

The End of The Mystery of The Riemann Hypothesis!

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Abstract

In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called the Riemann hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$. In this paper, I give the proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann hypothesis: the Dirichlet η function and the functional relation verified by the zeta function.

Keywords: Zeta function, non-trivial zeros of eta function, Riemann hypothesis, equivalence statements, the definition of limits of real sequences, the functional relation of zeta function.

MSC Classification: 11Axx , 11M26.

*This paper is dedicated to the memory of my father who taught me arithmetic,
To my wife, my daughter, my son and my granddaughter Rayhane
To my friend Ezzedine Djerbi*

*To the memory of the Prof. Mohamed AMARA, former Dean of the Faculty of Sciences
of Tunis*

'I feel that these aren't the right techniques to solve the Riemann hypothesis itself, it's going to need some big idea from somewhere else.'

1 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann hypothesis:

Conjecture 1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we use the functional relation verified by the zeta function so we obtain a contradiction and we arrive to give the proof that $\sigma = \frac{1}{2}$.

1.1 The function zeta(s)

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

In addition to the properties cited by the theorem 2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for

$s \in \mathbb{C} \setminus \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s) \quad (1.1)$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,$$

So, instead of using one detailed formula of the zeta function, we will use for our proof the function presented by G.H. Hardy [3] namely the Dirichlet eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 3. For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2 A Equivalent statement to the Riemann Hypothesis

Among the equivalent statements to the Riemann hypothesis is that of the Dirichlet eta function which is stated as follows [2]:

Equivalence 4. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad \sigma > 1 \quad (1.2)$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (1.2) is convergent, and represents $(1 - 2^{1-s}) \zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad \Re(s) = \sigma > 0 \quad (1.3)$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho \cdot e^{i\alpha} \implies \rho^2 = \eta(s) \cdot \overline{\eta(s)} \quad (1.4)$$

and $\eta(s) = 0 \iff \rho = 0$.

2 Preliminaries of the proof that the zeros of the function $\eta(s)$ are on the critical line

Proof. We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \iff (1 - 2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

$$\boxed{s, \text{ is one zero of } \eta(s) \text{ that falls in the critical strip, is also one zero of } \zeta(s) \text{ in the critical strip}} \quad (2.1)$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \implies \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s, \text{ is one zero of } \zeta(s) \text{ that falls in the critical strip, is also one zero of } \eta(s) \text{ in the critical strip}} \quad (2.2)$$

Let us write the function η :

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-s \operatorname{Log} n} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it) \operatorname{Log} n} = \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \operatorname{Log} n} \cdot e^{-it \operatorname{Log} n} \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \operatorname{Log} n} (\cos(t \operatorname{Log} n) - i \sin(t \operatorname{Log} n)) \end{aligned}$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. We define the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} - i \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let $s = \sigma + it$ with $0 < \sigma < 1$ be one zero of the function η , then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^N \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

It follows that we can write $\lim_{n \rightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} &= 0 \\ \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma} &= 0 \end{aligned}$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0 \exists n_r, \forall N > n_r, |\Re(\eta(s)_N)| < \epsilon_1 \implies \Re^2(\eta(s)_N) < \epsilon_1^2 \quad (2.3)$$

$$\forall \epsilon_2 > 0 \exists n_i, \forall N > n_i, |\Im(\eta(s)_N)| < \epsilon_2 \implies \Im^2(\eta(s)_N) < \epsilon_2^2 \quad (2.4)$$

Then:

$$0 < \sum_{k=1}^N \frac{\cos^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1; k \neq k'}^N \frac{(-1)^{k+k'} \cos(t \operatorname{Log} k) \cos(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_1^2 \quad (2.5)$$

$$0 < \sum_{k=1}^N \frac{\sin^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1; k \neq k'}^N \frac{(-1)^{k+k'} \sin(t \operatorname{Log} k) \sin(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_2^2 \quad (2.6)$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > \max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \quad (2.7)$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \quad (2.8)$$

or $\rho(s) = 0$.

3 Case $0 < \Re(s) < 1/2$

Suppose there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \implies \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \implies s$ lies inside the critical band. We write the equation (2.7):

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \quad (3.1)$$

or:

$$-\frac{1}{2} \sum_{k=1}^N \frac{1}{k^{2\sigma}} < \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < \epsilon^2 - \frac{1}{2} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \quad (3.2)$$

The middle term of the above inequality is a double sum:

$$\sum_{k=1}^{k=N-1} \frac{(-1)^k}{k^\sigma} \cdot \left(\sum_{k'=k+1}^{k'=N} \frac{(-1)^{k'} \cos(t \operatorname{Log}(k/k'))}{k'^\sigma} \right) \quad (3.3)$$

But $2\sigma < 1$, it follows that $\left(\lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \right) \rightarrow +\infty$ and then, we obtain necessarily from the equation (3.2):

$$\boxed{\sum_{k,k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} = -\infty} \quad (3.4)$$

4 Case $\Re(s) = 1/2$

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 5. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 6. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.7) is written for s_j :

$$0 < \sum_{k=1}^N \frac{1}{k} + 2 \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t_j \operatorname{Log}(k/k'))}{\sqrt{k} \sqrt{k'}} < 2\epsilon^2$$

or:

$$-\frac{1}{2} \sum_{k=1}^N \frac{1}{k} < \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t_j \operatorname{Log}(k/k'))}{\sqrt{k} \sqrt{k'}} < \epsilon^2 - \frac{1}{2} \sum_{k=1}^N \frac{1}{k} \quad (4.1)$$

If $N \rightarrow +\infty$, the series $\sum_{k=1}^N \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t_j \operatorname{Log}(k/k'))}{\sqrt{k} \sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{\lim_{N \rightarrow +\infty} \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \quad (4.2)$$

if not, we will have a contradiction with the fact that :

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \iff \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

5 Case $1/2 < \Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the section 3. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \implies \rho(s') = 0$. By applying (2.7), we get:

$$\begin{aligned} 0 &< \sum_{k=1}^N \frac{1}{k^{2\sigma'}} + 2 \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \\ -\frac{1}{2} \sum_{k=1}^N \frac{1}{k^{2\sigma'}} &< \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < \epsilon^2 - \frac{1}{2} \sum_{k=1}^N \frac{1}{k^{2\sigma'}} \end{aligned} \quad (5.1)$$

As $0 < \sigma < \frac{1}{2} \implies 2 > 2\sigma' = 2(1 - \sigma) > 1$, then the series $\sum_{k=1}^N \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\boxed{\sum_{k, k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty} \quad (5.2)$$

5.0.1 Case $t = 0$

We suppose that $t = 0 \implies t' = 0$. We known the following proposition:

Proposition 7. *For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.*

We deduce the contradiction with the hypothesis $s' = \sigma'$ is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (5.2) is false for the case } t' = t = 0.} \quad (5.3)$$

5.0.2 Case $t' = t \neq 0$

We suppose that $t' \neq 0$. As $s = \sigma + it$ is a zero of the function η , it follows $s' = \sigma' + it' = 1 - \sigma + it$ is a zero of $\eta(s)$, we have:

$$\eta(s') = 0 \implies \sum_{k, k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \quad (5.4)$$

We recall the expression of the functional relation (1.1) and the Dirichlet eta function (1.2):

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s) \quad (5.5)$$

$$\eta(s) = (1 - 2^{1-s}) \zeta(s) \quad (5.6)$$

We introduce the following notations:

$$\begin{aligned} A(s) &= (1 - 2^{1-s}) \implies a(s) = |A(s)| \implies \eta(s) = A(s) \zeta(s) \\ B(s) &= 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \implies b(s) = |B(s)| \implies \zeta(1-s) = B(s) \zeta(s) \end{aligned} \quad (5.7)$$

We know that $a(s) > 0$, $b(s) > 0$ for $\Re(s) \in]0, 1[$ and are bounded. We consider now $s = \sigma + it$ is the zero of $\eta(s)$ cited in the paragraph (3) "Case $0 < \Re(s) < 1/2$ ", it follows $\eta(s) = 0 = \zeta(s) = 0$ with $0 < \sigma < 1/2$. We introduce $\eta(s)$, we obtain the equation (3.1):

$$\begin{aligned} 0 &< \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2 \implies \\ -\frac{1}{2a^2(s)} \sum_{k=1}^N \frac{1}{k^{2\sigma}} &< \frac{1}{a^2(s)} \cdot \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^{\sigma} k'^{\sigma}} < \frac{\epsilon^2}{a^2(s)} - \frac{1}{2a^2(s)} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \end{aligned}$$

Finally, we obtain:

$$\frac{1}{a^2(s)} \cdot \sum_{k, k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty \quad (5.8)$$

As $\bar{s} = \sigma - it$ is also a zero of the zeta function, and $\cos \alpha = \cos(-\alpha)$, and $a(s) = |A(s)| = |A(\bar{s})| \implies a(\bar{s}) = a(s)$, from the equation $\zeta(\bar{s}) = \frac{\eta(\bar{s})}{a(\bar{s})} = 0$, it follows the equation above (5.8).

$\bar{s} = \sigma - it = \sigma - it'$ is a zero of ζ . We can write the functional relation of zeta function (5.7) as:

$$\begin{aligned}
\zeta(1 - \bar{s}) &= B(\bar{s})\zeta(\bar{s}) \implies \\
\zeta(1 - \sigma + it') &= B(\bar{s})\zeta(\sigma - it') \implies \\
\zeta(\sigma' + it') &= \zeta(s') = B(\bar{s})\zeta(\bar{s}) \implies \\
\frac{\eta(s')}{A(s')} &= B(\bar{s})\zeta(\bar{s}) \implies \zeta(\bar{s}) = \frac{\eta(s')}{A(s')B(\bar{s})} \implies \\
|\zeta(\bar{s})|^2 &= \frac{|\eta(s')|^2}{a^2(s')b^2(\bar{s})} = 0
\end{aligned} \tag{5.9}$$

Discussion:

I- From the equation (3.1) where $0 < \sigma < 1/2$, we obtain the equation above (5.8):

$$\frac{1}{a^2(s)} \cdot \sum_{k, k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^\sigma k'^\sigma} = -\infty \tag{5.10}$$

II- $\sigma' = 1 - \sigma \in]1/2, 1[$, from the equations cited above (5.4-5.9), we obtain :

$$\frac{\eta(s')}{A(s').B(\bar{s})} = 0 \implies \frac{1}{a^2(s).b^2(\bar{s})} \sum_{k, k'=1; k \neq k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{\zeta(2\sigma')}{2a^2(s).b^2(\bar{s})} > -\infty \tag{5.11}$$

The left member of the equation (5.9) implies the equation above (5.10) with the value $-\infty$, but for the right member of the equation (5.9), we obtain from the equation (5.11) the bounded value:

$$-\frac{1}{2a^2(s').b^2(\bar{s})} \cdot \zeta(2\sigma') > -\infty$$

It follows the contradiction. Then the limit of the series (5.4) is false for the case $\sigma' \in]1/2, 1[$ and $t' \neq 0$. It follows that the equation (5.2) is false and $\eta(s')$ does not vanish for $\sigma' \in]1/2, 1[$, it implies that $s \in]0, 1/2[$ is not a zero of the function η . It follows that the case of the section (3) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. Then $\sigma' = 1/2 \Rightarrow \sigma = 1 - \sigma' = 1/2 \implies \zeta(2\sigma') = \zeta(1) = +\infty$ and the equations (5.10, 5.11) coincide, up to positive multiplicative coefficients, with the equation (4.2).

From the equivalent statement (4), it follows that **the Riemann hypothesis is verified**.

□

We therefore announce the important theorem as follows:

Theorem 8. *The Riemann hypothesis is true:*

All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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