^{l+|m|}}{dx^{l+|m|}}(x^2-1)^l \qquad \text{(A.13)}$$
As an example, when l=1 and m=1, the (A.13) becomes:
$$A_1^1(x)=-\frac{1}{2}(1-x^2)^{1/2}\frac{d^2}{dx^2}(x^2-1)=-\sqrt{(1-x^2)},$$
and if we put it into the (A.10), we get 0=0.
After recalling that we had $Y(\theta,\varphi)=A(\theta)B(\varphi)$, we say: $Y_l^m(\theta,\varphi)=A_l^m(\theta)B_m(\varphi)$
Let's evaluate, through the (A.13), the first $A_l^m(x)$ ones: ($x=\cos\theta$)
$A_0^0(x)=1=const$, $A_1^0(x)=x=\cos\theta=const\cdot\cos\theta$, $A_1^{+1}(x)=-(1-x^2)^{1/2}=-\sin\theta=-const\cdot\sin\theta$
and so on, so having all the Y shapes of the orbitals, here reminded: ($B_m(\varphi)=e^{im\varphi}$)
$$Y_0^0=\frac{1}{\sqrt{4\pi}},\quad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\cdot e^{\pm i\varphi},\quad Y_2^0=\frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta-1),$$
$$Y_2^{\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\cdot e^{\pm i\varphi},\quad Y_2^{\pm2}=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\cdot e^{\pm i2\varphi}.$$
But now we ask ourselves why do we have those particular constants? ($\frac{1}{\sqrt{4\pi}},\sqrt{\frac{3}{4\pi}}$, etc) Well, you recall that the highest probability all over the space must be 1, from which we get the normalization to 1. So, as an example on case:
$Y_1^0(\theta,\varphi)=A_1^0(\theta)B_0(\varphi)=const\cdot\cos\theta\cdot e^{i0\varphi}=const\cdot\cos\theta=C_1^0\cos\theta$, and after considering that, with the spherical coordinates, θ is spanned by r, while φ is spanned by the projection of r over the plane x-y, that is $r\sin\theta$, the infinitesimal solid angle $d\Omega$ is given, as we know, by the ratio

between the infinitesimal spherical susface dS and r^2, that is:
$d\Omega = \frac{dS}{r^2} = \frac{rd\theta \cdot r\sin\theta d\varphi}{r^2} = d\varphi \sin\theta d\theta$. So, let's integrate over all Ω and let's request that the highest probability is 1: (θ goes from 0 amd π, while φ goes from 0 and 2π)

$$\int_0^{4\pi}(Y_1^0)^*(\theta,\varphi)\cdot Y_1^0(\theta,\varphi)d\Omega = \int_0^{2\pi}\int_0^{\pi}(C_1^0)^2\cos^2\theta d\varphi\sin\theta d\theta = \int_0^{2\pi}d\varphi\int_0^{\pi}(C_1^0)^2\cos^2\theta\sin\theta d\theta =$$

$$= -\int_0^{2\pi}d\varphi\int_0^{\pi}(C_1^0)^2\cos^2\theta d\cos\theta = -2\pi(C_1^0)^2\int_0^{\pi}\cos^2\theta d\cos\theta = -2\pi(C_1^0)^2\left|\frac{1}{3}\cos^3\theta\right|_0^{\pi} = \frac{4}{3}\pi(C_1^0)^2 = 1,$$

that is: $\frac{4}{3}\pi(C_1^0)^2 = 1$, or: $C_1^0 = \sqrt{\frac{3}{4\pi}}$, which is exactly the coefficient of Y_1^0. Similarly,

$$\int_0^{4\pi}(Y_1^1)^*(\theta,\varphi)\cdot Y_1^1(\theta,\varphi)d\Omega = \int_0^{2\pi}\int_0^{\pi}(C_1^1)^2\sin^2\theta\cdot e^{-i\varphi+i\varphi}d\varphi\sin\theta d\theta = \int_0^{2\pi}d\varphi\int_0^{\pi}(C_1^1)^2\sin^3\theta d\theta =$$

$= 2\pi(C_1^1)^2\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{8}{3}\pi(C_1^1)^2 = 1$, so: $\frac{8}{3}\pi(C_1^1)^2 = 1$, and so: $C_1^1 = \sqrt{\frac{3}{8\pi}}$, that is exactly the coefficient of Y_1^1. And so on.

The shapes of the orbitals would be explained just by what has been explained so far. Out of completeness, as the complete solution of Schrodinger's Equation for the atom is (see above): $\Psi(\vec{r}) = \Phi(r)Y(\theta,\varphi)$, we just have to give a representation of the pure radial function $\Phi(r)$.

Let's remind here the (A.5):

$R\Phi_{nl}(r) = -\Phi_{nl}(r)\frac{L^2Y(\theta,\varphi)}{Y(\theta,\varphi)} = -\lambda\Phi_{nl}(r) = -\hbar^2 l(l+1)\Phi_{nl}(r) = r^2\{p_r^2 + 2m[V(r)-E]\}\Phi_{nl}(r)$, or:

$-\hbar^2 l(l+1)\Phi_{nl}(r) = r^2\{p_r^2 + 2m[V(r)-E]\}\Phi_{nl}(r)$, from which:

$\frac{p_r^2}{2m}\Phi_{nl}(r) + [V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}]\Phi_{nl}(r) = E\Phi_{nl}(r)$ and after reminding also the (A.6) and the subsequent expressions for p_r, that are: $p_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = -\hbar^2(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r})$, we have:

$-\frac{\hbar^2}{2m}(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr})\Phi_{nl}(r) + [V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}]\Phi_{nl}(r) = E\Phi_{nl}(r)$, or:

$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + \{\frac{2m}{\hbar^2}[E - V(r)] - \frac{l(l+1)}{r^2}\}\Phi_{nl}(r) = 0$;

Now, considering that $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}$, we have:

$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + \{\frac{2m}{\hbar^2}[E + \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}] - \frac{l(l+1)}{r^2}\}\Phi_{nl}(r) = 0$, or also (after considering the Bohr's radius $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$): $\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + [\frac{2m}{\hbar^2}E + \frac{2Z}{a_0 r} - \frac{l(l+1)}{r^2}]\Phi_{nl}(r) = 0$

and after considering the total energy $E = -\frac{Z^2e^4m}{32\pi^2n^2\varepsilon_0^2\hbar^2}$ (and notice that here we have $\hbar = h/2\pi$ instead of h), we have:

$$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + [-\frac{Z^2e^4m^2}{16\pi^2n^2\varepsilon_0^2\hbar^4} + \frac{2Z}{a_0 r} - \frac{l(l+1)}{r^2}]\Phi_{nl}(r) = 0 \qquad \text{(A.14)}$$

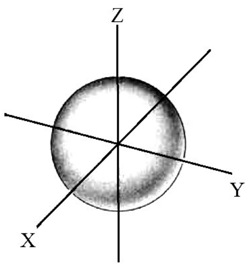
and we also realize that the following functions are solutions for the (A.14):

$$\Phi_{nl}(r)=\frac{1}{n}\sqrt{(\frac{Z}{a_0})\frac{(n-l-1)!}{(n+l)!}}\cdot e^{-Zr/na_0}\frac{1}{r}(\frac{2Zr}{na_0})^{l+1}\cdot L_{n-l-1}^{2l+1}(\frac{2Zr}{na_0}) \;, \tag{A.15}$$

(where $L_0^{\alpha}(x)=1$ and $L_1^{\alpha}(x)=\alpha+1-x$, $L_n^{\alpha}(x)=\frac{1}{n!}e^{x}x^{-\alpha}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}]$ Generalized Laguerre's Polinomials). Let's give some of them:

$$\Phi_{10}(r)=\sqrt{\frac{Z}{a_0}}\frac{2Z}{a_0}e^{-Zr/a_0}\;,\;\Phi_{20}(r)=\sqrt{\frac{Z}{8a_0}}\frac{Z}{a_0}(2-\frac{Zr}{a_0})e^{-Zr/2a_0}\;,\;\Phi_{21}(r)=\sqrt{\frac{Z}{24a_0}}(\frac{Z}{a_0})^2 re^{-Zr/2a_0} \tag{A.16}$$

The normalization has been carried out in the following way: $\int_0^{\infty}r^2|\Phi_{nl}(r)|^2dr$, as functions Y (see above) were normalized over $d\Omega=\frac{dS}{r^2}=\frac{rd\theta\cdot r\sin\theta d\varphi}{r^2}=d\varphi\sin\theta d\theta$, and in order to go from $d\Omega$ to dV we have to multiply $d\Omega$ by r^2dr, as $d\Omega$ x r^2 gives dS and dS x dr gives dV and so the remaining part of the normalization, that is r^2dr is taken by $\Phi_{nl}(r)$ indeed, as it is the radial function. As a simple crosscheck, you can see that the (A.15) is really a solution for the (A.14); in other words, put one by one every (A.16) in the (A.14) (and by respecting, time by time, values of n and l) and you will see you will always get 0=0. Finally, by considering the example on the s orbital (n=1, l=0, m=0),



it is a sphere, as its function $Y_l^m(\theta,\varphi)=Y_0^0=1/\sqrt{4\pi}$ is constant, so θ and φ don't show up , and so the "angular" probability P_a ($\propto|Y_l^m(\theta,\varphi)|^2$) does not depend on how we orientate, but its radial function is $\Phi_{nl}(r)=\Phi_{10}(r)=\sqrt{\frac{Z}{a_0}}\frac{2Z}{a_0}e^{-Zr/a_0}$, that is a constant by an exponential, so telling us that the radial "probability" P_r ($\propto|\Phi_{nl}(r)|^2$) to find the electron is not in the thin peel of that perfect spherical surface, but it's cloud-like spread and goes dimming as long as one gets away from the centre.

Schrodinger's Wave Function for the atom:

$\Psi(\vec{r})=Y_l^m(x=\cos\theta,\varphi)\Phi_{nl}(r)=[A_l^m(x=\cos\theta)B_m(\varphi)]\Phi_{nl}(r)$, or:

$$\boxed{\Psi(\vec{r})=\frac{(-1)^m}{2^l nl!}\cdot\sqrt{(\frac{Z}{a_0})\frac{(n-l-1)!}{(n+l)!}}(1-x^2)^{|m|/2}[\frac{d^{l+|m|}}{dx^{l+|m|}}(x^2-1)^l]e^{im\varphi}\cdot e^{-Zr/na_0}\frac{1}{r}(\frac{2Zr}{na_0})^{l+1}\cdot L_{n-l-1}^{2l+1}(\frac{2Zr}{na_0})}$$

-B-CYLINDRICAL COORDINATES

Aerodynamics and Kutta-Zhukowsky's Equation

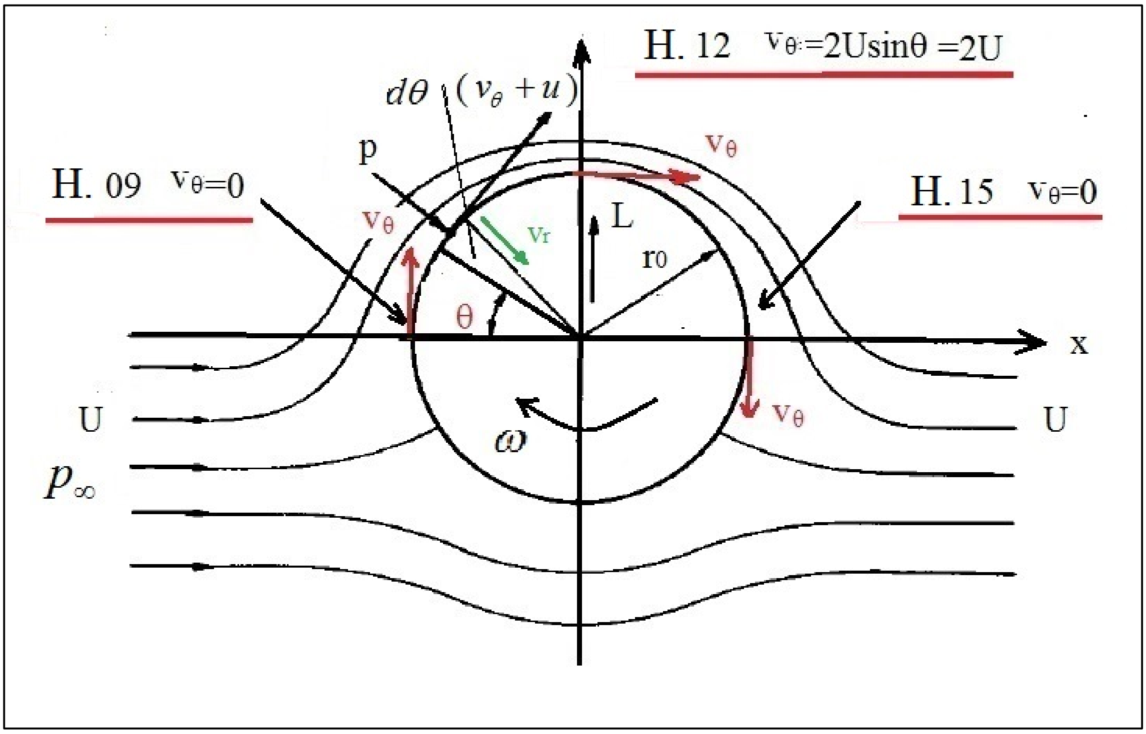


Fig. B.1.

We have a long cylinder in a uniform flow U and spinning with a constant angular speed ω. This makes the fluid over the cylinder surface have a speed component $u = r_0\omega$, because of the viscosity. Still on the cylinder surface there is the v_θ component from the U flow. We will prove below (eq. (B.8)) that the component v_θ at h. 12.00 is 2U, so around the cylinder at different θ angles, we will have: $v_\theta = 2U\sin\theta$. Therefore, around the cylinder we will have tangentially a $v_T = v_\theta + u = 2U\sin\theta + r_0\omega$.

After naming p_∞ the pressure far away from the cylinder and p that over the cylinder, according to the Bernoulli Equation: $p + \frac{1}{2}\rho v^2 + \rho gh = const$, that is: $p_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$ and being all at the same height h, we have, after a comparison between the flow on the left side of the cylinder and that over the cylinder:

$p_\infty + \frac{\rho}{2}U^2 = p + \frac{\rho}{2}(2U\sin\theta + r_0\omega)^2$. Therefore:

$$\frac{p - p_\infty}{\frac{\rho}{2}U^2} = 1 - (\frac{2U\sin\theta + r_0\omega}{U})^2 \qquad \text{(B.1)}$$

If we now consider the infinitesimal element of arc $r_0 d\theta$, it makes along an l (el) stretch of cylinder (which ideally extends into the sheet), a small surface $lr_0 d\theta$. Its projection over the orizontal plane is obviously: $lr_0\sin\theta d\theta$. Over it the pressure $(p - p_\infty)$ acts, and as it is drawn downwards in the drawing, the relevant force over the small surface is upwards and its sign will be opposite:

$dL_{[N]} = -(p - p_\infty) l r_0 \sin\theta d\theta$. Let's evaluate now the force per unit of length of the cylinder: $dL_{[N/m]} = dL = -(p - p_\infty) r_0 \sin\theta d\theta$ (with no more the small l - el).

About the total lift L, we will integrate from bottom to top of the cylinder, that is between $-\pi/2$ and $+\pi/2$; not only; we will multiply by two, as the cylinder has got two "sides", back and front:

$$L = L_{[N/m]} = 2\int_{-\pi/2}^{+\pi/2} -(p - p_\infty) r_0 \sin\theta d\theta = -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{2U\sin\theta + r_0\omega}{U})^2] \sin\theta d\theta =$$

$$= -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{r_0\omega}{U})^2 - \frac{4 r_0 \omega}{U}\sin\theta - 4\sin^2\theta] \sin\theta d\theta = 2\pi \cdot r_0^2 \omega \rho U = 2\pi \cdot r_0 u \rho U$$

or: $L = 2\pi \cdot r_0 u \rho U$ (B.2)

Now, as the peripheral speed u of the cylinder is all over the circumference, the circulation Γ (speed by space of action) is: $\Gamma = 2\pi \cdot r_0 u$, from which:

$L = \rho U \Gamma$ (Kutta-Joukowski Equation for the spinning cylinder) (B.3)

Therefore, we can state that when a circulation Γ is generated and not only by a spinning cylinder, but also by an aerofoil (wing), a lift L is generated.

– The Potential of the flow

About the speed v on the circumference of the spinning cylinder, and in particular about the component v_θ due to the external flow, whose horizontal speed far away is U, let's find a potential function ϕ so that $v = \nabla\phi$; from the Continuity Equation $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})$ we have, if ρ is constant, $\vec{\nabla}\cdot(\rho\vec{v}) = \rho\vec{\nabla}\cdot\vec{v} = 0$, so $\vec{\nabla}\cdot\vec{v} = 0$, and $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(\nabla\phi) = \Delta\phi = 0$ and $\Delta\phi = 0$ is the Laplace's Equation:

Laplacian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta\phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi = 0$.

Let's remind now the expression for the **Laplacian in cylindrical coordinates** (very suitable for us, as our spinning cylinder has got a cylindrical symmetry indeed):

$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$, and with reference to our spinning cylinder, let's consider the z axis as ideally penetrating into the sheet. Moreover, out of symmetry reasons of our long cylinder, the component with z is zero, so: $\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$ and for our new Laplace's Equation, we have: $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$, that is:

$r\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{\partial^2\phi}{\partial\theta^2} = 0$. (B.4)

Therefore, in cylindrical coordinates we can say that the v speed components around the cylinder and due to the speed of the external flow (U, from far away), will be formally three: v_θ, v_r and v_z, but out of geometric reasons and out of reasons of uniformity of the cylinder, we have that $v_z = 0$.

We realize that the function $\phi = A_n r^n \cos(n\theta)$ is a solution for the (B.4) both with either n=+1 or n=-1 (and A_n / A_{-n} = constants to be found), so we have at least two solutions: $\phi_1 = A_1 r\cos\theta$ and $\phi_2 = A_{-1} r^{-1}\cos(-\theta) = A_{-1} r^{-1}\cos\theta$. So, the sum of them will be a more general solution:

$\phi = \phi_1 + \phi_2 = A_1 r\cos\theta + A_{-1} r^{-1}\cos\theta$. (B.5)

With reference to the above figure and after suitably reconciling all the algebraic signs, in order to make the equations fit the figure, we see that a bit on the left of the point at h. 09 (as if it was about a clock…) the speed of the flow is U and is horizontal, from left to right, while over the cylinder, right at h. 09, the speed gets obviously zero, and it will eventually increase upwards (v_θ) as long as θ increases. The same is about the point at h. 15, on the right side of which the flow will then get back a U horizontal speed, towards right, that is -U.
Let's use those two points (h. 09 and h. 15) to evaluate ϕ and then, from the equation we will find, we will calculate the value of v_θ at h. 12:

$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(A_1 r\cos\theta + A_{-1}r^{-1}\cos\theta) = -A_1\sin\theta - A_{-1}r^{-2}\sin\theta$, from which:

$v_{\theta=0} = -A_1\sin 0 - A_{-1}r^{-2}\sin 0 = 0$. Same value for θ=180°.

Moreover, obviously: $v_r = \frac{\partial \phi}{\partial r} = \frac{\partial}{\partial r}(A_1 r\cos\theta + A_{-1}r^{-1}\cos\theta) = A_1\cos\theta - A_{-1}r^{-2}\cos\theta$, from which, as we must also have $v_{r_0} = 0$, we have: $0 = A_1\cos\theta - A_{-1}r_0^{-2}\cos\theta$, that is: $A_{-1} = r_0^2 A_1$, and so:

$\phi = A_1 r\cos\theta + r_0^2 A_1 r^{-1}\cos\theta = A_1\cos\theta(r + \frac{r_0^2}{r})$. Therefore:

$$\phi = A_1\cos\theta(r + \frac{r_0^2}{r}) \qquad \text{(B.6)}$$

Now, in order to figure out A1, let's get on the horizontal axis (θ=0 or θ=180°) and let's evaluate v_r, which must be, for $r \to +\infty$, respectively U and –U:

$v_r = \frac{\partial \phi}{\partial r} = A_1\cos\theta(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=0} = A_1(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=180°} = -A_1(1 - \frac{r_0^2}{r^2})$, that is:

$(v_r)_{\theta=0\_r\to+\infty} = A_1 = U$, $(v_r)_{\theta=180°\_r\to+\infty} = -A_1 = -U$ and so, finally:

$$\phi = U\cos\theta(r + \frac{r_0^2}{r}) \qquad \text{(B.7)}$$

At last, about v_θ evaluated in a generic θ and in θ=90°, we have:

$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}[U\cos\theta(r + \frac{r_0^2}{r})] = -U(1 + \frac{r_0^2}{r^2})\sin\theta$ and over the circumference of the cylinder (r=r0): $v_\theta = -2U\sin\theta$, and also:

$$v_{\theta=90°} = -2U \;. \qquad \text{(B.8)}$$

Moreover, still vertically (θ=90°), or at a certain θ= θ0, but far away from the cylinder, ($r \to \infty$) the flow must get back to the imperturbated flow when there's no cylinder, that is with v=U:

$v_\theta = -U(1 + \frac{r_0^2}{r^2})\sin\theta$, $v_r = U\cos\theta(1 - \frac{r_0^2}{r^2})$,

so, when θ=90° and $r \to \infty$, $v_\theta = -U$ and $v_r = 0$, that is, according to Pithagora: v=U.
On the contrary, for θ= θ0 and $r \to \infty$, $v_\theta = -U\sin\theta_0$ and $v_r = U\cos\theta_0$, that is, according to Pithagora: $v = \sqrt{[(U\cos\theta_0)^2 + (-U\sin\theta_0)^2]} = U$, so the (2.4) represents very well our flow, everywhere around the cylinder.

SUBAPPENDIXES

SUBAPPENDIX 1-Vector Identities:

At last, we remind also the following vector identities, which can be directly proved:
(V and W are scalar values and $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ vectors)

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \quad (A1.1)$$
$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B}) \quad (A1.2)$$
$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \quad (A1.3)$$

$$\left|\vec{A}\times\vec{B}\right|^2=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-(\vec{A}\cdot\vec{B})^2 \quad (A1.4)$$

In fact: $\vec{A}\times\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$ and $\vec{A}\cdot\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$, so: $\left|\vec{A}\times\vec{B}\right|^2=\left|\vec{A}\right|^2\left|\vec{B}\right|^2\sin^2\theta=\left|\vec{A}\right|^2\left|\vec{B}\right|^2(1-\cos^2\theta)=$
$=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-\left|\vec{A}\right|^2\left|\vec{B}\right|^2\cos^2\theta=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-(\vec{A}\cdot\vec{B})^2$ qed.

$$\vec{\nabla}(V+W)=\nabla V+\nabla W \quad (A1.5)$$
$$\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B} \quad (A1.6)$$
$$\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B} \quad (A1.7)$$
$$\vec{\nabla}(\vec{A}\cdot\vec{B})=(\vec{A}\cdot\vec{\nabla})\vec{B}+(\vec{B}\cdot\vec{\nabla})\vec{A}+\vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A}) \quad (A1.8)$$
$$\nabla(VW)=W\nabla V+V\nabla W \quad (A1.9)$$
$$\vec{\nabla}\times\nabla V=0 \quad (A1.10)$$
$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}=0 \quad (A1.11)$$
$$\vec{\nabla}\cdot(V\vec{A})=\nabla V\cdot\vec{A}+V(\vec{\nabla}\cdot\vec{A}) \quad (A1.12)$$
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B})=(\vec{\nabla}\times\vec{A})\cdot\vec{B}-\vec{A}\cdot(\vec{\nabla}\times\vec{B}) \quad (A1.13)$$
$$\vec{\nabla}\times(V\vec{B})=V(\vec{\nabla}\times\vec{B})+\nabla V\times\vec{B} \quad (A1.14)$$
$$\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{\nabla}\cdot\vec{B})\vec{A}-(\vec{\nabla}\cdot\vec{A})\vec{B}-\vec{B}(\vec{A}\cdot\vec{\nabla})+\vec{A}(\vec{B}\cdot\vec{\nabla}) \quad (A1.15)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\nabla(\vec{\nabla}\cdot\vec{A})\vec{A}-\vec{\Delta}\vec{A} \quad (A1.16)$$

where $\vec{\Delta}\vec{A}$ in the (A1.16) is the "very little known" <u>Vector Laplacian</u>.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\vec{\nabla}\times[\hat{i}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\hat{j}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\hat{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})]=$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}) & (\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}) & (\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})\end{vmatrix}=\hat{i}[\frac{\partial(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})}{\partial y}-\frac{\partial(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})}{\partial z}]+\hat{j}[\frac{\partial(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})}{\partial z}-\frac{\partial(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})}{\partial x}]+$$

$$+\hat{k}[\frac{\partial(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})}{\partial x}-\frac{\partial(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})}{\partial y}]=\hat{i}[\frac{\partial}{\partial y}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})-\frac{\partial}{\partial z}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})]+\hat{j}[\frac{\partial}{\partial z}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})-\frac{\partial}{\partial x}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})]+$$

$$+\hat{k}[\frac{\partial}{\partial x}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})-\frac{\partial}{\partial y}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})]=\hat{i}(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}-\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_x}{\partial x\partial y})+$$

$\hat{k}(\frac{\partial^2 A_x}{\partial x\partial z}-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_y}{\partial y\partial z})$. Now, in the first round brackets, we add the zero quantity:

$(\frac{\partial^2 A_x}{\partial x^2}-\frac{\partial^2 A_x}{\partial x^2})$, then in the second brackets we add $(\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial y^2})$ and in the third $(\frac{\partial^2 A_z}{\partial z^2}-\frac{\partial^2 A_z}{\partial z^2})$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\hat{i}(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_z}{\partial x\partial z}+\frac{\partial^2 A_x}{\partial x^2}-\frac{\partial^2 A_x}{\partial x^2})+\hat{j}(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}-\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_x}{\partial x\partial y}+\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial y^2})+$$

$$\hat{k}(\frac{\partial^2 A_x}{\partial x\partial z}-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_y}{\partial y\partial z}+\frac{\partial^2 A_z}{\partial z^2}-\frac{\partial^2 A_z}{\partial z^2})=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})-\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})]+$$

$$+[\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})-\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})]+[\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]-\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]=$$

$$=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})+\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]-$$

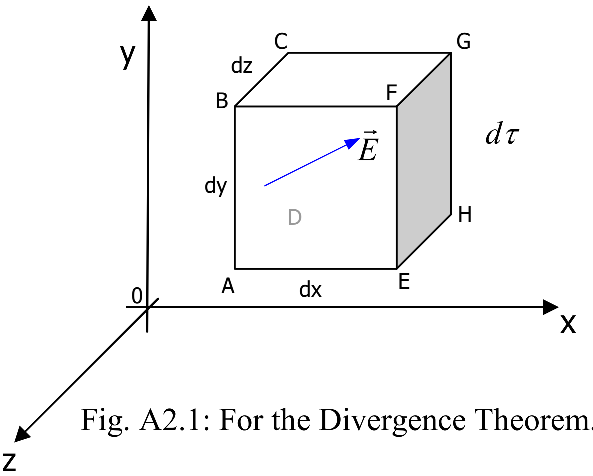
$-[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})+\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})+\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]=\nabla(\vec{\nabla}\cdot\vec{A})\vec{A}-\vec{\Delta}\vec{A}$, where:

$\nabla(\vec{\nabla}\cdot\vec{A})=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})+\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]$ and

$\boxed{\vec{\Delta}\vec{A}=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})+\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})+\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]}$ **Vector Laplacian**.

SUBAPPENDIX 2 – Divergence and Curl Theorems

Divergence Theorem (practical proof):



$$\boxed{\int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV}$$

Fig. A2.1: For the Divergence Theorem.

Name ϕ the flux of the vector $\vec{E}$; we have:

$d\phi_{ABCD} = \vec{E}\cdot d\vec{S} = -E_x(x,\bar{y},\bar{z})dydz$ ($\bar{y}$ means y "mean")

$d\phi_{EFGH} = E_x(x+dx,\bar{y},\bar{z})dydz$, but we obviously know that also: (as a development):

$E_x(x+dx,\bar{y},\bar{z}) = E_x(x,\bar{y},\bar{z}) + \dfrac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dx$ so:

$d\phi_{EFGH} = E_x(x,\bar{y},\bar{z})dydz + \dfrac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dxdydz$ and so:

$d\phi_{ABCD} + d\phi_{EFGH} = \dfrac{\partial E_x}{\partial x}dV$. We similarly act on axes y and z:

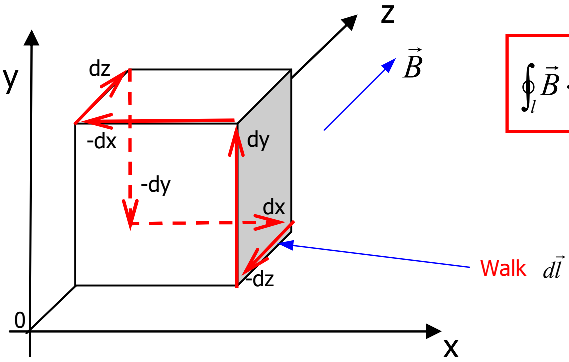
$d\phi_{AEHD} + d\phi_{BCGF} = \dfrac{\partial E_y}{\partial y}dV$ $\qquad d\phi_{ABFE} + d\phi_{CGHD} = \dfrac{\partial E_z}{\partial z}dV$

And then we sum up the fluxes so found, having totally:

$d\phi = (\dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z})dV = (div\cdot\vec{E})dV = (\vec{\nabla}\cdot\vec{E})dV$ therefore:

$\phi_S(\vec{E}) = \int_\phi d\phi = \int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV = \int_V (\vec{\nabla}\cdot\vec{E})\cdot dV$ that is the statement.

Curl or Stokes' Theorem (practical proof-**by Rubino!**):



$$\boxed{\oint_l \vec{B}\cdot d\vec{l} = \int_S curl\vec{B}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}}$$

Fig. A2.2: For the Curl Theorem (proof by Rubino).

Let's figure out $\vec{B}\cdot d\vec{l}$:

On dz B is B_z; on dx B is B_x; on dy B is B_y;

on -dz B is $B_z+\frac{\partial B_z}{\partial x}dx-\frac{\partial B_z}{\partial y}dy$, for 3-D Taylor's development and also because to go from the center of dz to that of –dz we go up along x, then we go down along y and nothing along z itself.

Similarly, on -dx B is $B_x-\frac{\partial B_x}{\partial z}dz+\frac{\partial B_x}{\partial y}dy$ and on -dy B is $B_y-\frac{\partial B_y}{\partial x}dx+\frac{\partial B_y}{\partial z}dz$.

By summing up all contributions:

$$\vec{B}\cdot d\vec{l} = B_z dz-(B_z+\frac{\partial B_z}{\partial x}dx-\frac{\partial B_z}{\partial y}dy)dz+B_x dx-(B_x-\frac{\partial B_x}{\partial z}dz+\frac{\partial B_x}{\partial y}dy)dx+B_y dy-$$

$$+(B_y-\frac{\partial B_y}{\partial x}dx+\frac{\partial B_y}{\partial z}dz)dy=(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})dydz+(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x})dxdz+(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y})dxdy=$$

$=curl\vec{B}\cdot d\vec{S}=\vec{\nabla}\times\vec{B}\cdot d\vec{S}$, where $d\vec{S}$ has got components $[\hat{x}(dydz),\hat{y}(dxdz),\hat{z}(dxdy)]$

that is, the statement: $\oint_l \vec{B}\cdot d\vec{l}=\int_S curl\vec{B}\cdot d\vec{S}=\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}$, after having reminded of:

$$curl\vec{B}=\vec{\nabla}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_x & B_y & B_z\end{vmatrix} .$$

COORDINATE SFERICHE E CILINDRICHE CON I TENSORI DELLA RELATIVITÀ

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Abstract: in fisica il passaggio alle coordinate sferiche, o cilindriche, è spesso l'unica via per poter sviluppare un argomento ad un livello di difficoltà accettabile, dal momento che tale passaggio consente di usufruire delle semplificazioni derivanti dalla simmetria esistente (sferica o cilindrica appunto). Ecco una trattazione effettuata con i tensori ed anche una trattazione classica, senza tensori.

1-PREMESSE SUI SISTEMI DI COORDINATE

-COORDINATE CARTESIANE:

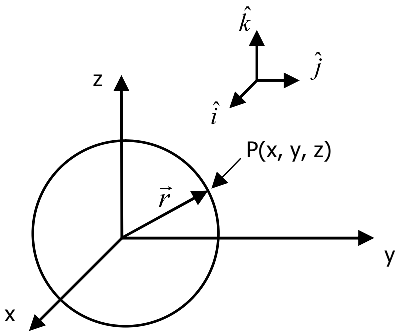


Fig. 1.1

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad , \tag{1.1}$$

$$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k} \quad , \tag{1.2}$$

V è uno scalare ed $\vec{F}$ è un vettore. La divergenza fornisce uno scalare, mentre il gradiente ed il rotore forniscono un vettore.

$$gradV = \nabla V = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}) \tag{1.3}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) \tag{1.4}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{j}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{1.5}$$

$$\Delta V = \nabla^2 V = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})V \tag{1.6}$$

-COORDINATE SFERICHE:

Si passa da (x,y,z) a (ρ, θ, φ). φ varia tra 0 e 2π, mentre θ varia tra 0 e π.

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\theta)\hat{\theta} + (\rho \sin\theta d\varphi)\hat{\varphi} \tag{1.7}$$

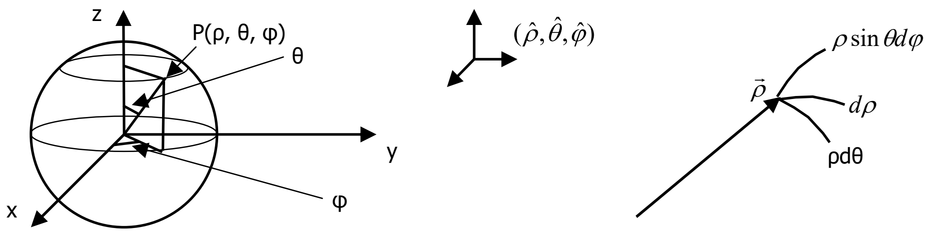


Fig. 1.2

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \tag{1.8}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}) \tag{1.9}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho^2 \sin\theta}[\frac{\partial}{\partial \rho}(\rho^2 \sin\theta F_\rho) + \frac{\partial}{\partial \theta}(\rho \sin\theta F_\theta) + \frac{\partial}{\partial \varphi}(\rho F_\varphi)] \tag{1.10}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \rho\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_\rho & \rho F_\theta & \rho \sin\theta F_\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho^2 \sin\theta}[\frac{\partial(\rho \sin\theta F_\varphi)}{\partial \theta} - \frac{\partial(\rho F_\theta)}{\partial \varphi}]\hat{\rho} + \frac{1}{\rho \sin\theta}[\frac{\partial F_\rho}{\partial \varphi} - \frac{\partial(\rho \sin\theta F_\varphi)}{\partial \rho}]\hat{\theta} + \frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}]\hat{\varphi} \tag{1.11}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} \tag{1.12}$$

-COORDINATE CILINDRICHE:

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

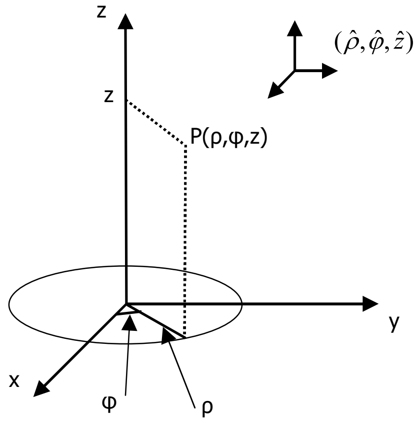


Fig. 1.3

$\varphi \in [0,2\pi)$

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\varphi)\hat{\varphi} + (dz)\hat{z} \tag{1.13}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z}) \tag{1.14}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \varphi}(F_\varphi) + \frac{\partial}{\partial z}(\rho F_z)] \tag{1.15}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} = \frac{1}{\rho}[\frac{\partial(F_z)}{\partial \varphi} - \frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho} + [\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}]\hat{\varphi} + \frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}]\hat{\varphi} \tag{1.16}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.17}$$

2-CALCOLO TENSORIALE

-**<u>derivazione di un 4-vettore</u>**:

si abbia un 4-vettore, ma, in generale, anche un vettore: $\vec{V} = V^{\mu}\vec{\tau}_{\mu}$ (V^{μ} sono le componenti, mentre i $\vec{\tau}_{\mu}$ sono i versori ed è sottintesa la sommatoria su μ, per la convenzione di Einstein, ossia:
$\vec{V} = V^{\mu}\vec{\tau}_{\mu} = \sum_{\mu} V^{\mu}\vec{\tau}_{\mu} = V^{x}\vec{\tau}_{x} + V^{y}\vec{\tau}_{y} + V^{z}\vec{\tau}_{z} = V^{x}\hat{x} + V^{y}\hat{y} + V^{z}\hat{z} \equiv V^{r}\hat{r} + V^{\theta}\hat{\theta} + V^{\varphi}\hat{\varphi}$).

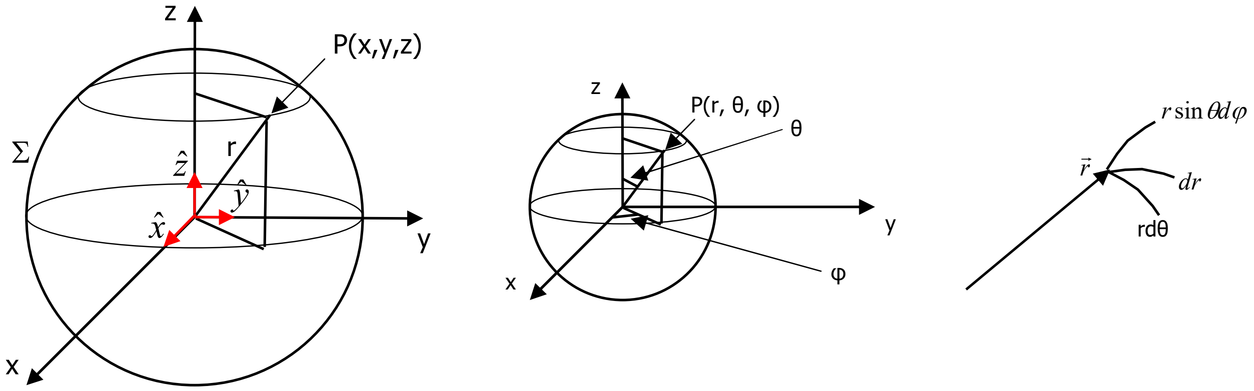


Fig. 2.1: coordinate cartesiane e polari sferiche.

Indicando ora, più convenientemente, i versori in questo modo:
$(\hat{x},\hat{y},\hat{z}) = (\hat{e}_1,\hat{e}_2,\hat{e}_3) = (\hat{e}_x,\hat{e}_y,\hat{e}_z)$ per le cartesiane e $(\hat{r},\hat{\theta},\hat{\varphi}) = (\hat{e}_r,\hat{e}_{\theta},\hat{e}_{\varphi})$ per le sferiche, nonché le componenti così: $(V^x,V^y,V^z) = (V^1,V^2,V^3)$ per le cartesiane e $(V^r,V^{\theta},V^{\varphi})$ per le sferiche (che coincidono con $[dr, rd\theta, (r\sin\theta)d\varphi]$ in caso ci si riferisca al vettore di posizione $d\vec{P}$), si avrà:

-**<u>per le cartesiane</u>**:

$\vec{V} = V^x\hat{x} + V^y\hat{y} + V^z\hat{z} = (\hat{e}_1V^1 + \hat{e}_2V^2 + \hat{e}_3V^3) = \sum_{i=1}^{3}\hat{e}_iV^i = \hat{e}_iV^i$, ossia:

$$\vec{V} = \hat{e}_iV^i \quad . \tag{2.1}$$

Se poi il vettore è il de vettore posizione $d\vec{P}$, allora si ha:
$d\vec{P} = dx\hat{x} + dy\hat{y} + dz\hat{z} = dr\hat{r} + rd\theta\hat{\theta} + d\varphi(r\sin\theta)\hat{\varphi} = (\hat{e}_1dx^1 + \hat{e}_2dx^2 + \hat{e}_3dx^3) = \sum_{i=1}^{3}\hat{e}_idx^i = \hat{e}_idx^i$, ossia:
$d\vec{P} = \hat{e}_idx^i$. Deriviamo ora il vettore (2.1) ($\vec{V} = \hat{e}_{\mu}V^{\mu}$) rispetto ad una coordinata x^{ν}:

$$\frac{d\vec{V}}{dx^{\nu}} = \frac{d(\hat{e}_{\mu}V^{\mu})}{dx^{\nu}} = \frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu} + V^{\sigma}\frac{d\hat{e}_{\sigma}}{dx^{\nu}}; \tag{2.2}$$

gli indici cambiano da un termine all'altro (μ→σ), perché essi variano su tutti i valori (vi è la sommatoria sottintesa sugli indici ripetuti) ed essi non devono necessariamente essere contemporaneamente gli stessi nei due termini.

Nel caso di coordinate cartesiane (piatte), la derivata resta comunque $\frac{d\vec{V}}{dx^{\nu}} = \frac{dV^{\mu}}{dx^{\nu}}\hat{e}_{\mu}$, poiché, trattandosi di geometria euclidea, i versori $(\hat{x},\hat{y},\hat{z})$, ossia gli $\hat{e}_i$, sono fissi e, dunque, le loro derivate sono nulle ($\frac{d\hat{e}_{\sigma}}{dx^{\nu}} = 0$). Diverso è il caso della geometria non euclidea.

Moltiplichiamo ora il secondo membro della (2.2) per la quantità unitaria $\hat{e}^{\mu}\hat{e}_{\mu}$, dove $\hat{e}^{\mu}$ è il versore controvariante, che evidentemente esiste, tale che il suo prodotto con quello covariante ($\hat{e}_{\mu}$) dia, per definizione, appunto l'unità:

$$\frac{d\vec{V}}{dx^\nu}=\frac{dV^\mu}{dx^\nu}\hat{e}_\mu+V^\sigma\frac{d\hat{e}_\sigma}{dx^\nu}=(\frac{dV^\mu}{dx^\nu}\hat{e}_\mu+V^\sigma\frac{d\hat{e}_\sigma}{dx^\nu})\hat{e}^\mu\hat{e}_\mu=\frac{dV^\mu}{dx^\nu}\hat{e}_\mu\hat{e}^\mu\hat{e}_\mu+V^\sigma\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu\hat{e}_\mu=[\frac{dV^\mu}{dx^\nu}+V^\sigma\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu]\hat{e}_\mu$$

bene, la quantità tra le quadre è la derivata covariante ed è un tensore:

$$V^\mu_{;\nu}=\frac{dV^\mu}{dx^\nu}+V^\sigma\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu=\frac{dV^\mu}{dx^\nu}+V^\sigma\Gamma^\mu_{\sigma\nu}\qquad\text{(derivata covariante)}\qquad(2.3)$$

con i $\Gamma^\mu_{\sigma\nu}=\frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu$ (2.4)

detti Simboli di Christoffel (connessione affine).

Nel caso di vettore espresso nella base reciproca ($\vec{V}=V_\mu\vec{\tau}^\mu$), o meglio: $\vec{V}=\hat{e}^\mu V_\mu$ si ha:

$\frac{d\vec{V}}{dx^\nu}=\frac{d(\hat{e}^\mu V_\mu)}{dx^\nu}=\frac{dV_\mu}{dx^\nu}\hat{e}^\mu+V_\sigma\frac{d\hat{e}^\sigma}{dx^\nu}$; moltiplichiamo ora il secondo membro per la quantità unitaria $\hat{e}_\mu\hat{e}^\mu$, dove $\hat{e}_\mu$ è il versore covariante, che evidentemente esiste, tale che il suo prodotto con quello controvariante ($\hat{e}^\mu$) dia, per definizione, appunto l'unità:

$$\frac{d\vec{V}}{dx^\nu}=\frac{dV_\mu}{dx^\nu}\hat{e}^\mu+V_\sigma\frac{d\hat{e}^\sigma}{dx^\nu}=(\frac{dV_\mu}{dx^\nu}\hat{e}^\mu+V_\sigma\frac{d\hat{e}^\sigma}{dx^\nu})\hat{e}_\mu\hat{e}^\mu=\frac{dV_\mu}{dx^\nu}\hat{e}^\mu\hat{e}_\mu\hat{e}^\mu+V_\sigma\frac{d\hat{e}^\sigma}{dx^\nu}\hat{e}_\mu\hat{e}^\mu=[\frac{dV_\mu}{dx^\nu}+V_\sigma\frac{d\hat{e}^\sigma}{dx^\nu}\hat{e}_\mu]\hat{e}^\mu,$$

ma: $\frac{\partial}{\partial x^\nu}(\hat{e}^\sigma\hat{e}_\mu)=\frac{\partial}{\partial x^\nu}\delta^\sigma_\mu=0=\frac{\partial\hat{e}^\sigma}{\partial x^\nu}\hat{e}_\mu+\hat{e}^\sigma\frac{\partial\hat{e}_\mu}{\partial x^\nu}$, da cui: $\frac{\partial\hat{e}^\sigma}{\partial x^\nu}\hat{e}_\mu=-\hat{e}^\sigma\frac{\partial\hat{e}_\mu}{\partial x^\nu}$, allora:

$\frac{d\vec{V}}{dx^\nu}=[\frac{dV_\mu}{dx^\nu}-V_\sigma\hat{e}^\sigma\frac{\partial\hat{e}_\mu}{\partial x^\nu}]\hat{e}^\mu=[\frac{dV_\mu}{dx^\nu}-V_\sigma\Gamma^\sigma_{\mu\nu}]\hat{e}^\mu=[V_{\mu;\nu}]\hat{e}^\mu$, dunque:

$$V_{\mu;\nu}=\frac{dV_\mu}{dx^\nu}-V_\sigma\Gamma^\sigma_{\mu\nu}\quad.$$

Possiamo quindi dire che: $\Gamma^k_{ij}=0\quad\rightarrow$ no curvatura.

-**<u>derivazione di un tensore</u>**:

si ha che: $T^{\mu\sigma}_{\lambda;\rho}=\frac{\partial}{\partial x^\rho}T^{\mu\sigma}_\lambda+\Gamma^\mu_{\rho\nu}T^{\nu\sigma}_\lambda+\Gamma^\sigma_{\rho\nu}T^{\mu\sigma}_\lambda-\Gamma^k_{\lambda\rho}T^{\mu\sigma}_k$ (2.5)

infatti ($T^{\mu\sigma}_\lambda$ sono solo le componenti, senza "versori" e inoltre $\hat{e}_\mu\hat{e}^\mu=\delta^\mu_\mu=1$):

T(tensore)= $T^{\mu\sigma}_\lambda\hat{e}_\mu\hat{e}_\sigma\hat{e}^\lambda$, da cui:

$$\frac{\partial}{\partial x^\rho}(T^{\mu\sigma}_\lambda\hat{e}_\mu\hat{e}_\sigma\hat{e}^\lambda)=(\frac{\partial}{\partial x^\rho}T^{\mu\sigma}_\lambda)(\hat{e}_\mu\hat{e}_\sigma\hat{e}^\lambda)+T^{\nu\sigma}_\lambda(\frac{\partial}{\partial x^\rho}\hat{e}_\nu)\hat{e}_\sigma\hat{e}^\lambda(\hat{e}_\mu\hat{e}^\mu)+T^{\mu\nu}_\lambda(\frac{\partial}{\partial x^\rho}\hat{e}_\nu)\hat{e}_\mu\hat{e}^\lambda(\hat{e}_\sigma\hat{e}^\sigma)+$$

$$-T^{\mu\sigma}_k(\frac{\partial}{\partial x^\rho}\hat{e}_\mu)\hat{e}^k\hat{e}_\sigma(\hat{e}_\lambda\hat{e}^\lambda)=\frac{\partial}{\partial x^\rho}T^{\mu\sigma}_\lambda+\Gamma^\mu_{\rho\nu}T^{\nu\sigma}_\lambda+\Gamma^\sigma_{\rho\nu}T^{\mu\sigma}_\lambda-\Gamma^k_{\lambda\rho}T^{\mu\sigma}_k=T^{\mu\sigma}_{\lambda;\rho}\quad\text{cvd.}$$

(in quanto : $\frac{\partial}{\partial x^\nu}\hat{e}^\mu\hat{e}_k=\frac{\partial}{\partial x^\nu}\delta^\mu_k=0=\frac{\partial\hat{e}^\mu}{\partial x^\nu}\hat{e}_k+\hat{e}^\mu\frac{\partial\hat{e}_k}{\partial x^\nu}$, da cui: $\frac{\partial\hat{e}^\mu}{\partial x^\nu}\hat{e}_k=-\hat{e}^\mu\frac{\partial\hat{e}_k}{\partial x^\nu}$)

Tornando al vettore posizione $d\vec{P}=\hat{e}_i dx^i$, dunque con componenti ($dx^1,dx^2,dx^3,(dx^4)$) nella base curvilinea ($\hat{e}_1,\hat{e}_2,\hat{e}_3,(\hat{e}_4)$), abbiamo detto che esiste anche una quaterna reciproca ($\hat{e}^1,\hat{e}^2,\hat{e}^3,(\hat{e}^4)$) tale che, per definizione: $\hat{e}_i\cdot\hat{e}^j=\delta^j_i$. Ora, se prendiamo un'altra base ($\hat{e}'_1,\hat{e}'_2,\hat{e}'_3,(\hat{e}'_4)$), allora, essendo $d\vec{P}$ sempre lo stesso, si avrà: $d\vec{P}=\hat{e}'_l dx'^l$ e

$\hat{e}_i dx^i=\hat{e}'_l dx'^l$, (2.6)

con le coordinate che passano da dx^i a dx'^l .
Moltiplicando ora entrambi i membri della (2.6) per $\hat{e}^i$, si ha: $dx^i = dx'^l \hat{e}'_l \cdot \hat{e}^i$, ma è anche vero che, grazie alle derivate parziali, si ha la seguente dipendenza ovvia: $dx^i = \frac{\partial x^i}{\partial x'^l} dx'^l$, da cui: $\frac{\partial x^i}{\partial x'^l} = \hat{e}'_l \cdot \hat{e}^i$,

ossia: $\hat{e}'_l = \hat{e}_i \frac{\partial x^i}{\partial x'^l}$ (2.7)

e la (2.7) è l'equazione per il cambiamento di base.

-**<u>legge di trasformazione delle componenti di un 4-vettore</u>**:
sia $\vec{V} = V^i \hat{e}_i$ un generico vettore espresso con le coordinate curvilinee nella base $\hat{e}_i$; nella base $\hat{e}'_l$ si avrà invece: $\vec{V} = V'^l \hat{e}'_l$ e, per la (2.7):

$\vec{V} = V'^l \hat{e}'_l = V'^l \frac{\partial x^i}{\partial x'^l} \hat{e}_i = V^i \hat{e}_i$, con:

$$V^i = V'^l \frac{\partial x^i}{\partial x'^l} \qquad (2.8)$$

che è la equazione di trasformazione delle componenti di un 4-vettore per cambiamento di base. Banalmente, la sua inversa è: $V'^i = V^l \frac{\partial x'^i}{\partial x^l}$

-**<u>legge di trasformazione delle componenti di un 4-tensore</u>**:
in Relatività Generale si ha che si ottiene un tensore T di rango n quando si moltiplicano tra loro le componenti di n vettori. Si abbiano dunque due vettori V ed S: (dove stiamo contemporaneamente ricordando come si trasformano le loro componenti)

$V'^\mu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu}$ e $S'^\lambda = S^\sigma \frac{\partial x'^\lambda}{\partial x^\sigma}$ >>>>

$$T'^{\mu\lambda} = V'^\mu S'^\lambda = V^\nu \frac{\partial x'^\mu}{\partial x^\nu} S^\sigma \frac{\partial x'^\lambda}{\partial x^\sigma} = V^\nu S^\sigma \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} = \boxed{T^{\nu\sigma} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} = T'^{\mu\lambda}} \qquad (2.9)$$

ed ecco come si trasformano le componenti di un tensore di rango 2. Si procede poi similmente per tensori di rango superiore.
Bene, adesso, dalla Fig. 2.1 si evince che per una sfera (siamo cioè in tre dimensioni) il modulo quadro dP^2 del vettore posizione è (essendo che $d\vec{P} = \hat{e}_i dx^i$):

$$dP^2 = (\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = (\hat{e}_\mu \hat{e}_\nu) dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu , \qquad (2.10)$$

e col passaggio alle polari sferiche: $\mu,\nu \to r,\theta,\varphi$

$$dP^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{rr} dx^r dx^r + g_{\theta\theta} dx^\theta dx^\theta + g_{\varphi\varphi} dx^\varphi dx^\varphi + \cancel{2g_{\theta\varphi} dx^\theta dx^\varphi} + \cancel{2g_{r\theta} dx^r dx^\theta} + \cancel{2g_{r\varphi} dx^r dx^\varphi} =$$

$= g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$ (semplice applicazione del Teorema di Pitagora sulla sfera) e qui si vede che non vi sono termini misti (tipo $d\theta d\varphi$) , dacché il tensore metrico g è diagonale, ossia i termini misti sono nulli ($g_{\theta\varphi} = g_{\varphi\theta} = g_{r\theta} = g_{\theta r} = g_{r\varphi} = g_{\varphi r} = 0$) e dunque ancora, essendo che:

$$\boxed{\begin{array}{l} g_{\mu\nu} g^{\mu\nu} = 1 \to g_{\mu\nu} = \frac{1}{g^{\mu\nu}} \to g_{rr} = 1 \text{ (e } g^{rr} = \frac{1}{g_{rr}} = 1) , \; g_{\theta\theta} = r^2 \text{ (e } g^{\theta\theta} = \frac{1}{r^2}), \\ g_{\varphi\varphi} = r^2 \sin^2\theta \text{ (e } g^{\varphi\varphi} = \frac{1}{r^2 \sin^2\theta}) \text{ e} \\ g_{\theta\varphi} = g_{\varphi\theta} = g^{\theta\varphi} = g^{\varphi\theta} = g_{r\theta} = g_{\theta r} = g^{r\theta} = g^{\theta r} = g_{r\varphi} = g_{\varphi r} = g^{r\varphi} = g^{\varphi r} = 0 . \end{array}} \qquad (2.11)$$

ed essendo che $g_{\mu\nu} = \hat{e}_\mu \hat{e}_\nu$ e $g^{\mu\nu} = \hat{e}^\mu \hat{e}^\nu$, segue che:

$$\hat{e}_x = 1 \ , \ \hat{e}_y = 1 \ , \ \hat{e}_z = 1 \ , \ \hat{e}^x = 1 \ , \ \hat{e}^y = 1 \ , \ \hat{e}^z = 1 \ ; \tag{2.12}$$

$$\hat{e}_r = 1 \ , \ \hat{e}_\theta = r \ , \ \hat{e}_\varphi = r\sin\theta \ , \ \hat{e}^r = 1 \ , \ \hat{e}^\theta = \frac{1}{r} \ , \ \hat{e}^\varphi = \frac{1}{r\sin\theta} \ . \tag{2.13}$$

Riprendendo poi un attimo la (2.4), ossia la: $\Gamma^\mu_{\sigma\nu} = \frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu$ e la (2.10), ossia la: $dP^2 = g_{\mu\nu}dx^\mu dx^\nu$; bene, tale (2.10) ha valore generale per uno spazio curvo, mentre per uno spazio dichiaratamente piatto (di Minkowski), essa assume la forma: $dP^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$, con $\eta_{\alpha\beta} = (1,1,1)$, poiché in uno spazio piatto si ha:
$dP^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 = 1\cdot(dx)^2 + 1\cdot(dy)^2 + 1\cdot(dz)^2 = dx^2 + dy^2 + dz^2$.
Esprimendo ora un versore generico $\hat{e}_i$ (curvilineo) in funzione dei versori della geometria piatta $(\hat{i},\hat{j},\hat{k})$, si ha ovviamente che: $\hat{e}_i = \frac{\partial\xi^1}{\partial x^i}\hat{i} + \frac{\partial\xi^2}{\partial x^i}\hat{j} + \frac{\partial\xi^3}{\partial x^i}\hat{k} = (\sum_\alpha)\eta_\alpha\frac{\partial\xi^\alpha}{\partial x^i} = \eta_\alpha\frac{\partial\xi^\alpha}{\partial x^i}$ e seguirà anche una forma duale per la terna reciproca $\hat{e}^j$ (tale che, per def. di reciprocità: $\hat{e}^j\hat{e}_j = \delta^j_i$): $\hat{e}^j = (\sum_\alpha)\eta^\alpha\frac{\partial x^j}{\partial\xi^\alpha} = \eta^\alpha\frac{\partial x^j}{\partial\xi^\alpha}$ e dunque il coefficiente $\Gamma^\mu_{\nu\sigma}$ $(=\Gamma^\mu_{\sigma\nu})$ della (2.4) possiamo anche esprimerlo nel seguente modo: $\Gamma^\mu_{\nu\sigma} = \frac{d\hat{e}_\sigma}{dx^\nu}\hat{e}^\mu = \frac{d}{dx^\nu}(\eta_\alpha\frac{\partial\xi^\alpha}{\partial x^\sigma})\eta^\alpha\frac{\partial x^\mu}{\partial\xi^\alpha} = \eta_\alpha\eta^\alpha\frac{\partial^2\xi^\alpha}{\partial x^\sigma\partial x^\nu}\frac{\partial x^\mu}{\partial\xi^\alpha}$, che può essere anche scritta in modo più semplice, senza i coefficienti unitari η:

$$\Gamma^\mu_{\nu\sigma} = \frac{\partial^2\xi^\alpha}{\partial x^\sigma\partial x^\nu}\frac{\partial x^\mu}{\partial\xi^\alpha} \quad . \tag{2.14}$$

-**<u>la relazione tra</u>** $g_{\mu\nu}$ **<u>e</u>** $\Gamma^\lambda_{\mu\nu}$:
eguagliando il tensore metrico (2.10) del caso curvilineo con quello per il caso di spazio piatto, si ha: $dP^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$, da cui: $g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}$; applichiamo ora a quest'ultima $\frac{\partial}{\partial x^\lambda}$: $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2\xi^\alpha}{\partial x^\lambda\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu}\eta_{\alpha\beta}$; ricordando ora che, per la (2.14): $\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu} = \Gamma^\lambda_{\mu\nu}\frac{\partial\xi^\alpha}{\partial x^\lambda}$, segue che:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu}\frac{\partial\xi^\alpha}{\partial x^\rho}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\rho_{\lambda\nu}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\alpha}{\partial x^\rho}\eta_{\alpha\beta} = \Gamma^\rho_{\lambda\mu}g_{\rho\nu} + \Gamma^\rho_{\lambda\nu}g_{\rho\mu} \tag{2.15}$$

Similmente, si calcolano poi $\frac{\partial g_{\lambda\nu}}{\partial x^\mu}$ e $\frac{\partial g_{\mu\lambda}}{\partial x^\nu}$, da cui: $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2g_{k\nu}\Gamma^k_{\lambda\mu}$; definendo ora una matrice (o tensore) reciproco $g^{\nu\sigma}$ tale che: $g^{\nu\sigma}g_{k\nu} = \delta^\sigma_k$, si avrà, dunque:

$$\Gamma^\sigma_{\lambda\mu} = \Gamma^\sigma_{\mu\lambda} = \frac{1}{2}g^{\nu\sigma}(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}) . \tag{2.16}$$

Adesso, nella (2.16) poniamo σ=μ , da cui: ($g_{AB} = g_{BA}$ e $g^{AB} = g^{BA}$)

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\nu\mu} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right) = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} \right) = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\nu\mu}}{\partial x^{\lambda}} = \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} . \tag{2.17}$$

Notiamo ora che: $\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^{\lambda}} = \frac{1}{2} \frac{\partial}{\partial x^{\lambda}} \ln g$, (2.18)

con g che è il determinante della matrice $g_{\rho\mu}$; infatti, esaminando il caso più semplice di matrice 2x2 , si ha: $g_{\rho\mu} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ e $g^{\rho\mu} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$ e per le formule di inversione di una matrice, si ha: $g^{11} = g_{22} / g$, $g^{12} = g^{21} = -g_{12} / g = -g_{21} / g$, $g^{22} = g_{11} / g$;

$$g = \left| g_{\rho\mu} \right| = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11} g_{22} - g_{12} g_{21} = g_{11} g_{22} - (g_{12})^2 ,$$

$$\frac{\partial g}{\partial x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}} [g_{11} g_{22} - (g_{12})^2] = \frac{\partial g_{11}}{\partial x^{\lambda}} g_{22} + \frac{\partial g_{22}}{\partial x^{\lambda}} g_{11} - 2 g_{12} \frac{\partial g_{12}}{\partial x^{\lambda}} = g [g^{11} \frac{\partial g_{11}}{\partial x^{\lambda}} + g^{22} \frac{\partial g_{22}}{\partial x^{\lambda}} + 2 g^{12} \frac{\partial g_{12}}{\partial x^{\lambda}}] =$$

$= g g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^{\lambda}}$; allora, per la (2.18), la (2.17) diviene:

$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} = \frac{1}{2} \frac{\partial}{\partial x^{\lambda}} \ln g = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^{\lambda}} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\lambda}} \sqrt{g}$. Il coinvolgimento del determinante g ci sarà utile poiché lasciando invece le espressioni classiche (senza g), si dovrebbero poi calcolare direttamente parecchi simboli di Christoffel, che, invece, con le espressioni in g, possiamo non calcolarli più poiché restano incapsulati.

Consideriamo ora la (2.3): $V^{\mu}_{;\nu} = \frac{dV^{\mu}}{dx^{\nu}} + V^{\sigma} \Gamma^{\mu}_{\sigma\nu}$ e la analizziamo nel caso specifico di μ=ν :

$V^{\mu}_{;\mu} = \frac{dV^{\mu}}{dx^{\mu}} + V^{\sigma} \Gamma^{\mu}_{\sigma\mu}$ ($\Gamma^{\mu}_{\sigma\mu} = \Gamma^{\mu}_{\mu\sigma}$) ed essendo σ un indice muto, possiamo cambiargli anche nome (ad esempio λ), da cui:

$$V^{\mu}_{;\mu} = \frac{\partial V^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda} V^{\lambda} \tag{2.19}$$

che, per la (2.18), diviene:

$$V^{\mu}_{;\mu} = \frac{\partial V^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda} V^{\lambda} = \frac{\partial V^{\mu}}{\partial x^{\mu}} + \left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\lambda}} \sqrt{g} \right) V^{\lambda} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\lambda}} (\sqrt{g} V^{\lambda}) . \tag{2.20}$$

Per ultimo, per completezza, passando alla derivata di un tensore di rango 2:

$T^{\mu\nu}_{;\lambda} = \frac{\partial}{\partial x^{\lambda}} T^{\mu\nu} + \Gamma^{\mu}_{\lambda k} T^{k\nu} + \Gamma^{\nu}_{\lambda k} T^{\mu k}$, (generica) e con λ=μ , si ha: (2.21)

$$T^{\mu\nu}_{;\mu} = \frac{\partial}{\partial x^{\mu}} T^{\mu\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\mu\lambda} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{g} T^{\mu\nu}) + \Gamma^{\mu}_{\mu\lambda} T^{\mu\lambda} . \tag{2.22}$$

-**l'Equazione della Geodetica**:

Un paracadutista in caduta libera non ha un pavimento su cui il suo corpo può gravare, dunque lui, tra sé e sé, non avverte accelerazione di gravità, e si sente come sospeso nel vuoto e si accorge che sta cadendo solo se guarda gli altri oggetti attorno che si muovono. Per una particella in caduta libera, dunque, vi è ovviamente un sistema di riferimento in cui $\vec{a} = 0$, ossia:

$$\frac{\partial^2 \xi^\alpha}{\partial \tau^2} = 0 \quad \text{(caduta libera in coordinate euclidee)} \tag{2.23}$$

con $d\tau^2 = -\eta_{ik} d\xi^i d\xi^k$ **.** Possiamo però vedere la (2.23) nel seguente modo:

$$\frac{\partial^2 \xi^\alpha}{\partial \tau^2} = 0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

Moltiplichiamo ora entrambi i membri per $\frac{\partial x^\lambda}{\partial \xi^\alpha}$ e teniamo conto del fatto che: $\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu$ si avrà: $\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ (caduta libera in coordinate curvilinee) (2.24)

(equazione della geodetica, dove geodetica, qui sulla Terra, è la rotta minima tra una località e l'altra) con $\Gamma^\lambda_{\mu\nu} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial \xi^\alpha}$ appunto**.**

Proseguendo il discorso, per trovare l'equazione di trasformazione della connessione affine, riscriviamo la stessa in un altro sistema di riferimento: $\Gamma'^\lambda_{\mu\nu} = \frac{\partial^2 \xi^\alpha}{\partial x'^\mu \partial x'^\nu}\frac{\partial x'^\lambda}{\partial \xi^\alpha}$, da cui:

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}\right) = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\left[\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial^2 \xi^\alpha}{\partial x^\tau \partial x^\sigma} + \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}\right] =$$

$$= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\ ; \tag{2.25}$$

differenziamo ora $\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu$:

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} = -\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}, \text{ da cui: } \Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}\ , \tag{2.26}$$

che è appunto l'equazione di trasformazione della connessione affine; vediamo che essa non si trasforma come un tensore, come succede, ad esempio, con la (2.8) $V^i = V'^l \frac{\partial x^i}{\partial x'^l}$, dunque non è prettamente un tensore. A proposito della (2.8), o meglio, della sua inversa: $V'^\mu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu}$, si ha, differenziando rispetto a x'^λ :

$$\frac{\partial V'^\mu}{\partial x'^\lambda} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu; \tag{2.27}$$

ora, riscriviamo legittimamente la (2.8) con nuovi indici: $V'^\kappa = V^\eta \frac{\partial x'^\kappa}{\partial x^\eta}$ e moltiplichiamo entrambi i membri per $\Gamma'^\mu_{\lambda\kappa}$: $\Gamma'^\mu_{\lambda\kappa} V'^\kappa = \Gamma'^\mu_{\lambda\kappa} V^\eta \frac{\partial x'^\kappa}{\partial x^\eta}$, ma la (2.26) con gli indici coerentemente riadattati per fornire appunto $\Gamma'^\mu_{\lambda\kappa}$, fornisce:

$$\Gamma'^\mu_{\lambda\kappa} V'^\kappa = \left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\kappa}\Gamma^\nu_{\rho\sigma} - \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\kappa}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\right)V^\eta \frac{\partial x'^\kappa}{\partial x^\eta} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma}V^\sigma - \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\lambda}V^\sigma \tag{2.28}$$

Sommando ora m.a.m. le (2.27) e (2.28) otteniamo:

$\frac{\partial V'^\mu}{\partial x'^\lambda} + \Gamma'^\mu_{\lambda\kappa} V'^\kappa = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\left(\frac{\partial V^\nu}{\partial x^\rho} + \Gamma^\nu_{\rho\sigma}V^\sigma\right)$ che ci conferma che la derivata covariante (2.3) è appunto un tensore, poiché si trasforma come un tensore.

Per completezza, ripetiamo il tutto riguardo la derivata controvariante: partendo dalla

$$V'_\mu = V_\rho \frac{\partial x^\rho}{\partial x'^\mu} \tag{2.29}$$

e differenziando rispetto a x'^ν : $\frac{\partial V'_\mu}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial V_\rho}{\partial x^\sigma} + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho$; (2.30)

ora, la (2.29) in λ dà: $V'_\lambda = V_\kappa \frac{\partial x^\kappa}{\partial x'^\lambda}$ e moltiplicando m.a.m. per $\Gamma'^\lambda_{\mu\nu}$:

$\Gamma'^\lambda_{\mu\nu} V'_\lambda = \Gamma'^\lambda_{\mu\nu} V_\kappa \frac{\partial x^\kappa}{\partial x'^\lambda}$, ma $\Gamma'^\lambda_{\mu\nu}$ dà, per la (2.25) con gli indici riadattati al caso:

$$\Gamma'^\lambda_{\mu\nu} V'_\lambda = \left(\frac{\partial x'^\lambda}{\partial x^\tau}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\tau_{\rho\sigma} + \frac{\partial x'^\lambda}{\partial x^\tau}\frac{\partial^2 x^\tau}{\partial x'^\mu \partial x'^\nu}\right) V_\kappa \frac{\partial x^\kappa}{\partial x'^\lambda} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\kappa_{\rho\sigma} V_\kappa + \frac{\partial^2 x^\kappa}{\partial x'^\mu \partial x'^\nu} V_\kappa ; \tag{2.31}$$

sottraendo ora la (2.31) dalla (2.30) si ha:

$\frac{\partial V'_\mu}{\partial x'^\nu} - \Gamma'^\lambda_{\mu\nu} V'_\lambda = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\left(\frac{\partial V_\rho}{\partial x^\sigma} - \Gamma^\kappa_{\rho\sigma} V_\kappa\right)$ c.v.d.

Vedere anche l'Appendice 1 per ulteriori dettagli sui tensori.

3-CAMBIO DI COORDINATE

Ricordiamo che per la (2.10), ossia la: $dP^2 = (\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = (\hat{e}_\mu \hat{e}_\nu) dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$, si ha che: $(\hat{e}_i \hat{e}_i) = g_{ii} = (\hat{e}_i)^2$, da cui:

$\hat{e}_i = \sqrt{g_{ii}}$ e parimenti, riguardo la terna reciproca:

$dP^2 = (\hat{e}^\mu dx_\mu)(\hat{e}^\nu dx_\nu) = (\hat{e}^\mu \hat{e}^\nu) dx_\mu dx_\nu = g^{\mu\nu} dx_\mu dx_\nu \rightarrow (\hat{e}^i \hat{e}^i) = g^{ii} = (\hat{e}^i)^2$, da cui:

$$\hat{e}^i = \sqrt{g^{ii}} \tag{3.1}$$

mentre per la definizione stessa di vettore $\vec{V}$, si ha:

$$\vec{V} = V_i \hat{e}^i = V_i \sqrt{g^{ii}}\,\hat{i} \left(= V^i \hat{e}_i = V^i \sqrt{g_{ii}}\,\hat{i}\right) , \tag{3.2}$$

-**<u>il GRADIENTE con i tensori</u>**:

si abbia uno scalare $\mathrm{V} = \mathrm{V}(\mathrm{x}^i) = \mathrm{V}(\mathrm{x}^1, \mathrm{x}^2, \ldots, \mathrm{x}^n)$; $\mathrm{V} = \mathrm{V}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ nel caso cartesiano e $\mathrm{V} = \mathrm{V}(\rho, \theta, \varphi)$ nel caso sferico; allora:

$$gradV = \nabla V = \hat{e}^i \frac{\partial V}{\partial x^i} = \hat{e}^i (\nabla V)_i \left(= \hat{e}^1 \frac{\partial V}{\partial x^1} + \hat{e}^2 \frac{\partial V}{\partial x^2} + \ldots + \hat{e}^n \frac{\partial V}{\partial x^n}\right)\left(\equiv \underline{\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}} \equiv\right)$$

$$\left(\equiv \hat{e}^r \frac{\partial V}{\partial r} + \hat{e}^\theta \frac{\partial V}{\partial \theta} + \hat{e}^\varphi \frac{\partial V}{\partial \varphi} = \right) \text{(per la (3.2))} \left(= \sqrt{g^{rr}}\frac{\partial V}{\partial r}\hat{r} + \sqrt{g^{\theta\theta}}\frac{\partial V}{\partial \theta}\hat{\theta} + \sqrt{g^{\varphi\varphi}}\frac{\partial V}{\partial \varphi}\hat{\varphi} = \right.$$

$$\left. = \sqrt{1}\frac{\partial V}{\partial r}\hat{r} + \sqrt{\frac{1}{r^2}}\frac{\partial V}{\partial \theta}\hat{\theta} + \sqrt{\frac{1}{r^2 \sin^2\theta}}\frac{\partial V}{\partial \varphi}\hat{\varphi} = \underline{\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}}\right)$$

e sappiamo così che l'operazione di gradiente (ovviamente su uno scalare) fornisce un vettore ed abbiamo ottenuto altresì le espressioni del gradiente cartesiano e di quello sferico:

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}}$$ (ossia la (1.3))

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}}$$ (ossia la (1.9))

Per ultimo, considerando il caso cilindrico, con riferimento alla Fig. 1.3, si ha:
$(d\vec{P} = \hat{e}_i dx^i)$, $dP^2 = (\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = (\hat{e}_\mu \hat{e}_\nu) dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$ e col passaggio alle cilindriche: $\mu, \nu \to r, \varphi, z$:

$$dP^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{rr} dx^r dx^r + g_{\varphi\varphi} dx^\varphi dx^\varphi + g_{zz} dx^z dx^z + \cancel{2g_{\varphi z} dx^\varphi dx^z} + \cancel{2g_{r\varphi} dx^r dx^\varphi} + \cancel{2g_{rz} dx^r dx^z} =$$

$= g_{rr} dr^2 + g_{\varphi\varphi} d\varphi^2 + g_{zz} dz^2 = dr^2 + r^2 d\varphi^2 + dz^2$, da cui:

$$\boxed{g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \ , \ g_{zz} = 1 \quad (\sqrt{g} = \sqrt{1 \cdot r^2 \cdot 1} = r) \quad \text{e} \quad g^{rr} = 1 \ , \ g^{\varphi\varphi} = \frac{1}{r^2} \ , \ g^{zz} = 1 \ .}$$

Dunque:

$$gradV = \nabla V = \hat{e}^i \frac{\partial V}{\partial x^i} = \hat{e}^i (\nabla V)_i \ \left(= \hat{e}^1 \frac{\partial V}{\partial x^1} + \hat{e}^2 \frac{\partial V}{\partial x^2} + \ldots + \hat{e}^n \frac{\partial V}{\partial x^n}\right) \left(\equiv \hat{e}^r \frac{\partial V}{\partial r} + \hat{e}^\varphi \frac{\partial V}{\partial \varphi} + \hat{e}^z \frac{\partial V}{\partial z} = \right)$$

(per la (3.2)) $\left(= \sqrt{g^{rr}} \frac{\partial V}{\partial r} \hat{r} + \sqrt{g^{\varphi\varphi}} \frac{\partial V}{\partial \varphi} \hat{\varphi} + \sqrt{g^{zz}} \frac{\partial V}{\partial z} \hat{z} = \right.$

$= \sqrt{1} \frac{\partial V}{\partial r} \hat{r} + \sqrt{\frac{1}{r^2}} \frac{\partial V}{\partial \varphi} \hat{\varphi} + \sqrt{1} \frac{\partial V}{\partial z} \hat{z} = \boxed{\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \varphi} \hat{\varphi} + \frac{\partial V}{\partial z} \hat{z} = \nabla V \ )}$, ossia proprio la (1.14)!!!

-**<u>la DIVERGENZA con i tensori</u>**:

Ripartiamo ora dalla (2.20): $V^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\lambda} (\sqrt{g} V^\lambda)$, dove il λ doppio a secondo membro, per la Convenzione di Einstein, ha il compito di forzare la sommatoria, dunque può tranquillamente diventare anch'esso μ, da cui:

$$V^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g} V^\mu) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} (\sqrt{g} V^1) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} (\sqrt{g} V^2) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} (\sqrt{g} V^3) \tag{3.3}$$

e ricordiamo inoltre, riguardo le coordinate cartesiane, che, come già esposto in precedenza:

$$dP^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 = 1 \cdot (dx)^2 + 1 \cdot (dy)^2 + 1 \cdot (dz)^2 = dx^2 + dy^2 + dz^2$$

da cui: $|g_{\mu\nu}| = |\eta_{\mu\nu}| = \begin{vmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zx} & \eta_{zy} & \eta_{zz} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & 0 & 0 \\ 0 & \eta_{yy} & 0 \\ 0 & 0 & \eta_{zz} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = g$, mentre riguardo il caso delle coordinate sferiche, per le (2.11), si ha che:

$$|g_{\mu\nu}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} g_{rr} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} = r^4 \sin^2\theta = g$$

e allora la (3.3) fornisce:

-**<u>per le cartesiane</u>**: $V^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} (\sqrt{g} V^1) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} (\sqrt{g} V^2) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} (\sqrt{g} V^3) =$

$= \frac{1}{\sqrt{1}} \frac{\partial}{\partial x} (\sqrt{1} V^x) + \frac{1}{\sqrt{1}} \frac{\partial}{\partial y} (\sqrt{1} V^y) + \frac{1}{\sqrt{1}} \frac{\partial}{\partial z} (\sqrt{1} V^z) = \boxed{V^\mu_{;\mu} = \frac{\partial}{\partial x} V^x + \frac{\partial}{\partial y} V^y + \frac{\partial}{\partial z} V^z = \vec{\nabla} \cdot \vec{V} \ ,}$ ossia la

classica divergenza cartesiana già fornita con la (1.4)!

-**per le sferiche**: $V^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^{1-sph})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^{2-sph})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^{3-sph})=$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})]; \qquad (3.4)$$

ora, per la (3.2), si ha:

$$\vec{V}=V_1\hat{e}^r+V_2\hat{e}^\theta+V_3\hat{e}^\varphi=V_i\hat{e}^i=V_i\sqrt{g^{ii}}\hat{i}=(V_r\sqrt{g^{rr}})\hat{r}+(V_\theta\sqrt{g^{\theta\theta}})\hat{\theta}+(V_\varphi\sqrt{g^{\varphi\varphi}})\hat{\varphi}=V^{1-sph}\hat{r}+V^{2-sph}\hat{\theta}+V^{3-sph}\hat{\varphi}$$

da cui si ha, per la (3.4):

$$V^{\mu}_{;\mu}=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V^{1-sph})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V^{2-sph})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V^{3-sph})]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r\sqrt{g^{rr}})+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V_\theta\sqrt{g^{\theta\theta}})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V_\varphi\sqrt{g^{\varphi\varphi}})]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r\cdot 1)+\frac{\partial}{\partial\theta}(r^2\sin\theta\cdot V_\theta\frac{1}{r})+\frac{\partial}{\partial\varphi}(r^2\sin\theta\cdot V_\varphi\frac{1}{r\sin\theta})]=$$

$$=\boxed{\frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\cdot V_r)+\frac{\partial}{\partial\theta}(r\sin\theta\cdot V_\theta)+\frac{\partial}{\partial\varphi}(r\cdot V_\varphi)]=div\cdot\vec{V}=\vec{\nabla}\cdot\vec{V}\ ,}$$ che è proprio la (1.10)

(divergenza sferica)!

-**per le cilindriche**: $V^{\mu}_{;\mu}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}(\sqrt{g}V^{1-cyl})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}V^{2-cyl})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}V^{3-cyl})=$

$$=\frac{1}{r}\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{1}{r}\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{1}{r}\frac{\partial}{\partial z}(rV^{3-cyl})=\frac{1}{r}[\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{\partial}{\partial z}(rV^{3-cyl})];$$

ora, per la (3.2), si ha:

$$\vec{V}=V_1\hat{e}^r+V_2\hat{e}^\varphi+V_3\hat{e}^z=V_i\hat{e}^i=V_i\sqrt{g^{ii}}\hat{i}=(V_r\sqrt{g^{rr}})\hat{r}+(V_\varphi\sqrt{g^{\varphi\varphi}})\hat{\varphi}+(V_z\sqrt{g^{zz}})\hat{z}=V^{1-cyl}\hat{r}+V^{2-cyl}\hat{\varphi}+V^{3-cyl}\hat{z}$$

da cui si ha:

$$V^{\mu}_{;\mu}=\frac{1}{r}[\frac{\partial}{\partial r}(rV^{1-cyl})+\frac{\partial}{\partial\varphi}(rV^{2-cyl})+\frac{\partial}{\partial z}(rV^{3-cyl})]=\frac{1}{r}[\frac{\partial}{\partial r}(rV_r\sqrt{g^{rr}})+\frac{\partial}{\partial\varphi}(rV_\varphi\sqrt{g^{\varphi\varphi}})+\frac{\partial}{\partial z}(rV_z\sqrt{g^{zz}})]=$$

$$=\frac{1}{r}[\frac{\partial}{\partial r}(rV_r\cdot 1)+\frac{\partial}{\partial\varphi}(rV_\varphi\frac{1}{r})+\frac{\partial}{\partial z}(rV_z\cdot 1)]=\boxed{\frac{1}{r}[\frac{\partial}{\partial r}(rV_r)+\frac{\partial}{\partial\varphi}(V_\varphi)+\frac{\partial}{\partial z}(rV_z)]=div\cdot\vec{V}=\vec{\nabla}\cdot\vec{V}\ ,}$$

che è proprio la (1.15) (divergenza cilindrica)!

-**LAPLACIANO con i tensori**:

si abbia uno scalare $V=V(x^i)=V(x^1,x^2,\ldots,x^n)$; $V=V(x,y,z)$ nel caso cartesiano e $V=V(\rho,\theta,\varphi)$ nel caso sferico; sappiamo che si ha, poi:

$div(gradV)=\nabla\cdot(\nabla V)=\nabla^2V=\Delta V$, che è il laplaciano (scalare) di V.

Ma andiamo a riprendere un attimo la (3.3) per la divergenza di un vettore $\vec{S}$: $S^{\mu}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}S^{\mu}) = \vec{\nabla}\cdot\vec{S}$ ed abbassiamo l'indice ad S^{μ} tramite (come d'uso e definizione) il tensore metrico: $S^{\mu} = g^{\eta\mu}S_{\eta}$, da cui:

$$S^{\mu}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}S^{\mu}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}g^{\eta\mu}S_{\eta}) = \vec{\nabla}\cdot\vec{S} \qquad (3.5)$$

ed ora, nella (3.5) andremo ad inserire, come vettore S_{η}, il (vettore) gradiente $(\nabla V)_{\eta}$ precedentemente trovato:

$$S^{\mu}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{g}g^{\eta\mu}S_{\eta}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\nabla V)_{\eta}] = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] = \nabla^2 V \ ; \qquad (3.6)$$

Adesso, ricordando le matrici sotto la (3.3), ossia:

$$|g_{\mu\nu}| = |\eta_{\mu\nu}| = \begin{vmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zx} & \eta_{zy} & \eta_{zz} \end{vmatrix} = \begin{vmatrix} \eta_{xx} & 0 & 0 \\ 0 & \eta_{yy} & 0 \\ 0 & 0 & \eta_{zz} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = g$$ per le cartesiane e per le sferiche:

$$|g_{\mu\nu}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} g_{rr} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta = g$$, segue, per la (3.6), che:

-**<u>per le cartesiane</u>**: $\nabla^2 V = \Delta V = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] = \frac{1}{\sqrt{1}}\frac{\partial}{\partial x^{\mu}}[\sqrt{1}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] = \frac{\partial}{\partial x^{\mu}}[g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] =$

$$= \frac{\partial}{\partial x^{\mu}}[g^{1\mu}(\frac{\partial V}{\partial x^1}) + g^{2\mu}(\frac{\partial V}{\partial x^2}) + g^{3\mu}(\frac{\partial V}{\partial x^3})] = \frac{\partial}{\partial x^{\mu}}[g^{1\mu}(\frac{\partial V}{\partial x}) + g^{2\mu}(\frac{\partial V}{\partial y}) + g^{3\mu}(\frac{\partial V}{\partial z})] =$$

$$= \frac{\partial}{\partial x^1}[g^{11}(\frac{\partial V}{\partial x}) + \cancel{g^{21}(\frac{\partial V}{\partial y})} + \cancel{g^{31}(\frac{\partial V}{\partial z})}] + \frac{\partial}{\partial x^2}[\cancel{g^{12}(\frac{\partial V}{\partial x})} + g^{22}(\frac{\partial V}{\partial y}) + \cancel{g^{32}(\frac{\partial V}{\partial z})}] +$$

$$+ \frac{\partial}{\partial x^3}[\cancel{g^{13}(\frac{\partial V}{\partial x})} + \cancel{g^{23}(\frac{\partial V}{\partial y})} + g^{33}(\frac{\partial V}{\partial z})] = \frac{\partial}{\partial x}[g^{11}(\frac{\partial V}{\partial x})] + \frac{\partial}{\partial y}[g^{22}(\frac{\partial V}{\partial y})] + \frac{\partial}{\partial z}[g^{33}(\frac{\partial V}{\partial z})] =$$

$$= g^{11}(\frac{\partial^2 V}{\partial x^2}) + g^{22}(\frac{\partial^2 V}{\partial y^2}) + g^{33}(\frac{\partial^2 V}{\partial z^2}) = \boxed{(\frac{\partial^2 V}{\partial x^2}) + (\frac{\partial^2 V}{\partial y^2}) + (\frac{\partial^2 V}{\partial z^2}) = \Delta V} \quad !!!! \quad \text{(ossia la (1.6))}$$

-**<u>per le sferiche</u>**:

$$\nabla^2 V = \Delta V = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] = \frac{1}{\sqrt{r^4\sin^2\theta}}\frac{\partial}{\partial x^{\mu}}[\sqrt{r^4\sin^2\theta}g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] =$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^{\mu}}[r^2\sin\theta\cdot g^{\eta\mu}(\frac{\partial V}{\partial x^{\eta}})] = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^{\mu}}[r^2\sin\theta\cdot g^{1\mu}(\frac{\partial V}{\partial x^1}) + r^2\sin\theta\cdot g^{2\mu}(\frac{\partial V}{\partial x^2}) + r^2\sin\theta\cdot g^{3\mu}(\frac{\partial V}{\partial x^3})] =$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^{\mu}}[r^2\sin\theta\cdot g^{r\mu}(\frac{\partial V}{\partial r}) + r^2\sin\theta\cdot g^{\theta\mu}(\frac{\partial V}{\partial\theta}) + r^2\sin\theta\cdot g^{\varphi\mu}(\frac{\partial V}{\partial\varphi})] =$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^r}[r^2\sin\theta\cdot g^{rr}(\frac{\partial V}{\partial r}) + \cancel{r^2\sin\theta\cdot g^{\theta r}(\frac{\partial V}{\partial\theta})} + \cancel{r^2\sin\theta\cdot g^{\varphi r}(\frac{\partial V}{\partial\varphi})}] +$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\theta}[r^2\sin\theta\cdot \cancel{g^{r\theta}(\frac{\partial V}{\partial r})}+r^2\sin\theta\cdot g^{\theta\theta}(\frac{\partial V}{\partial\theta})+r^2\sin\theta\cdot \cancel{g^{\varphi\theta}(\frac{\partial V}{\partial\varphi})}]+$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\varphi}[r^2\sin\theta\cdot \cancel{g^{r\varphi}(\frac{\partial V}{\partial r})}+r^2\sin\theta\cdot \cancel{g^{\theta\varphi}(\frac{\partial V}{\partial\theta})}+r^2\sin\theta\cdot g^{\varphi\varphi}(\frac{\partial V}{\partial\varphi})]=$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}[r^2\sin\theta\cdot g^{rr}(\frac{\partial V}{\partial r})]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[r^2\sin\theta\cdot g^{\theta\theta}(\frac{\partial V}{\partial\theta})]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}[r^2\sin\theta\cdot g^{\varphi\varphi}(\frac{\partial V}{\partial\varphi})]=$$

$$=\frac{\sin\theta}{r^2\sin\theta}\frac{\partial}{\partial r}[r^2\cdot 1\cdot(\frac{\partial V}{\partial r})]+\frac{r^2}{r^2\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\cdot\frac{1}{r^2}(\frac{\partial V}{\partial\theta})]+\frac{r^2\sin\theta}{r^2\sin\theta}\frac{\partial}{\partial\varphi}[\frac{1}{r^2\sin^2\theta}(\frac{\partial V}{\partial\varphi})]=$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\frac{\partial V}{\partial r})]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\cdot\frac{1}{r^2}(\frac{\partial V}{\partial\theta})]+\frac{\partial}{\partial\varphi}[\frac{1}{r^2\sin^2\theta}(\frac{\partial V}{\partial\varphi})]=$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\frac{\partial V}{\partial r})]+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(\frac{\partial V}{\partial\theta})]+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}[(\frac{\partial V}{\partial\varphi})]=$$

$$=(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial V}{\partial r}2r)+[\frac{1}{r^2}\frac{\sin\theta}{\sin\theta}\frac{\partial^2 V}{\partial\theta^2}+\frac{1}{r^2}\frac{1}{\sin\theta}(\frac{\partial V}{\partial\theta})\cos\theta]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=$$

$$\boxed{=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=\Delta V}$$ (ossia la (1.12))!!!

-**<u>per le cilindriche</u>**:

$$\nabla^2 V=\Delta V=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}[\sqrt{g}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{\sqrt{r^2}}\frac{\partial}{\partial x^\mu}[\sqrt{r^2}g^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{\eta\mu}(\frac{\partial V}{\partial x^\eta})]=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{1\mu}(\frac{\partial V}{\partial x^1})+rg^{2\mu}(\frac{\partial V}{\partial x^2})+rg^{3\mu}(\frac{\partial V}{\partial x^3})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial x^\mu}[rg^{r\mu}(\frac{\partial V}{\partial r})+rg^{\varphi\mu}(\frac{\partial V}{\partial\varphi})+rg^{z\mu}(\frac{\partial V}{\partial z})]=\frac{1}{r}\frac{\partial}{\partial x^r}[rg^{rr}(\frac{\partial V}{\partial r})+r\cancel{g^{\varphi r}(\frac{\partial V}{\partial\varphi})}+r\cancel{g^{zr}(\frac{\partial V}{\partial z})}]+$$

$$+\frac{1}{r}\frac{\partial}{\partial x^\varphi}[r\cancel{g^{r\varphi}(\frac{\partial V}{\partial r})}+rg^{\varphi\varphi}(\frac{\partial V}{\partial\varphi})+r\cancel{g^{z\varphi}(\frac{\partial V}{\partial z})}]+\frac{1}{r}\frac{\partial}{\partial x^z}[r\cancel{g^{rz}(\frac{\partial V}{\partial r})}+r\cancel{g^{\varphi z}(\frac{\partial V}{\partial\varphi})}+rg^{zz}(\frac{\partial V}{\partial z})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial r}[rg^{rr}(\frac{\partial V}{\partial r})]+\frac{1}{r}\frac{\partial}{\partial\varphi}[rg^{\varphi\varphi}(\frac{\partial V}{\partial\varphi})]+\frac{1}{r}\frac{\partial}{\partial z}[rg^{zz}(\frac{\partial V}{\partial z})]=$$

$$=\frac{1}{r}\frac{\partial}{\partial r}[r\cdot 1\cdot(\frac{\partial V}{\partial r})]+\frac{1}{r}\frac{\partial}{\partial\varphi}[r\frac{1}{r^2}(\frac{\partial V}{\partial\varphi})]+\frac{1}{r}\frac{\partial}{\partial z}[r\cdot 1\cdot(\frac{\partial V}{\partial z})]=$$

$$\boxed{=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\varphi^2}+\frac{\partial^2 V}{\partial z^2}=\nabla^2 V}$$, (ossia la (1.17))!!!

-**<u>rotore</u>**: $(rot\vec{V})^i=(\vec{\nabla}\times\vec{V})^i=\varepsilon^{ijk}\frac{\partial V_k}{\partial x^j}$

ε^{ijk} Tensore emisimmetrico di Levi-Civita, che cambia di segno ogni volta che due indici contigui vengono scambiati tra loro; esso vale +1 se ijk=123, 231, 321 , poi vale -1 se ijk=321, 132, 213 e vale 0 se due indici coincidono. Lo stesso dicasi per ε_{ijk} .

In precedenza dimostrammo la formula (2.9) di trasformazione delle componenti di un tensore: $T'^{\mu\lambda} = T^{\nu\sigma} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}}$ e allora per il passaggio dal tensore emisimmetrico cartesiano "ε" a quello sferico "e", si ha:

$$e'^{ijk} = \varepsilon^{lmn} \frac{\partial x'^{i}}{\partial x^{l}} \frac{\partial x'^{j}}{\partial x^{m}} \frac{\partial x'^{k}}{\partial x^{n}} \quad ; \tag{3.7}$$

e, nel caso contrario: $\varepsilon^{lmn} = e'^{ijk} \frac{\partial x^{l}}{\partial x'^{i}} \frac{\partial x^{m}}{\partial x'^{j}} \frac{\partial x^{n}}{\partial x'^{k}}$;

se poi, come d'uso, tramite il tensore metrico g alziamo ed abbassiamo indici del tensore emisimmetrico, allora si ha, ad esempio, l'emisimmetrico misto:

$$e'^{i}_{\ jk} = \varepsilon^{l}_{\ mn} \frac{\partial x'^{i}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{j}} \frac{\partial x^{n}}{\partial x'^{k}} \tag{3.8}$$

dove la seconda e la terza derivata parziale (in m ed n) sono invertite, poiché lì m ed n sono indici bassi, mentre l (elle) è rimasto alto. Ma analizziamo un attimo queste tre derivate parziali:

$$\frac{\partial x'^{i}}{\partial x^{l}}$$
$$\frac{\partial x^{m}}{\partial x'^{j}}$$
$$\frac{\partial x^{n}}{\partial x'^{k}}$$

tenendo conto della (2.7), ossia della $\hat{e}'_{l} = \hat{e}_{i} \frac{\partial x^{i}}{\partial x'^{l}}$, che riscriviamo con gli indici scambiati di nome: $\hat{e}'_{i} = \hat{e}_{l} \frac{\partial x^{l}}{\partial x'^{i}}$; allora, per quelle tre derivate parziali si ha, considerando la successione di valori 1,2,3 cartesiani ed r,θ,φ sferici nella (3.8), poiché essendoci una successione di lettere diverse (l,m,n)/(i,j,k), allora anche noi le poniamo in cascata appunto (1,2,3)(=(x,y,z)) / (r,θ,φ), da cui:

$$\frac{\partial x'^{i}}{\partial x^{l}} \equiv \hat{e}_{l}\hat{e}'^{i} = \sqrt{g_{ll}}\sqrt{g'^{ii}} = \sqrt{g_{11}}\sqrt{g'^{rr}} = \sqrt{g_{xx}}\sqrt{g'^{rr}} = 1 \cdot 1 = 1$$

$$\frac{\partial x^{m}}{\partial x'^{j}} = \hat{e}'_{j}\,\hat{e}^{m} = \sqrt{g'_{jj}}\sqrt{g^{mm}} = \sqrt{g_{22}}\sqrt{g'^{\theta\theta}} = \sqrt{g_{yy}}\sqrt{g'^{\theta\theta}} = 1 \cdot \frac{1}{r} = \frac{1}{r}$$

$$\frac{\partial x^{n}}{\partial x'^{k}} = \hat{e}'_{k}\,\hat{e}^{n} = \sqrt{g'_{kk}}\sqrt{g^{nn}} = \sqrt{g_{33}}\sqrt{g'^{\varphi\varphi}} = \sqrt{g_{zz}}\sqrt{g'^{\varphi\varphi}} = 1 \cdot \frac{1}{r\sin\theta} = \frac{1}{r\sin\theta}$$

e, quindi, l'eq. di trasformazione (3.8) fornisce:

$$e'^{i}_{\ jk} = \varepsilon^{l}_{\ mn} \frac{\partial x'^{i}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{j}} \frac{\partial x^{n}}{\partial x'^{k}} = \varepsilon^{l}_{\ mn}(1 \cdot \frac{1}{r}\frac{1}{r\sin\theta}) = \sqrt{g}\varepsilon^{l}_{\ mn} \quad . \tag{3.9}$$

Ovviamente, si ha poi anche che: $(rot\vec{V})_{i} = (\vec{\nabla} \times \vec{V})_{i} = \varepsilon_{ijk} \frac{\partial V^{k}}{\partial x_{j}}$.

-**<u>caso delle coordinate cartesiane</u>**:

$$(rot\vec{V})^{i} = (\vec{\nabla} \times \vec{V})^{i} = \varepsilon^{ijk} \frac{\partial V_{k}}{\partial x^{j}} = \varepsilon^{ij1} \frac{\partial V_{1}}{\partial x^{j}} + \varepsilon^{ij2} \frac{\partial V_{2}}{\partial x^{j}} + \varepsilon^{ij3} \frac{\partial V_{3}}{\partial x^{j}} =$$

$$= (\varepsilon^{i11} \frac{\partial V_{1}}{\partial x^{1}} + \varepsilon^{i12} \frac{\partial V_{2}}{\partial x^{1}} + \varepsilon^{i13} \frac{\partial V_{3}}{\partial x^{1}}) + (\varepsilon^{i21} \frac{\partial V_{1}}{\partial x^{2}} + \varepsilon^{i22} \frac{\partial V_{2}}{\partial x^{2}} + \varepsilon^{i23} \frac{\partial V_{3}}{\partial x^{2}}) + (\varepsilon^{i31} \frac{\partial V_{1}}{\partial x^{3}} + \varepsilon^{i32} \frac{\partial V_{2}}{\partial x^{3}} + \varepsilon^{i33} \frac{\partial V_{3}}{\partial x^{3}}) =$$

$$= (\varepsilon^{i11}\frac{\partial V_x}{\partial x} + \varepsilon^{i12}\frac{\partial V_y}{\partial x} + \varepsilon^{i13}\frac{\partial V_z}{\partial x}) + (\varepsilon^{i21}\frac{\partial V_x}{\partial y} + \varepsilon^{i22}\frac{\partial V_y}{\partial y} + \varepsilon^{i23}\frac{\partial V_z}{\partial y}) + (\varepsilon^{i31}\frac{\partial V_x}{\partial z} + \varepsilon^{i32}\frac{\partial V_y}{\partial z} + \varepsilon^{i33}\frac{\partial V_z}{\partial z})$$

da cui:

$$(rot\vec{V})^1 = (\vec{\nabla}\times\vec{V})^1 = (rot\vec{V})^x = (\vec{\nabla}\times\vec{V})^x = (\varepsilon^{111}\frac{\partial V_x}{\partial x} + \varepsilon^{112}\frac{\partial V_y}{\partial x} + \varepsilon^{113}\frac{\partial V_z}{\partial x}) + (\varepsilon^{121}\frac{\partial V_x}{\partial y} + \varepsilon^{122}\frac{\partial V_y}{\partial y} + \varepsilon^{123}\frac{\partial V_z}{\partial y}) +$$

$$+ (\varepsilon^{131}\frac{\partial V_x}{\partial z} + \varepsilon^{132}\frac{\partial V_y}{\partial z} + \varepsilon^{133}\frac{\partial V_z}{\partial z}) = (0\frac{\partial V_x}{\partial x} + 0\frac{\partial V_y}{\partial x} + 0\frac{\partial V_z}{\partial x}) + (0\frac{\partial V_x}{\partial y} + 0\frac{\partial V_y}{\partial y} + \varepsilon^{123}\frac{\partial V_z}{\partial y}) +$$

$$+ (0\frac{\partial V_x}{\partial z} + \varepsilon^{132}\frac{\partial V_y}{\partial z} + 0\frac{\partial V_z}{\partial z}) = \varepsilon^{123}\frac{\partial V_z}{\partial y} + \varepsilon^{132}\frac{\partial V_y}{\partial z} = (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})$$ e, similmente:

$$(rot\vec{V})^2 = (\vec{\nabla}\times\vec{V})^2 = (rot\vec{V})^y = (\vec{\nabla}\times\vec{V})^y = (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})$$, e:

$$(rot\vec{V})^3 = (\vec{\nabla}\times\vec{V})^3 = (rot\vec{V})^z = (\vec{\nabla}\times\vec{V})^z = (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})$$; in totale:

$$\boxed{rot\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}) + \hat{j}(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}) + \hat{k}(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})}$$, che è proprio la

(1.5) !!!

-**<u>caso delle coordinate sferiche</u>**:

$$(rot\vec{V})^i = (\vec{\nabla}\times\vec{V})^i = e^i_{jk}\frac{\partial V^k}{\partial x_j} =$$ (per la (3.9))

$$= \sqrt{g}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j} = \frac{1}{r^2\sin\theta}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j} = \frac{1}{r^2\sin\theta}(\varepsilon^i_{j1}\frac{\partial V^1}{\partial x_j} + \varepsilon^i_{j2}\frac{\partial V^2}{\partial x_j} + \varepsilon^i_{j3}\frac{\partial V^3}{\partial x_j}) =$$

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^i_{11}\frac{\partial V^1}{\partial x_1} + \varepsilon^i_{12}\frac{\partial V^2}{\partial x_1} + \varepsilon^i_{13}\frac{\partial V^3}{\partial x_1}) + (\varepsilon^i_{21}\frac{\partial V^1}{\partial x_2} + \varepsilon^i_{22}\frac{\partial V^2}{\partial x_2} + \varepsilon^i_{23}\frac{\partial V^3}{\partial x_2}) + (\varepsilon^i_{31}\frac{\partial V^1}{\partial x_3} + \varepsilon^i_{32}\frac{\partial V^2}{\partial x_3} + \varepsilon^i_{33}\frac{\partial V^3}{\partial x_3})] =$$

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^i_{11}\frac{\partial V^r}{\partial x_1} + \varepsilon^i_{12}\frac{\partial V^\theta}{\partial x_1} + \varepsilon^i_{13}\frac{\partial V^\varphi}{\partial x_1}) + (\varepsilon^i_{21}\frac{\partial V^r}{\partial x_2} + \varepsilon^i_{22}\frac{\partial V^\theta}{\partial x_2} + \varepsilon^i_{23}\frac{\partial V^\varphi}{\partial x_2}) + (\varepsilon^i_{31}\frac{\partial V^r}{\partial x_3} + \varepsilon^i_{32}\frac{\partial V^\theta}{\partial x_3} + \varepsilon^i_{33}\frac{\partial V^\varphi}{\partial x_3})]$$

da cui: $(rot\vec{V})^1 = (\vec{\nabla}\times\vec{V})^1 = (rot\vec{V})^r = (\vec{\nabla}\times\vec{V})^r =$ (anche per la complementare della (3.2))

$$= \frac{1}{r^2\sin\theta}[(\varepsilon^1_{11}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial r} + \varepsilon^1_{12}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r} + \varepsilon^1_{13}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r})] +$$

$$+ \frac{1}{r^2\sin\theta}[\varepsilon^1_{21}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta} + \varepsilon^1_{22}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\theta} + \varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}] +$$

$$+ \frac{1}{r^2\sin\theta}[\varepsilon^1_{31}\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi} + \varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi} + \varepsilon^1_{33}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\varphi}] =$$

$$= \frac{1}{r^2\sin\theta}[(0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial r} + 0\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r} + 0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r})] +$$

$$+ \frac{1}{r^2\sin\theta}[0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta} + 0\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\theta} + \varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}] +$$

$$+\frac{1}{r^2\sin\theta}[0\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}+\varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\varphi}]=$$

$$=\frac{1}{r^2\sin\theta}[\varepsilon^1_{23}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}+\varepsilon^1_{32}\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]=\frac{1}{r^2\sin\theta}[1\cdot\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}+(-1)\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]=$$

$$=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}-\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]$$ e, similmente:

$$(rot\vec{V})^2=(\vec{\nabla}\times\vec{V})^2=(rot\vec{V})^\theta=(\vec{\nabla}\times\vec{V})^\theta=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r}]$$

$$(rot\vec{V})^3=(\vec{\nabla}\times\vec{V})^3=(rot\vec{V})^\varphi=(\vec{\nabla}\times\vec{V})^\varphi=\frac{1}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta}]$$

ed in totale: (e ancora una volta per la complementare della (2.34))

$$rot\vec{V}=\vec{\nabla}\times\vec{V}=\hat{e}_r(\vec{\nabla}\times\vec{V})^r+\hat{e}_\theta(\vec{\nabla}\times\vec{V})^\theta+\hat{e}_\varphi(\vec{\nabla}\times\vec{V})^\varphi=$$

$$=\frac{1}{r^2\sin\theta}[\sqrt{g_{rr}}(\vec{\nabla}\times\vec{V})^r\hat{r}+\sqrt{g_{\theta\theta}}(\vec{\nabla}\times\vec{V})^\theta\hat{\theta}+\sqrt{g_{\varphi\varphi}}(\vec{\nabla}\times\vec{V})^\varphi\hat{\varphi}]=$$

$$=\frac{\sqrt{g_{rr}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial\theta}-\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial\varphi}]\hat{r}+\frac{\sqrt{g_{\theta\theta}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^\varphi)}{\partial r}]\hat{\theta}+$$

$$+\frac{\sqrt{g_{\varphi\varphi}}}{r^2\sin\theta}[\frac{\partial(\sqrt{g_{\theta\theta}}V^\theta)}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^r)}{\partial\theta}]\hat{\varphi}$$; ricordando ora che: $\sqrt{g^{rr}}=1$, $\sqrt{g^{\theta\theta}}=\frac{1}{r}$ e $\sqrt{g^{\varphi\varphi}}=\frac{1}{r\sin\theta}$,

nonché: $\sqrt{g_{rr}}=1$, $\sqrt{g_{\theta\theta}}=r$ e $\sqrt{g_{\varphi\varphi}}=r\sin\theta$, segue che:

$$rot\vec{V}=\frac{1}{r^2\sin\theta}[\frac{\partial(r\sin\theta V^\varphi)}{\partial\theta}-\frac{\partial(rV^\theta)}{\partial\varphi}]\hat{r}+\frac{r}{r^2\sin\theta}[\frac{\partial(1\cdot V^r)}{\partial\varphi}-\frac{\partial(r\sin\theta V^\varphi)}{\partial r}]\hat{\theta}+$$

$$+\frac{r\sin\theta}{r^2\sin\theta}[\frac{\partial(rV^\theta)}{\partial r}-\frac{\partial(1\cdot V^r)}{\partial\theta}]\hat{\varphi}=$$

$$\boxed{=\vec{\nabla}\times\vec{V}=\frac{1}{r^2\sin\theta}[\frac{\partial(r\sin\theta V^\varphi)}{\partial\theta}-\frac{\partial(rV^\theta)}{\partial\varphi}]\hat{r}+\frac{1}{r\sin\theta}[\frac{\partial V^r}{\partial\varphi}-\frac{\partial(r\sin\theta V^\varphi)}{\partial r}]\hat{\theta}+\frac{1}{r}[\frac{\partial(rV^\theta)}{\partial r}-\frac{\partial V^r}{\partial\theta}]\hat{\varphi}}$$

ossia proprio la (1.11)!!!

-**<u>caso delle coordinate cilindriche</u>**:

$$(rot\vec{V})^i=(\vec{\nabla}\times\vec{V})^i=e^i_{jk}\frac{\partial V^k}{\partial x_j}=$$ (per la (3.9))

$$=\sqrt{g}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j}=\frac{1}{r}\varepsilon^i_{jk}\frac{\partial V^k}{\partial x_j}=\frac{1}{r}(\varepsilon^i_{j1}\frac{\partial V^1}{\partial x_j}+\varepsilon^i_{j2}\frac{\partial V^2}{\partial x_j}+\varepsilon^i_{j3}\frac{\partial V^3}{\partial x_j})=$$

$$=\frac{1}{r}[(\varepsilon^i_{11}\frac{\partial V^1}{\partial x_1}+\varepsilon^i_{12}\frac{\partial V^2}{\partial x_1}+\varepsilon^i_{13}\frac{\partial V^3}{\partial x_1})+(\varepsilon^i_{21}\frac{\partial V^1}{\partial x_2}+\varepsilon^i_{22}\frac{\partial V^2}{\partial x_2}+\varepsilon^i_{23}\frac{\partial V^3}{\partial x_2})+(\varepsilon^i_{31}\frac{\partial V^1}{\partial x_3}+\varepsilon^i_{32}\frac{\partial V^2}{\partial x_3}+\varepsilon^i_{33}\frac{\partial V^3}{\partial x_3})]=$$

$$=\frac{1}{r}[(\varepsilon^i_{11}\frac{\partial V^r}{\partial x_1}+\varepsilon^i_{12}\frac{\partial V^\theta}{\partial x_1}+\varepsilon^i_{13}\frac{\partial V^\varphi}{\partial x_1})+(\varepsilon^i_{21}\frac{\partial V^r}{\partial x_2}+\varepsilon^i_{22}\frac{\partial V^\theta}{\partial x_2}+\varepsilon^i_{23}\frac{\partial V^\varphi}{\partial x_2})+(\varepsilon^i_{31}\frac{\partial V^r}{\partial x_3}+\varepsilon^i_{32}\frac{\partial V^\theta}{\partial x_3}+\varepsilon^i_{33}\frac{\partial V^\varphi}{\partial x_3})]$$

da cui: $(rot\vec{V})^1=(\vec{\nabla}\times\vec{V})^1=(rot\vec{V})^r=(\vec{\nabla}\times\vec{V})^r=$ (anche per la complementare della (3.2))

$$=\frac{1}{r}[(\varepsilon_{11}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial r}+\varepsilon_{12}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}+\varepsilon_{13}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r})]+\frac{1}{r}[\varepsilon_{21}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}+\varepsilon_{22}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial\varphi}+\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}]+$$

$$+\frac{1}{r}[\varepsilon_{31}^{1}\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}+\varepsilon_{33}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial z}]=\frac{1}{r}[(0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial r}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}+0\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r})]+$$

$$+\frac{1}{r}[0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}+0\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial\varphi}+\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}]+\frac{1}{r}[0\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}+0\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial z}]=$$

$$=\frac{1}{r}[\varepsilon_{23}^{1}\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}+\varepsilon_{32}^{1}\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]=\frac{1}{r}[1\cdot\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}+(-1)\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]=\frac{1}{r}[\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]$$

e, similmente:

$$(rot\vec{V})^{2}=(\vec{\nabla}\times\vec{V})^{2}=(rot\vec{V})^{\varphi}=(\vec{\nabla}\times\vec{V})^{\varphi}=\frac{1}{r}[\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}-\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r}]$$

$$(rot\vec{V})^{3}=(\vec{\nabla}\times\vec{V})^{3}=(rot\vec{V})^{z}=(\vec{\nabla}\times\vec{V})^{z}=\frac{1}{r}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}]$$

ed in totale: (e ancora una volta per la complementare della (2.34))

$$rot\vec{V}=\vec{\nabla}\times\vec{V}=\hat{e}_{r}(\vec{\nabla}\times\vec{V})^{r}+\hat{e}_{\varphi}(\vec{\nabla}\times\vec{V})^{\varphi}+\hat{e}_{z}(\vec{\nabla}\times\vec{V})^{z}=$$

$$=\frac{1}{r}[\sqrt{g_{rr}}(\vec{\nabla}\times\vec{V})^{r}\hat{r}+\sqrt{g_{\varphi\varphi}}(\vec{\nabla}\times\vec{V})^{\varphi}\hat{\varphi}+\sqrt{g_{zz}}(\vec{\nabla}\times\vec{V})^{z}\hat{z}]=$$

$$=\frac{\sqrt{g_{rr}}}{r}[\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial\varphi}-\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial z}]\hat{r}+\frac{\sqrt{g_{\varphi\varphi}}}{r}[\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial z}-\frac{\partial(\sqrt{g_{zz}}V^{z})}{\partial r}]\hat{\varphi}+$$

$$+\frac{\sqrt{g_{zz}}}{r}[\frac{\partial(\sqrt{g_{\varphi\varphi}}V^{\varphi})}{\partial r}-\frac{\partial(\sqrt{g_{rr}}V^{r})}{\partial\varphi}]\hat{z}$$; ricordando ora che: $\sqrt{g^{rr}}=1$, $\sqrt{g^{\varphi\varphi}}=\frac{1}{r}$ e $\sqrt{g^{zz}}=1$, nonché:

$\sqrt{g_{rr}}=1$, $\sqrt{g_{\varphi\varphi}}=r$ e $\sqrt{g_{zz}}=1$, segue che:

$$rot\vec{V}=\frac{1}{r}[\frac{\partial V^{z}}{\partial\varphi}-\frac{\partial(rV^{\varphi})}{\partial z}]\hat{r}+\frac{r}{r}[\frac{\partial V^{r}}{\partial z}-\frac{\partial V^{z}}{\partial r}]\hat{\varphi}+\frac{1}{r}[\frac{\partial(rV^{\varphi})}{\partial r}-\frac{\partial V^{r}}{\partial\varphi}]\hat{z}=$$

$$\boxed{=\frac{1}{r}[\frac{\partial V^{z}}{\partial\varphi}-\frac{\partial(rV^{\varphi})}{\partial z}]\hat{r}+[\frac{\partial V^{r}}{\partial z}-\frac{\partial V^{z}}{\partial r}]\hat{\varphi}+\frac{1}{r}[\frac{\partial(rV^{\varphi})}{\partial r}-\frac{\partial V^{r}}{\partial\varphi}]\hat{z}=\vec{\nabla}\times\vec{V}}$$

ossia proprio la duale della (1.16)!!!

Vedere anche l'Appendice 2 per ulteriori dettagli sul cambio di coordinate.

APPENDICE 1: ULTERIORI DETTAGLI SUL CALCOLO TENSORIALE.

Analizzeremo principalmente il caso del moto rotatorio in coordinate polari, ossia un argomento che contiene la curvatura, ma lo faremo anche con i tensori. E poi anche la sfera.

Ecco un elenco delle principali equazioni tensoriali della relatività generale:
-il Tensore Metrico: $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$
-l'equazione della geodetica: $\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ (caduta libera, in coord. curvilinee, su x^λ)
-i simboli di Christoffel: $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma})$ $(\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu})$.
-il Tensore di Curvatura di Riemann-Christoffel:

$$R^\lambda_{\mu\nu k} = (\Gamma^\lambda_{\mu\nu})_k - (\Gamma^\lambda_{\mu k})_\nu + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\nu\eta} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\nu\eta}$$

-il Tensore di Ricci $R_{\mu k} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\lambda\eta}$ $(R_{\mu k} = R_{k\mu})$, $R_{\mu k} = g^{\lambda\nu} R_{\lambda\mu\nu k}$
-la curvatura scalare R (di Ricci): $R = g^{\mu k} R_{\mu k}$
-le Equazioni di Einstein: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$
(Si ricorda che, da un contesto all'altro, i nomi degli indici possono cambiare, visto che essi variano entro un range di 4 valori, quante sono le coordinate, ma la coerenza tra loro va ovviamente preservata, nel senso che se, ad esempio, un μ diventa un ν, allora esso deve diventare tale, ossia appunto un ν, ovunque nell'equazione…)

<u>Moto rotatorio</u>

Ponendoci ora in due dimensioni e rappresentando in Fig. 1 contemporaneamente assi cartesiani x,y e polari r,θ , immaginiamo il punto P, estremo del vettore $\vec{P}$, che ruota in senso antiorario ed abbiamo identificato anche il versore $\hat{r}$ ed il versore tangente $\hat{t}$.

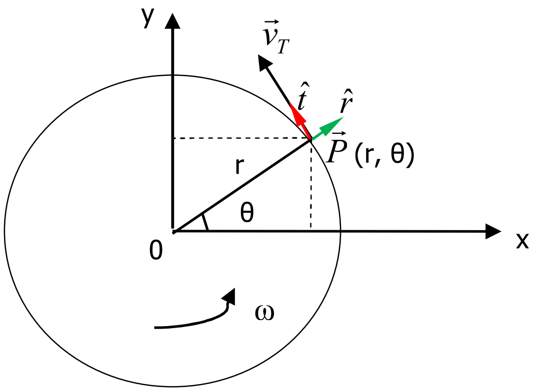


Fig. 1: Circonferenza.

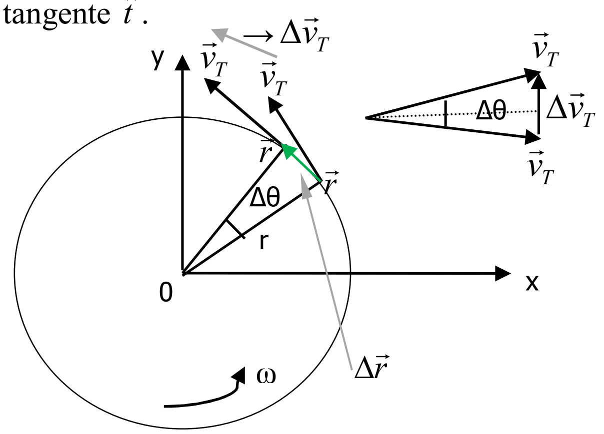


Fig. 2: Calcolo dell'accelerazione centripeta.

Ora, ricordiamo preliminarmente i seguenti fatti:

-velocità angolare ω: l'angolo giro è di 360° (2π) e, in caso di rotazione, per definizione, il tempo impiegato a compiere un ciclo (giro) è detto periodo T. Segue, per la velocità angolare:

$\omega = \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt} = \dfrac{2\pi}{T}$.

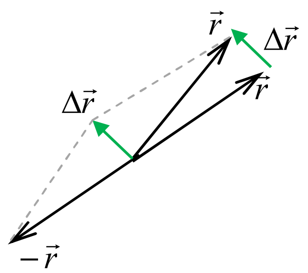
-riguardo invece la velocità periferica v_P, sappiamo che essa, se costante in modulo, ci dice la rapidità con cui la circonferenza (ciclo intero) è percorsa (nel tempo T di ciclo, ossia il periodo):

$v_p = \dfrac{2\pi r}{T} = \dfrac{2\pi}{T} r = \boxed{\omega r = v_p = v_T}$. (vettorialmente: $\vec{v}_P = \vec{v}_T = \vec{\omega} \times \vec{r}$) , cioè tale prodotto vettoriale si orienta come il versore tangente $\hat{t}$. Infatti, $\vec{v}_T$ è ortogonale ad $\vec{r}$, come si evince dalla Fig. 2, dove il vettorino verde $\Delta\vec{r}$ è appunto ortogonale ad $\vec{r}$; più specificamente, si ha che:

$\vec{v}_T = \dfrac{\Delta\vec{r}}{\Delta t} = \omega r \hat{t}$; infatti, con riferimento alla Fig. 2, e con il Δ che diventa "d":

$$\vec{v}_T = \frac{\Delta\vec{r}}{\Delta t} \to \frac{d\vec{r}}{dt} = \frac{2r\sin(d\theta/2)}{dt}\hat{t} = 2r\frac{\sin(d\theta/2)}{dt(d\theta/2)}(d\theta/2)\hat{t} = r\frac{\sin(d\theta/2)}{(d\theta/2)}(d\theta/dt)\hat{t} = \omega r\frac{\sin(d\theta/2)}{(d\theta/2)}\hat{t} \to \vec{\omega}\times\vec{r}$$

poiché: $\lim\limits_{\Delta\theta\to 0}\dfrac{\sin(\Delta\theta/2)}{(\Delta\theta/2)} = 1 \equiv \dfrac{\sin(d\theta/2)}{(d\theta/2)}$. Dunque, con la derivata di $\vec{r}$, ossia con $\dfrac{d\vec{r}}{dt}$, abbiamo ottenuto $\vec{v}_T$ che non è più rivolta come $\vec{r}$, ma come $\hat{t}$, che si ottiene tramite la rotazione antioraria di $\hat{r}$ di 90°; dunque: $\dfrac{d}{dt} \to$ direzione a 90° in senso antiorario.



Se invece il punto P si muove di moto qualsiasi, allora:

$$\frac{d\vec{P}}{dt} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cdot\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = v_r\hat{r} + r(\vec{\omega}\times\hat{r}) = \vec{v}_r + \omega r\hat{t} = \vec{v}_r + \vec{v}_T\,; \qquad (1)$$

ciò perché $\dfrac{d\hat{r}}{dt} = (\vec{\omega}\times\hat{r})$, in quanto $\vec{\omega}$ esce dal foglio e moltiplicandosi vettorialmente con $\hat{r}$, dà appunto $\hat{t}$ che, come prima previsto, è nella rotazione antioraria di 90° di $\hat{r}$ appunto.

-Per ultimo, la rotazione determina una accelerazione centripeta, in quanto $\vec{v}_T$ potrà anche essere costante in modulo, ma non lo è in direzione:

$\vec{a}_C = \dfrac{\Delta\vec{v}_T}{\Delta t} = -\omega^2 R\hat{r}$; infatti, con riferimento alla Fig. 2 (triangolo di vettori in alto a destra), e con il Δ che diventa "d":

$$\vec{a}_C = \frac{\Delta\vec{v}_T}{\Delta t} = \frac{2v_T\sin(d\theta/2)}{dt}(-\hat{r}) = 2v_T\frac{\sin(d\theta/2)}{dt(d\theta/2)}(d\theta/2)(-\hat{r}) = v_T\frac{\sin(d\theta/2)}{(d\theta/2)}(d\theta/dt)(-\hat{r}) \to -\vec{\omega}\times\vec{v}_T =$$

$= -\vec{\omega}\times(\vec{\omega}\times\vec{r})$ ($a_C = -\omega^2 r$), poiché: $\lim\limits_{\Delta\theta\to 0}\dfrac{\sin(\Delta\theta/2)}{(\Delta\theta/2)} = 1 \equiv \dfrac{\sin(d\theta/2)}{(d\theta/2)}$.

Ora, la (1) ci dice che: $\dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\cdot\hat{r}) = \dfrac{dr}{dt}\hat{r} + r\dfrac{d\hat{r}}{dt} = v_r\hat{r} + r(\vec{\omega}\times\hat{r}) = v_r\hat{r} + (\omega r)\hat{t} = \vec{v}_r + \vec{v}_T =$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$, poiché $\dot{r}$ è $\dfrac{dr}{dt}$, mentre $\hat{t}$ coincide appunto con la direzione tangenziale (antioraria) $\hat{\theta}$ delle coordinate polari e, per ultimo, ovviamente: $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$. Derivando ancora:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta}) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}(\dot{\theta}\hat{\theta}) + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} =$$

$= \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - \dot{\theta}^2 r\hat{r} = \hat{r}(\ddot{r} - \dot{\theta}^2 r) + \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta})$, ossia:

$\vec{a} = \hat{r}(\ddot{r} - \dot{\theta}^2 r) + \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta})$ (2)

(poiché $\frac{d\hat{r}}{dt}$, come dimostrato in precedenza, ci dà $\omega|\hat{r}|\hat{t} = \omega 1\hat{\theta} = \dot{\theta}\hat{\theta}$, ossia ω x il versore ottenuto ruotando $\hat{r}$ di 90° in senso antiorario, ossia $\hat{t}$, ossia ancora $\hat{\theta}$, mentre $\frac{d\hat{\theta}}{dt}$ ci dà $\omega|\hat{\theta}|(-\hat{r}) = -\omega 1\hat{r} = -\dot{\theta}\hat{r}$) e così via. La $(2\dot{r}\dot{\theta})\hat{\theta}$ è la componente (accelerazione) di **Coriolis**.

La (2) ci fornisce allora una componente tangenziale (//) ed una radiale/centripeta ($\perp$), da cui:

$$\boxed{\vec{a} = \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) = \vec{a}_{//} + \vec{a}_{\perp}} \qquad (= a^\lambda = a^\lambda_{//} + a^\lambda_{\perp}) \qquad (3)$$

Adesso, in caso di moto circolare, dove il punto ruota sempre stando a distanza r ed a velocità angolare fissa $\omega = \dot{\theta} = const$, è $\dot{r} = \ddot{r} = \ddot{\theta} = 0$, da cui, per la (3):

$\boxed{\vec{a}} = \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) = \hat{\theta}(2 \cdot 0 \cdot 0 + r) + \hat{r}(0 - \dot{\theta}^2 r) = -\dot{\theta}^2 r = \boxed{-\omega^2 r}$ (risultato già ottenuto).

L'Equazione della geodetica

Un paracadutista in caduta libera non ha un pavimento su cui il suo corpo può gravare, dunque lui, tra sé e sé, non avverte accelerazione di gravità, e si sente come sospeso nel vuoto e si accorge che sta cadendo solo se guarda gli altri oggetti attorno che si muovono. Per una particella in caduta libera, dunque, vi è ovviamente un sistema di riferimento in cui $\vec{a} = 0$, ossia:

$\vec{a} = \frac{\partial^2 \xi^\alpha}{\partial \tau^2} = 0$ (caduta libera in coordinate euclidee)

Notoriamente, se lo spazio è curvo (ad esempio, nel caso delle coordinate polari), allora compariranno i simboli di Christoffel $\Gamma^\lambda_{\mu\nu}$ a testimoniare la curvatura e la caduta libera (ossia il moto lungo una geodetica) sarà:

$\vec{a} = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$ (caduta libera in coordinate curvilinee)

(equazione della geodetica, dove geodetica, qui sulla Terra, è la rotta minima tra una località e l'altra)

Per fare dei calcoli che ci permettano di giungere alla (3) col calcolo tensoriale (equazione della geodetica), abbiamo dunque bisogno dei valori dei simboli di Christoffel $\Gamma^\lambda_{\mu\nu}$ e per avere questi ultimi, ossia: $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right)$ ($\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$) , serve conoscere la metrica $g^{\mu\nu}$, dunque:

-metrica: ricordiamo ora ancora le coordinate polari della Fig. 1, da cui, per il Teorema di Pitagora:

$d\tau^2 = -(\hat{e}_\mu dx^\mu)(\hat{e}_\nu dx^\nu) = -(\hat{e}_\mu \hat{e}_\nu) dx^\mu dx^\nu = -g_{\mu\nu} dx^\mu dx^\nu$, $\mu, \nu \to r, \theta$

$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = -g_{rr} dx^r dx^r - g_{\theta\theta} dx^\theta dx^\theta - 2g_{r\theta} dx^r dx^\theta = -g_{rr} dr^2 - g_{\theta\theta} d\theta^2 - \cancel{2g_{r\theta} dr d\theta} =$

$= -(dr^2 + r^2 d\theta^2)$ e qui si vede che non vi sono termini misti $drd\theta$, dacché il tensore metrico g è diagonale, ossia i termini misti sono nulli ($g_{r\theta} = g_{\theta r} = 0$) e dunque ancora, essendo che

$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu \to$ ($g_{\mu\nu} g^{\mu\nu} \equiv "1"$) $\to$ $g_{\mu\nu} \equiv \frac{1}{g^{\mu\nu}}$, $g_{rr} = 1$ (e $g^{rr} = 1^{-1} = 1$) $\to g_{\theta\theta} = r^2$ (e $g^{\theta\theta} = \frac{1}{r^2}$) , e $g_{r\theta} = g_{\theta r} = g^{r\theta} = g^{\theta r} = 0$.

-<u>simboli di Christoffel</u>: sappiamo dalla relatività generale e dalle pagine precedenti che:

$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\sigma\lambda}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu})$, da cui:

<u>caso 1 (λ=r)</u>: $\Gamma^{r}_{\mu\nu}=\frac{1}{2}g^{\sigma r}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{r}_{\mu\nu}=\Gamma^{r}_{\nu\mu})$ e allora:

(μ=ν=θ): $\boxed{\Gamma^{r}_{\theta\theta}}=\frac{1}{2}g^{\sigma r}(\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}}+\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}}-\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{rr}\frac{\partial g_{\theta r}}{\partial\theta}+\frac{1}{2}g^{\theta r}\frac{\partial g_{\theta\theta}}{\partial\theta})+(\frac{1}{2}g^{rr}\frac{\partial g_{\theta r}}{\partial\theta}+\frac{1}{2}g^{\theta r}\frac{\partial g_{\theta\theta}}{\partial\theta})-(\frac{1}{2}g^{rr}\frac{\partial g_{\theta\theta}}{\partial x^{r}}+\frac{1}{2}g^{\theta r}\frac{\partial g_{\theta\theta}}{\partial x^{\theta}})=$

$=(\frac{1}{2}1\cdot 0+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial\theta})+(\frac{1}{2}g^{rr}\frac{\partial 0}{\partial\theta}+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial\theta})-(\frac{1}{2}g^{rr}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial\theta})=-\frac{1}{2}g^{rr}\frac{\partial g_{\theta\theta}}{\partial r}=-\frac{1}{2}(1)\frac{\partial r^{2}}{\partial r}=\boxed{-r}$

(μ=ν=r): $\boxed{\Gamma^{r}_{rr}}=\frac{1}{2}g^{\sigma r}(\frac{\partial g_{r\sigma}}{\partial x^{r}}+\frac{\partial g_{r\sigma}}{\partial x^{r}}-\frac{\partial g_{rr}}{\partial x^{\sigma}})=(\frac{1}{2}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}g^{\sigma r}\frac{\partial g_{rr}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{rr}}{\partial x^{\sigma}})=(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta r}\frac{\partial g_{r\theta}}{\partial r})+(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta r}\frac{\partial g_{r\theta}}{\partial r})-$

$-(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial x^{r}}+\frac{1}{2}g^{\theta r}\frac{\partial g_{rr}}{\partial x^{\theta}})=(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}0\frac{\partial 0}{\partial r})+(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}0\frac{\partial 0}{\partial r})-(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}0\frac{\partial g_{rr}}{\partial\theta})=$

$=\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r}=\frac{1}{2}1\frac{\partial 1}{\partial r}=\boxed{0\,.}$

(μ=r e ν=θ o viceversa): $\boxed{\Gamma^{r}_{r\theta}}=\Gamma^{r}_{\theta r}=\frac{1}{2}g^{\sigma r}(\frac{\partial g_{\theta\sigma}}{\partial x^{r}}+\frac{\partial g_{r\sigma}}{\partial x^{\theta}}-\frac{\partial g_{\theta r}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta r}}{\partial x^{\sigma}})=(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}\sum_{\sigma}g^{\sigma r}\frac{\partial g_{\theta r}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial r}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial\theta}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta r}}{\partial x^{\sigma}}+\frac{1}{2}g^{\sigma r}\frac{\partial g_{\theta r}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{rr}\frac{\partial g_{\theta r}}{\partial r}+\frac{1}{2}g^{\theta r}\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}g^{\theta r}\frac{\partial g_{r\theta}}{\partial\theta})-(\frac{1}{2}g^{rr}\frac{\partial g_{\theta r}}{\partial x^{r}}+\frac{1}{2}g^{\theta r}\frac{\partial g_{\theta r}}{\partial x^{\theta}})=$

$=(\frac{1}{2}g^{rr}\frac{\partial 0}{\partial r}+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}0\frac{\partial 0}{\partial\theta})-(\frac{1}{2}g^{rr}\frac{\partial 0}{\partial r}+\frac{1}{2}0\frac{\partial 0}{\partial\theta})=\frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial\theta}=\frac{1}{2}1\frac{\partial 1}{\partial\theta}=\boxed{0\ .}$

E ancora (λ=θ):

<u>caso 2(λ=θ)</u>: $\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})$ $(\Gamma^{\theta}_{\mu\nu}=\Gamma^{\theta}_{\nu\mu})$ e allora:

(μ=ν =θ): $\boxed{\Gamma^{\theta}_{\theta\theta}}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}}+\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}}-\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}\sum_{\sigma}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}\sum_{\sigma}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}\sum_{\sigma}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}+\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$

$=(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})+(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})-(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta\theta}}{\partial x^{r}}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\theta}})=$

$$= (\frac{1}{2}0\frac{\partial 0}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})+(\frac{1}{2}0\frac{\partial 0}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})-(\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta})=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}=\frac{1}{2}(\frac{1}{r^2})\frac{\partial r^2}{\partial\theta}=\boxed{0}$$

(μ=ν =r): $\boxed{\Gamma^{\theta}_{rr}}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{r\sigma}}{\partial x^r}+\frac{\partial g_{r\sigma}}{\partial x^r}-\frac{\partial g_{rr}}{\partial x^\sigma})=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{rr}}{\partial x^\sigma})=$

$$=(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})+(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial r})-(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{rr}}{\partial x^\sigma})=(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial r})+(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial r})-$$

$$-(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial x^r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{rr}}{\partial x^\theta})=(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial r})+(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial r})-(\frac{1}{2}0\frac{\partial g_{rr}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 1}{\partial\theta})=\boxed{0\,.}$$

(μ=r e ν=θ o viceversa): $\boxed{\Gamma^{\theta}_{r\theta}}=\Gamma^{\theta}_{\theta r}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\theta\sigma}}{\partial x^r}+\frac{\partial g_{r\sigma}}{\partial x^\theta}-\frac{\partial g_{\theta r}}{\partial x^\sigma})=$

$$=(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta r}}{\partial x^\sigma})=(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial r})+(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{r\sigma}}{\partial\theta})-(\frac{1}{2}\sum_\sigma g^{\sigma\theta}\frac{\partial g_{\theta r}}{\partial x^\sigma})=$$

$$=(\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta 1}}{\partial r}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta 2}}{\partial r})+(\frac{1}{2}g^{1\theta}\frac{\partial g_{r1}}{\partial\theta}+\frac{1}{2}g^{2\theta}\frac{\partial g_{r2}}{\partial\theta})-(\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta r}}{\partial x^1}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta r}}{\partial x^2})=$$

$$=(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}g^{r\theta}\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{r\theta}}{\partial\theta})-(\frac{1}{2}g^{r\theta}\frac{\partial g_{\theta r}}{\partial x^r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta r}}{\partial x^\theta})=$$

$$=(\frac{1}{2}0\frac{\partial 0}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r})+(\frac{1}{2}0\frac{\partial g_{rr}}{\partial\theta}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial\theta})-(\frac{1}{2}0\frac{\partial g_{\theta r}}{\partial r}+\frac{1}{2}g^{\theta\theta}\frac{\partial 0}{\partial\theta})=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r}=\frac{1}{2}\frac{1}{r^2}\frac{\partial r^2}{\partial r}=\boxed{\frac{1}{r}\,.}$$

Riassumendo:

$$\boxed{\Gamma^{r}_{\theta\theta}=-r\ ,\ \Gamma^{r}_{rr}=0\ ,\ \Gamma^{r}_{r\theta}=\Gamma^{r}_{\theta r}=0\ ,\ \Gamma^{\theta}_{\theta\theta}=0\ ,\ \Gamma^{\theta}_{rr}=0\ ,\ \Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=\frac{1}{r}\ .}$$

-Equazione della Geodetica: $\vec{a}=\frac{d^2x^\lambda}{d\tau^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0\ ;$ (4)

abbiamo: (λ=r): $a^r=\frac{d^2x^r}{d\tau^2}+\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0\ \rightarrow$ (dτ→dt poiché dτ=dt/γ=t, visto che v<<c)

$$\rightarrow\ \frac{d^2r}{dt^2}+\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0=\frac{d^2r}{dt^2}+\sum_{\mu,\nu}\Gamma^{r}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=$$

$$=\frac{d^2r}{dt^2}+\sum_\mu[\Gamma^{r}_{\mu 1}\frac{dx^\mu}{dt}\frac{dx^1}{dt}+\Gamma^{r}_{\mu 2}\frac{dx^\mu}{dt}\frac{dx^2}{dt}]=\frac{d^2r}{dt^2}+\sum_\mu[\Gamma^{r}_{\mu r}\frac{dx^\mu}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{\mu\theta}\frac{dx^\mu}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{1r}\frac{dx^1}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{2r}\frac{dx^2}{dt}\frac{dx^r}{dt}]+[\Gamma^{r}_{1\theta}\frac{dx^1}{dt}\frac{dx^\theta}{dt}+\Gamma^{r}_{2\theta}\frac{dx^2}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}\frac{dx^r}{dt}\frac{dx^r}{dt}+\Gamma^{r}_{\theta r}\frac{dx^\theta}{dt}\frac{dx^r}{dt}]+[\Gamma^{r}_{r\theta}\frac{dx^r}{dt}\frac{dx^\theta}{dt}+\Gamma^{r}_{\theta\theta}\frac{dx^\theta}{dt}\frac{dx^\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}\frac{dr}{dt}\frac{dr}{dt}+\Gamma^{r}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{r}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{r}_{\theta\theta}\frac{d\theta}{dt}\frac{d\theta}{dt}]=$$

$$=\frac{d^2r}{dt^2}+[\Gamma^{r}_{rr}(\frac{dr}{dt})^2+\Gamma^{r}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{r}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{r}_{\theta\theta}(\frac{d\theta}{dt})^2]=$$

$$=\frac{d^2r}{dt^2}+[0(\frac{dr}{dt})^2+0\frac{d\theta}{dt}\frac{dr}{dt}]+[0\frac{dr}{dt}\frac{d\theta}{dt}+(-r)(\frac{d\theta}{dt})^2]=\frac{d^2r}{dt^2}-r(\frac{d\theta}{dt})^2=\boxed{\ddot{r}-r\dot{\theta}^2=a^r=0\,.}\qquad(5)$$

(λ=θ): $a^\theta=\frac{d^2x^\theta}{d\tau^2}+\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0\ \rightarrow$

$$\rightarrow\ \frac{d^2\theta}{dt^2}+\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0=\frac{d^2\theta}{dt^2}+\sum_{\mu,\nu}\Gamma^{\theta}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=\frac{d^2\theta}{dt^2}+\sum_\mu[\Gamma^{\theta}_{\mu 1}\frac{dx^\mu}{dt}\frac{dx^1}{dt}+\Gamma^{\theta}_{\mu 2}\frac{dx^\mu}{dt}\frac{dx^2}{dt}]=$$

$$=\frac{d^2\theta}{dt^2}+\sum_{\mu}[\Gamma^{\theta}_{\mu r}\frac{dx^{\mu}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{\mu\theta}\frac{dx^{\mu}}{dt}\frac{dx^{\theta}}{dt}]=$$
$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{1r}\frac{dx^{1}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{2r}\frac{dx^{2}}{dt}\frac{dx^{r}}{dt}]+[\Gamma^{\theta}_{1\theta}\frac{dx^{1}}{dt}\frac{dx^{\theta}}{dt}+\Gamma^{\theta}_{2\theta}\frac{dx^{2}}{dt}\frac{dx^{\theta}}{dt}]=$$
$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}\frac{dx^{r}}{dt}\frac{dx^{r}}{dt}+\Gamma^{\theta}_{\theta r}\frac{dx^{\theta}}{dt}\frac{dx^{r}}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dx^{r}}{dt}\frac{dx^{\theta}}{dt}+\Gamma^{\theta}_{\theta\theta}\frac{dx^{\theta}}{dt}\frac{dx^{\theta}}{dt}]=$$
$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}\frac{dr}{dt}\frac{dr}{dt}+\Gamma^{\theta}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{\theta}_{\theta\theta}\frac{d\theta}{dt}\frac{d\theta}{dt}]=$$
$$=\frac{d^2\theta}{dt^2}+[\Gamma^{\theta}_{rr}(\frac{dr}{dt})^2+\Gamma^{\theta}_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\Gamma^{\theta}_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt}+\Gamma^{\theta}_{\theta\theta}(\frac{d\theta}{dt})^2]=$$
$$=\frac{d^2\theta}{dt^2}+[0(\frac{dr}{dt})^2+\frac{1}{r}\frac{d\theta}{dt}\frac{dr}{dt}]+[\frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt}+0(\frac{d\theta}{dt})^2]=\frac{d^2\theta}{dt^2}+\frac{2}{r}\frac{d\theta}{dt}\frac{dr}{dt}=\ddot{\theta}+\frac{2}{r}\dot{\theta}\dot{r}=$$

$\boxed{=\frac{1}{r}(r\ddot{\theta}+2\dot{\theta}\dot{r})=a^{\theta}=0}$ (la $(2\dot{r}\dot{\theta})\hat{\theta}$ è la componente (accelerazione) di **Coriolis**). (6)

Allora, dalle (4), (5) e (6) si ha che:

$\boxed{\vec{a}=\vec{a}_{//}+\vec{a}_{\perp}=a^{\lambda}=a^{r}+a^{\theta}=\hat{r}(\ddot{r}-\dot{\theta}^2 r)+\frac{\hat{\theta}'}{r}(2\dot{r}\dot{\theta}+r\ddot{\theta})=\hat{r}(\ddot{r}-\dot{\theta}^2 r)+\hat{\theta}(2\dot{r}\dot{\theta}+r\ddot{\theta})}$, **ossia proprio la (3)!**

($\vec{a}=0$ è richiesto solo se si esige la caduta libera)
($\hat{\theta}'=r\hat{\theta}$ poiché, in questo contesto tensoriale, tutti i versori devono avere la stessa unità di misura, per coerenza; dunque, se $[\hat{r}]=metro$, allora lo stesso si deve dire per $\hat{\theta}'$, dunque $\hat{\theta}$ si ritroverà moltiplicato per r)

Richiamiamo un attimo le componenti appena ottenute per l'accelerazione nel moto:
$a^{r}=(\ddot{r}-\dot{\theta}^2 r)\hat{r}$ e $a^{\theta}=(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\theta}$. Ora, ponendoci in un campo gravitazionale, si ha che la forza F=ma è radiale, dunque:

$F^{r}=ma^{r}=m(\ddot{r}-\dot{\theta}^2 r)\hat{r}=-G\frac{Mm}{r^2}\hat{r}$ e

$F^{\theta}=ma^{\theta}=m(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\theta}=0$

La F^{θ} può essere così vista: $(2\dot{r}\dot{\theta}+r\ddot{\theta})=0=\frac{d}{dt}(r^2\dot{\theta})$, in quanto: $\frac{d}{dt}(r^2\dot{\theta})=(2r\dot{r})\dot{\theta}+r^2\ddot{\theta}=0$, da cui: $(2\dot{r})\dot{\theta}+r\ddot{\theta}=0$, ma se $\frac{d}{dt}(r^2\dot{\theta})=0\rightarrow$

$\rightarrow \boxed{r^2\dot{\theta}=const}$,

che altro non è che l'espressione della **Seconda Legge di Keplero**:

II-*Il moto dei pianeti avviene con velocità areolare costante: ciò significa che il raggio vettore, scelto con origine nel Sole, spazza aree uguali in tempi uguali.*
Infatti, con riferimento alla Fig. 3, si ha, per l'areola A spazzata da r e la relativa velocità areolare A':

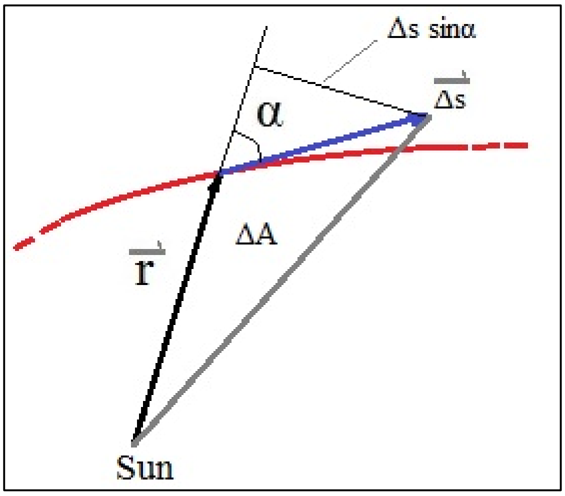


Fig. 3: Sulla velocità areolare.

$$\Delta A=\frac{1}{2}r\cdot\Delta s\sin\alpha=\frac{1}{2}(\vec r\times\Delta\vec s)\ \text{ e }\ \dot A=\frac{dA}{dt}=\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t}=\lim_{\Delta t\to 0}\frac{1}{2}(\vec r\times\frac{\Delta\vec s}{\Delta t})=\frac{1}{2}(\vec r\times\vec v)=\frac{1}{2}(rv\sin\alpha)(\hat\theta)=$$

$=\frac{1}{2}rv^{\theta}(\hat\theta)=\frac{1}{2}r\omega r(\hat\theta)=\frac{1}{2}r^2\dot\theta(\hat\theta)$, che per la precedente $r^2\dot\theta=const$, è appunto costante:

$$\boxed{\left|\frac{dA}{dt}\right|=\frac{1}{2}r^2\dot\theta=const}$$

-Valutiamo, per ultimo, il <u>Tensore di Ricci</u> e lo <u>scalare di curvatura (di Ricci)</u> R; con riferimento alla Fig. 1, si ha:

$$R_{\mu k}=\frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^k}-\frac{\partial\Gamma^{\lambda}_{\mu k}}{\partial x^{\lambda}}+\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta}-\Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\lambda\eta}\qquad(R_{\mu k}=R_{k\mu})$$

(μ=k=r): $\boxed{R_{rr}}=\frac{\partial\Gamma^{\lambda}_{r\lambda}}{\partial x^r}-\frac{\partial\Gamma^{\lambda}_{rr}}{\partial x^{\lambda}}+\Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta}-\Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta}=(\frac{\partial\Gamma^{1}_{r1}}{\partial r}+\frac{\partial\Gamma^{2}_{r2}}{\partial r})-(\frac{\partial\Gamma^{1}_{rr}}{\partial x^1}+\frac{\partial\Gamma^{2}_{rr}}{\partial x^2})+\sum_{\lambda,\eta}\Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta}-\sum_{\lambda,\eta}\Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta}=$

$$=(\frac{\partial\Gamma^{r}_{rr}}{\partial r}+\frac{\partial\Gamma^{\theta}_{r\theta}}{\partial r})-(\frac{\partial\Gamma^{r}_{rr}}{\partial r}+\frac{\partial\Gamma^{\theta}_{rr}}{\partial\theta})+\sum_{\lambda}[\Gamma^{1}_{r\lambda}\Gamma^{\lambda}_{r1}+\Gamma^{2}_{r\lambda}\Gamma^{\lambda}_{r2}]-\sum_{\lambda}[\Gamma^{1}_{rr}\Gamma^{\lambda}_{\lambda1}+\Gamma^{2}_{rr}\Gamma^{\lambda}_{\lambda2}]=$$

$$=(\frac{\partial 0}{\partial r}+\frac{\partial(1/r)}{\partial r})-(\frac{\partial 0}{\partial r}+\frac{\partial 0}{\partial\theta})+\sum_{\lambda}[\Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{rr}+\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}]-\sum_{\lambda}[\Gamma^{r}_{rr}\Gamma^{\lambda}_{\lambda r}+\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}]=$$

$$=\frac{\partial(1/r)}{\partial r}+\sum_{\lambda}[0\Gamma^{\lambda}_{rr}+\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}]-\sum_{\lambda}[0\Gamma^{\lambda}_{\lambda r}+\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}]=-\frac{1}{r^2}+\sum_{\lambda}\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}-\sum_{\lambda}\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}=$$

$$=-\frac{1}{r^2}+[\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}+\Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta}]-[\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}+\Gamma^{\theta}_{rr}\Gamma^{\lambda}_{\lambda\theta}]=-\frac{1}{r^2}+[\Gamma^{\theta}_{r1}\Gamma^{1}_{r\theta}+\Gamma^{\theta}_{r2}\Gamma^{2}_{r\theta}]-[\Gamma^{\theta}_{rr}\Gamma^{1}_{1\theta}+\Gamma^{\theta}_{rr}\Gamma^{2}_{2\theta}]=$$

$$=-\frac{1}{r^2}+[\Gamma^{\theta}_{rr}\Gamma^{r}_{r\theta}+\Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{r\theta}]-[\Gamma^{\theta}_{rr}\Gamma^{r}_{r\theta}+\Gamma^{\theta}_{rr}\Gamma^{\theta}_{\theta\theta}]=-\frac{1}{r^2}+[0\Gamma^{r}_{r\theta}+\frac{1}{r}\frac{1}{r}]-[0\Gamma^{r}_{r\theta}+0\Gamma^{\theta}_{\theta\theta}]=\boxed{0\,.}$$

(μ=k=θ)

$$\boxed{R_{\theta\theta}}=\frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial x^{\theta}}-\frac{\partial\Gamma^{\lambda}_{\theta\theta}}{\partial x^{\lambda}}+\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}-\Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta}=(\frac{\partial\Gamma^{1}_{\theta1}}{\partial\theta}+\frac{\partial\Gamma^{2}_{\theta2}}{\partial\theta})-(\frac{\partial\Gamma^{1}_{\theta\theta}}{\partial x^1}+\frac{\partial\Gamma^{2}_{\theta\theta}}{\partial x^2})+\sum_{\lambda,\eta}\Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta}-\sum_{\lambda,\eta}\Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta}=$$

$$=(\frac{\partial\Gamma^{r}_{\theta r}}{\partial\theta}+\frac{\partial\Gamma^{\theta}_{\theta\theta}}{\partial\theta})-(\frac{\partial\Gamma^{r}_{\theta\theta}}{\partial x^r}+\frac{\partial\Gamma^{\theta}_{\theta\theta}}{\partial x^{\theta}})+\sum_{\lambda}[\Gamma^{1}_{\theta\lambda}\Gamma^{\lambda}_{\theta1}+\Gamma^{2}_{\theta\lambda}\Gamma^{\lambda}_{\theta2}]-\sum_{\lambda}[\Gamma^{1}_{\theta\theta}\Gamma^{\lambda}_{\lambda1}+\Gamma^{2}_{\theta\theta}\Gamma^{\lambda}_{\lambda2}]=$$

$$=(\frac{\partial 0}{\partial\theta}+\frac{\partial 0}{\partial\theta})-(\frac{\partial(-r)}{\partial r}+\frac{\partial 0}{\partial\theta})+\sum_{\lambda}[\Gamma^{r}_{\theta\lambda}\Gamma^{\lambda}_{\theta r}+\Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta}]-\sum_{\lambda}[\Gamma^{r}_{\theta\theta}\Gamma^{\lambda}_{\lambda r}+\Gamma^{\theta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\theta}]=$$

$$=1+\sum_{\lambda}[\Gamma^{r}_{\theta\lambda}\Gamma^{\lambda}_{\theta r}+\Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta}]-\sum_{\lambda}[(-r)\Gamma^{\lambda}_{\lambda r}+0\Gamma^{\lambda}_{\lambda\theta}]=$$

$$=1+\sum_{\lambda}[\Gamma^r_{\theta\lambda}\Gamma^\lambda_{\theta r}+\Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\theta\theta}]+\sum_{\lambda} r\Gamma^\lambda_{\lambda r}=1+[\Gamma^r_{\theta 1}\Gamma^1_{\theta r}+\Gamma^r_{\theta 2}\Gamma^2_{\theta r}]+[\Gamma^\theta_{\theta 1}\Gamma^1_{\theta\theta}+\Gamma^\theta_{\theta 2}\Gamma^2_{\theta\theta}]+[r\Gamma^1_{1r}+r\Gamma^2_{2r}]=$$

$$=1+[\Gamma^r_{\theta r}\Gamma^r_{\theta r}+\Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}]+[\Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta}+\Gamma^\theta_{\theta\theta}\Gamma^\theta_{\theta\theta}]+[r\Gamma^r_{rr}+r\Gamma^\theta_{\theta r}]=1+[0\cdot 0+(-r)\frac{1}{r}]+[\frac{1}{r}(-r)+0\cdot 0]+[r\cdot 0+r\frac{1}{r}]=$$

$$=1+[0\cdot 0-1]+[-1+0\cdot 0]+[0+1]=\boxed{0\ .}$$

(μ=r e k=θ o viceversa): $\boxed{R_{r\theta}=R_{\theta r}}=\frac{\partial\Gamma^\lambda_{r\lambda}}{\partial x^\theta}-\frac{\partial\Gamma^\lambda_{r\theta}}{\partial x^\lambda}+\Gamma^\eta_{r\lambda}\Gamma^\lambda_{\theta\eta}-\Gamma^\eta_{r\theta}\Gamma^\lambda_{\lambda\eta}=$

$$=\frac{\partial\Gamma^1_{r1}}{\partial\theta}+\frac{\partial\Gamma^2_{r2}}{\partial\theta}-\frac{\partial\Gamma^1_{r\theta}}{\partial x^1}-\frac{\partial\Gamma^2_{r\theta}}{\partial x^2}+\sum_{\lambda,\eta}\Gamma^\eta_{r\lambda}\Gamma^\lambda_{\theta\eta}-\sum_{\lambda,\eta}\Gamma^\eta_{r\theta}\Gamma^\lambda_{\lambda\eta}=\frac{\partial\Gamma^r_{rr}}{\partial\theta}+\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}-\frac{\partial\Gamma^r_{r\theta}}{\partial x^r}-\frac{\partial\Gamma^\theta_{r\theta}}{\partial x^\theta}+$$

$$+\sum_{\lambda}[\Gamma^1_{r\lambda}\Gamma^\lambda_{\theta 1}+\Gamma^2_{r\lambda}\Gamma^\lambda_{\theta 2}]-\sum_{\lambda}[\Gamma^1_{r\theta}\Gamma^\lambda_{\lambda 1}+\Gamma^2_{r\theta}\Gamma^\lambda_{\lambda 2}]=\frac{\partial\Gamma^r_{rr}}{\partial\theta}+\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}-\frac{\partial\Gamma^r_{r\theta}}{\partial r}-\frac{\partial\Gamma^\theta_{r\theta}}{\partial\theta}+$$

$$+\sum_{\lambda}[\Gamma^r_{r\lambda}\Gamma^\lambda_{\theta r}+\Gamma^\theta_{r\lambda}\Gamma^\lambda_{\theta\theta}]-\sum_{\lambda}[\Gamma^r_{r\theta}\Gamma^\lambda_{\lambda r}+\Gamma^\theta_{r\theta}\Gamma^\lambda_{\lambda\theta}]=\frac{\partial 0}{\partial\theta}+\frac{\partial(1/r)}{\partial\theta}-\frac{\partial 0}{\partial r}-\frac{\partial(1/r)}{\partial\theta}+$$

$$+\sum_{\lambda}[\Gamma^r_{r\lambda}\Gamma^\lambda_{\theta r}+\Gamma^\theta_{r\lambda}\Gamma^\lambda_{\theta\theta}]-\sum_{\lambda}[0\Gamma^\lambda_{\lambda r}+\frac{1}{r}\Gamma^\lambda_{\lambda\theta}]=0+0-0-0+[\Gamma^r_{r1}\Gamma^1_{\theta r}+\Gamma^r_{r2}\Gamma^2_{\theta r}]+[\Gamma^\theta_{r1}\Gamma^1_{\theta\theta}+\Gamma^\theta_{r2}\Gamma^2_{\theta\theta}]-[\frac{1}{r}\Gamma^1_{1\theta}+\frac{1}{r}\Gamma^2_{2\theta}]=$$

$$=[\Gamma^r_{rr}\Gamma^r_{\theta r}+\Gamma^r_{r\theta}\Gamma^\theta_{\theta r}]+[\Gamma^\theta_{rr}\Gamma^r_{\theta\theta}+\Gamma^\theta_{r\theta}\Gamma^\theta_{\theta\theta}]-[\frac{1}{r}\Gamma^r_{r\theta}+\frac{1}{r}\Gamma^\theta_{\theta\theta}]=[0\cdot 0+0\frac{1}{r}]+[0\Gamma^r_{\theta\theta}+\Gamma^\theta_{r\theta}0]-[\frac{1}{r}0+\frac{1}{r}0]=\boxed{0\,.}$$

Riassumendo:

$$\boxed{R_{rr}=0\ ,\quad R_{\theta\theta}=0\quad ,\quad R_{r\theta}=R_{\theta r}=0\ .}$$

Ed infine, riguardo lo scalare di curvatura R:

$$\boxed{R}=g^{\mu k}R_{\mu k}=g^{rr}R_{rr}+g^{\theta\theta}R_{\theta\theta}=1\cdot 0+\frac{1}{r^2}0=\boxed{0\ .}$$

Abbiamo ottenuto R=0 poiché, in effetti, il piano polare (r,θ) su cui ci siamo basati è, in definitiva, appunto “piano”, stando esso comodamente sul foglio di carta.

-<u>Considerando invece la superficie di una sfera e stando sempre in due dimensioni (θ,φ; r=const), si ha</u>:

$$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$$

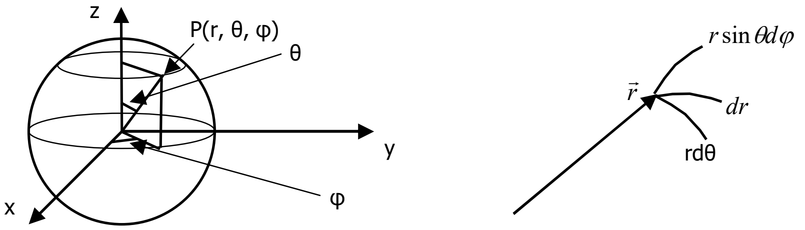


Fig. 4 a) e b)

φ varia tra 0 e 2π , mentre θ varia tra 0 e π . Inoltre, ovviamente:

$$\begin{cases} r^2=x^2+y^2+z^2 \\ \varphi=arctg\,\frac{y}{x} \\ \theta=\arccos\frac{z}{r} \end{cases}$$

Bene, adesso, dalla Fig. 4 b), si evince che sulla superficie della sfera, che è effettivamente ciò che costituisce la sfera stessa (siamo cioè in due dimensioni), il modulo quadro $d\tau^2$ di un vettore è: ($d\tau^2=-g_{\mu\nu}dx^\mu dx^\nu$, $\mu,\nu\to\theta,\varphi$)

$$d\tau^2=-g_{\mu\nu}dx^\mu dx^\nu=-g_{\theta\theta}dx^\theta dx^\theta-g_{\varphi\varphi}dx^\varphi dx^\varphi-2g_{\theta\varphi}dx^\theta dx^\varphi=-g_{\theta\theta}d\theta^2-g_{\varphi\varphi}d\varphi^2-2g_{\theta\varphi}\cancel{d\theta}d\varphi=$$

$= -r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$ (semplice applicazione del Teorema di Pitagora sulla superficie sferica) e qui si vede che non vi sono termini misti $d\theta d\varphi$, dacché il tensore metrico g è diagonale, ossia i termini misti sono nulli ($g_{\theta\varphi} = g_{\varphi\theta} = 0$) e dunque ancora, essendo che $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho \rightarrow$ ($g_{\mu\nu} g^{\mu\nu} \equiv$"1")

$\rightarrow \quad g_{\mu\nu} \equiv \dfrac{1}{g^{\mu\nu}} \quad \rightarrow g_{\theta\theta} = r^2$ (e $g^{\theta\theta} = \dfrac{1}{r^2}$) , $g_{\varphi\varphi} = r^2 \sin^2\theta$ (e $g^{\varphi\varphi} = \dfrac{1}{r^2 \sin^2\theta}$) e $g_{\theta\varphi} = g_{\varphi\theta} = g^{\theta\varphi} = g^{\varphi\theta} = 0$.

Esploriamo subito la variabilità su λ ; per $\lambda = \varphi$ nella espressione per i simboli di Christoffel (dove φ è una delle nostre due coordinate θ, φ , sul caso specifico della sfera), si ha allora:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) = \Gamma^\varphi_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) \ . \qquad (7)$$

Adesso, con $\mu, \nu \rightarrow \theta, \varphi$ nella (7), segue che:

$$\Gamma^\varphi_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) = \Gamma^\varphi_{\theta\varphi} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\varphi\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \varphi} - \frac{\partial g_{\varphi\theta}}{\partial x^\sigma}\right) = \Gamma^\varphi_{\varphi\theta} =$$

$= [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\varphi\sigma}}{\partial \theta}] + [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \varphi}] - [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^\sigma}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è l'indice variabile che è ripetuto):

$$\boxed{\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta}} = [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\varphi 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\varphi 2}}{\partial \theta}] + [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \varphi} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \varphi}] - [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^1} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^2}] =$$

$$= [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\varphi\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\varphi\varphi}}{\partial \theta}] + [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \varphi}] - [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^\theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\varphi\theta}}{\partial x^\varphi}] =$$

$$= [\frac{1}{2} 0 \cdot 0 + \frac{1}{2} \frac{1}{(r^2 \sin^2\theta)} \frac{\partial (r^2 \sin^2\theta)}{\partial \theta}] + [\frac{1}{2} 0 \frac{\partial (r^2)}{\partial \varphi} + \frac{1}{2} g^{\varphi\varphi} 0] - [\frac{1}{2} 0 \cdot 0 + \frac{1}{2} g^{\varphi\varphi} 0] =$$

$$= \frac{1}{2} \frac{1}{r^2 \sin^2\theta} 2r^2 \sin\theta \cos\theta = \boxed{\cot\theta \, .}$$

Se invece nella (7) poniamo $\mu, \nu \rightarrow \theta, \theta$, segue che:

$$\Gamma^\varphi_{\mu\nu} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) = \Gamma^\varphi_{\theta\theta} = \frac{1}{2} g^{\sigma\varphi} \left(\frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^\sigma}\right) =$$

$= [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \theta}] + [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\sigma}}{\partial \theta}] - [\frac{1}{2} g^{\sigma\varphi} \frac{\partial g_{\theta\theta}}{\partial x^\sigma}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è ripetuto):

$$\boxed{\Gamma^\varphi_{\theta\theta}} = [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \theta}] + [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta 1}}{\partial \theta} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta 2}}{\partial \theta}] - [\frac{1}{2} g^{1\varphi} \frac{\partial g_{\theta\theta}}{\partial x^1} + \frac{1}{2} g^{2\varphi} \frac{\partial g_{\theta\theta}}{\partial x^2}] =$$

$$= [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \theta}] + [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\varphi}}{\partial \theta}] - [\frac{1}{2} g^{\theta\varphi} \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi}] =$$

$$= [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} 0] + [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} 0] - [\frac{1}{2} 0 \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi}] = -\frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial \varphi} =$$

$$= -\frac{1}{2} g^{\varphi\varphi} \frac{\partial (r^2)}{\partial \varphi} = \boxed{0 \ .}$$

Se, per ultimo, nella (7) poniamo $\mu, \nu \rightarrow \varphi, \varphi$, segue che:

$$\Gamma^{\varphi}_{\mu\nu}=\frac{1}{2}g^{\sigma\varphi}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\varphi}_{\varphi\varphi}=\frac{1}{2}g^{\sigma\varphi}(\frac{\partial g_{\varphi\sigma}}{\partial\varphi}+\frac{\partial g_{\varphi\sigma}}{\partial\varphi}-\frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}})=$$

$=[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]+[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]-[\frac{1}{2}g^{\sigma\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{\sigma}}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è ripetuto):

$$\boxed{\Gamma^{\varphi}_{\varphi\varphi}}=[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi 1}}{\partial\varphi}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi 2}}{\partial\varphi}]+[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi 1}}{\partial\varphi}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi 2}}{\partial\varphi}]-[\frac{1}{2}g^{1\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{1}}+\frac{1}{2}g^{2\varphi}\frac{\partial g_{\varphi\varphi}}{\partial x^{2}}]=$$

$$=[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]+[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=$$

$$=[\frac{1}{2}0\cdot 0+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]+[\frac{1}{2}0\cdot 0+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]-[\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\theta}+\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}]=\frac{1}{2}g^{\varphi\varphi}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}=$$

$$=\frac{1}{2}\frac{1}{r^2\sin^2\theta}\frac{\partial(r^2\sin^2\theta)}{\partial\varphi}=\frac{1}{2}\frac{1}{r^2\sin^2\theta}0=\boxed{0\ .}$$

Poniamo ora, nella espressione per $\Gamma^{\lambda}_{\mu\nu}$, invece $\lambda=\theta$ (dove θ è una delle nostre due coordinate θ,φ , sul caso specifico della sfera); si ha allora:

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\sigma\lambda}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})\qquad(8)$$

Adesso, con $\mu,\nu\rightarrow\theta,\varphi$ nella (8), segue che:

$$\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\theta\varphi}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\varphi\sigma}}{\partial\theta}+\frac{\partial g_{\theta\sigma}}{\partial\varphi}-\frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\varphi\theta}=$$

$=[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\sigma}}{\partial\theta}]+[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\varphi}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{\sigma}}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è ripetuto):

$$\boxed{\Gamma^{\theta}_{\theta\varphi}=\Gamma^{\theta}_{\varphi\theta}}=[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi 1}}{\partial\theta}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi 2}}{\partial\theta}]+[\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\varphi}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\varphi}]-[\frac{1}{2}g^{1\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{1}}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\varphi\theta}}{\partial x^{2}}]=$$

$$=[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta}]+[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\varphi}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\varphi\theta}}{\partial\varphi}]=$$

$$=[\frac{1}{2}g^{\theta\theta}0+\frac{1}{2}0\frac{\partial g_{\varphi\varphi}}{\partial\theta}]+[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}+\frac{1}{2}0\cdot 0]-[\frac{1}{2}g^{\theta\theta}0+\frac{1}{2}0\cdot 0]=\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}=\frac{1}{2}\frac{1}{r^2}\frac{\partial r^2}{\partial\varphi}\boxed{=0\ .}$$

Se invece nella (8) poniamo $\mu,\nu\rightarrow\theta,\theta$, segue che:

$$\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}-\frac{\partial g_{\nu\mu}}{\partial x^{\sigma}})=\Gamma^{\theta}_{\theta\theta}=\frac{1}{2}g^{\sigma\theta}(\frac{\partial g_{\theta\sigma}}{\partial\theta}+\frac{\partial g_{\theta\sigma}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}})=$$

$=[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]+[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}]=[g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\theta}]-[\frac{1}{2}g^{\sigma\theta}\frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è ripetuto):

$$\boxed{\Gamma^{\theta}_{\theta\theta}}=[g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\theta}+g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\theta}]-[\frac{1}{2}g^{1\theta}\frac{\partial g_{\theta\theta}}{\partial x^{1}}+\frac{1}{2}g^{2\theta}\frac{\partial g_{\theta\theta}}{\partial x^{2}}]=[g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\theta}]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{1}{2}g^{\varphi\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi}]=$$

$$=[g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+0\cdot 0]-[\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{1}{2}0\frac{\partial g_{\theta\theta}}{\partial\varphi}]=-\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}=-\frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2)}{\partial\theta}\boxed{=0\ .}$$

Se, per ultimo, nella (8) poniamo $\mu,\nu \to \varphi,\varphi$, segue che:

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2} g^{\sigma\theta}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) = \Gamma^\theta_{\varphi\varphi} = \frac{1}{2} g^{\sigma\theta}\left(\frac{\partial g_{\varphi\sigma}}{\partial \varphi} + \frac{\partial g_{\theta\sigma}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial x^\sigma}\right) =$$

$=[\frac{1}{2} g^{\sigma\theta}\frac{\partial g_{\varphi\sigma}}{\partial\varphi}]+[\frac{1}{2} g^{\sigma\theta}\frac{\partial g_{\theta\sigma}}{\partial\varphi}]-[\frac{1}{2} g^{\sigma\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^\sigma}]$ e valendo ora la convenzione di Einstein sulla sommatoria sottintesa sugli indici ripetuti in un termine, si ha (poiché solo σ, effettivamente, è ripetuto):

$$\boxed{\Gamma^\theta_{\varphi\varphi}} = [\frac{1}{2} g^{1\theta}\frac{\partial g_{\varphi 1}}{\partial\varphi} + \frac{1}{2} g^{2\theta}\frac{\partial g_{\varphi 2}}{\partial\varphi}] + [\frac{1}{2} g^{1\theta}\frac{\partial g_{\theta 1}}{\partial\varphi} + \frac{1}{2} g^{2\theta}\frac{\partial g_{\theta 2}}{\partial\varphi}] - [\frac{1}{2} g^{1\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^1} + \frac{1}{2} g^{2\theta}\frac{\partial g_{\varphi\varphi}}{\partial x^2}] =$$

$$=[\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\varphi\theta}}{\partial\varphi} + \frac{1}{2} g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}] + [\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi} + \frac{1}{2} g^{\varphi\theta}\frac{\partial g_{\theta\varphi}}{\partial\varphi}] - [\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta} + \frac{1}{2} g^{\varphi\theta}\frac{\partial g_{\varphi\varphi}}{\partial\varphi}] =$$

$$=[\frac{1}{2} g^{\theta\theta} 0 + \frac{1}{2} 0 \frac{\partial g_{\varphi\varphi}}{\partial\varphi}] - [\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi} + \frac{1}{2} 00] - [\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta} + \frac{1}{2} 0 \frac{\partial g_{\varphi\varphi}}{\partial\varphi}] =$$

$$=\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\varphi} - \frac{1}{2} g^{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial\theta} = \frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2)}{\partial\varphi} - \frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2\sin^2\theta)}{\partial\theta} = 0 - \frac{1}{2}\frac{1}{r^2}\frac{\partial(r^2\sin^2\theta)}{\partial\theta} = -\sin\theta\cos\theta =$$

$$=\boxed{-\frac{1}{2}\sin 2\theta\,.}$$

Riassumendo, abbiamo ottenuto che, riguardo il tensore metrico g:

$$g_{\theta\theta} = r^2 \ ,\ g^{\theta\theta} = \frac{1}{r^2} \quad ,\quad g_{\varphi\varphi} = r^2\sin^2\theta \ ,\ g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta} \ ,\ g_{\theta\varphi} = g_{\varphi\theta} = g^{\theta\varphi} = g^{\varphi\theta} = 0 \ ,$$

mentre riguardo i Simboli di Christoffel Γ, abbiamo ottenuto che:

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta \quad ,\quad \Gamma^\theta_{\varphi\varphi} = -\frac{1}{2}\sin 2\theta \quad ,\quad \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\theta\theta} = \Gamma^\theta_{\theta\theta} = 0 \ .$$

A questo punto valutiamo il Tensore di Ricci $R_{\mu k} = \frac{\partial\Gamma^\lambda_{\mu\lambda}}{\partial x^k} - \frac{\partial\Gamma^\lambda_{\mu k}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\lambda\eta}$:

$$\boxed{R_{\theta\theta}} = \frac{\partial\Gamma^\lambda_{\theta\lambda}}{\partial\theta} - \frac{\partial\Gamma^\lambda_{\theta\theta}}{\partial x^\lambda} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - \Gamma^\eta_{\theta\theta}\Gamma^\lambda_{\lambda\eta} = \frac{\partial\Gamma^\lambda_{\theta\lambda}}{\partial\theta} - \frac{\partial 0}{\partial x^\lambda} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - 0\Gamma^\lambda_{\lambda\eta} = [\frac{\partial\Gamma^\lambda_{\theta\lambda}}{\partial\theta}] + [\Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta}] =$$

($\Gamma^\lambda_{\theta\theta} = \Gamma^\eta_{\theta\theta} = 0$ poiché $\Gamma^?_{\theta\theta}$ è sempre zero)

$$=[\sum_\lambda\frac{\partial\Gamma^\lambda_{\theta\lambda}}{\partial\theta}] + [\sum_{\lambda,\eta}\Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta}] = [\sum_\lambda\frac{\partial\Gamma^\lambda_{\theta\lambda}}{\partial\theta}] + \sum_\lambda\sum_\eta\Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} = [\frac{\partial\Gamma^1_{\theta 1}}{\partial\theta} + \frac{\partial\Gamma^2_{\theta 2}}{\partial\theta}] + [\sum_\eta\Gamma^\eta_{\theta 1}\Gamma^1_{\theta\eta} + \sum_\eta\Gamma^\eta_{\theta 2}\Gamma^2_{\theta\eta}] =$$

$$=[\frac{\partial\Gamma^\theta_{\theta\theta}}{\partial\theta} + \frac{\partial\Gamma^\varphi_{\theta\varphi}}{\partial\theta}] + [\sum_\eta\Gamma^\eta_{\theta\theta}\Gamma^{\theta 1}_{\theta\eta} + \sum_\eta\Gamma^\eta_{\theta\varphi}\Gamma^\varphi_{\theta\eta}] = [0 + \frac{\partial\cot\theta}{\partial\theta}] + [\sum_\eta 0\Gamma^{\theta 1}_{\theta\eta} + \sum_\eta\Gamma^\eta_{\theta\varphi}\Gamma^\varphi_{\theta\eta}] =$$

$$=-\frac{1}{\sin^2\theta} + \Gamma^1_{\theta\varphi}\Gamma^\varphi_{\theta 1} + \Gamma^2_{\theta\varphi}\Gamma^\varphi_{\theta 2} = -\frac{1}{\sin^2\theta} + \Gamma^\theta_{\theta\varphi}\Gamma^\varphi_{\theta\theta} + \Gamma^\varphi_{\theta\varphi}\Gamma^\varphi_{\theta\varphi} = -\frac{1}{\sin^2\theta} + \Gamma^\theta_{\theta\varphi}0 + (\Gamma^\varphi_{\theta\varphi})^2 = -\frac{1}{\sin^2\theta} + \cot^2\theta =$$

$$=-\frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos^2\theta - 1}{\sin^2\theta} = \boxed{-1\,.}$$

$$\boxed{R_{\varphi\varphi}} = \frac{\partial\Gamma^\lambda_{\varphi\lambda}}{\partial x^\varphi} - \frac{\partial\Gamma^\lambda_{\varphi\varphi}}{\partial x^\lambda} + \Gamma^\eta_{\varphi\lambda}\Gamma^\lambda_{\varphi\eta} - \Gamma^\eta_{\varphi\varphi}\Gamma^\lambda_{\lambda\eta} = \frac{\partial\Gamma^1_{\varphi 1}}{\partial\varphi} + \frac{\partial\Gamma^2_{\varphi 2}}{\partial\varphi} - \frac{\partial\Gamma^1_{\varphi\varphi}}{\partial x^1} - \frac{\partial\Gamma^2_{\varphi\varphi}}{\partial x^2} + \sum_\lambda\sum_\eta\Gamma^\eta_{\varphi\lambda}\Gamma^\lambda_{\varphi\eta} - \sum_\lambda\sum_\eta\Gamma^\eta_{\varphi\varphi}\Gamma^\lambda_{\lambda\eta} =$$

$$=[\frac{\partial\Gamma^\theta_{\varphi\theta}}{\partial\varphi} + \frac{\partial\Gamma^\varphi_{\varphi\varphi}}{\partial\varphi} - \frac{\partial\Gamma^\theta_{\varphi\varphi}}{\partial\theta} - \frac{\partial\Gamma^\varphi_{\varphi\varphi}}{\partial\varphi}] + [\sum_\eta\Gamma^\eta_{\varphi 1}\Gamma^1_{\varphi\eta} + \sum_\eta\Gamma^\eta_{\varphi 2}\Gamma^2_{\varphi\eta}] - [\sum_\eta\Gamma^\eta_{\varphi\varphi}\Gamma^1_{1\eta} + \sum_\eta\Gamma^\eta_{\varphi\varphi}\Gamma^2_{2\eta}] =$$

$$= [0 + 0 - \frac{\partial \Gamma^{\theta}_{\varphi\varphi}}{\partial \theta} - 0] + [\sum_{\eta} \Gamma^{\eta}_{\varphi\theta}\Gamma^{\theta}_{\varphi\eta} + \sum_{\eta} \Gamma^{\eta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\eta}] - [\sum_{\eta} \Gamma^{\eta}_{\varphi\varphi}\Gamma^{\theta}_{\theta\eta} + \sum_{\eta} \Gamma^{\eta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\eta}] =$$

$$= -\frac{\partial(-\frac{1}{2}\sin 2\theta)}{\partial \theta} + [(\Gamma^{1}_{\varphi\theta}\Gamma^{\theta}_{\varphi 1} + \Gamma^{2}_{\varphi\theta}\Gamma^{\theta}_{\varphi 2}) + (\Gamma^{1}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 2})] - [(\Gamma^{1}_{\varphi\varphi}\Gamma^{\theta}_{\theta 1} + \Gamma^{2}_{\varphi\varphi}\Gamma^{\theta}_{\theta 2}) + (\Gamma^{1}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi 2})] =$$

$$= \cos 2\theta + [(\Gamma^{\theta}_{\varphi\theta}\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi})] - [(\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\theta}_{\theta\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi})] =$$

$$= \cos 2\theta + [(0 \cdot 0 + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + 0 \cdot 0)] - [(\Gamma^{\theta}_{\varphi\varphi}0 + 0\Gamma^{\theta}_{\theta\varphi}) + (\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} + 0 \cdot 0)] = \cos 2\theta + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi} =$$

$$= \cos 2\theta - \frac{1}{2}\cot\theta \sin 2\theta = (2\cos^2\theta - 1) - \frac{1}{2}\cot\theta(2\sin\theta\cos\theta) = 2\cos^2\theta - 1 - \cos^2\theta = \boxed{-\sin^2\theta\ .}$$

$$\boxed{R_{\theta\varphi}} = R_{\varphi\theta} = \frac{\partial \Gamma^{\lambda}_{\theta\lambda}}{\partial x^{\varphi}} - \frac{\partial \Gamma^{\lambda}_{\theta\varphi}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\varphi\eta} - \Gamma^{\eta}_{\theta\varphi}\Gamma^{\lambda}_{\lambda\eta} = \frac{\partial \Gamma^{1}_{\theta 1}}{\partial \varphi} + \frac{\partial \Gamma^{2}_{\theta 2}}{\partial \varphi} - \frac{\partial \Gamma^{1}_{\theta\varphi}}{\partial x^{1}} - \frac{\partial \Gamma^{2}_{\theta\varphi}}{\partial x^{2}} + \sum_{\lambda}\sum_{\eta} \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\varphi\eta} - \sum_{\lambda}\sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\lambda}_{\lambda\eta} =$$

$$= \frac{\partial \Gamma^{\theta}_{\theta\theta}}{\partial \varphi} + \frac{\partial \Gamma^{\varphi}_{\theta\varphi}}{\partial \varphi} - \frac{\partial \Gamma^{\theta}_{\theta\varphi}}{\partial \theta} - \frac{\partial \Gamma^{\varphi}_{\theta\varphi}}{\partial \varphi} + [\sum_{\eta} \Gamma^{\eta}_{\theta 1}\Gamma^{1}_{\varphi\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta 2}\Gamma^{2}_{\varphi\eta}] - [\sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{1}_{1\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{2}_{2\eta}] =$$

$$= 0 + \frac{\partial \cot\theta}{\partial \varphi} - 0 - \frac{\partial \cot\theta}{\partial \varphi} + [\sum_{\eta} \Gamma^{\eta}_{\theta\theta}\Gamma^{\theta}_{\varphi\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] - [\sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\theta}_{\theta\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] =$$

$$= 0 + [\sum_{\eta} \Gamma^{\eta}_{\theta\theta}\Gamma^{\theta}_{\varphi\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] - [\sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\theta}_{\theta\eta} + \sum_{\eta} \Gamma^{\eta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\eta}] =$$

$$= [\Gamma^{1}_{\theta\theta}\Gamma^{\theta}_{\varphi 1} + \Gamma^{2}_{\theta\theta}\Gamma^{\theta}_{\varphi 2} + \Gamma^{1}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 2}] - [\Gamma^{1}_{\theta\varphi}\Gamma^{\theta}_{\theta 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\theta}_{\theta 2} + \Gamma^{1}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 1} + \Gamma^{2}_{\theta\varphi}\Gamma^{\varphi}_{\varphi 2}] =$$

$$= [\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\varphi\varphi} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\varphi}] - [\Gamma^{\theta}_{\theta\varphi}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\theta\varphi} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\varphi}] =$$

$$= [0 \cdot 0 + 0 \cdot 0 + 0\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}0] - [0 \cdot 0 + \Gamma^{\varphi}_{\theta\varphi}0 + 0\Gamma^{\varphi}_{\varphi\theta} + \Gamma^{\varphi}_{\theta\varphi}0] = \boxed{0\ .}$$

Riassumendo ancora: $R_{\theta\theta} = -1$, $R_{\varphi\varphi} = -\sin^2\theta$ e $R_{\theta\varphi} = R_{\varphi\theta} = 0$.

Finalmente possiamo calcolare lo scalare R, contraendo il tensore di Ricci col tensore metrico g:

$$\boxed{R} = g^{\mu k}R_{\mu k} = g^{1k}R_{1k} + g^{2k}R_{2k} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} = g^{\theta\theta}R_{\theta\theta} + g^{\theta\varphi}R_{\theta\varphi} + g^{\varphi\theta}R_{\varphi\theta} + g^{\varphi\varphi}R_{\varphi\varphi} =$$

$$= \frac{1}{r^2}(-1) + 0 \cdot 0 + 0 \cdot 0 + (\frac{1}{r^2\sin^2\theta})(-\sin^2\theta) = -\frac{1}{r^2} - \frac{\sin^2\theta}{r^2\sin^2\theta} = \boxed{-\frac{2}{r^2}}$$

; dunque, la curvatura R decresce all'aumentare di r.

TRASPORTO PARALLELO E TENSORE DI RIEMANN

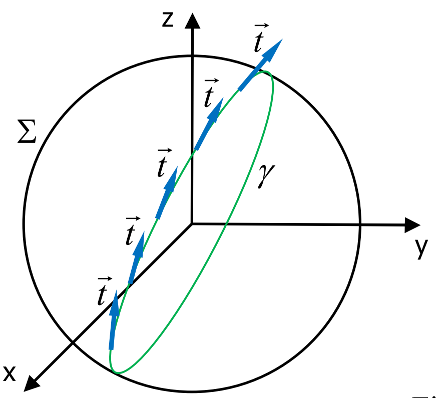


Fig. 5: trasporto parallelo.

Consideriamo una formichina monodimensionale che abita sulla curva γ (la quale si trova sulla superficie Σ della sfera); bene, per tale formichina, che non sa nulla delle altre dimensioni superiori, andare dritto significa muoversi come si muove il vettore tangente $\vec{t}$; infatti, in ogni posizione della curva, la formichina assume sempre le stesse caratteristiche di posizione, ossia quelle della tangente, anche perché il suo unico versore del suo sistema di riferimento locale/personale è proprio $\vec{t}$; dunque, così facendo, essa si trasporta parallelamente. Diverso è il discorso per un osservatore tridimensionale solidale col sistema x,y,z, secondo cui la formichina, nel muoversi, cambia continuamente direzione.

Se la formichina compie il giro completo, ritornando al punto di partenza, anche il vettore $\vec{t}$, nel compiere anch'esso il giro, si ritroverà come se stesso nel punto di partenza. E se il percorso γ fosse una retta, nel percorso vedremmo $\vec{t}$ assumere posizioni realmente parallele tra loro.

Volendo scrivere la derivata di un vettore lungo una direzione generica $\vec{u}$ (di componenti u^ν), basterà ovviamente proiettare la derivata $V^\mu_{;\nu}$ lungo appunto $\vec{u}$:

$$\nabla_{\vec{u}}(\vec{V}) = u^\nu (V^\mu_{;\nu}\hat{e}_\mu) = u^\nu (\frac{dV^\mu}{dx^\nu} + V^\sigma \Gamma^\mu_{\sigma\nu})\hat{e}_\mu$$

e allora, per quanto finora detto, il trasporto parallelo di un vettore $\vec{V}$ lungo una direzione $\vec{u}$ implica che $V^\mu_{;\nu} = 0 \rightarrow \nabla_{\vec{u}}(\vec{V}) = 0$.

In uno spazio curvo (non euclideo) il trasporto parallelo di un vettore su un anello chiuso infinitesimo non riporterà al punto di partenza un vettore parallelo a quello che iniziò il percorso; immaginiamo, a tal proposito, di muovere un vettore $\vec{V} = V^1\hat{e}_1 + V^2\hat{e}_2$ lungo un anello chiuso infinitesimo δγ , partendo da A, come in Fig. 6:

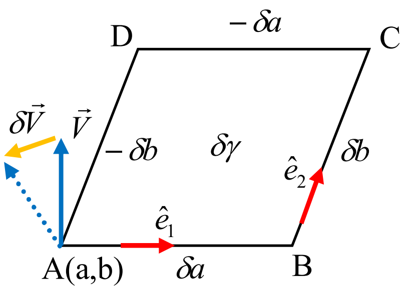


Fig. 6: anello chiuso infinitesimo.

Se al ritorno al punto A il vettore $\vec{V}$ risulterà ruotato, allora potremo dire che la superficie contenuta in δγ è curva:

il primo passo è il trasporto parallelo, da A(a,b) a B, di $\vec{V}$, lungo $\hat{e}_1$ e trasporto parallelo, per quanto finora detto, implica che: $\nabla_{\hat{e}_1}(\vec{V}) = 0$, ossia: $\nabla_{\hat{e}_1}(\vec{V}) = const(V^\mu_{;1}\hat{e}_\mu) = const(\frac{dV^\mu}{dx^1} + V^\sigma \Gamma^\mu_{\sigma 1})\hat{e}_\mu = 0 \rightarrow$

$\rightarrow \frac{dV^\mu}{dx^1} = -V^\sigma \Gamma^\mu_{\sigma 1}$, da cui: $dV^\mu = -V^\sigma \Gamma^\mu_{\sigma 1} dx^1 \rightarrow$

$\rightarrow \int_A^B dV^\mu = V^\mu(B) - V^\mu(A) = -\int_A^B V^\sigma \Gamma^\mu_{\sigma 1} dx^1$; dunque, in B, V^μ sarà:

$$V^\mu(B) = V^\mu(A) + (\int_A^B dx^1 \frac{\partial V^\mu}{\partial x^1})_{with\_x^2=b} = V^\mu(A) - \int_a^{a+\delta a} dx^1 V^\sigma \Gamma^\mu_{\sigma 1} \; ; \qquad (9)$$

(l'integrale, infatti, è in dx^1, tra A e B, ed x^1 vale a in A e a+δa in B)

Inoltre, riguardo il salto da B a C lungo $\hat{e}_2$, si ha: $(\frac{dV^\mu}{dx^2} + V^\sigma \Gamma^\mu_{\sigma 2}) = 0$, poiché, muovendoci questa volta lungo $\hat{e}_2$, la coordinata spaziale di derivazione, ossia di variazione, sarà ovviamente x^2, e dunque: $dV^\mu = -[V^\sigma \Gamma^\mu_{\sigma 2}]dx^2$, ma non è finita, poiché non bisogna qui trascurare il fatto che sul tratto BC su cui stiamo lavorando, si ha che $x^1 = b+\delta b$ e allora la quantità tra le parentesi quadre va aggiornata di conseguenza, tramite Taylor lungo x^1 appunto (lungo δa), ossia:

$[V^\sigma \Gamma^\mu_{\sigma 2}] \rightarrow [V^\sigma \Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a]$ e dunque, similmene al caso precedente:

$$V^\mu(C) = V^\mu(B) - (\int_B^C V^\sigma \Gamma^\mu_{\sigma 2} dx^2)_{with\_x^1=b+\delta b} = V^\mu(B) - \int_b^{b+\delta b} [V^\sigma \Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2 \ ; \qquad (10)$$

(l'integrale, infatti, è in dx^2, tra B e C, ed x^2 vale b in B e b+δb in C)

similmente, riguardo il salto C→D e ricordando che qui ci si muove lungo $-\hat{e}_1$:

$$V^\mu(D) = V^\mu(C) + (\int_C^D dx^1 \frac{\partial V^\mu}{\partial x^1})_{x^2=b+\delta b} = V^\mu(C) - (\int_{a+\delta a}^a dx^1 V^\sigma \Gamma^\mu_{\sigma 1})_{x^2=b+\delta b} =$$

(sempre per Taylor, lungo x^2 (lungo δb), poiché in tale tratto, x^2 non vale solo b, ma bensì b+δb →)

$$= V^\mu(C) - \int_{a+\delta a}^a [V^\sigma \Gamma^\mu_{\sigma 1} + \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1 \qquad (11)$$

e infine, sul salto D→A e ricordando che qui ci si muove lungo $-\hat{e}_2$:

$$V^\mu(A_{END}) = V^\mu(D) + (\int_D^{A_{END}} dx^2 \frac{\partial V^\mu}{\partial x^2})_{x^1=a} = V^\mu(D) - \int_{b+\delta b}^b dx^2 V^\sigma \Gamma^\mu_{\sigma 2} \quad . \qquad (12)$$

(e nessuno sviluppo di Taylor lungo x^1, poiché su questo tratto x^1 vale sempre e solo a e non a+δa)

Dunque, riguardo la presunta variazione $\delta\vec{V}$: $\delta\vec{V} = V^\mu(A_{END}) - V^\mu(A_{START})$ e $V^\mu(A_{END})$ è dato dalla (12), mentre $V^\mu(A_{START})$ è ricavato dalla (9), ossia: $V^\mu(A) = V^\mu(B) + \int_a^{a+\delta a} V^\sigma \Gamma^\mu_{\sigma 1} dx^1 \rightarrow$

$\rightarrow \ \delta\vec{V} = V^\mu(D) - \int_{b+\delta b}^b V^\sigma \Gamma^\mu_{\sigma 2} dx^2 - V^\mu(B) - \int_a^{a+\delta a} V^\sigma \Gamma^\mu_{\sigma 1} dx^1$ ed usando le (10) e (11) rispettivamente per avere $V^\mu(B)$ e $V^\mu(D)$:

$$V^\mu(B) = V^\mu(C) + \int_b^{b+\delta b} [V^\sigma \Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2 \quad \text{(ricavata dalla (10))}$$

$$V^\mu(D) = V^\mu(C) - \int_{a+\delta a}^a [V^\sigma \Gamma^\mu_{\sigma 1} + \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1 \quad \text{(la (11) intonsa)}$$

da cui, per sostituzione, si ha:

$$\delta\vec{V} = \{V^\mu(D)\} - \int_{b+\delta b}^b V^\sigma \Gamma^\mu_{\sigma 2} dx^2 - \{V^\mu(B)\} - \int_a^{a+\delta a} V^\sigma \Gamma^\mu_{\sigma 1} dx^1 = \{\cancel{V^\mu(C)} - \int_{a+\delta a}^a [V^\sigma \Gamma^\mu_{\sigma 1} + \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1\} -$$

$$- \int_{b+\delta b}^b V^\sigma \Gamma^\mu_{\sigma 2} dx^2 - \{\cancel{V^\mu(C)} - \int_b^{b+\delta b} [V^\sigma \Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2\} - \int_a^{a+\delta a} V^\sigma \Gamma^\mu_{\sigma 1} dx^1 =$$

$$= -\int_{a+\delta a}^a [V^\sigma \Gamma^\mu_{\sigma 1} + \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1 - \int_{b+\delta b}^b V^\sigma \Gamma^\mu_{\sigma 2} dx^2 - \int_b^{b+\delta b} [V^\sigma \Gamma^\mu_{\sigma 2} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2 - \int_a^{a+\delta a} V^\sigma \Gamma^\mu_{\sigma 1} dx^1 =$$

$$= \int_a^{a+\delta a} [\cancel{V^\sigma \Gamma^\mu_{\sigma 1}} + \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1 - \int_a^{a+\delta a} \cancel{V^\sigma \Gamma^\mu_{\sigma 1}} dx^1 + \int_b^{b+\delta b} \cancel{V^\sigma \Gamma^\mu_{\sigma 2}} dx^2 - \int_b^{b+\delta b} [\cancel{V^\sigma \Gamma^\mu_{\sigma 2}} + \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2 =$$

$$= \int_a^{a+\delta a} [\frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] dx^1 - \int_b^{b+\delta b} [\frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] dx^2 = [\frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta b] \cdot x^1 \Big|_{x^1=a}^{a+\delta a} - [\frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a] \cdot x^2 \Big|_{x^2=b}^{b+\delta b} =$$

$$= \frac{\partial}{\partial x^2}(V^\sigma \Gamma^\mu_{\sigma 1})\delta a \delta b - \frac{\partial}{\partial x^1}(V^\sigma \Gamma^\mu_{\sigma 2})\delta a \delta b = \delta a \delta b [\frac{\partial \Gamma^\mu_{\sigma 1}}{\partial x^2} V^\sigma + \frac{\partial V^\eta}{\partial x^2} \Gamma^\lambda_{\eta 1} - \frac{\partial \Gamma^\mu_{\sigma 2}}{\partial x^1} V^\sigma - \frac{\partial V^\eta}{\partial x^1} \Gamma^\lambda_{\eta 2}]$$

(nello svolgere le derivate di prodotti di funzioni, appena qui sopra, si nota che sono nati nuovi indici, fatto, questo, tipico del calcolo tensoriale, a testimoniare il fatto che tra un termine e l'altro non vige il vincolo dell'eguaglianza contemporanea dei valori degli indici stessi) ed avendo in precedenza dimostrato (col dovuto riadattamento degli indici) che:

$\frac{dV^\eta}{dx^1} = -V^\sigma \Gamma^\eta_{\sigma 1}$ e $\frac{dV^\eta}{dx^2} = -V^\sigma \Gamma^\eta_{\sigma 2}$, segue che:

$$\delta\vec{V} = \delta a \delta b[\frac{\partial \Gamma^\mu_{\sigma 1}}{\partial x^2} V^\sigma - \frac{\partial \Gamma^\mu_{\sigma 2}}{\partial x^1} V^\sigma + \frac{\partial V^\eta}{\partial x^2} \Gamma^\mu_{\eta 1} - \frac{\partial V^\eta}{\partial x^1} \Gamma^\mu_{\eta 2}] = \delta a \delta b[\frac{\partial \Gamma^\mu_{\sigma 1}}{\partial x^2} V^\sigma - \frac{\partial \Gamma^\mu_{\sigma 2}}{\partial x^1} V^\sigma - V^\sigma \Gamma^\eta_{\sigma 2} \Gamma^\mu_{\eta 1} + V^\sigma \Gamma^\eta_{\sigma 1} \Gamma^\mu_{\eta 2}] =$$

$$= \delta a \delta b[\frac{\partial \Gamma^\mu_{\sigma 1}}{\partial x^2} - \frac{\partial \Gamma^\mu_{\sigma 2}}{\partial x^1} - \Gamma^\eta_{\sigma 2} \Gamma^\mu_{\eta 1} + \Gamma^\eta_{\sigma 1} \Gamma^\mu_{\eta 2}] V^\sigma .$$

Prendiamo ora la parte tra parentesi: $R^\mu_{\sigma 21} = [\frac{\partial \Gamma^\mu_{\sigma 1}}{\partial x^2} - \frac{\partial \Gamma^\mu_{\sigma 2}}{\partial x^1} - \Gamma^\eta_{\sigma 2} \Gamma^\mu_{\eta 1} + \Gamma^\eta_{\sigma 1} \Gamma^\mu_{\eta 2}]$ e, per coerenza, attribuiamo indici letterali (ν, λ) anche al posto di 1 e 2:

$$\boxed{R^\mu_{\sigma\lambda\nu} = [\frac{\partial \Gamma^\mu_{\sigma\nu}}{\partial x^\lambda} - \frac{\partial \Gamma^\mu_{\sigma\lambda}}{\partial x^\nu} - \Gamma^\eta_{\sigma\lambda} \Gamma^\mu_{\eta\nu} + \Gamma^\eta_{\sigma\nu} \Gamma^\mu_{\eta\lambda}] \ ;} \qquad (13)$$

bene, abbiamo ottenuto il <u>Tensore di Curvatura di Riemann-Christoffel</u> che, infatti, se è nullo, determina un $\delta\vec{V} = 0$, ossia il vettore $\vec{V}$ è rimasto invariato nel trasporto parallelo, ossia lo spazio è piatto (ossia no curvatura), ossia, come appena detto, è il caso in cui $R^\mu_{\sigma\lambda\nu} = 0$.

Effettivamente, nel caso della Fig. 6, ossia di un parallelogramma (piatto), è auspicabile che il vettore $\vec{V}$, trasportato parallelamente, ritorni esattamente nello stesso posto, determinando un $d\vec{V} = 0$ e, dunque, una curvatura $R^\mu_{\sigma\lambda\nu} = 0$; diverso sarebbe il caso di una superficie più simile ad una calotta sferica, invece che ad un parallelogramma.

APPENDICE 2: I CALCOLI CLASSICI (NON TENSORIALI) PER IL CAMBIO DI COORDINATE.

1-LE FORMULE **2**-DEDUZIONE DIRETTA DEL SOLO GRADIENTE **3**-DEDUZIONE COMPLETA DI GRAD, DIV, ROT E LAPLACIANO **4**-DEDUZIONE ALTERNATIVA SUPPLEMENTARE **5**-L'IMPORTANZA DELLE COORDINATE SFERICHE E CILINDRICHE SUBAPPENDICI

1-LE FORMULE
-COORDINATE CARTESIANE:

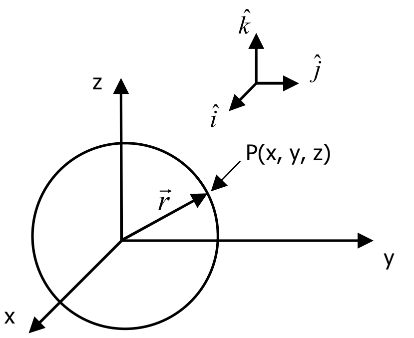


Fig. 1.1

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad , \tag{1.1}$$

$$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k} \quad , \tag{1.2}$$

V è uno scalare ed $\vec{F}$ è un vettore. La divergenza fornisce uno scalare, mentre il gradiente ed il rotore forniscono un vettore.

$$gradV = \nabla V = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}) \tag{1.3}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) \tag{1.4}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{j}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{1.5}$$

$$\Delta V = \nabla^2 V = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})V \tag{1.6}$$

-COORDINATE SFERICHE:
Si passa da (x,y,z) a (ρ,θ,φ). φ varia tra 0 e 2π, mentre θ varia tra 0 e π.

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\theta)\hat{\theta} + (\rho \sin\theta d\varphi)\hat{\varphi} \tag{1.7}$$

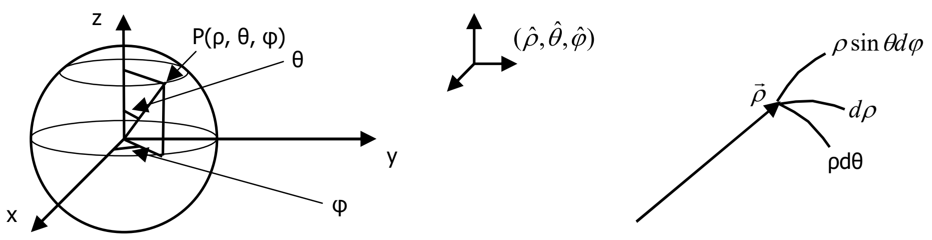


Fig. 1.2

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \tag{1.8}$$

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}) \tag{1.9}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho^2 \sin\theta}[\frac{\partial}{\partial \rho}(\rho^2 \sin\theta F_\rho) + \frac{\partial}{\partial \theta}(\rho \sin\theta F_\theta) + \frac{\partial}{\partial \varphi}(\rho F_\varphi)] \tag{1.10}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \rho \sin\theta \hat{\varphi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_\rho & \rho F_\theta & \rho \sin\theta F_\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho^2 \sin\theta}[\frac{\partial(\rho \sin\theta F_\varphi)}{\partial \theta} - \frac{\partial(\rho F_\theta)}{\partial \varphi}]\hat{\rho} + \frac{1}{\rho \sin\theta}[\frac{\partial F_\rho}{\partial \varphi} - \frac{\partial(\rho \sin\theta F_\varphi)}{\partial \rho}]\hat{\theta} + \frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}]\hat{\varphi} \tag{1.11}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} \tag{1.12}$$

-COORDINATE CILINDRICHE:

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$\varphi \in [0, 2\pi)$

$$d\vec{r} = (d\rho)\hat{\rho} + (\rho d\varphi)\hat{\varphi} + (dz)\hat{z} \tag{1.13}$$

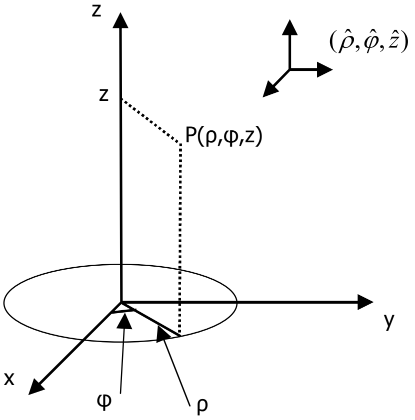


Fig. 1.3

$$gradV = \nabla V = (\frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{\varphi} + \frac{\partial V}{\partial z}\hat{z}) \tag{1.14}$$

$$div \cdot \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \varphi}(F_\varphi) + \frac{\partial}{\partial z}(\rho F_z)] \tag{1.15}$$

$$rot\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} = \frac{1}{\rho}[\frac{\partial(F_z)}{\partial \varphi} - \frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho} + [\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}]\hat{\varphi} + \frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}]\hat{\varphi} \tag{1.16}$$

$$\Delta V = \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.17}$$

2-DEDUZIONE DIRETTA DEL SOLO GRADIENTE
(esempio di metodo diretto, ma poco efficace, che abbandoniamo subito)
Sappiamo che:

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho\cos\theta \end{cases} \qquad (2.1)$$

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 & (2.2) \\ \varphi = arctg\dfrac{y}{x} & (2.3) \\ \theta = \arccos\dfrac{z}{\rho} & (2.4) \end{cases}$$

e dunque: $\dfrac{\partial}{\partial x} = \dfrac{\partial\rho}{\partial x}\dfrac{\partial}{\partial\rho} + \dfrac{\partial\theta}{\partial x}\dfrac{\partial}{\partial\theta} + \dfrac{\partial\varphi}{\partial x}\dfrac{\partial}{\partial\varphi}$ (2.5)

$$\frac{\partial}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi} \qquad (2.6) \qquad\qquad \frac{\partial}{\partial z} = \frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi} \qquad (2.7)$$

Differenziamo ora la (2.2): $\rho d\rho = xdx + ydy + zdz$ (2.8)

Differenziando invece la (2.3): $d\varphi = -\dfrac{y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy$ (2.9)

(genericamente: $\partial\varphi = \dfrac{\partial\varphi}{\partial x}dx + \dfrac{\partial\varphi}{\partial y}dy + \cancel{\dfrac{\partial\varphi}{\partial z}dz}$) e differenziando la (2.4):

$$-\sin\theta d\theta = -\frac{zx}{\rho^3}dx - \frac{zy}{\rho^3}dy + (\frac{1}{\rho} - \frac{z^2}{\rho^3})dz\,. \qquad (2.10)$$

(il $-\dfrac{1}{\sin\theta}$, poi portato a 1° membro, sarebbe la derivata dell'arccos)
Tenendo ora conto delle (2.1) e delle (2.8), (2.9), e (2.10), si ha:

$$\begin{cases} \dfrac{\partial\rho}{\partial x} = \sin\theta\cos\varphi & (2.11) \\ \dfrac{\partial\rho}{\partial y} = \sin\theta\sin\varphi & (2.12) \\ \dfrac{\partial\rho}{\partial z} = \cos\theta & (2.13) \end{cases} \qquad \begin{cases} \dfrac{\partial\varphi}{\partial x} = -\dfrac{\sin\varphi}{\rho\sin\theta} & (2.14) \\ \dfrac{\partial\varphi}{\partial y} = \dfrac{\cos\varphi}{\rho\sin\theta} & (2.15) \\ \dfrac{\partial\varphi}{\partial z} = 0 & (2.16) \end{cases}$$

$$\begin{cases} \dfrac{\partial\theta}{\partial x} = \dfrac{\cos\theta\cos\varphi}{\rho} & (2.17) \\ \dfrac{\partial\theta}{\partial y} = \dfrac{\cos\theta\sin\varphi}{\rho} & (2.18) \\ \dfrac{\partial\theta}{\partial z} = -\dfrac{\sin\theta}{\rho} & (2.19) \end{cases}$$

Infatti, riguardo il primo sistema, ad esempio la (2.11) sarebbe la (2.8) con dy=dz=0 e considerando che per ottenere x devo prima moltiplicare ρ per $\sin\theta$ per proiettarlo sul piano xy, ottenendo la proiezione ρ_p e poi tale proiezione va moltiplicata per $\cos\varphi$ per ottenere appunto x. In altre parole, la (2.8) con dy=dz=0 ci dice che $\dfrac{d\rho}{dx} = \dfrac{x}{\rho}$, che altro non è che $(\sin\theta\cos\varphi)$ per il discorso di

proiezioni appena fatto. Riguardo invece il secondo sistema, la (2.14) sarebbe la (2.9) con dy=0 e sempre tenendo conto delle proiezioni di cui sopra.
Per ultimo, sul terzo sistema, la (2.17) sarebbe la (2.10) con dy=dz=0 e ($z=\rho\cos\theta$) e ($x=\rho\sin\theta\cos\varphi$).
Ricordiamo, per ultimo, gradiente e laplaciano in coordinate sferiche:
per definizione, si ha: $gradV=\nabla V=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$; se ora usiamo le (2.5), (2.6) e (2.7) per esprimere le $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ e $\frac{\partial}{\partial z}$ e poi raccogliamo tutti i termini contenenti $\frac{\partial}{\partial\rho}$, otteniamo:

$$(\nabla)_\rho=\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\hat{i}+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}\hat{j}+\frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho}\hat{k}=\sin\theta\cos\varphi\frac{\partial}{\partial\rho}\hat{i}+\sin\theta\sin\varphi\frac{\partial}{\partial\rho}\hat{j}+\cos\theta\frac{\partial}{\partial\rho}\hat{k}=$$

$=\frac{\partial}{\partial\rho}(\hat{i}\sin\theta\cos\varphi+\hat{j}\sin\theta\sin\varphi+\hat{k}\cos\theta)=\hat{\rho}\frac{\partial}{\partial\rho}$. Analogamente, $(\nabla)_\theta=\frac{1}{\rho}\frac{\partial}{\partial\theta}\hat{\theta}$ e $(\nabla)_\varphi=\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$, da cui, riassumendo, le componenti del gradiente in coordinate polari sferiche:

$$(\nabla)_\rho=\hat{\rho}\frac{\partial}{\partial\rho}\quad,\quad(\nabla)_\theta=\frac{1}{\rho}\frac{\partial}{\partial\theta}\hat{\theta}\quad\text{e}\quad(\nabla)_\varphi=\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\quad.$$

Per giungere invece al laplaciano scalare, la definizione in coordinate cartesiane è:
$\nabla^2=\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ e per calcolare ciascuno di questi addendi, innanzitutto si deriveranno ulteriormente le (2.5), (2.6) e (2.7) e le si utilizzeranno nell'espressione qui sopra per il laplaciano, ottenendo:

$\boxed{\Delta=\frac{1}{\rho^2}(\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})}$. Ma noi non procederemo per questa via, in quanto essa è poco strutturata; bensì, utilizzeremo i metodi più generali che seguono.

3-DEDUZIONE COMPLETA DI GRAD, DIV, ROT E LAPLACIANO

Ricordiamo preliminarmente che, in coordinate cartesiane:
$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ e $d\vec{r}=(dx)\hat{i}+(dy)\hat{j}+(dz)\hat{k}$; poi, se voglio passare dalle coordinate cartesiane (x,y,z)-($\hat{i},\hat{j},\hat{k}$) ad un altro sistema di coordinate curvilinee generico (u,v,w)-($\hat{I},\hat{J},\hat{K}$), avrò delle equazioni di passaggio: x=f(u,v,w), y=g(u,v,w), z=h(u,v,w). Adesso:

$$d\vec{r}=\frac{\partial\vec{r}}{\partial u}du+\frac{\partial\vec{r}}{\partial v}dv+\frac{\partial\vec{r}}{\partial w}dw\ ;\tag{3.1}$$

ora, $\frac{\partial\vec{r}}{\partial u}$, $\frac{\partial\vec{r}}{\partial v}$ e $\frac{\partial\vec{r}}{\partial w}$ non hanno necessariamente modulo unitario, dunque, esprimendoli in funzione di versori: $\frac{\partial\vec{r}}{\partial u}=\left|\frac{\partial\vec{r}}{\partial u}\right|\hat{I}=h_u\hat{I}$, $\frac{\partial\vec{r}}{\partial v}=\left|\frac{\partial\vec{r}}{\partial v}\right|\hat{J}=h_v\hat{J}$ e $\frac{\partial\vec{r}}{\partial w}=\left|\frac{\partial\vec{r}}{\partial w}\right|\hat{K}=h_w\hat{K}$ da cui:

$$d\vec{r}=h_udu\hat{I}+h_vdv\hat{J}+h_wdw\hat{K}\tag{3.2}$$

e ancora: $d\vec{r}\cdot d\vec{r}=ds^2=h_u^2du^2+h_v^2dv^2+h_w^2dw^2$ (3.3)
e riguardo il volume infinitesimo (che è l'area di un parallelogramma per un'altezza), si ha:

$$dV_{ol}=(h_vdv\hat{J}\times h_wdw\hat{K})\cdot(h_udu\hat{I})=h_uh_vh_wdudvdw\ .\tag{3.4}$$

-<u>Gradiente</u>: nel nuovo sistema di riferimento ($\hat{I}, \hat{J}, \hat{K}$) esso avrà l'espressione:

$$\nabla V = (\nabla V)_u \hat{I} + (\nabla V)_v \hat{J} + (\nabla V)_w \hat{K} \ , \tag{3.5}$$

con i vari $((\nabla V)_u, (\nabla V)_v, (\nabla V)_w)$ da determinare.

Adesso, così come in coordinate cartesiane si ha:

$$\nabla V \cdot d\vec{r} = (\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k})[(dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}] = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV \ ,$$

così, nel nuovo sistema di riferimento, si avrà:

$$dV = \nabla V \cdot d\vec{r} = [(\nabla V)_u \hat{I} + (\nabla V)_v \hat{J} + (\nabla V)_w \hat{K}] \cdot (h_u du \hat{I} + h_v dv \hat{J} + h_w dw \hat{K}) =$$

$$= (\nabla V)_u h_u du + (\nabla V)_v h_v dv + (\nabla V)_w h_w dw \ , \tag{3.6}$$

ma essendo che, per il calcolo differenziale, si ha anche banalmente che:

$$dV = \frac{\partial V}{\partial u}du + \frac{\partial V}{\partial v}dv + \frac{\partial V}{\partial w}dw, \tag{3.7}$$

allora, dalle (3.6) e (3.7) si ha: $(\nabla V)_u = \frac{1}{h_u}\frac{\partial V}{\partial u}$, $(\nabla V)_v = \frac{1}{h_v}\frac{\partial V}{\partial v}$, $(\nabla V)_w = \frac{1}{h_w}\frac{\partial V}{\partial w}$

e allora per la (3.5) si ha il nostro gradiente in coordinate generiche:

$$\boxed{\nabla V = \frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I} + \frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J} + \frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K}} \tag{3.8}$$

-<u>Divergenza</u>:

$$\vec{F} = F_u \hat{I} + F_v \hat{J} + F_w \hat{K} = F_u(\hat{J} \times \hat{K}) + F_v(\hat{K} \times \hat{I}) + F_w(\hat{I} \times \hat{J}) \ , \tag{3.9}$$

ma la (3.8) applicata ad u (u piccolo) dà:

$\nabla u = \frac{1}{h_u}\frac{\partial u}{\partial u}\hat{I} + \frac{1}{h_v}\frac{\partial u}{\partial v}\hat{J} + \frac{1}{h_w}\frac{\partial u}{\partial w}\hat{K} = \frac{1}{h_u}\frac{\partial u}{\partial u}\hat{I} + 0 + 0 = \frac{1}{h_u}\hat{I}$ e similmente: $\nabla v = \frac{1}{h_v}\hat{J}$, $\nabla w = \frac{1}{h_w}\hat{K}$,

che riassumiamo: $\nabla u = \frac{1}{h_u}\hat{I}$, $\nabla v = \frac{1}{h_v}\hat{J}$ e $\nabla w = \frac{1}{h_w}\hat{K}$ (3.10)

grazie alle quali la (3.8) può essere così riscritta:

$$\nabla V = \nabla u \frac{\partial V}{\partial u} + \nabla v \frac{\partial V}{\partial v} + \nabla w \frac{\partial V}{\partial w} \tag{3.11}$$

Riprendendo ora la (3.9), si ha:

$$\vec{F} = F_u(\hat{J} \times \hat{K}) + F_v(\hat{K} \times \hat{I}) + F_w(\hat{I} \times \hat{J}) = F_u h_v h_w (\nabla v \times \nabla w) + F_v h_w h_u (\nabla w \times \nabla u) + F_w h_u h_v (\nabla u \times \nabla v) \tag{3.12}$$

ed applicando la divergenza alla (3.12), si ha:

$$\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w) + F_v h_w h_u (\nabla w \times \nabla u) + F_w h_u h_v (\nabla u \times \nabla v)] = \vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] +$$

$$+ \vec{\nabla} \cdot [F_v h_w h_u (\nabla w \times \nabla u)] + \vec{\nabla} \cdot [F_w h_u h_v (\nabla u \times \nabla v)] \tag{3.13}$$

Valutiamo ora le singole componenti della (3.13):

$$\begin{aligned} &\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] \\ &\vec{\nabla} \cdot [F_v h_w h_u (\nabla w \times \nabla u)] \\ &\vec{\nabla} \cdot [F_w h_u h_v (\nabla u \times \nabla v)] \end{aligned} \tag{3.14}$$

si ha dalla Subappendice 1 che vale l'identità vettoriale: $\vec{\nabla} \cdot (V\vec{A}) = \nabla V \cdot \vec{A} + V(\vec{\nabla} \cdot \vec{A})$, dunque:

$$\begin{aligned} &\vec{\nabla} \cdot [F_u h_v h_w (\nabla v \times \nabla w)] = \nabla(F_u h_v h_w) \cdot (\nabla v \times \nabla w) + (F_u h_v h_w) \vec{\nabla} \cdot (\nabla v \times \nabla w) \\ &\vec{\nabla} \cdot [F_v h_w h_u (\nabla w \times \nabla u)] = \nabla(F_v h_w h_u) \cdot (\nabla w \times \nabla u) + (F_v h_w h_u) \vec{\nabla} \cdot (\nabla w \times \nabla u) \end{aligned} \tag{3.15}$$

$\vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)]=\nabla(F_w h_u h_v)\cdot(\nabla u\times\nabla v)+(F_w h_u h_v)\vec{\nabla}\cdot(\nabla u\times\nabla v)$

e sempre dalla Subappendice 1 si ha che vale l'identità vettoriale: $\vec{\nabla}\cdot(\vec{A}\times\vec{B})=(\vec{\nabla}\times\vec{A})\cdot\vec{B}-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$ e nel nostro caso $\vec{A}$ e $\vec{B}$ sono i vari ∇u , ∇v e ∇w , che sono dei gradienti, non dimentichiamolo, ed il rotore di un gradiente è nullo, da cui, nel nostro caso specifico, $\vec{\nabla}\cdot(\vec{A}\times\vec{B})=0$, quindi:

$$\begin{aligned}
&\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]=\nabla(F_u h_v h_w)\cdot(\nabla v\times\nabla w)\\
&\vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)]=\nabla(F_v h_w h_u)\cdot(\nabla w\times\nabla u)\\
&\vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)]=\nabla(F_w h_u h_v)\cdot(\nabla u\times\nabla v)
\end{aligned}\tag{3.16}$$

Valutiamo ad esempio il termine $\nabla(F_u h_v h_w)$ nella prima delle (3.16), avvalendoci della (3.11):

$$\nabla(F_u h_v h_w)=\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}+\nabla v\frac{\partial(F_u h_v h_w)}{\partial v}+\nabla w\frac{\partial(F_u h_v h_w)}{\partial w},\tag{3.17}$$

da cui, sempre la prima delle (3.16) diventa: $\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]=(\nabla v\times\nabla w)\cdot\nabla(F_u h_v h_w)=$ $=(\nabla v\times\nabla w)\cdot[\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}+\nabla v\frac{\partial(F_u h_v h_w)}{\partial v}+\nabla w\frac{\partial(F_u h_v h_w)}{\partial w}]=(\nabla v\times\nabla w)\cdot[\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}]$, dove il solo termine superstite della (3.17) è stato $[\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}]$, poiché l'unico prodotto a tre nabla che non si annulla è appunto il primo: $(\nabla v\times\nabla w)\cdot\nabla u\neq 0$, mentre gli altri due sono nulli: $(\nabla v\times\nabla w)\cdot\nabla v=(\nabla v\times\nabla w)\cdot\nabla w=0$, in quanto sfociano in prodotti scalari tra vettori ortogonali, in quanto, per le (3.10), gli $(\nabla v,\nabla w,\nabla u)$ ricalcano gli $(\hat{I},\hat{J},\hat{K})$.

Perciò, le (3.16) divengono:

$$\begin{aligned}
&\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]=(\nabla v\times\nabla w)\cdot[\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}]\qquad\text{(e similmente)}\\
&\vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)]=(\nabla w\times\nabla u)\cdot[\nabla v\frac{\partial(F_v h_w h_u)}{\partial v}]\\
&\vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)]=(\nabla u\times\nabla v)\cdot[\nabla w\frac{\partial(F_w h_u h_v)}{\partial w}]
\end{aligned}\tag{3.18}$$

Sviluppiamo ora la prima delle (3.18):

$\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]=(\nabla v\times\nabla w)\cdot[\nabla u\frac{\partial(F_u h_v h_w)}{\partial u}]=[(\frac{1}{h_v}\hat{J})\times(\frac{1}{h_w}\hat{K})]\cdot[\frac{1}{h_u}\hat{I}\frac{\partial(F_u h_v h_w)}{\partial u}]=$ $=\frac{1}{h_u h_v h_w}[(\hat{J}\times\hat{K})\cdot\hat{I}]\frac{\partial(F_u h_v h_w)}{\partial u}=\frac{1}{h_u h_v h_w}[\hat{I}\cdot\hat{I}]\frac{\partial(F_u h_v h_w)}{\partial u}=\frac{1}{h_u h_v h_w}\frac{\partial(F_u h_v h_w)}{\partial u}$, sicché le (3.18) divengono:

$$\begin{aligned}
&\vec{\nabla}\cdot[F_u h_v h_w(\nabla v\times\nabla w)]=\frac{1}{h_u h_v h_w}\frac{\partial(F_u h_v h_w)}{\partial u}\qquad\text{(e similmente)}\\
&\vec{\nabla}\cdot[F_v h_w h_u(\nabla w\times\nabla u)]=\frac{1}{h_u h_v h_w}\frac{\partial(F_v h_w h_u)}{\partial v}\\
&\vec{\nabla}\cdot[F_w h_u h_v(\nabla u\times\nabla v)]=\frac{1}{h_u h_v h_w}\frac{\partial(F_w h_u h_v)}{\partial w}
\end{aligned}\tag{3.19}$$

In totale, la (3.13) fornisce finalmente:

$$\boxed{\vec{\nabla}\cdot\vec{F}=\frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u}+\frac{\partial(F_v h_w h_u)}{\partial v}+\frac{\partial(F_w h_u h_v)}{\partial w}]}\tag{3.20}$$

-Rotore:

$\vec{F} = F_u\hat{I} + F_v\hat{J} + F_w\hat{K} = h_uF_u\nabla u + h_vF_v\nabla v + h_wF_w\nabla w$

$$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times(h_uF_u\nabla u + h_vF_v\nabla v + h_wF_w\nabla w) = \vec{\nabla}\times[(h_uF_u)\nabla u] + \vec{\nabla}\times[(h_vF_v)\nabla v] + \vec{\nabla}\times[(h_wF_w)\nabla w] \quad (3.21)$$

Ricordiamo ora l'identità vettoriale riportata in Subappendice 1: $\vec{\nabla}\times(V\vec{B}) = V(\vec{\nabla}\times\vec{B}) + \nabla V\times\vec{B}$ e se in questa $\vec{B}$ scaturisce da un gradiente di un altro scalare A ($\vec{B} = \nabla A$), allora:

$$\vec{\nabla}\times(V\nabla A) = V(\vec{\nabla}\times\nabla A) + \nabla V\times\nabla A \ . \quad (3.22)$$

Scriviamo ora le tre componenti della (3.21):

$$\begin{aligned}&\vec{\nabla}\times[(h_uF_u)\nabla u]\\&\vec{\nabla}\times[(h_vF_v)\nabla v]\\&\vec{\nabla}\times[(h_wF_w)\nabla w]\end{aligned} \quad (3.23)$$

e sviluppiamole secondo la (3.22):

$$\begin{aligned}&\vec{\nabla}\times[(h_uF_u)\nabla u] = \nabla(h_uF_u)\times\nabla u + (h_uF_u)\vec{\nabla}\times\nabla u = \nabla(h_uF_u)\times\nabla u\\&\vec{\nabla}\times[(h_vF_v)\nabla v] = \nabla(h_vF_{vu})\times\nabla v + (h_vF_v)\vec{\nabla}\times\nabla v = \nabla(h_vF_v)\times\nabla v\\&\vec{\nabla}\times[(h_wF_w)\nabla w] = \nabla(h_wF_w)\times\nabla w + (h_wF_w)\vec{\nabla}\times\nabla w = \nabla(h_wF_w)\times\nabla w\end{aligned} \quad (3.24)$$

(tutti i secondi termini infatti sono nulli, in quanto rotori di gradienti)

ma per la (3.11), ossia ($\nabla V = \nabla u\frac{\partial V}{\partial u} + \nabla v\frac{\partial V}{\partial v} + \nabla w\frac{\partial V}{\partial w}$), segue che ad esempio la prima delle (3.24) diviene: $\nabla(h_uF_u)\times\nabla u = [\frac{\partial(h_uF_u)}{\partial u}\nabla u + \frac{\partial(h_uF_u)}{\partial v}\nabla v + \frac{\partial(h_uF_u)}{\partial w}\nabla w]\times\nabla u =$

$$= 0 + \frac{\partial(h_uF_u)}{\partial v}(\nabla v\times\nabla u) + \frac{\partial(h_uF_u)}{\partial w}(\nabla w\times\nabla u) = \frac{\partial(h_uF_u)}{\partial v}(\frac{\hat{J}}{h_v}\times\frac{\hat{I}}{h_u}) + \frac{\partial(h_uF_u)}{\partial w}(\frac{\hat{K}}{h_w}\times\frac{\hat{I}}{h_u}) =$$

$$= \frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}(\hat{J}\times\hat{I}) + \frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}(\hat{K}\times\hat{I}) = -\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}\hat{K} + \frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}\hat{J}$$

e discorso simile per le altre due delle (3.24):

$$\nabla(h_vF_v)\times\nabla v = -\frac{1}{h_vh_w}\frac{\partial(h_vF_v)}{\partial w}\hat{I} + \frac{1}{h_uh_v}\frac{\partial(h_vF_v)}{\partial u}\hat{K}$$

$$\nabla(h_wF_w)\times\nabla w = -\frac{1}{h_uh_w}\frac{\partial(h_wF_w)}{\partial u}\hat{J} + \frac{1}{h_vh_w}\frac{\partial(h_wF_w)}{\partial v}\hat{I}$$

e dunque le (3.24) divengono:

$$\begin{aligned}&\vec{\nabla}\times[(h_uF_u)\nabla u] = -\frac{1}{h_uh_v}\frac{\partial(h_uF_u)}{\partial v}\hat{K} + \frac{1}{h_uh_w}\frac{\partial(h_uF_u)}{\partial w}\hat{J}\\&\vec{\nabla}\times[(h_vF_v)\nabla v] = -\frac{1}{h_vh_w}\frac{\partial(h_vF_v)}{\partial w}\hat{I} + \frac{1}{h_uh_v}\frac{\partial(h_vF_v)}{\partial u}\hat{K}\\&\vec{\nabla}\times[(h_wF_w)\nabla w] = -\frac{1}{h_uh_w}\frac{\partial(h_wF_w)}{\partial u}\hat{J} + \frac{1}{h_vh_w}\frac{\partial(h_wF_w)}{\partial v}\hat{I}\end{aligned} \quad (3.25)$$

((((Parentesi: riguardo la comparsa dei segni "-", essi si presentano notoriamente quando si inverte l'ordine naturale:

1,2,3,1,2,3,1,2,3,……..

x,y,z,x,y,z,x,y,z,……..

u,v,w,u,v,w,u,v,w,……

$\hat{i},\hat{j},\hat{k},\hat{i},\hat{j},\hat{k},\hat{i},\hat{j},\hat{k}$,........

$\hat{I},\hat{J},\hat{K},\hat{I},\hat{J},\hat{K},\hat{I},\hat{J},\hat{K}$,........

Cioè se si passa da 1 a 2, o da 2 a 3, o da 3 ad 1 e così via, si ha il “+”, mentre se si passa, ad esempio, da 1 a 3, saltando il 2, o da 2 ad 1, saltando il 3, o da 3 a 2, saltando l’1, allora si eredita il “-“.))))

Riprendendo il discorso, grazie alle (3.25), la (3.21) fornisce:

$$\vec{\nabla}\times\vec{F}=[-\frac{1}{h_u h_v}\frac{\partial(h_u F_u)}{\partial v}\hat{K}+\frac{1}{h_u h_w}\frac{\partial(h_u F_u)}{\partial w}\hat{J}]+[-\frac{1}{h_v h_w}\frac{\partial(h_v F_v)}{\partial w}\hat{I}+\frac{1}{h_u h_v}\frac{\partial(h_v F_v)}{\partial u}\hat{K}]+$$

$$+[-\frac{1}{h_u h_w}\frac{\partial(h_w F_w)}{\partial u}\hat{J}+\frac{1}{h_v h_w}\frac{\partial(h_w F_w)}{\partial v}\hat{I}]=\frac{1}{h_v h_w}[\frac{\partial(h_w F_w)}{\partial v}-\frac{\partial(h_v F_v)}{\partial w}]\hat{I}+\frac{1}{h_u h_w}[\frac{\partial(h_u F_u)}{\partial w}-\frac{\partial(h_w F_w)}{\partial u}]\hat{J}+$$

$$+\frac{1}{h_u h_v}[\frac{\partial(h_v F_v)}{\partial u}-\frac{\partial(h_u F_u)}{\partial v}]\hat{K}=\boxed{\frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K}\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix}=\vec{\nabla}\times\vec{F}} \tag{3.26}$$

-<u>Laplaciano</u>:

Usando la (3.8): $(\nabla V=\frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I}+\frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J}+\frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K})$ nella (3.20):

$(\vec{\nabla}\cdot\vec{F}=\frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u}+\frac{\partial(F_v h_w h_u)}{\partial v}+\frac{\partial(F_w h_u h_v)}{\partial w}])$, si ottiene il Laplaciano:

$$\boxed{\vec{\nabla}\cdot\nabla V=\nabla^2 V=\Delta V=\frac{1}{h_u h_v h_w}[\frac{\partial}{\partial u}(\frac{h_v h_w}{h_u}\frac{\partial V}{\partial u})+\frac{\partial}{\partial v}(\frac{h_u h_w}{h_v}\frac{\partial V}{\partial v})+\frac{\partial}{\partial w}(\frac{h_u h_v}{h_w}\frac{\partial V}{\partial w})]} \tag{3.27}$$

-<u>Riassunto</u>:

Raduniamo le espressioni ottenute per tutti i nostri operatori:

$$\boxed{\begin{aligned}
&\nabla V=\frac{1}{h_u}\frac{\partial V}{\partial u}\hat{I}+\frac{1}{h_v}\frac{\partial V}{\partial v}\hat{J}+\frac{1}{h_w}\frac{\partial V}{\partial w}\hat{K}\\
&\vec{\nabla}\cdot\vec{F}=\frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u}+\frac{\partial(F_v h_w h_u)}{\partial v}+\frac{\partial(F_w h_u h_v)}{\partial w}]\\
&\vec{\nabla}\times\vec{F}=\frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K}\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix}\\
&\nabla^2 V=\frac{1}{h_u h_v h_w}[\frac{\partial}{\partial u}(\frac{h_v h_w}{h_u}\frac{\partial V}{\partial u})+\frac{\partial}{\partial v}(\frac{h_u h_w}{h_v}\frac{\partial V}{\partial v})+\frac{\partial}{\partial w}(\frac{h_u h_v}{h_w}\frac{\partial V}{\partial w})]\\
&d\vec{r}=h_u du\hat{I}+h_v dv\hat{J}+h_w dw\hat{K}
\end{aligned}}$$

-<u>caso delle coordinate cartesiane</u>: in esse sappiamo che: $d\vec{r}=(dx)\hat{i}+(dy)\hat{j}+(dz)\hat{k}$, da cui si evince che: (u,v,w)→(x,y,z) , $(\hat{I},\hat{J},\hat{K})\to(\hat{i},\hat{j},\hat{k})$ e $(h_u h_v h_w)=(h_x h_y h_z)=(1{,}1{,}1)$ e con tali tre sostituzioni nelle equazioni qui sopra otteniamo proprio le (1.3), (1.4), (1.5) ed (1.6):

$$gradV=\nabla V=(\frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k})$$

$$div\cdot\vec{F}=\vec{\nabla}\cdot\vec{F}=(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z})$$

$$rot\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}=\hat{i}(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z})+\hat{j}(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x})+\hat{k}(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y})$$

$$\nabla^2 V=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})V$$

-<u>caso delle coordinate sferiche</u>: in esse sappiamo che: $d\vec{r}=(d\rho)\hat{\rho}+(\rho d\theta)\hat{\theta}+(\rho\sin\theta d\varphi)\hat{\varphi}$, da cui si evince che: (u,v,w)→(ρ,θ,φ) , $(\hat{I},\hat{J},\hat{K})\to(\hat{\rho},\hat{\theta},\hat{\varphi})$ e $(h_u h_v h_w)=(h_\rho h_\theta h_\varphi)=(1,\rho,\rho\sin\theta)$ e con tali tre sostituzioni nelle equazioni generiche più sopra, otteniamo proprio le (1.9), (1.10), (1.11) ed (1.12):

$$gradV=\nabla V=(\frac{\partial V}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{\rho\sin\theta}\frac{\partial V}{\partial\varphi}\hat{\varphi})$$

$$div\cdot\vec{F}=\vec{\nabla}\cdot\vec{F}=\frac{1}{\rho^2\sin\theta}[\frac{\partial}{\partial\rho}(\rho^2\sin\theta F_\rho)+\frac{\partial}{\partial\theta}(\rho\sin\theta F_\theta)+\frac{\partial}{\partial\varphi}(\rho F_\varphi)]$$

$$rot\vec{F}=\vec{\nabla}\times\vec{F}=\frac{1}{\rho^2\sin\theta}\begin{vmatrix}\hat{\rho} & \rho\hat{\theta} & \rho\sin\theta\hat{\varphi}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ F_\rho & \rho F_\theta & \rho\sin\theta F_\varphi\end{vmatrix}=$$

$$=\frac{1}{\rho^2\sin\theta}[\frac{\partial(\rho\sin\theta F_\varphi)}{\partial\theta}-\frac{\partial(\rho F_\theta)}{\partial\varphi}]\hat{\rho}+\frac{1}{\rho\sin\theta}[\frac{\partial F_\rho}{\partial\varphi}-\frac{\partial(\rho\sin\theta F_\varphi)}{\partial\rho}]\hat{\theta}+\frac{1}{\rho}[\frac{\partial(\rho F_\theta)}{\partial\rho}-\frac{\partial F_\rho}{\partial\theta}]\hat{\varphi}$$

$$\nabla^2 V=\frac{1}{\rho^2\sin\theta}[\frac{\partial}{\partial\rho}(\rho^2\sin\theta\frac{\partial V}{\partial\rho})+\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{\partial}{\partial\varphi}(\frac{1}{\sin\theta}\frac{\partial V}{\partial\varphi})]=$$

$$=\frac{\partial^2 V}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{\cot\theta}{\rho^2}\frac{\partial V}{\partial\theta}+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

-<u>caso delle coordinate cilindriche</u>: in esse sappiamo che: $d\vec{r}=(d\rho)\hat{\rho}+(\rho d\varphi)\hat{\varphi}+(dz)\hat{z}$, da cui si evince che: (u,v,w)→(ρ,φ,z) , $(\hat{I},\hat{J},\hat{K})\to(\hat{\rho},\hat{\varphi},\hat{z})$ e $(h_u h_v h_w)=(h_\rho h_\varphi h_z)=(1,\rho,1)$ e con tali tre sostituzioni nelle equazioni generiche più sopra, otteniamo proprio le (1.14), (1.15), (1.16) e (1.17):

$$gradV=\nabla V=(\frac{\partial V}{\partial \rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{\varphi}+\frac{\partial V}{\partial z}\hat{z})$$
$$div\cdot\vec{F}=\vec{\nabla}\cdot\vec{F}=\frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho F_\rho)+\frac{\partial}{\partial \varphi}(F_\varphi)+\frac{\partial}{\partial z}(\rho F_z)]$$
$$rot\vec{F}=\vec{\nabla}\times\vec{F}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\varphi} & \hat{z}\\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z}\\ F_\rho & \rho F_\varphi & F_z\end{vmatrix}=\frac{1}{\rho}[\frac{\partial (F_z)}{\partial \varphi}-\frac{\partial(\rho F_\varphi)}{\partial z}]\hat{\rho}+[\frac{\partial F_\rho}{\partial z}-\frac{\partial F_z}{\partial \rho}]\hat{\varphi}+\frac{1}{\rho}[\frac{\partial(\rho F_\varphi)}{\partial \rho}-\frac{\partial F_\rho}{\partial \varphi}]\hat{\varphi}$$
$$\nabla^2 V=\frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho\frac{\partial V}{\partial \rho})+\frac{\partial}{\partial \varphi}(\frac{1}{\rho}\frac{\partial V}{\partial \varphi})+\frac{\partial}{\partial z}(\rho\frac{\partial V}{\partial z})=$$
$$=\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2}+\frac{\partial^2 V}{\partial z^2}$$

4-DEDUZIONE ALTERNATIVA SUPPLEMENTARE

-<u>Calcolo della divergenza sulla base del Teorema della Divergenza</u>:

Valutiamo preliminarmente i flussi delle componenti del vettore $\vec{F}=F_u\hat{I}+F_v\hat{J}+F_w\hat{K}$ (espresso nel suo sistema curvilineo $(u,v,w)--(\hat{I},\hat{J},\hat{K})$) attraverso le corrispondenti superfici:

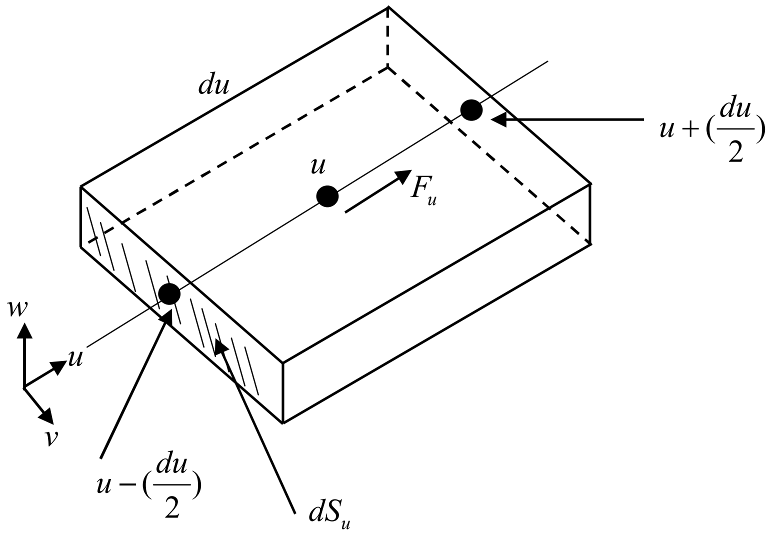


Fig. 4.1.

Con riferimento alla Fig. 4.1, valutiamo il flusso $d\Phi_u$ di F_u attraverso dS_u:

$dS_u=h_v dv h_w dw$ (vedere la (3.2))

$$d\Phi_u=F_u dS_u\big|_{in\_[u+\frac{du}{2}],v,w}-F_u dS_u\big|_{in\_[u-\frac{du}{2}],v,w}=F_u h_v h_w dvdw\big|_{in\_[u+\frac{du}{2}],v,w}-F_u h_v h_w dvdw\big|_{in\_[u-\frac{du}{2}],v,w}=$$

(moltiplicando sopra e sotto per du) $=d\Phi_u=\frac{\partial(F_u h_v h_w)}{\partial u}dudvdw$

e relazioni analoghe per le altre due componenti, ossia, in totale:

$$\boxed{\begin{aligned}d\Phi_u&=\frac{\partial(F_u h_v h_w)}{\partial u}dudvdw\\ d\Phi_v&=\frac{\partial(F_v h_u h_w)}{\partial v}dudvdw\\ d\Phi_w&=\frac{\partial(F_w h_u h_v)}{\partial w}dudvdw\end{aligned}}\qquad (4.1)$$

Adesso, per il Teorema della Divergenza esposto in Subappendice 2, si ha:
$\int_V div\vec{F}\cdot dV = \int_S \vec{F}\cdot d\vec{S}$ $\quad (d\vec{S} = \hat{n}dS)$

Considerando ora il valor medio di $div\vec{F}$, ossia $\overline{div\vec{F}}$, possiamo scrivere:

$\int_V div\vec{F}\cdot dV = \overline{div\vec{F}}\int_V dV = \overline{div\vec{F}}\Delta V = \int_S \vec{F}\cdot d\vec{S}$, da cui: $\overline{div\vec{F}} = \frac{\int_S \vec{F}\cdot d\vec{S}}{\Delta V}$; se adesso effettuiamo il limite per ΔV→0, allora $\overline{div\vec{F}} \to div\vec{F}$, da cui:

$$div\vec{F} = \lim_{\Delta V\to 0}\frac{\int_S \vec{F}\cdot d\vec{S}}{\Delta V} \tag{4.2}$$

ossia ancora: $div\vec{F} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\lim_{\Delta V\to 0}\int_S \vec{F}\cdot d\vec{S} \equiv \frac{1}{dV}\lim_{\Delta V\to 0}\int_S \vec{F}\cdot d\vec{S} \equiv \frac{1}{dV}(\vec{F}\cdot d\vec{S}) \equiv \frac{1}{dV}d\Phi$ (4.3)

poiché se il volume si riduce ad un infinitesimo, la sommatoria rappresentata dall'integrale (di infinitesimi) è un infinitesimo solo.
E ricordiamo altresì che, per la (3.4), si ha: $dV = h_u h_v h_w du dv dw$.

Allora: $div\vec{F} = \frac{1}{dV}d\Phi = \frac{1}{dV}(d\Phi_u + d\Phi_v + d\Phi_w) = \frac{1}{h_u h_v h_w dudvdw}[\frac{\partial(F_u h_v h_w)}{\partial u}dudvdw +$
$+\frac{\partial(F_v h_u h_w)}{\partial v}dudvdw + \frac{\partial(F_w h_u h_v)}{\partial w}dudvdw] = \frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_u h_w)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial w}]$,
ossia, riscrivendola:

$$\boxed{div\vec{F} = \frac{1}{h_u h_v h_w}[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_u h_w)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial w}]} \tag{4.4}$$

che è esattamente la (3.20) già ottenuta.

-<u>Calcolo del rotore sulla base del Teorema del Rotore</u>:

Come dimostrato in Subappendice 2, vale il seguente Teorema del Rotore:

$\oint_l \vec{F}\cdot d\vec{l} = \int_S rot\vec{F}\cdot d\vec{S}$ e facendo ora un semplice ragionamento analogo a quello che ci ha condotti alla (4.2), possiamo scrivere che:

$$rot\vec{F} = \lim_{\Delta S\to 0}\frac{\oint_l \vec{F}\cdot d\vec{l}}{\Delta S} \tag{4.5}$$

non solo; così come siamo giunti alla (4.3), anche qui possiamo scrivere che:

$$rot\vec{F} = \lim_{\Delta S\to 0}\frac{\oint_l \vec{F}\cdot d\vec{l}}{\Delta S} = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\lim_{\Delta S\to 0}\oint_l \vec{F}\cdot d\vec{l} \equiv \frac{1}{dS}\overrightarrow{(\vec{F}\cdot d\vec{l})} \tag{4.6}$$

poiché se la superficie ΔS (delineata, nel suo contorno, da $\Delta\vec{l}$) si riduce a dS (delineata da $d\vec{l}$), allora la sommatoria rappresentata dall'integrale (di infinitesimi) è un infinitesimo solo.
Valutiamo allora la quantità $(\vec{F}\cdot d\vec{l})$ del vettore $\vec{F} = F_u\hat{I} + F_v\hat{J} + F_w\hat{K}$ (espresso nel suo sistema curvilineo $(u,v,w) - -(\hat{I},\hat{J},\hat{K})$) attraverso il corrispondente percorso infinitesimo chiuso.

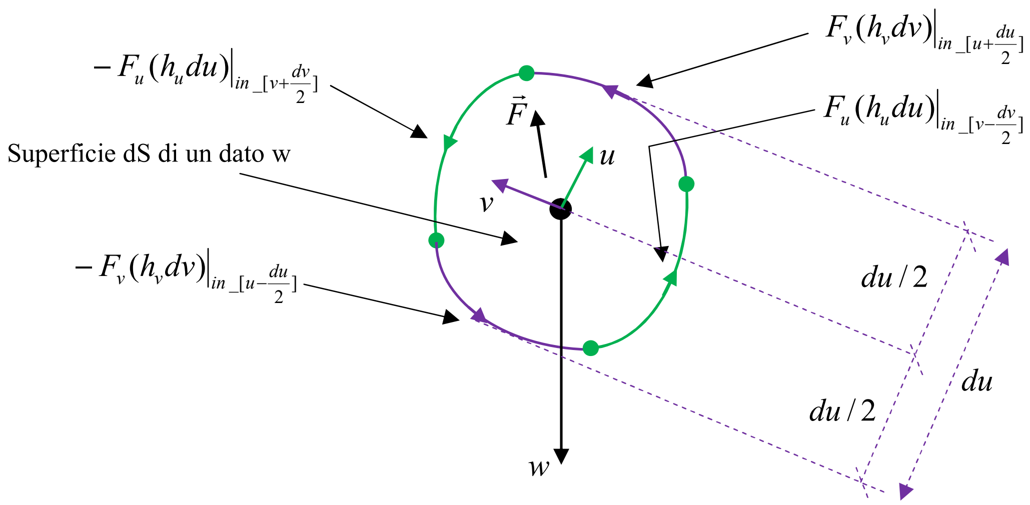


Fig. 4.2.

Per la (3.2), si ha che $d\vec{r} = (h_u du)\hat{I} + (h_v dv)\hat{J} + (h_w dw)\hat{K}$, dunque i vari percorsi valgono $(h_u du),(h_v dv),(h_w dw)$, mentre i contributi di superficie sono ovviamente:
$(h_u h_v du dv),(h_u h_w du dw),(h_v h_w dv dw)$.

Con riferimento alla Fig. 4.2, valutiamo su un dato w il contributo $(\vec{F}\cdot d\vec{l})_w$:

$$(\vec{F}\cdot d\vec{l})_w = [F_v(h_v dv)\big|_{in\_[u+\frac{du}{2}]} - F_v(h_v dv)\big|_{in\_[u-\frac{du}{2}]}] - [F_u(h_u du)\big|_{in\_[v+\frac{dv}{2}]} - F_u(h_u du)\big|_{in\_[v-\frac{dv}{2}]}] =$$
$$= [\partial(F_v h_v)dv] - [\partial(F_u h_u)du] = \partial(F_v h_v)dv - \partial(F_u h_u)du \; .$$

E calcolo simile per gli altri due assi, da cui, in totale:

$$\begin{aligned}
(\vec{F}\cdot d\vec{l})_u &= \partial(F_w h_w)dw - \partial(F_v h_v)dv \\
(\vec{F}\cdot d\vec{l})_v &= \partial(F_u h_u)du - \partial(F_w h_w)dw \\
(\vec{F}\cdot d\vec{l})_w &= \partial(F_v h_v)dv - \partial(F_u h_u)du
\end{aligned} \tag{4.7}$$

e allora, per la (4.6), si ha:

$$rot\vec{F} = \frac{1}{dS}\overrightarrow{(\vec{F}\cdot d\vec{l})} = \frac{1}{h_v h_w dv dw}(\vec{F}\cdot d\vec{l})_u\hat{I} + \frac{1}{h_u h_w du dw}(\vec{F}\cdot d\vec{l})_v\hat{J} + \frac{1}{h_u h_v du dv}(\vec{F}\cdot d\vec{l})_w\hat{K} =$$
$$= \frac{1}{h_v h_w dv dw}[\partial(F_w h_w)dw - \partial(F_v h_v)dv]\hat{I} + \frac{1}{h_u h_w du dw}[\partial(F_u h_u)du - \partial(F_w h_w)dw]\hat{J} +$$
$$+ \frac{1}{h_u h_v du dv}[\partial(F_v h_v)dv - \partial(F_u h_u)du]\hat{K} =$$
$$= \frac{1}{h_v h_w}[\frac{\partial(h_w F_w)}{\partial v} - \frac{\partial(h_v F_v)}{\partial w}]\hat{I} + \frac{1}{h_u h_w}[\frac{\partial(h_u F_u)}{\partial w} - \frac{\partial(h_w F_w)}{\partial u}]\hat{J} + \frac{1}{h_u h_v}[\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v}]\hat{K} =$$
$$= \frac{1}{h_u h_v h_w}[\frac{\partial(h_w F_w)}{\partial v} - \frac{\partial(h_v F_v)}{\partial w}]h_u\hat{I} + \frac{1}{h_u h_v h_w}[\frac{\partial(h_u F_u)}{\partial w} - \frac{\partial(h_w F_w)}{\partial u}]h_v\hat{J} + \frac{1}{h_u h_v h_w}[\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v}]h_w\hat{K} =$$

$$\boxed{= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{I} & h_v\hat{J} & h_w\hat{K} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix} = rot\vec{F} = \vec{\nabla}\times\vec{F}}$$

che è esattamente la (3.26) già ottenuta.

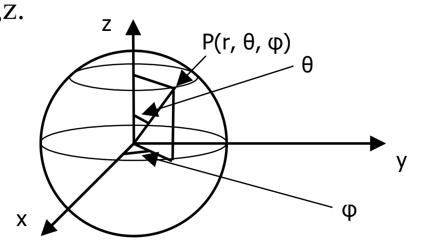
5-L'IMPORTANZA DELLE COORDINATE SFERICHE E CILINDRICHE

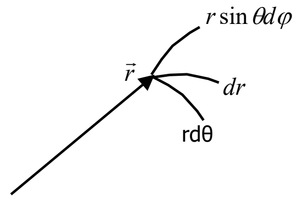
-A-COORDINATE SFERICHE

Generalità

Lavoreremo in coordinate polari sferiche (r,θ,φ) poiché così le cose si semplificano di molto, rispetto alle cartesiane x,y,z.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$





(φ varia tra 0 e 2π, mentre θ varia tra 0 e π).
Infatti, come esempio, sul piano polare l'equazione del cerchio centrato nell'origine è $r = R$ (indipendentemente da θ e φ), mentre in coordinate cartesiane si parte dalla equazione implicita $x^2 + y^2 = R^2$, da cui, per il primo quadrante xy, si ha:

$y = \sqrt{R^2 - x^2}$ (ben più complicata) (*)

nonchè, per calcolare l'area del cerchio, in coordinate polari si considera la coroncina di spessore dr a distanza r dal centro, la cui area è $dA = 2\pi r \cdot dr$ e si integra tra 0 ed R, ottenendo:

$A = 2\pi\int_0^R r\cdot dr = \pi R^2$, mentre, lavorando in coordinate cartesiane, dovrei integrare la (*) tra 0 ed R per avere un quarto di A (area sottesa), da cui:

$A = 4\int_0^R \sqrt{R^2 - x^2}\,dx = 4\left|\frac{x\sqrt{R^2 - x^2}}{2} + \frac{R^2}{2}\arcsin\frac{x}{R}\right|_0^R = \pi R^2$, (lavoro leggermente più complicato).

Il momento angolare nell'atomo

Sappiamo che $\vec{p} = -i\hbar\vec{\nabla}$ e $E = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\Delta$. Ora , riguardo il momento angolare L=mvr sappiamo che se una massa puntiforme m orbita a distanza r da un centro e lo fa con velocità v, si ha (p=mv):

$\vec{L} = \vec{r}\times\vec{p}$, $L = r\times p = -i\hbar r\times\nabla$ (prodotto vettoriale) e visto che, ovviamente, $-i\nabla = (-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z})$, allora: $L_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$, $L_y = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$, $L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ e

$$L^2 = \vec{L}\vec{L} = L_x^2 + L_y^2 + L_z^2. \tag{A.1}$$

Considerando l'espressione qui sopra per L_z e considerando la (A.1) e le (2.11)……(2.19), si ottiene:

$$L_z = -i\hbar[r\sin\theta\cos\varphi(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}) - r\sin\theta\sin\varphi(\sin\theta\cos\varphi\frac{\partial}{\partial r} +$$

$$+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi})]$$, che si semplifica molto per eliminazioni reciproche degli addendi,

diventando: $L_z = -i\hbar\frac{\partial}{\partial\varphi}$. Analogamente, si ottengono le seguenti:

$L_x = i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi})$, $L_y = -i\hbar(\cos\varphi\frac{\partial}{\partial\theta} - ctg\theta\sin\varphi\frac{\partial}{\partial\varphi})$ e finalmente, dalla (A.1) calcoliamo: $L^2 = -\hbar^2(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}) = -\hbar^2(\frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$ (A.2)

L'ultima eguaglianza è dovuta allo sviluppo ovvio $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} = \frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta}$.

Per giungere alla (A.2), facciamo solo l'esempio del calcolo L_x^2 e si eseguano i prodotti uno ad uno, senza usare le formule fatte sul quadrato di a+b, ad esempio, visto che qui si ha a che fare con operatori:

$$L_x^2 = i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi})\cdot i\hbar(\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}) = -\hbar^2(\sin\varphi\frac{\partial}{\partial\theta}\sin\varphi\frac{\partial}{\partial\theta} +$$

$$+\sin\varphi\frac{\partial}{\partial\theta}ctg\theta\cos\varphi\frac{\partial}{\partial\varphi} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\theta} + ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}ctg\theta\cos\varphi\frac{\partial}{\partial\varphi}) =$$

$$= -\hbar^2[\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cos\varphi(1+ctg^2\theta)\frac{\partial}{\partial\varphi} + ctg\theta\cos^2\varphi\frac{\partial}{\partial\theta} +$$

$$-\sin\varphi\cos\varphi ctg^2\theta\frac{\partial}{\partial\varphi} + \cos^2\varphi ctg^2\theta\frac{\partial^2}{\partial\varphi^2}]$$

ed un'espressione simile per L_y^2 ed una cortissima per L_z^2. Sommando il tutto, molti termini si uniranno od elideranno, conducendoci appunto alla (A.2).
Si noti che $[\varphi, L_z] = i\hbar$.

Riguardo l'operatore energia cinetica T, sappiamo che $\vec{p} = -i\hbar\vec{\nabla}$ e $T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\Delta$, ma sappiamo anche che l'espressione del **Laplaciano in coordinate sferiche** è:

$\Delta = \frac{1}{r^2}(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$ e confrontando tale relazione con la (A.2), si vede che:

$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$, da cui:

$$p^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{L^2}{r^2} = p_r^2 + \frac{L^2}{r^2}, \quad \text{(A.3)}$$

avendo denominato $p_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = -\hbar^2(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r})$, dopo aver sviluppato la derivata di un prodotto: $\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$.

Equazione di Schrodinger dell'atomo

Siamo ovviamente in un campo di forze a simmetria centrale generato dal nucleo, dunque:

$V(\vec{r}) = V(r)$, $H = T + V = \frac{p^2}{2m} + V(r)$, $p^2 = p_r^2 + \frac{L^2}{r^2}$ e, per Schrodinger, $H\Psi(\vec{r}) = E\Psi(\vec{r})$, ossia:

$\frac{1}{2m}(p_r^2 + \frac{L^2}{r^2})\Psi + V(r)\Psi = E\Psi$, cioè:

$$r^2[p_r^2 + 2m(V(r) - E)]\Psi(\vec{r}) = -L^2\Psi(\vec{r}), \quad \text{(A.4)}$$

con, ricordiamolo, $L^2 = -\hbar^2(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}) = -\hbar^2(\frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$.

Cerchiamo soluzioni della (A.4) del tipo a variabili separate: $\Psi(\vec{r}) = \Phi(r)Y(\theta,\varphi)$; sostituendo nella (A.4) e denominando con $R = r^2[p_r^2 + 2m(V(r) - E)]$ il fattore f(r) della (A.4) stessa, si ha: $R\Phi(r)Y(\theta,\varphi) = -L^2\Phi(r)Y(\theta,\varphi)$, da cui:

$$R\Phi(r) = -\Phi(r)\frac{L^2Y(\theta,\varphi)}{Y(\theta,\varphi)} \qquad (A.5)$$

e possiamo denominare: $\lambda = \frac{L^2Y(\theta,\varphi)}{Y(\theta,\varphi)}$, che è palesemente costante rispetto ad r, in quanto sia Y che L^2 non dipendono da r. Si ha:

$$L^2Y(\theta,\varphi) = \lambda Y(\theta,\varphi), \qquad (A.6)$$

che è l'equazione agli autovalori per L^2. Come già detto, le $Y(\theta,\varphi)$ ci daranno informazioni concrete sulla forma degli orbitali, mentre l'altra funzione in r, ossia la $\Phi(r)$, che anche espliciteremo, ci dirà solo quanto alta è la probabilità di trovare l'elettrone nell'orbitale, allontanandosi dal centro od avvicinandosi al centro lungo r, a parità di θ e φ.

Bene, esplicitando la (A.6), si ha:

$$\frac{\partial^2Y(\theta,\varphi)}{\partial\theta^2} + ctg\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2Y(\theta,\varphi)}{\partial\varphi^2} + \frac{\lambda}{\hbar^2}Y(\theta,\varphi) = 0$$

Per una questione di comodità, che sarà più evidente in seguito, poniamo $\lambda = \hbar^2\nu(\nu+1)$, con ν ovviamente altrettanto costante come λ, da cui:

$$\frac{\partial^2Y(\theta,\varphi)}{\partial\theta^2} + ctg\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2Y(\theta,\varphi)}{\partial\varphi^2} + \nu(\nu+1)Y(\theta,\varphi) = 0. \qquad (A.7)$$

Cerchiamo anche qui soluzioni a variabili disgiunte: $Y(\theta,\varphi) = A(\theta)B(\varphi)$; sostituendo nella (A.7):

$A''B + ctg\theta A'B + \frac{1}{\sin^2\theta}AB'' + \nu(\nu+1)AB = 0$. Moltiplicando ora entrambi i membri per $\frac{\sin^2\theta}{AB}$:

$$\sin^2\theta\frac{A''}{A} + \sin\theta\cos\theta\frac{A'}{A} + \nu(\nu+1)\sin^2\theta + \frac{B''}{B} = 0. \qquad (A.8)$$

Per una questione dimensionale, si vede subito che deve essere $\frac{B''}{B} = const$, in quanto $\frac{B''}{B}$ è l'unico ad essere funzione solo di φ, mentre gli altri addendi sono funzioni della sola θ. Perciò, considerando anche che Y è periodica rispetto all'angolo φ, possiamo scrivere: $\frac{B''}{B} = -m^2$ (con $B(\varphi) = e^{im\varphi}$ ed m=0,±1, ±2,…). Tra l'altro, essendo $L_z = -i\hbar\partial/\partial\varphi$ ed applicando L_z a $B(\varphi)$, si ha: $L_zB(\varphi) = -i\hbar\partial/\partial\varphi e^{im\varphi} = \hbar me^{im\varphi} = \hbar mB(\varphi)$, che è un'equazione agli autovalori per L_z, da cui il riferimento ad L_z del numero quantico m. Inserendo l'espressione trovata per B''/B nella (A.8), si ha: $\sin^2\theta A'' + \sin\theta\cos\theta A' + [\nu(\nu+1)\sin^2\theta - m^2]A = 0$. Ponendo ora:

$\cos\theta = x$, (da cui $\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$, $\frac{d^2}{d\theta^2} = -\cos\theta\frac{d}{dx} + \sin^2\theta\frac{d^2}{dx^2}$), si ha:

$\sin^2\theta(\sin^2\theta\frac{d^2A}{dx^2} - \cos\theta\frac{dA}{dx}) - \sin^2\theta\cos\theta\frac{dA}{dx} + [\nu(\nu+1)\sin^2\theta - m^2]A = 0$; dividendo infine per $\sin^2\theta = 1 - x^2$, si ottiene:

$$(1-x^2)\frac{d^2A}{dx^2}-2x\frac{dA}{dx}+[\nu(\nu+1)-\frac{m^2}{(1-x^2)}]A=0 \qquad (A.9)$$

detta Equazione differenziale di Legendre; essa ha tre singolarità (dette fuchsiane) in $x_0=\pm 1$ (ed all'infinito). Quelle su $x_0=\pm 1$ emergono dal "pericoloso" tendere a zero del denominatore di $\frac{1}{(1-x^2)}$ (mentre riguardo il punto all'infinito noi non ci interesseremo particolarmente).

Riscriviamo un attimo la (A.9) come segue:

$$\frac{d}{dx}[(1-x^2)\frac{dA}{dx}]+[\nu(\nu+1)-\frac{m^2}{(1-x^2)}]A=0 \qquad (A.10)$$

ed iniziamo col considerare il caso più semplice di m=0:

$$\frac{d}{dx}[(1-x^2)\frac{dA}{dx}]+\nu(\nu+1)A=0 \qquad (A.11)$$

Cerchiamo una soluzione in forma di serie di potenze, del tipo:

$A(x)=x^s\sum_{n=0}^{+\infty}a_nx^n$ e sostituiamo nella (A.11):

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}\sum_n a_nx^{n+s}]+\nu(\nu+1)\sum_n a_nx^{n+s}=0$$

$$\frac{d}{dx}[(1-x^2)\sum_n a_n(n+s)x^{n+s-1}]+\nu(\nu+1)\sum_n a_nx^{n+s}=0$$

$$\frac{d}{dx}[\sum_n a_n(n+s)x^{n+s-1}-\sum_n a_n(n+s)x^{n+s+1}]+\nu(\nu+1)\sum_n a_nx^{n+s}=0$$

$$\sum_n a_n(n+s)(n+s-1)x^{n+s-2}-\sum_n a_n(n+s+1)(n+s)x^{n+s}+\nu(\nu+1)\sum_n a_nx^{n+s}=0$$

Ora, nella prima sommatoria, ribattezziamo n con (n-2):

$$\sum_n a_{n+2}(n+s+2)(n+s+1)x^{n+s}-\sum_n a_n(n+s+1)(n+s)x^{n+s}+\nu(\nu+1)\sum_n a_nx^{n+s}=0$$

$$\sum_n[a_{n+2}(n+s+2)(n+s+1)-a_n(n+s+1)(n+s)+\nu(\nu+1)a_n]x^{n+s}=0$$

$$a_{n+2}(n+s+2)(n+s+1)-a_n[(n+s+1)(n+s)-\nu(\nu+1)]=0$$

$$a_{n+2}=a_n\frac{(n+s+1)(n+s)-\nu(\nu+1)}{(n+s+2)(n+s+1)}$$

La funzione A è limitata in x=1 (ossia $\theta=0$), dunque si deve prima o poi avere: $(n+s+1)(n+s)-\nu(\nu+1)=0$ per non avere divergenza, con tutti i coefficienti a_n diversi da zero, e visto che n ed s sono interi, tale deve essere ν, che ora chiameremo l, da cui: $\lambda=\hbar^2l(l+1)$.

Poi, dal momento che +m e –m giocano un ruolo di simmetria, si avrà: $-l\le m\le l$, o $|m|\le l$.

Ci accorgiamo altresì (per sostituzione diretta) che è soluzione della (A.11) la seguente:

$$A_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \qquad (A.12)$$

così come è soluzione della (A.10) la seguente:

$$A_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{|m|/2}\frac{d^{l+|m|}}{dx^{l+|m|}}(x^2-1)^l \qquad (A.13)$$

Ad esempio, con l=1 ed m=1, la (A.13) diventa: $A_1^1(x)=-\frac{1}{2}(1-x^2)^{1/2}\frac{d^2}{dx^2}(x^2-1)=-\sqrt{(1-x^2)}$, che sostituita appunto nella (A.10), ci dà 0=0.

Ricordando ora che avevamo posto $Y(\theta,\varphi)=A(\theta)B(\varphi)$, si ha: $Y_l^m(\theta,\varphi)=A_l^m(\theta)B_m(\varphi)$

Valutiamo, tramite la (A-13), le prime $A_l^m(x)$: ($x=\cos\theta$)

$A_0^0(x) = 1 = const$, $A_1^0(x) = x = \cos\theta = const \cdot \cos\theta$, $A_1^{+1}(x) = -(1-x^2)^{1/2} = -\sin\theta = -const \cdot \sin\theta$
e così via, ottenendo le note forme Y degli orbitali, che qui riportiamo: $(B_m(\varphi) = e^{im\varphi})$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta \cdot e^{\pm i\varphi}, \quad Y_2^0 = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1),$$

$$Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta \cdot e^{\pm i\varphi}, \quad Y_2^{\pm2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta \cdot e^{\pm i2\varphi}.$$

Ma come mai abbiamo quelle particolari costanti? ($\frac{1}{\sqrt{4\pi}}, \sqrt{\frac{3}{4\pi}}$, eccetera) Beh, se ricordate, la probabilità massima in tutto lo spazio deve essere 1, da cui la normalizzazione ad 1. Allora, ad esempio per il caso di $Y_1^0(\theta,\varphi) = A_1^0(\theta)B_0(\varphi) = const \cdot \cos\theta \cdot e^{i0\varphi} = const \cdot \cos\theta = C_1^0 \cos\theta$, e considerando che, con le coordinate polari sferiche, l'angolo θ è spazzato da r, mentre l'angolo φ è spazzato dalla proiezione di r sul piano x-y, ossia da $r\sin\theta$, l'angolo solido infinitesimo $d\Omega$ è dato, come noto, dal rapporto tra la superficie sferica infinitesima dS ed r^2, ossia: $d\Omega = \frac{dS}{r^2} = \frac{rd\theta \cdot r\sin\theta d\varphi}{r^2} = d\varphi \sin\theta d\theta$. Integriamo allora su tutto Ω ed imponiamo la probabilità massima pari ad 1: (θ varia tra 0 e π, mentre φ varia tra 0 e 2π)

$$\int_0^{4\pi} (Y_1^0)^*(\theta,\varphi) \cdot Y_1^0(\theta,\varphi) d\Omega = \int_0^{2\pi}\int_0^{\pi} (C_1^0)^2 \cos^2\theta d\varphi \sin\theta d\theta = \int_0^{2\pi} d\varphi \int_0^{\pi} (C_1^0)^2 \cos^2\theta \sin\theta d\theta =$$

$$= -\int_0^{2\pi} d\varphi \int_0^{\pi} (C_1^0)^2 \cos^2\theta d\cos\theta = -2\pi (C_1^0)^2 \int_0^{\pi} \cos^2\theta d\cos\theta = -2\pi (C_1^0)^2 \left|\frac{1}{3}\cos^3\theta\right|_0^{\pi} = \frac{4}{3}\pi (C_1^0)^2 = 1,$$

cioè: $\frac{4}{3}\pi (C_1^0)^2 = 1$, ossia: $C_1^0 = \sqrt{\frac{3}{4\pi}}$, cioè proprio il coefficiente di Y_1^0. Similmente,

$$\int_0^{4\pi} (Y_1^1)^*(\theta,\varphi) \cdot Y_1^1(\theta,\varphi) d\Omega = \int_0^{2\pi}\int_0^{\pi} (C_1^1)^2 \sin^2\theta \cdot e^{-i\varphi+i\varphi} d\varphi \sin\theta d\theta = \int_0^{2\pi} d\varphi \int_0^{\pi} (C_1^1)^2 \sin^3\theta d\theta =$$

$= 2\pi (C_1^1)^2 \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{8}{3}\pi (C_1^1)^2 = 1$, cioè: $\frac{8}{3}\pi (C_1^1)^2 = 1$, da cui: $C_1^1 = \sqrt{\frac{3}{8\pi}}$, ossia proprio il coefficiente di Y_1^1. E così via.

E le forme degli orbitali sarebbero già giustificate così. Per completezza, visto che la soluzione completa della Equazione di Schrodinger dell'atomo è (vedi più sopra): $\Psi(\vec{r}) = \Phi(r)Y(\theta,\varphi)$, resta solo da dare una forma alla funzione puramente radiale $\Phi(r)$.

Ricordiamo qui la (A.5):

$R\Phi_{nl}(r) = -\Phi_{nl}(r)\frac{L^2 Y(\theta,\varphi)}{Y(\theta,\varphi)} = -\lambda\Phi_{nl}(r) = -\hbar^2 l(l+1)\Phi_{nl}(r) = r^2\{p_r^2 + 2m[V(r) - E]\}\Phi_{nl}(r)$, ossia:

$-\hbar^2 l(l+1)\Phi_{nl}(r) = r^2\{p_r^2 + 2m[V(r)-E]\}\Phi_{nl}(r)$, da cui:

$\frac{p_r^2}{2m}\Phi_{nl}(r) + [V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}]\Phi_{nl}(r) = E\Phi_{nl}(r)$ e ricordando anche la (A.6) e le successive espressioni per p_r, ossia: $p_r^2 = -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} = -\hbar^2 (\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r})$, si ha:

$-\frac{\hbar^2}{2m}(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr})\Phi_{nl}(r) + [V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}]\Phi_{nl}(r) = E\Phi_{nl}(r)$, ovvero:

$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + \{\frac{2m}{\hbar^2}[E - V(r)] - \frac{l(l+1)}{r^2}\}\Phi_{nl}(r) = 0$;

Considerando ora che $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}$, si ha:

$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + \{\frac{2m}{\hbar^2}[E + \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}] - \frac{l(l+1)}{r^2}\}\Phi_{nl}(r) = 0$, ossia ancora (ricordando il raggio di Bohr $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$): $\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + [\frac{2m}{\hbar^2}E + \frac{2Z}{a_0 r} - \frac{l(l+1)}{r^2}]\Phi_{nl}(r) = 0$ e ricordando ora l'energia totale $E = -\frac{Z^2e^4m}{32\pi^2n^2\varepsilon_0^2\hbar^2}$ (e notare che abbiamo qui $\hbar = h/2\pi$ in luogo di h), si ha:

$$\Phi_{nl}''(r) + \frac{2}{r}\Phi_{nl}'(r) + [-\frac{Z^2e^4m^2}{16\pi^2n^2\varepsilon_0^2\hbar^4} + \frac{2Z}{a_0 r} - \frac{l(l+1)}{r^2}]\Phi_{nl}(r) = 0 \qquad (A.14)$$

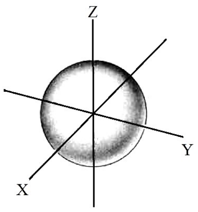
e ci accorgiamo che sono soluzioni della (A.14) le seguenti funzioni:

$$\Phi_{nl}(r) = \frac{1}{n}\sqrt{(\frac{Z}{a_0})\frac{(n-l-1)!}{(n+l)!}} \cdot e^{-Zr/na_0}\,\frac{1}{r}(\frac{2Zr}{na_0})^{l+1} \cdot L_{n-l-1}^{2l+1}(\frac{2Zr}{na_0}) \ , \qquad (A.15)$$

(con $L_0^\alpha(x) = 1$ e $L_1^\alpha(x) = \alpha + 1 - x$, $L_n^\alpha(x) = \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}]$ Polinomi generalizzati di Laguerre). Esplicitandone un po':

$$\Phi_{10}(r) = \sqrt{\frac{Z}{a_0}}\frac{2Z}{a_0}e^{-Zr/a_0} \ , \ \Phi_{20}(r) = \sqrt{\frac{Z}{8a_0}}\frac{Z}{a_0}(2 - \frac{Zr}{a_0})e^{-Zr/2a_0} \ , \ \Phi_{21}(r) = \sqrt{\frac{Z}{24a_0}}(\frac{Z}{a_0})^2 re^{-Zr/2a_0} \qquad (A.16)$$

La normalizzazione è stata effettuata nel seguente modo: $\int_0^\infty r^2|\Phi_{nl}(r)|^2 dr$, in quanto le funzioni Y (vedi sopra) erano state normalizzate su $d\Omega = \frac{dS}{r^2} = \frac{rd\theta \cdot r\sin\theta d\varphi}{r^2} = d\varphi\sin\theta d\theta$, e per passare da $d\Omega$ a dV bisogna moltiplicare $d\Omega$ per r^2dr, poichè $d\Omega$ x r^2 dà dS e dS x dr dà dV e dunque la parte rimanente della normalizzazione, ossia r^2dr la prende appunto $\Phi_{nl}(r)$ che è la funzione radiale. Come breve verifica, fate una prova per vedere che effettivamente la (A.15) è soluzione della (A.14); sostituite cioè una alla volta le (A.16) nella (A.14) (e rispettando, di volta in volta, i valori di n ed l) e vedrete che otterrete sempre 0=0. Per finire, considerando ad esempio il caso dell'orbitale s (n=1, l=0, m=0),



esso è una sfera, in quanto la sua funzione $Y_l^m(\theta,\varphi) = Y_0^0 = 1/\sqrt{4\pi}$ è costante, ossia non compaiono θ e φ, e quindi la probabilità "angolare" P_a ($\propto |Y_l^m(\theta,\varphi)|^2$) è indipendente da come mi oriento, ma la sua funzione radiale è $\Phi_{nl}(r) = \Phi_{10}(r) = \sqrt{Z/a_0}\,(2Z/a_0) \cdot e^{-Zr/a_0}$, cioè una costante per un esponenziale, dicendoci che la probabilità "radiale" P_r ($\propto |\Phi_{nl}(r)|^2$) di trovare l'elettrone non è nella pellicina sottile della superficie sferica perfetta, ma che è diffusa, a nube, e va scemando con l'allontanarsi dal centro. **Funzione d'onda di Schrodinger dell'atomo:**

$\Psi(\vec{r}) = Y_l^m(x = \cos\theta, \varphi)\Phi_{nl}(r) = [A_l^m(x = \cos\theta)B_m(\varphi)]\Phi_{nl}(r)$, ossia:

$$\Psi(\vec{r}) = \frac{(-1)^m}{2^l nl!} \cdot \sqrt{(\frac{Z}{a_0})\frac{(n-l-1)!}{(n+l)!}}(1-x^2)^{|m|/2}[\frac{d^{l+|m|}}{dx^{l+|m|}}(x^2-1)^l]e^{im\varphi} \cdot e^{-Zr/na_0} \frac{1}{r}(\frac{2Zr}{na_0})^{l+1} \cdot L_{n-l-1}^{2l+1}(\frac{2Zr}{na_0})$$

-B-COORDINATE CILINDRICHE

Aerodinamica ed Equazione di Kutta Zhukowsky

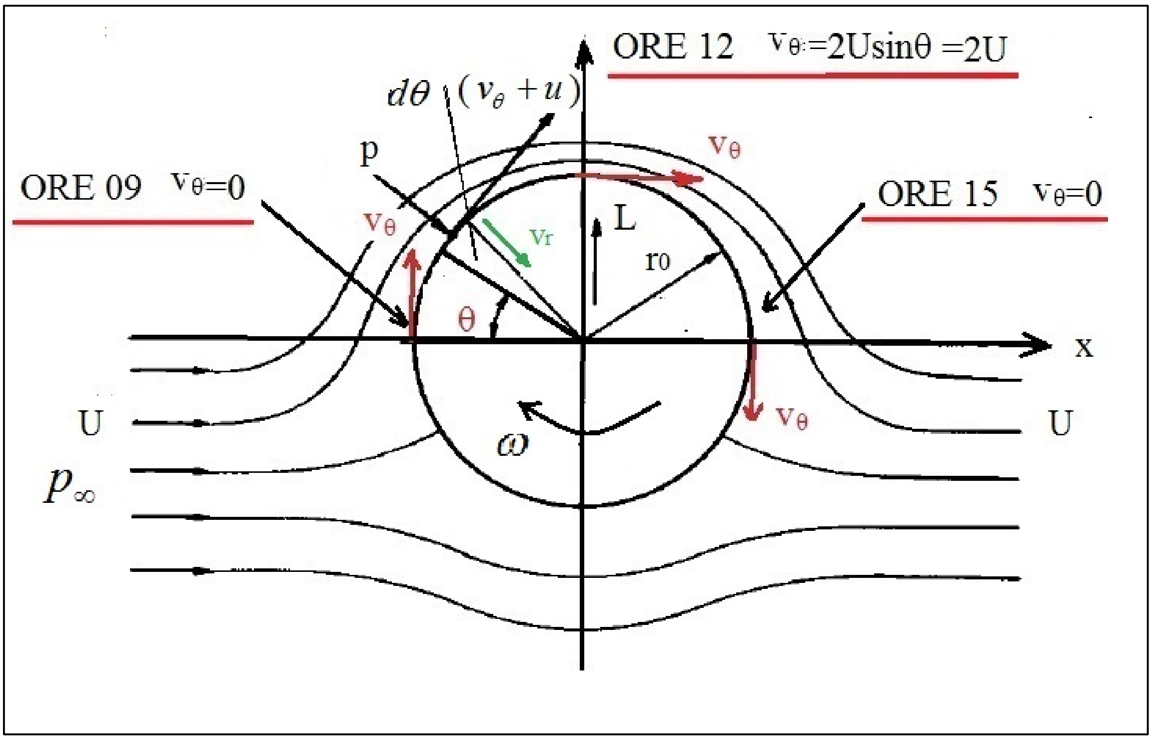


Fig. B.1.

Abbiamo un lungo cilindro posto in un flusso uniforme U e che ruota a velocità angolare ω. Ciò imprime al fluido sulla superficie del cilindro, una componente di velocità pari ad $u = r_0\omega$, a causa della viscosità. Sempre sulla superficie del cilindro c'è poi la componente di velocità v_θ dovuta al flusso U. Viene dimostrato più sotto (eq. (B.8)) che la componente v_θ ad ore 12 vale 2U e, dunque, intorno al cilindro ad angoli θ differenti, si avrà: $v_\theta = 2U\sin\theta$. In totale, intorno al cilindro, si avrà tangenzialmente una $v_T = v_\theta + u = 2U\sin\theta + r_0\omega$.

Chiamando p_∞ la pressione lontano dal cilindro e p quella sul cilindro, per l'Equazione di Bernoulli:

$p + \frac{1}{2}\rho v^2 + \rho g h = const$, cioè: $p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$ ed avvenendo il tutto praticamente alla stessa altezza h, si avrà, dal confronto tra flusso alla sinistra del cilindro e quello sul cilindro:

$p_\infty + \frac{\rho}{2}U^2 = p + \frac{\rho}{2}(2U\sin\theta + r_0\omega)^2$. Perciò:

$$\frac{p - p_\infty}{\frac{\rho}{2}U^2} = 1 - (\frac{2U\sin\theta + r_0\omega}{U})^2 \quad \text{(B.1)}$$

Considerando ora l'elemento infinitesimo di arco $r_0 d\theta$, esso determina lungo un tratto l (elle) di cilindro, che si estende entrando dentro il foglio, una areola $lr_0 d\theta$. La proiezione sul piano orizzontale è ovviamente: $lr_0 \sin\theta d\theta$. Su essa agisce la pressione $(p - p_\infty)$, e dal momento che essa nel disegno è schematizzata verso il basso, la forza sull'areola che essa determina invece verso l'alto avrà il segno cambiato: $dL_{[N]} = -(p - p_\infty) lr_0 \sin\theta d\theta$. Valutando ora la forza per unità di lunghezza di cilindro: $dL_{[N/m]} = dL = -(p - p_\infty) r_0 \sin\theta d\theta$ (senza più la elle minuscola).

Per la portanza (L-lift) totale, si integrerà tra il basso e l'alto del cilindro, ossia tra $-\pi/2$ e $+\pi/2$; non solo: si moltiplicherà per due, avendo il cilindro due "lati", ossia un davanti ed un dietro:

$$L = L_{[N/m]} = 2\int_{-\pi/2}^{+\pi/2} -(p - p_\infty) r_0 \sin\theta d\theta = -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{2U\sin\theta + r_0\omega}{U})^2] \sin\theta d\theta =$$

$$= -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{r_0\omega}{U})^2 - \frac{4r_0\omega}{U}\sin\theta - 4\sin^2\theta] \sin\theta d\theta = 2\pi \cdot r_0^2 \omega\rho U = 2\pi \cdot r_0 u\rho U$$

ossia: $L = 2\pi \cdot r_0 u \rho U$ (B.2)

Ora, dal momento che la velocità periferica u del cilindro si applica lungo tutta la circonferenza, la circolazione Γ (velocità per spazio di applicazione) è: $\Gamma = 2\pi \cdot r_0 u$, da cui:

$L = \rho U \Gamma$ (Equazione di Kutta Joukowski del cilindro rotante) (B.3)

Possiamo dunque dire che quando, anche ad opera di un'ala e non solo di un cilindro rotante, si genera una circolazione Γ, si genera una portanza L.

– Potenziale del flusso

Riguardo la velocità v sul contorno del cilindro rotante, ed in particolare la componente v_θ dovuta al flusso esterno, che lontano ha velocità orizzontale U, valutiamo la forma di una funzione potenziale ϕ tale che $v = \nabla\phi$; dall'Equazione di Continuità $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})$ si ha, nell'ipotesi di ρ costante, $\vec{\nabla}\cdot(\rho\vec{v}) = \rho\vec{\nabla}\cdot\vec{v} = 0$, ossia $\vec{\nabla}\cdot\vec{v} = 0$, ossia ancora $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(\nabla\phi) = \Delta\phi = 0$ e $\Delta\phi = 0$ è l'Equazione di Laplace:

Laplaciano: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta\phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi = 0$.

Ricordiamo ora l'espressione del **Laplaciano in coordinate cilindriche** (per noi comodissime, vista la simmetria col nostro cilindro rotante):

$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$, e con riferimento al nostro cilindro rotante, consideriamo l'asse z entrante nel foglio. Inoltre, per ragioni di simmetria del nostro lungo cilindro, la componente con z è nulla, da cui: $\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$ e per la nostra nuova Equazione di Laplace, si ha:

$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$, cioè:

$r\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{\partial^2\phi}{\partial\theta^2} = 0$. (B.4)

Dunque, in coordinate cilindriche, avremo che le componenti di velocità v intorno al cilindro e dovute alla velocità del flusso esterno (U, da lontano), saranno formalmente tre: v_θ, v_r e v_z, ma per ragioni geometriche e di uniformità del cilindro, si ha che $v_z = 0$.

Ci accorgiamo che la funzione $\phi = A_n r^n \cos(n\theta)$ è soluzione della (B.4) sia con n=+1 che con n=-1 (e A_n / A_{-n} = costanti di proporzionalità da determinare), sicchè abbiamo almeno due soluzioni: $\phi_1 = A_1 r\cos\theta$ e $\phi_2 = A_{-1} r^{-1}\cos(-\theta) = A_{-1} r^{-1}\cos\theta$. Allora, la somma sarà una soluzione ancora più generale: $\phi = \phi_1 + \phi_2 = A_1 r\cos\theta + A_{-1} r^{-1}\cos\theta$. (B.5)

Con riferimento alla figura qui sopra e conciliando opportunamente i segni algebrici nell'adattare le equazioni alla figura, poco a sinistra del punto ad ore 09 la velocità del flusso è U ed è orizzontale, da sx a dx, mentre contro il cilindro, proprio ad ore 09, ovviamente si annulla, riservandosi di incrementarsi in verticale (v_θ) al crescere di θ. Stessa cosa per il punto ad ore 15, alla destra del quale il flusso riprenderà poi ad avere velocità orizzontale U verso destra, ossia -U.
Sfruttiamo questi due punti (ore 09 ed ore 15) per dare una forma a ϕ e poi, dall'equazione ottenuta, ci faremo dire il valore di v_θ ad ore 12:

$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}(A_1 r\cos\theta + A_{-1} r^{-1}\cos\theta) = -A_1\sin\theta - A_{-1} r^{-2}\sin\theta$, da cui:

$v_{\theta=0} = -A_1\sin 0 - A_{-1} r^{-2}\sin 0 = 0$. Stesso valore per θ=180°.

Inoltre, ovviamente: $v_r = \frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}(A_1 r\cos\theta + A_{-1} r^{-1}\cos\theta) = A_1\cos\theta - A_{-1} r^{-2}\cos\theta$, da cui, dovendo essere $v_{r_0} = 0$, segue che: $0 = A_1\cos\theta - A_{-1} r_0^{-2}\cos\theta$, ossia: $A_{-1} = r_0^2 A_1$, e cioè:

$\phi = A_1 r\cos\theta + r_0^2 A_1 r^{-1}\cos\theta = A_1\cos\theta(r + \frac{r_0^2}{r})$. Dunque:

$$\phi = A_1\cos\theta(r + \frac{r_0^2}{r}) \qquad \text{(B.6)}$$

Ora, per valutare A_1, poniamoci sull'asse orizzontale (θ=0 o θ=180°) e valutiamo v_r, che per $r \to +\infty$ dovrà valere rispettivamente U e –U:

$v_r = \frac{\partial\phi}{\partial r} = A_1\cos\theta(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=0} = A_1(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=180°} = -A_1(1 - \frac{r_0^2}{r^2})$, ossia:

$(v_r)_{\theta=0\_r\to+\infty} = A_1 = U$, $(v_r)_{\theta=180°\_r\to+\infty} = -A_1 = -U$ e, dunque, in definitiva:

$$\phi = U\cos\theta(r + \frac{r_0^2}{r}) \qquad \text{(B.7)}$$

Infine, per quanto riguarda v_θ valutato ad un θ generico e per θ=90°, si ha:

$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}[U\cos\theta(r + \frac{r_0^2}{r})] = -U(1 + \frac{r_0^2}{r^2})\sin\theta$ e sulla circonferenza cilindro ($r=r_0$):

$v_\theta = -2U\sin\theta$, nonche:

$$v_{\theta=90°} = -2U . \qquad \text{(B.8)}$$

Per ultimo, sempre sulla verticale (θ=90°), oppure ad un tal θ= θ_0, ma lontano dal cilindro ($r \to \infty$) il flusso deve tornare quello imperturbato di assenza cilindro, ossia con v=U:

$v_\theta = -U(1 + \frac{r_0^2}{r^2})\sin\theta$, $v_r = U\cos\theta(1 - \frac{r_0^2}{r^2})$,

ossia, con θ=90° ed $r \to \infty$, $v_\theta = -U$ e $v_r = 0$, cioè, per Pitagora: v=U.

Invece, per θ= θ_0 e $r \to \infty$, $v_\theta = -U\sin\theta_0$ e $v_r = U\cos\theta_0$, cioè, per Pitagora:

$v = \sqrt{[(U\cos\theta_0)^2 + (-U\sin\theta_0)^2]} = U$, cioè la (B.7) rispecchia proprio bene il nostro flusso, ovunque, intorno al cilindro.

SUBAPPENDICI

SUBAPPENDICE 1-Identità vettoriali:

Per ultimo, ricordiamo anche le seguenti identità vettoriali, di dimostrazione diretta:
(V e W sono scalari ed $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ vettori)

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \qquad \text{(A1.1)}$$
$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B}) \qquad \text{(A1.2)}$$
$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \qquad \text{(A1.3)}$$

$$\left|\vec{A}\times\vec{B}\right|^2=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-(\vec{A}\cdot\vec{B})^2 \qquad \text{(A1.4)}$$

Infatti: $\vec{A}\times\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$ e $\vec{A}\cdot\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$, da cui: $\left|\vec{A}\times\vec{B}\right|^2=\left|\vec{A}\right|^2\left|\vec{B}\right|^2\sin^2\theta=\left|\vec{A}\right|^2\left|\vec{B}\right|^2(1-\cos^2\theta)=$
$=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-\left|\vec{A}\right|^2\left|\vec{B}\right|^2\cos^2\theta=\left|\vec{A}\right|^2\left|\vec{B}\right|^2-(\vec{A}\cdot\vec{B})^2$ cvd.

$$\vec{\nabla}(V+W)=\nabla V+\nabla W \qquad \text{(A1.5)}$$
$$\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B} \qquad \text{(A1.6)}$$
$$\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B} \qquad \text{(A1.7)}$$
$$\vec{\nabla}(\vec{A}\cdot\vec{B})=(\vec{A}\cdot\vec{\nabla})\vec{B}+(\vec{B}\cdot\vec{\nabla})\vec{A}+\vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A}) \qquad \text{(A1.8)}$$
$$\nabla(VW)=W\nabla V+V\nabla W \qquad \text{(A1.9)}$$
$$\vec{\nabla}\times\nabla V=0 \qquad \text{(A1.10)}$$
$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}=0 \qquad \text{(A1.11)}$$
$$\vec{\nabla}\cdot(V\vec{A})=\nabla V\cdot\vec{A}+V(\vec{\nabla}\cdot\vec{A}) \qquad \text{(A1.12)}$$
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B})=(\vec{\nabla}\times\vec{A})\cdot\vec{B}-\vec{A}\cdot(\vec{\nabla}\times\vec{B}) \qquad \text{(A1.13)}$$
$$\vec{\nabla}\times(V\vec{B})=V(\vec{\nabla}\times\vec{B})+\nabla V\times\vec{B} \qquad \text{(A1.14)}$$
$$\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{\nabla}\cdot\vec{B})\vec{A}-(\vec{\nabla}\cdot\vec{A})\vec{B}-\vec{B}(\vec{A}\cdot\vec{\nabla})+\vec{A}(\vec{B}\cdot\vec{\nabla}) \qquad \text{(A1.15)}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\nabla(\vec{\nabla}\cdot\vec{A})\vec{A}-\vec{\Delta}\vec{A} \qquad \text{(A1.16)}$$

con l' $\vec{\Delta}\vec{A}$ nella (A1.16) che è il "poco trattato" <u>Laplaciano Vettoriale</u>.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\vec{\nabla}\times[\hat{i}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\hat{j}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\hat{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})]=$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}) & (\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}) & (\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})\end{vmatrix}=\hat{i}[\frac{\partial(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})}{\partial y}-\frac{\partial(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})}{\partial z}]+\hat{j}[\frac{\partial(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})}{\partial z}-\frac{\partial(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})}{\partial x}]+$$

$$+\hat{k}[\frac{\partial(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})}{\partial x}-\frac{\partial(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})}{\partial y}]=\hat{i}[\frac{\partial}{\partial y}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})-\frac{\partial}{\partial z}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})]+\hat{j}[\frac{\partial}{\partial z}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})-\frac{\partial}{\partial x}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})]+$$

$+\hat{k}[\frac{\partial}{\partial x}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})-\frac{\partial}{\partial y}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})]=\hat{i}(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}-\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_x}{\partial x\partial y})+$ $\hat{k}(\frac{\partial^2 A_x}{\partial x\partial z}-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_y}{\partial y\partial z})$. Adesso, nella prima parentesi tonda aggiungiamo la quantità nulla $(\frac{\partial^2 A_x}{\partial x^2}-\frac{\partial^2 A_x}{\partial x^2})$, poi nella seconda parentesi aggiungiamo $(\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial y^2})$ e nella terza $(\frac{\partial^2 A_z}{\partial z^2}-\frac{\partial^2 A_z}{\partial z^2})$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\hat{i}(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_z}{\partial x\partial z}+\frac{\partial^2 A_x}{\partial x^2}-\frac{\partial^2 A_x}{\partial x^2})+\hat{j}(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}-\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_x}{\partial x\partial y}+\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial y^2})+$$

$$\hat{k}(\frac{\partial^2 A_x}{\partial x\partial z}-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_y}{\partial y\partial z}+\frac{\partial^2 A_z}{\partial z^2}-\frac{\partial^2 A_z}{\partial z^2})=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})-\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})]+$$

$$+[\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})-\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})]+[\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]-\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]=$$

$$=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})+\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]-$$

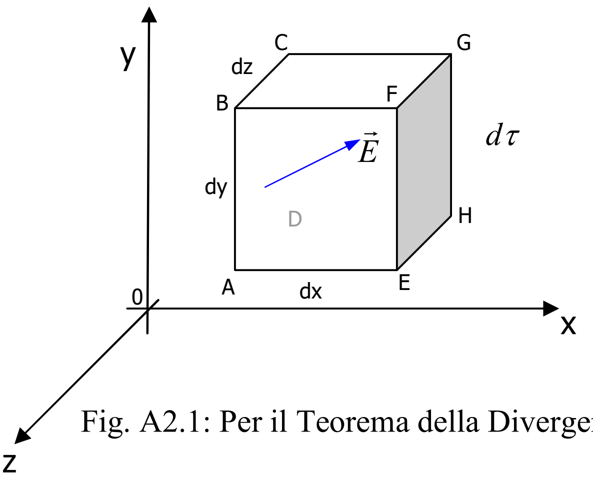
$-[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})+\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})+\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]=\nabla(\vec{\nabla}\cdot\vec{A})\vec{A}-\Delta\vec{A}$, con:

$\nabla(\vec{\nabla}\cdot\vec{A})=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z})+\hat{j}(\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y})+\hat{k}(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z})]$ e

$\boxed{\Delta\vec{A}=[\hat{i}(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2})+\hat{j}(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2})+\hat{k}(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2})]}$ **Laplaciano Vettoriale**.

SUBAPPENDICE 2 - Teoremi-Divergenza e Rotore

Teorema della Divergenza (dimostrazione pratica):



$$\boxed{\int_S \vec{E}\cdot d\vec{S}=\int_V div\vec{E}\cdot dV}$$

Fig. A2.1: Per il Teorema della Divergenza.

Denominiamo ϕ il flusso del vettore $\vec{E}$; si ha:

$d\phi_{ABCD}=\vec{E}\cdot d\vec{S}=-E_x(x,\bar{y},\bar{z})dydz$ ($\bar{y}$ significa y medio)

$d\phi_{EFGH}=E_x(x+dx,\bar{y},\bar{z})dydz$, ma si ha anche che, ovviamente (come sviluppo):

$E_x(x+dx,\bar{y},\bar{z})=E_x(x,\bar{y},\bar{z})+\frac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dx$ da cui:

$d\phi_{EFGH} = E_x(x,\bar{y},\bar{z})dydz + \frac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dxdydz$ e dunque:

$d\phi_{ABCD} + d\phi_{EFGH} = \frac{\partial E_x}{\partial x}dV$. Si agisce poi con analogia sugli altri assi y e z:

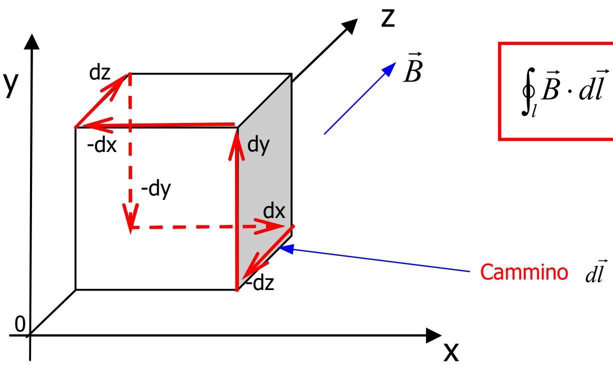
$d\phi_{AEHD} + d\phi_{BCGF} = \frac{\partial E_y}{\partial y}dV$, $d\phi_{ABFE} + d\phi_{CGHD} = \frac{\partial E_z}{\partial z}dV$

e si sommano tali flussi trovati, ottenendo, in totale:

$d\phi = (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})dV = (div \cdot \vec{E})dV = (\vec{\nabla} \cdot \vec{E})dV$ e dunque:

$\phi_S(\vec{E}) = \int_\phi d\phi = \int_S \vec{E} \cdot d\vec{S} = \int_V div\vec{E} \cdot dV = \int_V (\vec{\nabla} \cdot \vec{E}) \cdot dV$ e cioè l'asserto.

Teorema del Rotore o di Stokes (dimostrazione pratica-**by Rubino!**):



$$\boxed{\oint_l \vec{B} \cdot d\vec{l} = \int_S rot\vec{B} \cdot d\vec{S} = \int_S \vec{\nabla} \times \vec{B} \cdot d\vec{S}}$$

Fig. A2.2: Per il Teorema del Rotore (dim. by Rubino).

Valutiamo $\vec{B} \cdot d\vec{l}$:

Su dz B vale B_z; su dx B vale B_x; su dy B vale B_y;

su -dz B vale $B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy$, per lo sviluppo di Taylor in 3D ed anche per il fatto che per arrivare dal centro di dz e quello di –dz si sale su x, si scende su y e nulla lungo z stesso.

Analogamente, su -dx B vale $B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy$ e su -dy B vale $B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz$.

Sommando tutti i contributi:

$$\vec{B} \cdot d\vec{l} = B_z dz - (B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy)dz + B_x dx - (B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy)dx + B_y dy -$$

$$+ (B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz)dy = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})dydz + (\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x})dxdz + (\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})dxdy =$$

$= rot\vec{B} \cdot d\vec{S} = \vec{\nabla} \times \vec{B} \cdot d\vec{S}$, dove qui $d\vec{S}$ ha componenti $[\hat{x}(dydz), \hat{y}(dxdz), \hat{z}(dxdy)]$

e cioè l'asserto: $\oint_l \vec{B} \cdot d\vec{l} = \int_S rot\vec{B} \cdot d\vec{S} = \int_S \vec{\nabla} \times \vec{B} \cdot d\vec{S}$, dopo aver ricordato che:

$$rot\vec{B} = \vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} .$$