Ramanujan primes as stability thresholds

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Abstract

Ramanujan primes encode a persistent form of prime abundance: after the *n*th threshold, every interval of the form *(x/2,x]* contains at least *n* primes. Existing work establishes their existence, global bounds, and leading asymptotic equivalence to the *2n* th prime, yet the exact discrete mechanism and the next asymptotic correction are usually treated separately. This work provides a general framework for the analysis of stability and thresholds. It uses a prime window counting process as an event-driven step function and describes the latest deficit as well as the finite sieve process with a known upper bound guaranteeing its completeness. Asymptotic inversion of the prime-counting expansion formula provides the second-order approximation of the threshold with respect to a constant window value c from the interval (0, 1). In the classical case, the approximation shows that the difference between the 2n-th prime number and 2n converges to log 2. Exact computation of the first 100,000 Ramanujan primes confirms the predicted scale; at *n=100,000*, the second-order approximation has relative error 3.37×10^{-4}. The framework separates rigorous asymptotics, certified finite computation, and historical terminology, providing a reproducible basis for further explicit estimates.

Keywords: *Ramanujan primes; generalized Ramanujan primes; prime-counting function*

1. Introduction

The distribution of prime numbers may be studied globally through the prime-counting function and locally through the number of primes contained in a moving interval. Ramanujan primes sit exactly between these viewpoints. They are defined by a global persistence condition, but the condition concerns the local interval *(x/2,x]*. The resulting threshold is therefore not the first point at which the interval contains *n* primes. It is the first point after which the count never again falls below *n*. This distinction is the organizing principle of the present work.

The historical part begins with Bertrand, who in 1845 conjectured that every integer *m>1* has a prime strictly between *m* and *2m* [1]. Chebyshev proved the result through estimates for prime-counting functions [2]. Hadamard [3] and de la Vallée Poussin [4] independently proved the prime number theorem in 1896, providing the asymptotic scale needed to understand how the number of primes in proportional intervals grows. In 1919 Ramanujan gave a short proof of Bertrand's theorem and, at the end of the same paper, recorded the successive thresholds (2,11,17,29,41) for one, two, three, four, and five primes in *(x/2,x]* [5]. Rosser's explicit work on the *n* th prime later supplied inequalities that became useful in obtaining effective bounds [6].

Ramanujan also studied several other prime-counting formulae. Berndt's critical edition of the notebooks documents both correct identities and arguments whose conclusions exceeded their justification [7]. One formula appearing there is the eventual inequality

$$\pi(x)^2 < \frac{ex}{\log x}\,\pi\left(\frac{x}{e}\right), \quad (1)$$

where *e* is Euler's number and $\pi(x)$ counts primes not exceeding *x*. Equation (1) is now called Ramanujan's prime-counting inequality. It is not the definition of a Ramanujan prime. The shared name

has caused avoidable confusion, especially when a historical discussion of Ramanujan's work on primes is treated as though it were a paper about the sequence R_n.

The modern theory of the sequence began when Sondow introduced the term *Ramanujan prime*, proved $R_n \sim p_{2n}$, and established explicit upper and lower bounds [8]. Here p_j denotes the j th prime. Laishram proved Sondow's conjecture $R_n < p_{3n}$ for every $n>1$ [9]. Sondow, Nicholson, and Noe sharpened that estimate to the optimal inequality $R_n \leq (41/47)\, p_{3n}$, analysed runs, twins, and gaps, and described a fast computation method [10]. Amersi, Beckwith, Miller, Ronan, and Sondow then introduced c-Ramanujan primes for intervals $(cx,x]$, proved their existence for fixed $0<c<1$, and obtained the leading relation $R_{c,n} \sim p_{n/(1-c)}$ [11].

In a parallel direction, Dudek and Platt investigated Equation (1), proving an unconditional large-x range and determining its final integer counterexample under the Riemann hypothesis [12]. Explicit estimates and structural properties of the generalized Ramanujan primes were proved by Axler ([13]). Further, Yang and Togbé gave sharp bounds and the logarithmic factor in $\pi(R_n)$ [14]. Later on, Axler gave more precise bounds for $\pi(R_n)$ and similar counting functions [15]. In his further study on Equation (1), he sharpened the unconditional upper bound in the respective inequality [16]. Overall, all these results reflect the appearance of two interconnected theories derived from Ramanujan's studies on prime numbers: stability bounds in proportional intervals and inequality of $\pi(x)$ and $\pi(x/e)$. They use related tools but answer different questions.

This paper leaves room for a synthesis at three levels. First, the defining persistence condition is often stated without isolating the exact event structure of the underlying step function. Second, computational lists are frequently reported without an explicit finite certificate showing that no later deficit can alter the values. Third, the leading equivalence $R_{c,n} \sim p_{n/(1-c)}$ does not exhibit the displacement between the two sequences. That displacement is visible numerically and begins one order below the principal term.

This paper addresses these points through four contributions:

1. **Discrete structure:** It formulates Ramanujan primes as recovery points of an event-driven prime-window count and proves an exact last-deficit identity for the classical sequence.
2. **Certified computation:** It gives a finite sieve procedure whose completeness follows from the published bound $R_n < p_{3n}$, so every computed value is exact rather than provisionally observed.
3. **Second-order analysis.** It derives a second-order asymptotic expansion for fixed-c Ramanujan thresholds. For $c=1/2$, the expansion yields the explicit limit in Equation (2).
4. **Independent validation.** It validates the theory through exact computation of $R_1, \ldots, R_{100000}$, with symbolic coefficient checks and an independent Wolfram verification of the first 1,000 values.

The classical displacement limit just mentioned is

$$\frac{R_n - p_{2n}}{2n} \to \log 2. \quad (2)$$

2. Prime-window counts and stability thresholds

2.1. Notation and definitions

All logarithms are natural. The set of positive integers is denoted by $N=\{1,2,3,\ldots\}$, and $N_0 = N \cup \{0\}$. **Table 1** collects the notation used throughout this work.

Table 1. *Principal notation.*

Symbol	Definition
$\pi(x)$	Number of primes $p \le x$, for real $x \ge 0$
p_j	j th prime number
c	Fixed proportional-window parameter, $0<c<1$
$W_c(x)$	Prime count in $(cx, x]$
$R_{c,n}$	n th c-Ramanujan prime
$W(x)$	Classical count $W_{1/2}(x)$
R_n	Classical Ramanujan prime $R_{1/2,n}$
h	$\log(1/c)$
q	$1-c$
L	$\log x$
M	$\log(n/q)$ in the generalized asymptotic analysis

Definition 1 (prime-window count). For fixed $c \in (0,1)$ and real $x \ge 0$, we define

$$W_c(x) := \pi(x) - \pi(cx). \quad (3)$$

Thus $W_c(x)$ is exactly the number of primes in the half-open interval $(cx, x]$. The half-open convention is important since a prime equal to cx is excluded, whereas a prime equal to x is included.

Definition 2 (c-Ramanujan prime). For $n \in N$, the n th c-Ramanujan prime is the least positive integer $R_{c,n}$ such that

$$W_c(x) \ge n \text{ for every real } x \ge R_{c,n}. \quad (4)$$

When $c=1/2$, we write $W(x) := W_{1/2}(x)$ and $R_n := R_{1/2,n}$. The prime number theorem implies $W_c(x) \to \infty$ for every fixed $c \in (0,1)$, so the threshold in Definition 2 exists.

2.2. Event structure

The function W_c is not monotone. The appearance of a new prime at the high end causes an increase in the number, while the passage of a prime at the low end leads to a decrease in the number in the window. This is what makes the first-hitting rule inadequate.

Lemma 3 (event structure). Let $c \in (0,1)$ be fixed. The function $W_c(x)$ is right-continuous and piecewise constant. Its only possible upward jumps occur at prime values $x=p$, and its only possible downward jumps occur at values $x=p/c$, where p is prime. Every individual jump has magnitude at most one in each direction.

Proof. The function $\pi(x)$ is right-continuous and increases by one precisely when x crosses a prime p. Therefore the first term of Equation (3) can create only a *+1* jump at $x=p$. The second term, $-\pi(cx)$, decreases by one precisely when cx crosses a prime, equivalently when $x=p/c$. If an upper and a lower event coincide, the two changes cancel and the net jump is zero. No other change is possible because both prime-counting terms are constant between their event points.

Lemma 3 turns the definition into a recovery problem. A threshold $R_{c,n}$ is the final upward event that restores the count to level n after its last excursion below that level.

Theorem 4 (threshold recovery). For every fixed $c \in (0,1)$ and $n \in \mathbb{N}$, the number $R_{c,n}$ is prime and satisfies

$$W_c(R_{c,n}) = n. \quad (5)$$

Proof. Let $A := \{x \geq 0 : W_c(x) < n\}$. The prime number theorem makes A bounded, while $0 \in A$, so $T := {}^{A}¿$ is finite. For every $\varepsilon > 0$, the definition of the supremum gives a point $x \in A$ with $T-\varepsilon < x \leq T$. Moreover, $W_c(T) \geq n$: otherwise, right-continuity and local constancy at T would place points of A strictly to the right of T, contradicting the definition of T. Thus the event at T contains a positive jump from a value at most $n-1$ to a value at least n. By Lemma 3, such a jump can occur only at a prime and has positive magnitude one. Consequently, T is prime and $W_c(T) = n$. Finally, $W_c(x) \geq n$ for every $x > T$, whereas every number smaller than T precedes a later deficit. Hence T is precisely the least persistent threshold $R_{c,n}$. This proves Equation (5).

For $c=1/2$, all lower events occur at even integers $2p$. Consequently, the entire event structure can be sampled exactly on $\mathbb{N}_0$. Define the integer window count by

$$d(k) := \pi(k) - \pi(\lfloor k/2 \rfloor), k \in \mathbb{N}_0. \quad (6)$$

Because $\pi(k/2) = \pi(\lfloor k/2 \rfloor)$ for integer k, Equation (6) equals $W(k)$.

Corollary 5 (last-deficit identity). For every $n \in \mathbb{N}$,

$$R_n = 1 + \max\{k \in \mathbb{N}_0 : d(k) = n-1\}. \quad (7)$$

Proof. Let K be the largest integer for which $d(K) < n$. Such a largest integer exists because $d(k) \to \infty$. The definition of K gives $d(k) \geq n$ for every integer $k \geq K+1$. Between consecutive integers, every downward event for $c=1/2$ occurs at an integer, so the same inequality holds for every real $x \geq K+1$. Thus $R_n = K+1$. The increment from K to $K+1$ cannot exceed one. It follows that $d(K) = n-1$. Conversely, any later occurrence of $n-1$ would violate the persistence condition. Hence K is the maximum appearing in Equation (7).

Example 1. For $n=4$, the last integer at which the window count equals three is $k=28$. Equation (7) therefore gives $R_4 = 29$. Although the count reached four earlier, later lower-endpoint events returned it to three. The value 29 is the final recovery, not the first hitting time.

This mechanism for the first ten thresholds is displayed in **Figure 1**. Upward jumps at primes compete with downward jumps at twice a prime. The marked points are precisely the last recoveries at their respective levels.

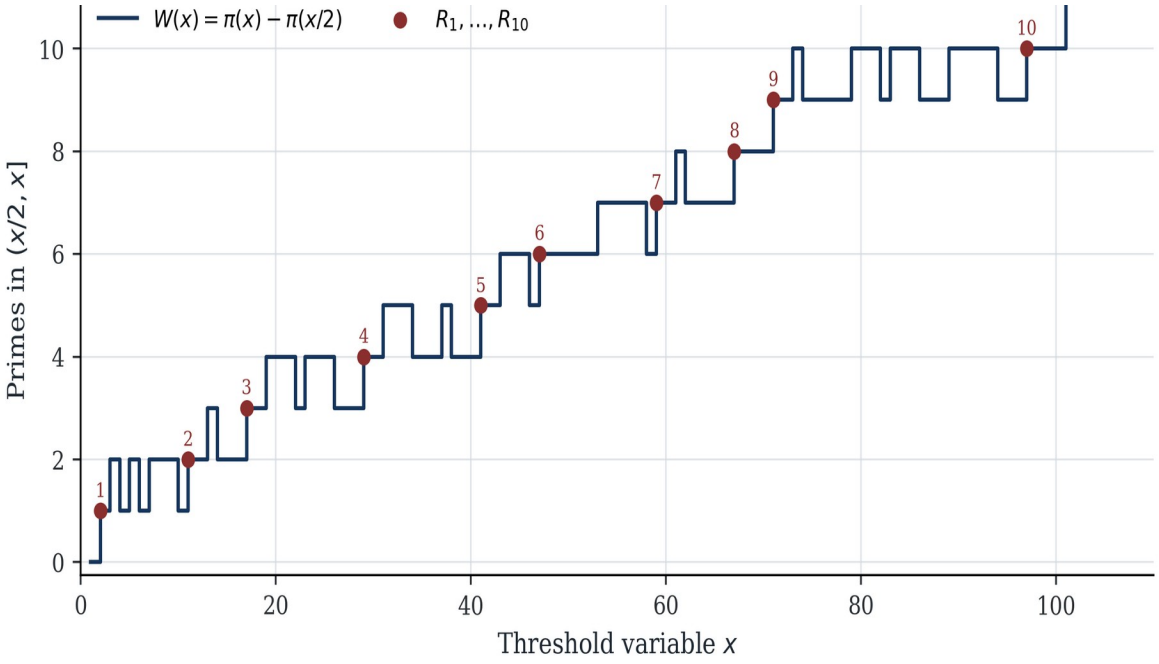

Figure 1. *The integer prime-window count* $W(x)=\pi(x)-\pi(x/2)$ *for* $1\le x\le 110$. *Red markers identify* $R_1,\ldots,R_{10}$.

Remark 1. The last-deficit identity is stronger than the statement that R_n is prime. It converts an infinite tail condition into the location of a final finite event, provided an independent upper certificate is available. Section 4 supplies that certificate.

3. Second-order asymptotics

3.1. Expansion of the generalized window count

The prime number theorem with its classical error term implies, for each fixed positive integer (J), the asymptotic expansion

$$\pi(x)=x\sum_{j=0}^{J-1}\frac{j!}{\log^{j+1}x}+O\left(\frac{x}{\log^{J+1}x}\right)(x\to\infty).\quad(8)$$

Only the first three terms are needed below. Fix $c\in(0,1)$, and define $q:=1-c$, $h:=\log(1/c)$, and $L:=\log x$. Since $\log(cx)=L-h$, Equation (8) gives

$$\pi(cx)=cx\left[\frac{1}{L-h}+\frac{1}{(L-h)^2}+\frac{2}{(L-h)^3}+O\left(\frac{1}{L^4}\right)\right].\quad(9)$$

The denominators in Equation (9) must be expanded in powers of *1/L*. Directly,

$$\begin{aligned}\frac{1}{L-h} &= \frac{1}{L}+\frac{h}{L^2}+\frac{h^2}{L^3}+O\left(L^{-4}\right),\\ \frac{1}{(L-h)^2} &= \frac{1}{L^2}+\frac{2h}{L^3}+O\left(L^{-4}\right), \qquad (10)\\ \frac{2}{(L-h)^3} &= \frac{2}{L^3}+O\left(L^{-4}\right).\end{aligned}$$

Substituting Equation (10) into Equation (9) and subtracting the result from the corresponding expansion of $\pi(x)$ yields the following proposition.

Proposition 6 (window-count expansion). For fixed $c \in (0,1)$, let

$$b_1 := 1-\frac{ch}{q}, b_2 := 2-\frac{c\left(2h+h^2\right)}{q}. \quad (11)$$

Then the generalized window count has the expansion

$$W_c(x) = \frac{qx}{L}\left(1+\frac{b_1}{L}+\frac{b_2}{L^2}+O\left(L^{-3}\right)\right)(x \to \infty). \quad (12)$$

Proof. From Equations (9) and (10),

$$\pi(cx) = cx\left[\frac{1}{L}+\frac{1+h}{L^2}+\frac{2+2h+h^2}{L^3}+O\left(L^{-4}\right)\right]. \quad (13)$$

Equation (8) with *J=3* gives

$$\pi(x) = x\left[\frac{1}{L}+\frac{1}{L^2}+\frac{2}{L^3}+O\left(L^{-4}\right)\right]. \quad (14)$$

Subtracting Equation (13) from Equation (14), the coefficients of L^{-1}, L^{-2}, and L^{-3} are respectively

$$q, q-ch, 2q-c\left(2h+h^2\right). \quad (15)$$

Factoring *qx/L* from the resulting series gives Equation (12) with the coefficients defined in Equation (11).

3.2. Asymptotic inversion

The leading term of Equation (12) recovers $R_{c,n} \sim p_{n/q}$. The next terms determine the displacement from that prime. To state the result compactly, set

$$y := \frac{n}{q}, M := \log y, l := \log M. \quad (16)$$

The parameter *c* is fixed while *n*→∞, so b_1 and b_2 are constants in the following inversion.

Theorem 7 (second-order generalized threshold). For every fixed $c \in (0,1)$,

$$R_{c,n} = y\left[M+l-b_1+\frac{l+b_1^2-b_1-b_2}{M}+O\left(\frac{l^2}{M^2}\right)\right](n \to \infty), \quad (17)$$

where *y,M,l* are defined in Equation (16), and b_1, b_2 are defined in Equation (11).

Proof. We write $R := R_{c,n}$ and $L := \log R$. Theorem 4 gives $W_c(R) = n$. Applying Proposition 6 at $x=R$ and dividing by q gives

$$y = \frac{R}{L}\left(1 + \frac{b_1}{L} + \frac{b_2}{L^2} + O\left(L^{-3}\right)\right). \quad (18)$$

Let $S := R/y$. Since the leading term in Equation (18) implies $S \sim \log y = M$, one has

$$L = \log R = M + \log S. \quad (19)$$

Solving Equation (18) algebraically for S, and using the geometric expansion of the reciprocal factor, gives

$$S = L - b_1 + \frac{b_1^2 - b_2}{L} + O\left(L^{-2}\right). \quad (20)$$

We now substitute the asymptotic form

$$S = M + l - b_1 + \frac{u}{M} + O\left(\frac{l^2}{M^2}\right), \quad (21)$$

where u is of order 1 and is to be determined. Equation (21) gives

$$\log S = l + \frac{l - b_1}{M} + O\left(\frac{l^2}{M^2}\right). \quad (22)$$

Combining Equations (19) and (22),

$$L = M + l + \frac{l - b_1}{M} + O\left(\frac{l^2}{M^2}\right). \quad (23)$$

Substitution of Equation (23) into the right-hand side of Equation (20) yields

$$S = M + l - b_1 + \frac{l - b_1 + b_1^2 - b_2}{M} + O\left(\frac{l^2}{M^2}\right). \quad (24)$$

Comparison of Equations (21) and (24) gives $u = l + b_1^2 - b_1 - b_2$. Finally, $R = yS$, which proves Equation (17).

Theorem 7 refines the established first-order relation for fixed c. Its coefficients have a direct interpretation. The term $-b_1$ measures the first effect of moving the lower endpoint from zero to cx, while the M^{-1} term combines the logarithmic inversion with the curvature of the prime-counting expansion.

3.3. The classical Ramanujan sequence

For $c=1/2$, one has $q=1/2$ and $h=\log 2$. Write $a := \log 2$. Equation (11) becomes

$$b_1 = 1 - a, \; b_2 = 2 - 2a - a^2. \quad (25)$$

Since $y=2n$, Theorem 7 immediately gives the classical expansion.

Corollary 8 (second-order formula for R_n). Let

$$M := \log(2n), \; l := \log M, \; a := \log 2. \quad (26)$$

Then

$$R_n = 2n\left[M + l + a - 1 + \frac{l + a - 2 + 2a^2}{M} + O\left(\frac{l^2}{M^2}\right)\right]. \quad (27)$$

The standard expansion for the $2n$ th prime, written using the same M and l, is

$$p_{2n} = 2n\left[M + l - 1 + \frac{l - 2}{M} + O\left(\frac{l^2}{M^2}\right)\right]. \quad (28)$$

Subtracting Equation (28) from Equation (27) isolates the displacement between the two sequences.

Corollary 9 (scaled displacement). As $n \to \infty$,

$$R_n - p_{2n} = 2n\left[a + \frac{a + 2a^2}{M} + O\left(\frac{l^2}{M^2}\right)\right]. \quad (29)$$

Consequently, Equation (2) holds.

Proof. Equation (29) follows by termwise subtraction of Equations (27) and (28). Division by $2n$ leaves a plus terms tending to zero, so the limit is $a = \log 2$.

Remark 2. The convergence in Equation (2) is slow. At the largest index studied here, $n = 100000$, the exact scaled displacement is 0.8319, still well above $\log 2 \approx 0.693147$. The second term in Equation (29), rather than a failure of the asymptotic law, explains most of this difference.

4. Certified finite computation

A finite scan of $d(k)$ does not by itself prove that an apparent threshold is final. A later lower-endpoint event could, in principle, create another deficit. Corollary 5 becomes a finite algorithm only when the scan endpoint is known to lie beyond every desired R_n.

Laishram's theorem gives exactly this information: $R_n < p_{3n}$ for $n > 1$ [9]. Therefore, if the first N Ramanujan primes are required and

$$B := p_{3N}, \quad (30)$$

then $R_n < B$ for every $2 \le n \le N$. The exceptional value $R_1 = 2$ is known directly. The stronger constant $41/47$ from Sondow, Nicholson, and Noe could reduce the scan range [10], but Equation (30) is simpler and already efficient at the present scale.

4.1. Algorithm

Algorithm 1 (certified last-deficit sieve).

Input: a positive integer N.

Output: the exact values $R_1, \ldots, R_N$.

1. First we compute $B = p_{3N}$.
2. Then use the sieve of Eratosthenes on $0, 1, \ldots, B$ to determine primality and the cumulative array $P(k) = \pi(k)$.
3. For every integer $0 \le k \le B$, compute
$$d(k) = P(k) - P(\lfloor k/2 \rfloor). \quad (31)$$
4. For each level $r = 0, 1, \ldots, N-1$, store

$$L_r = max\{k \le B : d(k) = r\}. (32)$$

5. Return $R_{r+1} = L_r + 1$ for $r=0,1,\ldots,N-1$.

Theorem 10 (algorithmic correctness). Algorithm 1 returns the exact first N Ramanujan primes.

Proof. For $n=1$, Equation (7) gives $R_1 = 2$. Fix $2 \le n \le N$. Laishram's bound gives $R_n < p_{3n} \le p_{3N} = B$. Hence the global last occurrence of the deficit level $n-1$ lies below B. Step 4 therefore finds the same maximum as the unrestricted maximum in Equation (7). Step 5 then returns R_n exactly. Since this argument holds for each $n \le N$, the entire output is exact.

The sieve requires $O(B \log\log B)$ arithmetic operations and $O(B)$ memory. The subsequent cumulative count and last-occurrence scan are linear in B. No probabilistic primality test, floating-point comparison, or fitted parameter enters the exact sequence computation.

4.2. Reproducibility and validation protocol

The computation used $N=100000$. Equation (30) then gives

$$B = p_{300000} = 4,256,233. (33)$$

Boolean primality values were generated by an Eratosthenes sieve; cumulative prime counts and all threshold indices were stored as exact integers. Three internal checks were applied:

1. every returned R_n was prime;
2. every returned value satisfied $d(R_n) = n$;
3. every $n>1$ satisfied $R_n < p_{3n}$.

An independent Wolfram Language calculation reconstructed the sequence from the defining prime counts up to $p_{3000} = 27449$ and agreed at every index $1 \le n \le 1000$. The symbolic series coefficients in Equations (12), (17), and (27) were also expanded independently. The computational comparison therefore tests both the discrete sequence and the analytic algebra, while the proof of Theorem 10 supplies the finite completeness certificate.

5. Results

5.1. Exact values and prime indices

Representative values from the certified dataset are given in **Table 2**. The column $\pi(R_n)/(2n)$ measures the position of R_n among all primes relative to the leading prediction $2n$. The final column reports the signed percentage error $\varepsilon_2(n) := 100\left[A_2(n) - R_n\right]/R_n$ of the second-order approximation $A_2(n)$ defined by the explicit part of Equation (27):

$$A_2(n) := 2n\left[M + l + a - 1 + \frac{l + a - 2 + 2a^2}{M}\right]. (34)$$

Table 2. *Certified Ramanujan primes and second-order accuracy.*

n	R_n	p_{2n}	R_n/p_{2n}	$\pi(R_n)$	**Index ratio**	$\varepsilon_2(n)$
1	2	3	0.666667	1	0.500000	-200.8081%
10	97	71	1.366197	25	1.250000	-16.7662%

n	R_n	p_{2n}	R_n/p_{2n}	$\pi(R_n)$	**Index ratio**	$\varepsilon_2(n)$
100	1,439	1,223	1.176615	228	1.140000	-3.9852%
1,000	19,403	17,389	1.115820	2,197	1.098500	-1.6271%
10,000	242,057	224,737	1.077068	21,394	1.069700	-0.1383%
100,000	2,916,539	2,750,159	1.060498	211,219	1.056095	-0.0337%

The small-index values demonstrate why an asymptotic formula must not be promoted to a global inequality. At *n=1*, even the ordering $p_{2n}<R_n$ has its known exception. From *n=100* onward, the second-order approximation improves steadily at the displayed milestones. At *n=100000*, its absolute error is about 9.82×10^{2}, only *0.0337%* of the exact threshold.

5.2. Leading and second-order convergence

The left panel of **Figure 2** shows R_n/p_{2n}. The ratio decreases toward one, in agreement with Sondow's theorem [8], but the finite-range gap remains visible. The right panel rescales the difference according to Corollary 9. The exact curve fluctuates because both sequences are discrete and prime gaps are irregular. Its central tendency is nevertheless tracked by the second-order curve

$$\log 2+\frac{\log 2+2(\log 2)^2}{\log(2n)}.\quad(35)$$

Equation (35) is the explicit part of Equation (29) after division by *2n*. It explains why the exact scaled displacement remains above *log2* throughout most of the computed range.

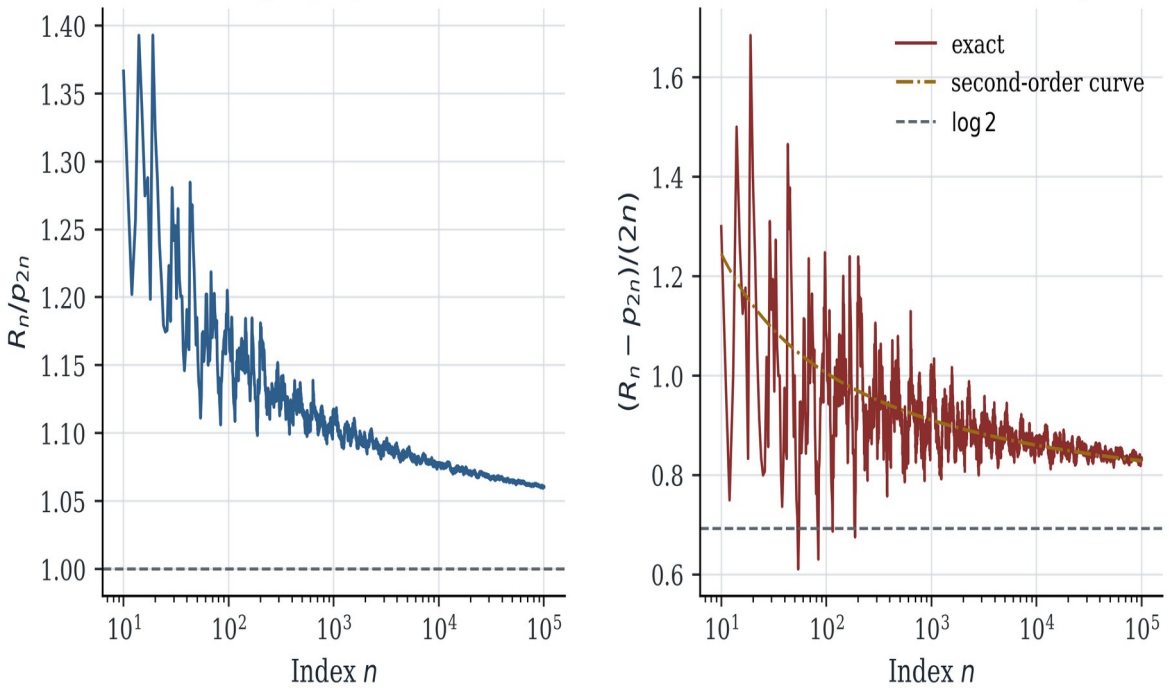

***Figure 2**. Left: the certified ratio R_n/p_{2n} for 10≤n≤100000. Right: the exact scaled displacement $(R_n - p_{2n})/(2n)$, the second-order curve from Equation (35), and its limiting value log2.*

5.3. Error hierarchy

To separate the value of successive terms, three approximations were compared:

$$\begin{aligned} A_0(n) &:= 2n[M + l - 1], \\ A_1(n) &:= 2n[M + l + a - 1], \\ A_2(n) &:= 2n\left[M + l + a - 1 + \frac{l + a - 2 + 2a^2}{M}\right]. \end{aligned} \tag{36}$$

Here A_0 is the standard two-term prime-index scale, A_1 adds the constant displacement *2nlog2*, and A_2 includes the complete M^{-1} correction from Corollary 8. **Figure 3** plots the absolute relative errors. The second-order expression gives the smallest typical error and an envelope that decreases markedly faster than the preceding approximations.

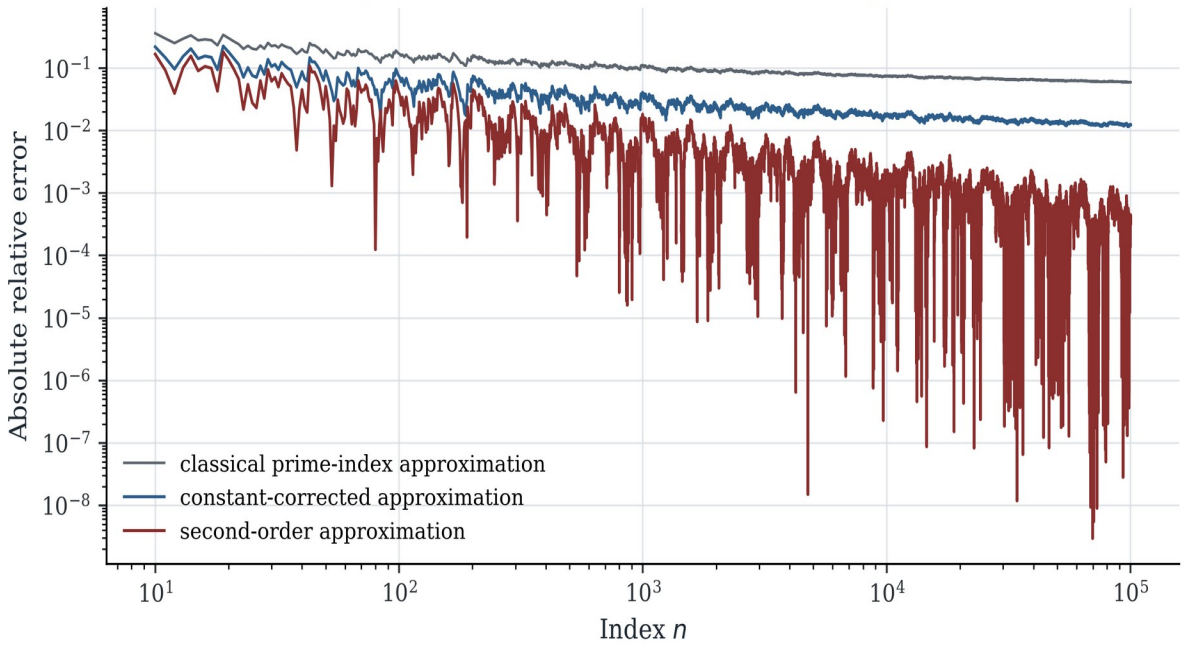

Figure 3. *Absolute relative errors of the three approximations in Equation (36) over $10 \le n \le 100000$.*

For all computed $n \ge 1000$, the maximum absolute relative error of A_2 was *1.6271%*. Restricting to $n \ge 10000$ reduced the maximum to *0.4517%*. These are descriptive finite-range results, not consequences of the big-O term in Equation (27). A rigorous explicit error bound would require numerical constants in the remainder of Equation (8) and a fully effective inversion.

5.4. Recovery structure and the optimal p_{3n} ratio

Within the first 100,000 values, the maximum of R_n / p_{3n} occurred at $n=5$ and equalled

$$\frac{R_5}{p_{15}} = \frac{41}{47} = 0.8723404255\ldots. \quad (37)$$

This exactly reproduces the global optimum proved by Sondow, Nicholson, and Noe [10]. It also acts as a useful end-to-end check: a shifted index, an incorrect endpoint convention, or a first-hitting algorithm would generally fail to recover the extremal equality in Equation (37).

6. Discussion

The stability formulation changes how R_n should be read. The quantity $W(x)$ records the occupancy of a moving multiplicative window. A prime entering at x increases occupancy; a prime leaving through $x/2$ decreases it. The Ramanujan prime is the last recovery from level $n-1$ to level n. This interpretation explains simultaneously why R_n is prime, why the sequence is sensitive to local prime gaps, and why an unverified finite scan is insufficient.

The asymptotic calculation identifies a systematic displacement that the leading equivalence conceals. The main term of $W(x)$ is one half of $x/\log x$, so matching $W(x)$ with n naturally places R_n near the $2n$ th prime. The lower endpoint $x/2$, however, samples the prime density at the smaller logarithm $\log x - \log 2$. This difference in density leads to the presence of 2n log 2 in Equation (29). Thus, log 2 is not a fitting parameter; instead, it represents the analytic factor related to halving the end point of the interval. In a

more general case when we have any fixed number c, the factor will be h = log(1/c) with the window width q = 1 - c. The factors b_1 and b_2 in Equation (11) contain this dependence on c. As c tends to 1, the window becomes narrower, and the factors become larger; however, the expansion holds for any fixed c < 1. A uniform narrow-window theory would require a joint limit in n and c, which is outside the present analysis.

Theorem 7 strengthens the resolution of the generalized leading relation $R_{c,n} \sim p_{n/(1-c)}$ proved by Amersi and co-authors [11]. It is consistent with the refined prime-index estimates developed by Axler [13,15] and Yang and Togbé [14]. The present derivation differs in emphasis and it works directly with the asymptotic expansion of the moving-window count and then performs an explicit inversion. This makes the origin of every c-dependent coefficient visible.

The certified algorithm is conceptually related to the computation methods described by Sondow, Nicholson, and Noe [10], but its purpose here is proof transparency rather than speed alone. Equation (7) identifies the exact datum to be stored, while Equation (30) shows why a finite endpoint suffices. The same principle can support faster segmented or bit-packed implementations without changing the proof of completeness.

6.1. Distinguishing the two Ramanujan prime-counting problems

The attached historical source centres on Ramanujan's broader theory of prime numbers and includes Equation (1). That equation and Ramanujan primes should not be merged. Equation (1) compares a quadratic expression in $\pi(x)$ with a scaled value of $\pi(x/e)$. Its principal problem is to determine a final validity threshold. Definition 2 concerns the number of primes in *(cx,x]*. Its principal objects are the recovery thresholds $R_{c,n}$. Dudek and Platt found that, under the Riemann hypothesis, *38,358,837,682* is the largest integer counterexample to Equation (1) [12]. Axler later obtained an unconditional upper threshold of $\exp(3158.442)$ [16]. Neither statement gives a value or bound for R_n. Conversely, the inequalities $R_n < p_{3n}$ and $R_n \le (41/47)\, p_{3n}$ do not determine the final threshold in Equation (1). The two subjects share the prime-counting expansion, but their quantifiers and event structures are different.

7. Conclusions

Ramanujan primes are best understood as stability thresholds of a non-monotone prime-window count. This viewpoint gives an exact last-deficit identity, explains the primality of each threshold, and leads to a finite computation whose completeness can be proved before the calculation begins. Using the certified endpoint p_{3N}, the first 100,000 values were computed exactly and independently checked through index 1,000. The analytic development refines the known leading equivalence. For fixed *0<c<1*, Theorem 7 gives a second-order expansion of $R_{c,n}$ with explicit c-dependent coefficients. In the classical case, it proves that the difference between R_n and p_{2n} is asymptotic to *2nlog2*, with a further correction of order *n/logn*. The numerical data follow this scale and show that the full second-order expression is already accurate to *0.0337%* at *n=100000*. The results lie in a clean separation of exact discrete structure, certified computation, and asymptotic analysis which also prevents confusion with Ramanujan's distinct prime-counting inequality and provides a rigorous base for future effective estimates.

Dedicated to…

*To the most brilliant mathematician, dear **Srinivasa Ramanujan**, who devoted his life to mathematics with purest imagination and faith in hidden order of numbers*

References

[1] J. Bertrand, “Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu’elle renferme,” *Journal de l’École Royale Polytechnique*, vol. 18, no. 30, pp. 123–140, 1845.

[2] P. L. Tchebichef, “Mémoire sur les nombres premiers,” *Journal de Mathématiques Pures et Appliquées*, ser. 1, vol. 17, pp. 366–390, 1852.

[3] J. Hadamard, “Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques,” *Bulletin de la Société Mathématique de France*, vol. 24, pp. 199–220, 1896. https://doi.org/10.24033/bsmf.545

[4] C.-J. de la Vallée Poussin, “Recherches analytiques sur la théorie des nombres premiers. Première partie,” *Annales de la Société Scientifique de Bruxelles*, vol. 20, pp. 183–256, 1896.

[5] S. Ramanujan, “A proof of Bertrand’s postulate,” *Journal of the Indian Mathematical Society*, vol. 11, pp. 181–182, 1919.

[6] J. B. Rosser, “The *n*-th prime is greater than *nlogn*,” *Proceedings of the London Mathematical Society*, ser. 2, vol. 45, pp. 21–44, 1938. https://doi.org/10.1112/plms/s2-45.1.21

[7] B. C. Berndt, *Ramanujan’s Notebooks, Part IV*. New York: Springer, 1994, ch. 24, pp. 111–137. https://doi.org/10.1007/978-1-4612-0879-2

[8] J. Sondow, “Ramanujan primes and Bertrand’s postulate,” *American Mathematical Monthly*, vol. 116, no. 7, pp. 630–635, 2009. https://doi.org/10.1080/00029890.2009.11920980

[9] S. Laishram, “On a conjecture on Ramanujan primes,” *International Journal of Number Theory*, vol. 6, no. 8, pp. 1869–1873, 2010. https://doi.org/10.1142/S1793042110003848

[10] J. Sondow, J. W. Nicholson, and T. D. Noe, “Ramanujan primes: bounds, runs, twins, and gaps,” *Journal of Integer Sequences*, vol. 14, Art. 11.6.2, 2011.

[11] N. Amersi, O. Beckwith, S. J. Miller, R. Ronan, and J. Sondow, “Generalized Ramanujan primes,” in *Combinatorial and Additive Number Theory: CANT 2011 and 2012*, Springer Proceedings in Mathematics & Statistics, vol. 101, pp. 1–13, 2014. https://doi.org/10.1007/978-1-4939-1601-6_1

[12] A. W. Dudek and D. J. Platt, “On solving a curious inequality of Ramanujan,” *Experimental Mathematics*, vol. 24, no. 3, pp. 289–294, 2015. https://doi.org/10.1080/10586458.2014.990118

[13] C. Axler, “On generalized Ramanujan primes,” *Ramanujan Journal*, vol. 39, no. 1, pp. 1–30, 2016. https://doi.org/10.1007/s11139-015-9693-9

[14] S. Yang and A. Togbé, “On the estimates of the upper and lower bounds of Ramanujan primes,” *Ramanujan Journal*, vol. 40, no. 2, pp. 245–255, 2016. https://doi.org/10.1007/s11139-015-9706-8

[15] C. Axler, “On Ramanujan primes,” *Functiones et Approximatio Commentarii Mathematici*, vol. 63, no. 1, pp. 67–93, 2020. https://doi.org/10.7169/facm/1824

[16] C. Axler, “On Ramanujan’s prime counting inequality,” *Mathematical Inequalities & Applications*, vol. 25, no. 4, pp. 1147–1154, 2022. https://doi.org/10.7153/mia-2022-25-71