Title: A Characterization of Gaussian Compositeness in Arithmetic Progressions: A Generalized Rational Sieve

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Abstract:

This paper introduces an algebraic sieve for identifying Gaussian primes within arithmetic progressions of the form N = ax + b. By mapping the primality of N onto a one-dimensional integer index x, we establish a parameterization x = (ap + b)n + p. We extend the domain of the parameters p and n to the field of Gaussian rationals $\mathbb{Q}(i)$, subject to specific integrality constraints on the resulting factors. We provide a formal proof of the sieve's completeness, demonstrating that the existence of a valid parameter pair (p, n) is both a necessary and sufficient condition for the compositeness of ax + b within the ring of Gaussian integers $\mathbb{Z}[i]$.

1. Introduction

The identification of prime numbers within arithmetic progressions is a foundational pursuit in number theory. While the Sieve of Sundaram offers a deterministic approach for identifying odd primes in the real integers, a similar structural sieve for the complex plane has remained less documented in elementary literature. This paper generalizes the Sundaram-type sieve structure to the Gaussian integers $\mathbb{Z}[i]$, providing a robust methodology for sieving Gaussian primes of the form ax + b, where x is a real integer index.

2. Theoretical Framework

Let a and b be fixed Gaussian integers. We consider the Gaussian integer N = ax + b. We define N as Gaussian composite if there exists a factorization N = αβ, where α and β are non-unit elements of $\mathbb{Z}[i]$ (i.e., Norm(α) > 1 and Norm(β) > 1).

We propose that the compositeness of N is fully characterized by the representation of its index x as:

x = (ap + b)n + p

Under the following constraints:

i. The parameters p and n belong to the field of Gaussian rationals $\mathbb{Q}(i)$.

ii. The term $\alpha = (ap + b)$ is an element of the Gaussian integers $\mathbb{Z}[i]$.

iii. The term $\beta = (an + 1)$ is an element of the Gaussian integers $\mathbb{Z}[i]$.

iv. Neither α nor β is a unit in $\mathbb{Z}[i]$.

3. Algebraic Derivation

The identity is established through the expansion of the product of two Gaussian integers α and β, where we assume α exists in the residue class of b modulo a:

$\alpha\beta = (ap + b)(an + 1)$

$\alpha\beta = a^2pn + ap + abn + b$

$\alpha\beta = a[(ap + b)n + p] + b$

By setting $N = \alpha\beta$ and $N = ax + b$, we find:

$ax + b = a[(ap + b)n + p] + b$

Subtracting the residue b and dividing by the modulus a (where $a \neq 0$) yields the sieve index:

$x = (ap + b)n + p$

4. Proof of Completeness (The Converse Theorem)

To establish the sieve as a complete characterization, we prove that any composite $N = ax + b$ admits a representation in this form.

Theorem: For every $N = \alpha\beta$ such that $N = ax + b$, there exists a pair (p, n) in $\mathbb{Q}(i)$ satisfying the parameterization.

Proof:

Given any non-unit factorization $\alpha\beta = ax + b$, we define the parameters as:

$p = (\alpha - b) / a$

$n = (\beta - 1) / a$

Substituting these definitions into the sieve formula $x = (ap + b)n + p$:

$x = (a * [(\alpha - b) / a] + b) * [(\beta - 1) / a] + (\alpha - b) / a$

$x = (\alpha - b + b) * [(\beta - 1) / a] + (\alpha - b) / a$

$x = [\alpha(\beta - 1) + \alpha - b] / a$

$x = (\alpha\beta - \alpha + \alpha - b) / a$

$x = (\alpha\beta - b) / a$

Since $\alpha\beta = ax + b$:

$x = (ax + b - b) / a = x$

This identity confirms that the parameterization captures all possible factorizations within the ring $\mathbb{Z}[i]$, ensuring that the sieve is algebraically complete.

5. Application Example: $a = 1$, $b = i$

We apply the sieve to the progression $N = x + i$. In this domain, the residue condition mod a is vacuous. The sieve simplifies to:

$x = (p + i)n + p$

Consider $x = 3$ ($N = 3 + i$). Given $3 + i = (1 + i)(2 - i)$:

$p = (1 + i) - i = 1$

$n = (2 - i) - 1 = 1 - i$

Applying the formula: $x = (1 + i)(1 - i) + 1 = 2 + 1 = 3$.

The existence of rational p, n resulting in integer factors α, β confirms that x = 3 is a composite index, and thus 3 + i is not a Gaussian prime.

6. Conclusion

This paper demonstrates that Gaussian primality can be determined through a one-dimensional sieve of indices. By allowing parameters to exist in the rational extension $\mathbb{Q}(i)$ while maintaining integrality constraints on the factors, we bridge the gap between real arithmetic progressions and the distribution of primes in the complex plane.