Perturbative String Theory and Its Non-Perturbative Probes

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Abstract

Perturbative string theory treats a moving string as a two-dimensional surface and studies the quantum field theory of the fields living on it. Removing the two gauge freedoms of that surface, the freedom to relabel its points and the freedom to rescale its metric, by the Faddeev–Popov method produces a nilpotent BRST charge Q_B, and its cohomology $H(Q_B)$ is the physical spectrum of the string. The condition $Q_B^2 = 0$ fixes the number of spacetime dimensions at twenty-six for the bosonic string and ten for the superstring. At one loop the surface is a torus, and its modular invariance chooses the allowed spectrum, the GSO projection of the superstring, and the charge lattices the theory can carry. The same degeneration limit gives unitarity, since the amplitude factorizes on physical intermediate states. When that intermediate state is a massless one emitted into the vacuum, its coefficient is a tadpole, and requiring the tadpole to vanish is what picks out an allowed background. For the unoriented open string this condition, equivalently the cancellation of gauge anomalies, forces the gauge group $SO(32)$. These results, and the tree and loop amplitudes built from them, make up the perturbative theory.

Perturbation theory is limited to weak coupling, and reaching strong coupling relies on a set of non-perturbative probes that supersymmetry protects. A single cylinder stretched between two D-branes can be understood in two ways, as a loop of open string and as a closed string exchanged across the gap, and equating the two fixes the tension of a D-brane and shows that its Ramond–Ramond charge is quantized in the smallest unit Dirac allows. The masses of the bound states the branes form, the (p, q) strings and the D0-brane threshold bound states, are set exactly by the supersymmetry algebra. Because these tensions, charges and masses do not move with the coupling they can be compared across two descriptions of the physics. The (p, q) strings fill out the $SL(2, \mathbb{Z})$ multiplet behind the S-duality of type IIB, and the D0-branes become the momentum modes of an eleventh dimension that opens up in strongly coupled type IIA. The same protected states count the microstates of a black hole and reproduce its Bekenstein–Hawking entropy, and D-branes sourcing a warped throat give a four-dimensional gauge theory that confines while keeping one massless particle.

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Abbreviations

CFT	conformal field theory
BRST	Becchi–Rouet–Stora–Tyutin
GSO	Gliozzi–Scherk–Olive
NS	Neveu–Schwarz
RR	Ramond–Ramond
SUSY	supersymmetry
BPS	Bogomol'nyi–Prasad–Sommerfield
WZW	Wess–Zumino–Witten
IR	infrared
VEV	vacuum expectation value
CY	Calabi–Yau
AdS	anti–de Sitter
QFT	quantum field theory
EFT	effective field theory
YM	Yang–Mills
PCO	picture-changing operator
ADE	A–D–E classification
OPE	operator product expansion
OCQ	old covariant quantization
CKG	conformal Killing group
R	Ramond
NSNS	Neveu–Schwarz–Neveu–Schwarz
SUGRA	supergravity
DBI	Dirac–Born–Infeld
KK	Kaluza–Klein
UV	ultraviolet
EOM	equation of motion
BFSS	Banks–Fischler–Shenker–Susskind
SCFT	superconformal field theory
QCD	quantum chromodynamics
RG	renormalization group
CKV	conformal Killing vector
BCJ	Bern–Carrasco–Johansson
M2/M5	M-theory membrane and fivebrane

Introduction

The incompatibility of general relativity with quantum mechanics is the defining problem of modern theoretical physics, and perturbative quantum gravity shows the obstruction already by dimensional analysis. In D spacetime dimensions Newton's constant has mass dimension $2-D$, so graviton loops are controlled by the dimensionless combination $G_N E^{D-2}$. This grows with energy, so each order in perturbation theory requires a new counterterm and the expansion loses predictive power in the ultraviolet. The difficulty comes from locality. A point particle interacts at a single spacetime point, and it is exactly at short distances that the gravitational coupling becomes strong [10, 363].

String theory removes this short-distance problem by changing what the path integral sums over. A point particle traces a one-dimensional worldline; a string traces a two-dimensional worldsheet, and the splitting or joining of strings is a smooth change in the topology of that surface. Because no single point carries the interaction, the ultraviolet region that spoils point-particle gravity has no counterpart here, and high-energy amplitudes are correspondingly soft. The dynamics of the embedding $X^\mu(\sigma)$ follows from the Polyakov action,

$$S_P = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{h}\,h^{ab}\partial_a X^\mu \partial_b X_\mu + \cdots\,, \tag{1}$$

which describes D worldsheet scalars X^μ coupled to a two-dimensional metric h_{ab}. After dividing by worldsheet diffeomorphisms and Weyl rescalings, h_{ab} carries no local degrees of freedom, and all that survives is the finite set of complex moduli of the surface. A scattering amplitude is then an integral over the moduli space of Riemann surfaces, with a two-dimensional conformal field theory and the bc ghosts inserted on each surface [1, 10, 360].

The particle content of the theory can be read directly from this worldsheet spectrum, and gravity appears already at the massless level. The closed string contains a massless spin-two excitation, the graviton. At level $N = \widetilde{N} = 1$ the corresponding closed-string vertex operator is

$$V_{\epsilon,k}(z,\bar z) = \epsilon_{\mu\nu}\,\partial X^\mu \bar\partial X^\nu e^{ik\cdot X}. \tag{2}$$

Requiring $V_{\epsilon,k}$ to have conformal weight $(1,1)$ gives $k^2 = 0$. BRST invariance and null-state equivalence impose

$$k^\mu \epsilon_{\mu\nu} = k^\nu \epsilon_{\mu\nu} = 0, \qquad \epsilon_{\mu\nu} \sim \epsilon_{\mu\nu} + k_\mu \xi_\nu + \widetilde{\xi}_\mu k_\nu. \tag{3}$$

The symmetric traceless part of $\epsilon_{\mu\nu}$ is the polarization of a massless spin-two field, and its low-energy couplings reproduce Einstein gravity in the α'-expansion. The antisymmetric part is the Kalb–Ramond two-form, and the trace is the dilaton.

The conditions just imposed on $V_{\epsilon,k}$ are a special case of a general rule. Gauge fixing the worldsheet metric produces a nilpotent BRST charge, $Q_B^2 = 0$, and the physical spectrum is its cohomology $H(Q_B)$; the weight-$(1,1)$ and transversality conditions above are the massless-level cohomology conditions written out. At one loop the torus has large diffeomorphisms, so its modulus is integrated over a fundamental domain, not over the full upper half-plane. When the surface degenerates at the boundary of moduli space, factorization inserts a complete set of physical intermediate states. In spacetime language these statements become the mass shell, gauge invariance, unitarity, tadpole cancellation, and anomaly cancellation [359, 360].

The link between worldsheet consistency and spacetime dynamics becomes clearer in a curved background. Coupling the worldsheet to slowly varying fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, and $\Phi(X)$ gives a nonlinear sigma model, and quantum Weyl invariance requires its beta functions to vanish. To first order in α',

$$\beta^G_{\mu\nu} = \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}\right) + O(\alpha'^2), \tag{4}$$

so $\beta^G_{\mu\nu} = 0$ is the Einstein equation, corrected by the two-form and the dilaton. The worldsheet theory therefore both quantizes the string and selects the spacetime backgrounds in which it can consistently propagate.

The five ten-dimensional superstring theories and eleven-dimensional supergravity are understood as limitations of M-theory, reached from one another by varying couplings and radii. These relations are non-perturbative, and establishing them requires protected quantities that survive beyond the formal expansion around $g_s = 0$.

The perturbative expansion that these non-perturbative relations must eventually go beyond is the genus expansion. For closed strings, a genus-h amplitude with n external insertions has the form

$$\mathcal{A}_{h,n} \sim g_s^{2h-2+n} \int_{\mathcal{M}_{h,n}} d\mu \left\langle \prod_{i=1}^{n} V_i \right\rangle_{\Sigma_h}, \tag{5}$$

where $\mathcal{M}_{h,n}$ is the space of inequivalent punctured surfaces left after diffeomorphisms and Weyl rescalings have been divided out. The power of g_s counts topology through the Euler characteristic. The measure $d\mu$ is generated by gauge fixing. The operators V_i must be physical vertex operators, of conformal weight $(1,1)$ for closed strings and weight one on the boundary for open strings. This weight condition is the spacetime mass-shell condition. A BRST-exact change of V_i, or a total derivative on the worldsheet, is a spacetime gauge transformation. When a surface degenerates, the amplitude factorizes through physical intermediate states.

The same things recur throughout perturbation theory. Gauge fixing the sum over metrics introduces the bc ghosts and leads to BRST cohomology. Modular invariance of the torus selects the allowed spin structures, GSO projections, and charge lattices. Degeneration of Riemann surfaces gives unitarity and explains why interacting open strings also require closed strings, since the cylinder can be read as either an open-string loop or a closed-string exchange. Tadpole and anomaly cancellation decide whether the background solves the spacetime equations of motion. In type I this fixes the gauge group to $SO(32)$. In the heterotic string it leaves precisely the $SO(32)$ and $E_8 \times E_8$ choices.

Perturbation theory only works when the coupling g_s is small, so reaching strong coupling can be treated differently by a small set of quantities that supersymmetry protects: their values are fixed once and do not change as the coupling is turned up, so they can be computed in the easy weak-coupling regime and then trusted where no direct calculation is possible. A non-perturbative probe, in what follows, means exactly such a protected quantity used to test a strong-coupling claim. The objects that carry these quantities are the D-branes, the surfaces on which open strings end. Each D-brane has a definite tension and a Ramond–Ramond charge that satisfies Dirac quantization, both fixed at weak coupling, and the masses of its bound states with strings and with other branes are set exactly by the supersymmetry algebra. Comparing one of these protected values in two different descriptions of the same system is how a duality is established. For instance, strongly coupled type IIA behaves as an eleven-dimensional theory in which a new circular dimension of radius $R_{11} = g_s \ell_s$ opens up, and the D0-branes are precisely the states carrying momentum around that circle. The same counting reappears for black holes, where the states of a bound system of branes can be enumerated one by one, and their number matches the Bekenstein–Hawking entropy.

Compactification is also a statement about the worldsheet CFT. For a string on a circle, the spectrum is unchanged under

$$R \longleftrightarrow \frac{\alpha'}{R}, \qquad n \longleftrightarrow w, \tag{6}$$

so a small radius and a large radius can describe the same physics, with momentum and winding exchanged. Geometry is read from the worldsheet spectrum, and it need not be unique. Orbifolds, rational conformal field theories, Gepner models, and Calabi–Yau sigma models give further examples, where spacetime geometry is sometimes only a large-radius description and the conformal field theory gives a more precise definition.

The same D-brane systems give a concrete way to study strongly coupled gauge theory. A stack of N D3-branes is described at once by a four-dimensional gauge theory on its worldvolume and by the curved throat geometry it sources, and the equivalence of the two descriptions is the gauge/gravity correspondence. Deforming the throat realizes an $\mathcal{N} = 1$ gauge theory that flows to strong coupling and confines, with the infrared end of the geometry closing off smoothly at a size set by dimensional transmutation. Confinement and a mass gap are not the same statement. In the Klebanov–Strassler background a baryonic symmetry is spontaneously broken, and the resulting Goldstone boson stays massless even though the theory confines, so the smooth cap of the geometry does not by itself imply a gap in the spectrum.

The notes proceed as follows. Chapters 1–11 build the perturbative theory from the Polyakov path integral, BRST cohomology, the no-ghost theorem, tree amplitudes, one-loop modular invariance, compactification, superstrings, heterotic strings, and anomaly cancellation. Chapters 12–13 then use D-branes and BPS charges to connect the weak-coupling constructions to T-duality, S-duality, Matrix theory, and M-theory. The remaining chapters develop the conformal-field-theory and geometric methods needed for compactification and four-dimensional effective physics. The path-integral and Gaussian-determinant conventions applied to two-dimensional fields are collected at the start of Chapter 1, and a single appendix gathers the dimension-by-dimension spinor and supersymmetry facts that the superstring chapters use repeatedly.

Sources. The development follows the standard treatment of Polchinski, *String Theory* Vols. I–II [1, 2], whose organization it broadly shares, and uses the lecture notes of Tong [10] and Minwalla [3], together with the primary literature cited at each result.

Chapter 1

The Polyakov path integral and worldsheet gauge symmetry

The perturbative theory is founded on the Polyakov path integral. This is a weighted sum over worldsheet geometries and over their embeddings in spacetime. Its definition requires care because the worldsheet metric is a redundant variable. Two configurations related by a diffeomorphism or a Weyl rescaling describe the same physics. The unrestricted integral therefore diverges by the volume of this local symmetry group. The Faddeev–Popov procedure removes the redundancy by replacing the gauge volume with a functional determinant. The anticommuting b, c ghost system represents this determinant. Its central charge, $c_{bc} = -26$, enters the Weyl anomaly together with the matter contribution and fixes the critical dimension. The topological term weights a surface of Euler characteristic χ by $g_s^{-\chi}$, so the perturbation series is an expansion in worldsheet genus. Physical external states are represented by integrated vertex operators of conformal weight $(1, 1)$, and the spacetime mass-shell conditions follow from that weight. Slowly varying massless polarizations convert the free worldsheet theory into a nonlinear sigma model. Weyl invariance then requires the beta functions to vanish, giving the Einstein equations together with their α' corrections. The cylinder identity shows that an interacting open-string theory necessarily contains closed strings [1, 10].

Start with the worldsheet itself. For one free string the answer is precisely the two-dimensional surface swept out by the string as it moves through spacetime. The embedding fields are maps

$$X^\mu : \Sigma \longrightarrow \mathbb{R}^{1,D-1},$$

where Σ is the abstract two-dimensional surface and X^μ tells us where each point of that surface sits in target spacetime. Strictly speaking, one often allows immersions, not just embeddings,, because a surface in the path integral need not be globally without self-intersections.

For an open string one may use coordinates

$$-\infty < \tau < \infty, \qquad 0 \le \sigma \le \pi,$$

so $\sigma = 0$ and $\sigma = \pi$ are the two endpoints. For a closed string the spatial coordinate is periodic,

$$-\infty < \tau < \infty, \qquad \sigma \sim \sigma + 2\pi,$$

and each constant-τ slice is a circle. Thus τ labels how the string evolves along the surface, while σ labels position along the string at a fixed instant of worldsheet time. This is the elementary σ, τ picture: a particle sweeps out a line, but a string sweeps out a surface [1, 3, 10].

1.1 Sums over worldsheets

The free-cylinder picture is only the starting point. Once interactions are allowed, strings split and join, and the worldsheet is no longer just one cylinder or one strip. The joining region is still smooth as a two-dimensional surface. There is no preferred point at which the interaction "happens". A different choice of worldsheet time can cut the same surface in a different way, as Figure 1.1 illustrates. This is the geometric reason string perturbation theory replaces point-particle vertices by smooth worldsheets.

The surfaces used in scattering amplitudes are therefore best thought of as Riemann surfaces

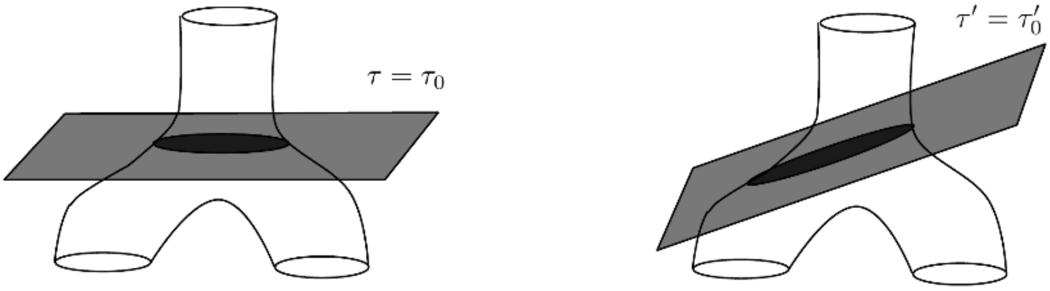


Figure 1.1: The interaction region on a worldsheet has no invariant time. One slicing by $\tau = \tau_0$ cuts the joining region in one way, while another allowed worldsheet coordinate system (σ', τ') places the same smooth process at a different slice τ'_0. The splitting and joining of strings is therefore part of the geometry of the surface, not a local vertex inserted at a distinguished time. From Ref. [17].

with punctures or boundaries. The semi-infinite external cylinders represent incoming and outgoing strings. After Wick rotation, a cylinder can be described by a complex coordinate; for example,

$$(\sigma, \tau_E) \longmapsto z = e^{\tau_E - i\sigma}.$$

The far past and far future become punctures at $z = 0$ and $z = \infty$, and the corresponding external states are inserted there as vertex operators. Thus the perturbative sum is not over the free worldsheets $\mathbb{R} \times S^1$ or $\mathbb{R} \times [0, \pi]$. It is a sum over all allowed two-dimensional topologies obtained by sewing such pieces together, with the external ends represented by marked points or boundary insertions. This is the practical meaning of the "sum over worldsheets" in string perturbation theory.

Open strings cannot be separated from closed strings once interactions are included. The annulus can be read in two equivalent ways.

1. An open-string loop also has a closed-string description. The annulus amplitude may be written either as a trace over open-string states or, after the modular change of variable $\ell = 1/t$, as propagation of a closed string between two boundary states:

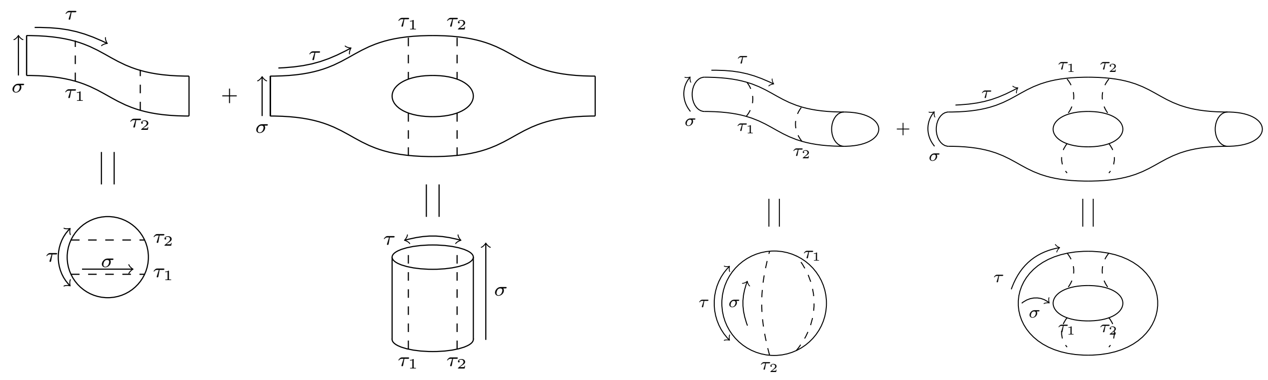
$$\mathcal{A}_{\text{cyl}} = \int_0^\infty \frac{dt}{2t} \operatorname{Tr}_{\mathcal{H}_{\text{open}}} \left(e^{-2\pi t (L_0 - a)}\right) = \int_0^\infty d\ell \, \langle B | e^{-\pi \ell (L_0 + \widetilde{L}_0 - 2a)} | B \rangle. \tag{1.1}$$

   Poles in the second expression are closed-string poles. Thus the closed-string spectrum is already present in the factorization of the one-loop open-string amplitude [10, 40].

2. Forbidding this closed-string interpretation would not be a local rule on the worldsheet. The annulus is a perfectly allowed open-string diagram, and the same Riemann surface has the closed-string degeneration described in Eq. (1.1). Removing the closed-string poles would spoil factorization of the amplitude on physical intermediate states. The consistent perturbative expansion therefore contains both open and closed sectors whenever open strings interact.

A closed string is a circle, so its free propagation sweeps out a cylinder, and the elementary interaction of one string splitting into two is a pair of pants. The coordinate σ runs around the string, while τ runs along the propagation direction. Constant-time slices such as τ_1 and τ_2 cut the surface into copies of the string at different instants of worldsheet time.

If the external legs are left open, these surfaces describe amplitudes with incoming and outgoing closed strings. If each external leg is capped by a disc, the same pictures become compact surfaces: the cylinder closes into a sphere, and the one-loop double pair of pants closes into a torus (Figure 1.2). This is also the geometric origin of radial quantization. Under the radial map, a constant-τ circle is sent to a circle in the complex plane, so a state living on a circle can equivalently be represented by a local operator inserted at the corresponding puncture.



(a) External legs left open.

(b) External legs capped by discs.

Figure 1.2: Closed-string worldsheet cobordisms. A propagating closed string gives a cylinder, and a splitting or joining gives a pair of pants. Leaving the external legs open represents scattering states, while capping the legs produces compact surfaces: the sphere at genus zero and the torus at genus one.

1.2 Path-integral conventions and Gaussian integrals

Before applying the path integral to the worldsheet it is worth recording the Gaussian integrals it rests on. Their outputs, $(\det A)^{-1/2}$ for a bosonic field and $\det D$ for a fermionic one, enter all determinant factors in string perturbation theory, and the sign difference between the two is what lets worldsheet fermions cancel bosonic modes in a supersymmetric partition function. The same functional integral reproduces the Hilbert-space evolution operator, the bridge between the covariant and operator descriptions used later [8, 10].

1.2.1 Bosonic Gaussian integrals

For a real bosonic field with quadratic operator A, the Gaussian path integral gives the determinant rule

$$\int \mathcal{D}\phi \, \exp\left[-\frac{1}{2}\phi A\phi + J\phi\right] \propto (\det A)^{-1/2} \exp\left[\frac{1}{2}JA^{-1}J\right]. \tag{1.2}$$

This is the origin of the determinants that appear in gauge fixing, ghost path integrals, and one-loop string amplitudes.[1] [8].

1.2.2 Relation to the Hilbert-space formalism

The path integral computes the kernel of the time-evolution operator. For a particle coordinate q,

$$\langle q_f, t_f | q_i, t_i\rangle = \langle q_f | e^{-iH(t_f - t_i)} | q_i\rangle = \int_{q_i}^{q_f} \mathcal{D}q \, e^{iS[q]}. \tag{1.3}$$

The derivation by time slicing is the finite-dimensional ancestor of the worldsheet path integral, obtained by inserting complete sets of states, evaluating short-time kernels, and taking the continuum limit.[2] [8, 10].

[1] Perturbation theory follows by expanding around the Gaussian integral. With A the quadratic operator and V an interaction insertion, $\log\det(A+V) = \log\det A + \mathrm{Tr}(A^{-1}V) - \frac{1}{2}\mathrm{Tr}(A^{-1}VA^{-1}V) + \cdots$, where A^{-1} is the propagator and the traces are loop diagrams.

[2] A Wick rotation $t = -i\tau$ turns oscillatory amplitudes into statistical weights, $e^{iS_M} \to e^{-S_E}$ and $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$, which is why the Polyakov path integral is usually written with a Euclidean worldsheet metric, where the measure is better behaved and the modular parameters become complex-structure moduli.

1.2.3 Fermionic Gaussian integrals

Grassmann integration reverses the determinant power relative to bosons,

$$\int d\bar{\psi}\, d\psi\, \exp(-\bar{\psi}D\psi) = \det D, \qquad \int d\psi\, \exp\left(-\frac{1}{2}\psi A\psi\right) = \mathrm{Pf}(A). \tag{1.4}$$

This sign and determinant structure is what makes worldsheet fermions cancel bosonic degrees of freedom in supersymmetric partition functions. [121].

1.3 The Polyakov path integral

The Polyakov action is written with a Euclidean worldsheet metric.[3] Including the topological term, it is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^{\mu}\partial_b X_{\mu} + \lambda\chi \\ &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^{\mu}\partial_b X_{\mu} + \frac{\lambda}{4\pi}\int_M d^2\sigma\sqrt{g}R + \frac{\lambda}{2\pi}\int_{\partial M} dsk \end{aligned} \tag{3.2.1}$$

where χ is the Euler characteristic and k is the geodesic curvature. The expression for χ for genus (number of handles) g and b boundaries is as follows:

$$\chi = 2 - 2g - b$$

Using Eq. (3.2.1), the Polyakov path integral is as follows:

$$\int [dX dg] e^{-S} \tag{3.2.2}$$

The worldsheet metric can be eliminated classically, before any gauge fixing, and doing so recovers the Nambu–Goto action and shows that h_{ab} is an auxiliary field. In Lorentzian signature write

$$\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu, \qquad S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-h}\,h^{ab}\gamma_{ab}.$$

Varying h^{ab} gives the metric equation

$$T_{ab} = 0, \qquad T_{ab} = \gamma_{ab} - \frac{1}{2}h_{ab}\,h^{cd}\gamma_{cd}.$$

This equation is traceless identically in two dimensions. It therefore fixes only the conformal class of h_{ab}. Taking the determinant of the equation gives

$$\det\gamma = \det h\left(\frac{1}{2}h^{cd}\gamma_{cd}\right)^2, \qquad \sqrt{-\det\gamma} = \frac{1}{2}\sqrt{-h}\,h^{cd}\gamma_{cd},$$

with the sign chosen for the same Lorentzian orientation. Substituting this relation into S_P gives

$$S_P = -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\sqrt{-\det\gamma} = S_{\mathrm{NG}}.$$

Thus h_{ab} is auxiliary, but it is not literally the induced metric; its equation gives $h_{ab} = e^{2\omega}\gamma_{ab}$. In conformal gauge the remaining equations are

$$\partial_+\partial_- X^\mu = 0, \qquad T_{++} = (\partial_+ X)^2 = 0, \qquad T_{--} = (\partial_- X)^2 = 0.$$

The $+-$ component is the trace equation and carries no new condition. This is the classical origin of the Virasoro constraints imposed later in the operator and BRST treatments [3, 10].

[3] Conventions used from this point on are as follows. The target-space metric is mostly plus, $\eta_{\mu\nu} = \mathrm{diag}(-,+,\ldots,+)$, with spacetime indices $\mu,\nu = 0,\ldots,D-1$. Worldsheet indices are $a,b \in \{1,2\}$ in real coordinates σ^1,σ^2, and the Euclidean worldsheet is used in all path-integral and modular arguments. Complex coordinates are $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$, with $d^2z \equiv 2\,d\sigma^1 d\sigma^2$, so $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Holomorphic and antiholomorphic quantities are distinguished by a tilde, for example L_n versus $\widetilde{L}_n$ and α_n^μ versus $\widetilde{\alpha}_n^\mu$. The Regge slope α' is kept explicit; $\ell_s = \sqrt{\alpha'}$ and $g_s = e^{\Phi_0}$ for constant dilaton background Φ_0. The free-scalar OPE and stress tensor are normalized by $X^\mu(z,\bar{z})X^\nu(0,0) \sim -(\alpha'/2)\eta^{\mu\nu}\ln|z|^2$ and $T(z) = -(1/\alpha'){:}\partial X^\mu \partial X_\mu{:}$. A holomorphic weight-$h$ primary obeys $T(z)\mathcal{O}(0) \sim hz^{-2}\mathcal{O}(0) + z^{-1}\partial\mathcal{O}(0)$. A weight-$\lambda$ first-order bc system has $c_\lambda = -3(2\lambda-1)^2 + 1$; for the reparametrization ghosts $\lambda = 2$, hence $c_{bc} = -26$.

For perturbative amplitudes one usually works with the Euclidean worldsheet path integral obtained by analytic continuation from Lorentzian signature. This Euclidean form is used because the genus expansion is then a sum over compact Riemann surfaces and their boundaries. After the division by diffeomorphisms and Weyl transformations, Eq. (3.2.2) becomes

$$Z = \sum_{h,b} e^{-\lambda\chi(h,b)} \int \frac{[dX\,dg]}{\mathrm{Vol}(\mathrm{Diff}\times\mathrm{Weyl})} \exp\left[-\frac{1}{4\pi\alpha'}\int_{\Sigma_{h,b}} d^2\sigma\sqrt{g}\,g^{ab}\partial_a X^\mu\partial_b X_\mu\right], \qquad \chi(h,b) = 2-2h-b. \tag{1.5}$$

If we define $g_s = e^\lambda$, the topological factor is

$$e^{-\lambda\chi} = g_s^{-\chi}. \tag{1.6}$$

This convention gives the standard perturbative counting. Adding one boundary changes

$$\chi(h,b+1) = \chi(h,b)-1, \qquad g_s^{-\chi(h,b+1)} = g_s^{-\chi(h,b)}g_s. \tag{1.7}$$

Adding one handle changes

$$\chi(h+1,b) = \chi(h,b)-2, \qquad g_s^{-\chi(h+1,b)} = g_s^{-\chi(h,b)}g_s^2. \tag{1.8}$$

Thus an extra open-string loop carries one extra power of g_s, while an extra closed-string loop carries two powers of g_s. With the usual normalization of external vertices this gives

$$g_o^2 \propto g_s, \qquad g_c \propto g_s, \qquad g_o^2 \propto g_c. \tag{1.9}$$

External open strings should be treated as vertex insertions on a boundary, not by modifying the Euler characteristic with a term proportional to the number of endpoints. The Euler characteristic counts the topology of the worldsheet itself. Punctures and boundary insertions change the amplitude through the vertex-operator normalization and through the moduli integration, not through a correction such as $-n_c/4$ in χ.

1.4 Gauge fixing

Eq. (3.2.2) becomes the following if we divide by the volume of redundant transformations;

$$\int \frac{[dX dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S[X,g]} \tag{3.3.1}$$

We want the metric to have some functional form (called the fiducial metric $\hat{g}_{ab}$).

1.4.1 The Faddeev–Popov determinant

To make the metric path integral well defined, the redundant integration over gauge-equivalent metrics must be factored out. This is achieved by inserting a resolution of the identity that fixes the metric to a fiducial representative $\hat{g}_{ab}(m)$ on each gauge orbit, where m labels the residual moduli. simply, one writes

$$1 = \Delta_{\mathrm{FP}}[\hat{g}] \int D\xi\, D\omega\, \delta\Big(g_{ab}^{\xi,\omega} - \hat{g}_{ab}(m)\Big), \tag{1.10}$$

where ξ^a is the infinitesimal diffeomorphism parameter and ω is the infinitesimal Weyl parameter. The linearized metric variation is

$$\delta g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a + 2\omega g_{ab} = (P_1\xi)_{ab} + 2\widetilde{\omega}g_{ab}, \tag{1.11}$$

where $(P_1\xi)_{ab}$ is the traceless part of $\nabla_a\xi_b + \nabla_b\xi_a$. The Jacobian for the change of variables from (ξ,ω) to metric deformations is the Faddeev–Popov determinant. For a finite-dimensional matrix M,

$$\det M = \int d\bar{\eta}\, d\eta\, \exp(-\bar{\eta}M\eta), \tag{1.12}$$

with anticommuting variables $\eta, \bar{\eta}$. Applying this identity to the operator P_1 gives the b, c ghost path integral,

$$\Delta_{\mathrm{FP}} = \int Db\, Dc\, \exp\left[-\frac{1}{2\pi}\int d^2z\, \big(b\,\bar{\partial}c + \bar{b}\,\partial\bar{c}\big)\right]. \tag{1.13}$$

The quick check is the sign of the statistics: ξ^a is a bosonic gauge parameter, so its determinant is represented by anticommuting ghosts. This ghost system has central charge $c_{bc} = -26$, which is the number used in the Weyl-anomaly calculation below [10].

The index structure of the ghosts follows from what P_1 relates. The gauge parameter ξ^a is a worldsheet vector, while its image $(P_1\xi)_{ab}$ is a symmetric traceless tensor, so representing the determinant of P_1 as a Grassmann integral introduces an anticommuting ghost c^a carrying the vector index of ξ^a and an anticommuting antighost b_{ab} carrying the symmetric traceless index structure of the metric deformation it is contracted against. In conformal gauge $\hat{g}_{ab} = \delta_{ab}$ a symmetric traceless tensor has only the components b_{zz} and $b_{\bar{z}\bar{z}}$, and a vector only c^z and $c^{\bar{z}}$; writing $b \equiv b_{zz}$, $c \equiv c^z$ together with their antiholomorphic partners $\bar{b} \equiv b_{\bar{z}\bar{z}}$, $\bar{c} \equiv c^{\bar{z}}$, the operator P_1 reduces to $\bar{\partial}$ acting on c (and ∂ acting on $\bar{c}$), which is exactly the holomorphically factorized action in Eq. (1.13).

The central charge of this system is fixed by the conformal weights of b and c. An anticommuting first-order system with b of weight λ and c of weight $1 - \lambda$ has central charge

$$c_\lambda = -3(2\lambda - 1)^2 + 1. \tag{1.14}$$

The reparametrization antighost b_{ab} is a weight-2 field, so $\lambda = 2$ and c has weight -1, giving

$$c_{bc} = -3(2 \cdot 2 - 1)^2 + 1 = -26. \tag{1.15}$$

Together with the contribution $c_X = D$ of the D free bosons X^μ, this is the number that enters the Weyl-anomaly cancellation of the next section and fixes $D = 26$ [1, 10].

1.5 The Weyl anomaly

Classically, the Polyakov action is invariant under a local Weyl rescaling $g_{ab} \mapsto e^{2\omega} g_{ab}$. After quantization the functional measure may acquire a Weyl dependence. In conformal gauge the matter fields contribute $c_X = D$, while the reparametrization ghosts contribute $c_{bc} = -26$. The coefficient of the Weyl anomaly is therefore

$$c_{\text{tot}} = c_X + c_{bc} = D - 26. \tag{1.16}$$

The bosonic string requires $c_{\text{tot}} = 0$, hence $D = 26$.

We can see the same coefficient from the stress tensor. For a two-dimensional CFT, a conformal map $z \mapsto w(z)$ gives

$$T(w) = \left(\frac{dz}{dw}\right)^2 \left[T(z) - \frac{c}{12}\{w, z\}\right], \qquad \{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2. \tag{1.17}$$

The Schwarzian term is absent in a purely classical tensor transformation. It is the local signal of the anomaly. Equivalently, the effective action changes under $g_{ab} = e^{2\omega}\hat{g}_{ab}$ as

$$W[e^{2\omega}\hat{g}] - W[\hat{g}] = -\frac{c_{\text{tot}}}{24\pi} \int d^2\sigma \sqrt{\hat{g}} \left[(\hat{\nabla}\omega)^2 + \hat{R}\,\omega\right], \tag{1.18}$$

up to convention-dependent signs in the definition of $W = -\log Z$. The Weyl factor disappears from physical amplitudes only when the coefficient in Eq. (1.18) vanishes.

For integrated vertex operators one must also impose the residual conformal-weight condition. If a closed-string local operator has weights $(h, \bar{h})$, then

$$d^2z\, \mathcal{V}_{h,\bar{h}}(z, \bar{z}) \mapsto \left(\frac{dw}{dz}\right)^{h-1} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}-1} d^2w\, \mathcal{V}_{h,\bar{h}}(w, \bar{w}). \tag{1.19}$$

The integrated insertion is invariant only for $h = \bar{h} = 1$. This is why the same Weyl-anomaly calculation must be used together with the vertex-operator mass-shell conditions [10].

1.5.1 Inclusion of boundaries

For an open string the worldsheet has a boundary, so the Polyakov action must be varied with boundary terms included. The topological part uses the Gauss–Bonnet formula

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g}\, R + \frac{1}{2\pi} \int_{\partial\Sigma} ds\, k. \tag{1.20}$$

The kinetic term gives the boundary variation

$$\delta S_P\big|_{\partial\Sigma} = \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\,\delta X_\mu\,\partial_n X^\mu. \tag{1.21}$$

This vanishes either by imposing Neumann boundary conditions $\partial_n X^\mu = 0$ or Dirichlet boundary conditions $\delta X^\mu = 0$ in a chosen target-space direction. The same boundary choice makes the normal flow of worldsheet momentum vanish,

$$T_{\sigma\tau}|_{\partial\Sigma} = 0. \tag{1.22}$$

Thus the boundary discussion follows directly from the Polyakov action in Eq. (3.2.1): the boundary term in topology counts open-string worldsheets, while the boundary variation fixes the allowed endpoint conditions [10, 107].

Scattering amplitudes.

String scattering amplitudes are obtained by inserting physical vertex operators and integrating over the moduli of the punctured worldsheet. For n closed-string insertions at genus h one writes schematically

$$\mathcal{A}_{h,n} \sim g_s^{2h-2+n}\int_{\mathcal{M}_{h,n}} d\mu\,\left\langle \prod_{i=1}^{n} V_i(k_i)\right\rangle_{\Sigma_h}. \tag{1.23}$$

The powers of g_s come from Eq. (1.6) together with the normalization of the external closed-string vertices. Degeneration limits of $\mathcal{M}_{h,n}$ give the expected poles from intermediate physical string states, which is the worldsheet form of factorization [10, 17].

1.6 Vertex operators

The vertex operator of a closed string tachyon must have a factor of g_c because an extra string is added. We adopt a normalization to define the closed string tachyon operator as follows:

$$V_0 = 2g_c\int d^2\sigma\,\sqrt{g}\,{:}e^{ik\cdot X}{:}\,. \tag{1.24}$$

On a flat Euclidean worldsheet we pass to the complex coordinates

$$z = \sigma^1 + i\sigma^2, \qquad \bar z = \sigma^1 - i\sigma^2, \tag{1.25}$$

and adopt the standard area measure $d^2z \equiv 2\,d\sigma^1 d\sigma^2$, so that $\int d^2z = 2\int d^2\sigma$ in the flat gauge $\sqrt{g} = 1$ used here. With this measure the factor of two in Eq. (1.24) is absorbed exactly, with no residual normalization ambiguity, and the same operator reads

$$V_0 = g_c\int d^2z\,{:}e^{ik\cdot X(z,\bar z)}{:}\,. \tag{1.26}$$

After conformal gauge fixing, the residual transformations are holomorphic and antiholomorphic changes of coordinates. A local primary field with weights $(h,\bar h)$ transforms as

$$\mathcal{V}_{h,\bar h}(z,\bar z) = \left(\frac{dw}{dz}\right)^{h}\left(\frac{d\bar w}{d\bar z}\right)^{\bar h}\mathcal{V}_{h,\bar h}(w,\bar w). \tag{1.27}$$

The measure transforms with weight $(-1,-1)$, so the integrated closed-string vertex $\int d^2z\,\mathcal{V}_{h,\bar h}$ is invariant only when $h = \bar h = 1$, as in Eq. (1.19). For the free boson,

$$:e^{ik\cdot X}: \quad \text{has} \quad h = \bar h = \frac{\alpha' k^2}{4}. \tag{1.28}$$

The tachyon vertex is therefore physical when

$$\frac{\alpha' k^2}{4} = 1, \qquad m^2 = -k^2 = -\frac{4}{\alpha'}. \tag{1.29}$$

The first massless closed-string vertex has one left-moving and one right-moving oscillator:

$$V_\epsilon = \frac{2g_c}{\alpha'}\int d^2z\,\epsilon_{\mu\nu} : \partial X^\mu \bar\partial X^\nu e^{ik\cdot X} : . \tag{1.30}$$

Since ∂X^μ has weight $(1,0)$ and $\bar\partial X^\nu$ has weight $(0,1)$, the total weights are

$$h = 1 + \frac{\alpha' k^2}{4}, \qquad \bar h = 1 + \frac{\alpha' k^2}{4}. \tag{1.31}$$

The condition $h = \bar h = 1$ gives

$$k^2 = 0. \tag{1.32}$$

This is the massless shell condition for the graviton, Kalb–Ramond two-form, and dilaton polarizations contained in $\epsilon_{\mu\nu}$ [10].

Vertex operators in the Polyakov approach.

A vertex operator is the worldsheet operator corresponding to an external string state. The state–operator map gives the local operator, and the integration over its insertion point gives the amplitude. For a closed-string insertion,

$$V(k) = \int d^2 z\, \mathcal{V}_{h,\bar h}(z,\bar z), \qquad h = \bar h = 1. \tag{1.33}$$

Equations (1.28)–(1.32) show explicitly how this condition gives the tachyon and massless mass-shell equations. If k^2 is moved away from the on-shell value, the conformal weights change and the integrated operator is no longer invariant under the residual conformal transformations. Off-shell continuations therefore require extra input, such as a choice of local coordinates at the punctures or a Liouville dressing, instead of just replacing k^2 by an arbitrary value [12, 13].

Massless closed-string polarizations.

The polarization tensor in Eq. (1.30) has a unique decomposition into irreducible Lorentz pieces:

$$\epsilon_{\mu\nu} = \left(\epsilon_{(\mu\nu)} - \frac{1}{D}\eta_{\mu\nu}\epsilon^\rho{}_\rho\right) + \epsilon_{[\mu\nu]} + \frac{1}{D}\eta_{\mu\nu}\epsilon^\rho{}_\rho. \tag{1.34}$$

The first term is symmetric and traceless and is identified with the graviton polarization. The antisymmetric term is the Kalb–Ramond two-form polarization. The trace is the dilaton. The physical-state conditions also impose transversality, schematically $k^\mu\epsilon_{\mu\nu} = k^\nu\epsilon_{\mu\nu} = 0$, and null states shift the symmetric and antisymmetric polarizations by gauge transformations. These are the linearized gauge symmetries that later appear in the spacetime action in Eq. (1.41) [10].

Open-string vertex operators.

Open-string insertions are placed on the boundary of the worldsheet. A boundary integral $\int d\tau\, \mathcal{V}(\tau)$ is invariant under boundary reparametrizations when $\mathcal{V}$ has conformal weight 1. The massless vector vertex is

$$V_A = e_\mu \int_{\partial\Sigma} d\tau\, \dot X^\mu e^{ik\cdot X}, \qquad k^2 = 0, \qquad k\cdot e = 0. \tag{1.35}$$

The equivalence $e_\mu \sim e_\mu + \lambda k_\mu$ follows directly from the integrated vertex. If $e_\mu = k_\mu$, then

$$k_\mu \dot X^\mu e^{ik\cdot X} = \frac{1}{i}\frac{d}{d\tau} e^{ik\cdot X}. \tag{1.36}$$

For a boundary without endpoints, the integral of the total derivative vanishes. Hence a polarization proportional to k_μ does not create a distinct physical state. In spacetime notation this is the linearized gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu\Lambda$. On a D-brane the same open-string vector becomes the gauge field living on the brane worldvolume [10, 352].

1.7 Strings in curved spacetime

The flat-space action is obtained by taking the target-space metric to be $\eta_{\mu\nu}$. The massless closed-string vertex in Eq. (1.30) shows how to deform this theory. If the polarization is promoted to a slowly varying field, the linear coupling is

$$\delta S = \frac{1}{4\pi\alpha'}\int d^2 z\, h_{\mu\nu}(X)\,\partial X^\mu \bar\partial X^\nu. \tag{1.37}$$

Adding this to the flat action is equivalent, to first order, to replacing $\eta_{\mu\nu}$ by $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$. The antisymmetric polarization and the trace polarization similarly introduce $B_{\mu\nu}(X)$

and $\Phi(X)$. The finite form is the nonlinear sigma model

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\,\left(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}\,\alpha' R^{(2)}\Phi(X). \tag{1.38}$$

The background fields are coupling functions in a two-dimensional quantum field theory. Their beta functions must vanish for the Weyl factor of the worldsheet metric to decouple. The leading metric beta function is

$$\beta^G_{\mu\nu} = \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}\right) + O(\alpha'^2), \qquad H = dB. \tag{1.39}$$

A quick derivation of the first term comes from the background-field expansion in Riemann normal coordinates. The one-loop logarithmic divergence of the sigma model contains

$$\Gamma^{(1)}_{\text{div}} = -\frac{1}{4\pi\epsilon}\int d^2\sigma\sqrt{h}\,R_{\mu\nu}(X)h^{ab}\partial_a X^\mu \partial_b X^\nu, \tag{1.40}$$

so renormalizing $G_{\mu\nu}$ gives $\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + \cdots$. The torsion term comes from the B-field coupling, and the $2\nabla_\mu\nabla_\nu\Phi$ term comes from the dilaton coupling to $R^{(2)}$ [10].

Spacetime action.

The beta-function equations are reproduced by varying the leading string-frame action

$$S_{\text{eff}} = \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + O(\alpha')\right]. \tag{1.41}$$

Varying with respect to $B_{\mu\nu}$ gives

$$\nabla_\rho\left(e^{-2\Phi}H^\rho{}_{\mu\nu}\right) = 0. \tag{1.42}$$

Varying with respect to the metric and using the dilaton equation to simplify the trace gives

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} = 0, \tag{1.43}$$

which is the vanishing of Eq. (1.39) at leading order. Varying with respect to Φ gives

$$4\nabla^2\Phi - 4(\nabla\Phi)^2 + R - \frac{1}{12}H^2 = 0 \tag{1.44}$$

for the critical bosonic string at this order. Thus the graviton, two-form, and dilaton first identified through the closed-string vertex operator obey the field equations obtained from S_{eff} [10].

Compactification as a worldsheet spectrum.

A string background is specified by a conformal field theory and by its operator spectrum. For a closed string on a circle of radius R,

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}, \qquad N - \widetilde{N} + nw = 0. \tag{1.45}$$

The spectrum is invariant under

$$R \longleftrightarrow \frac{\alpha'}{R}, \qquad n \longleftrightarrow w. \tag{1.46}$$

The equality of the two spectra is the basic worldsheet statement of T-duality. A geometric radius in spacetime is therefore read from the spectrum of the underlying conformal field theory [10,350].

Chapter 2

Physical states, BRST cohomology, and the no-ghost theorem

Having gauge-fixed the path integral, we must now say which states of the oscillator Fock space are physical, and the difficulty is immediate: the time-like oscillators create negative-norm states, so the naive Hilbert space is not a Hilbert space at all. This chapter resolves the problem twice, by two methods whose agreement is itself a consistency check. First, old covariant quantization imposes the positive Virasoro constraints and the L_0 mass-shell condition by hand, and we show explicitly at the lowest levels how null states decouple and how the values $D = 26$, $a = 1$ are singled out. Second, we derive the BRST charge from scratch, for a general gauge theory, then for the point particle, then for the string, and prove that physical states are cohomology classes of a nilpotent operator, $\mathcal{H}_{\text{phys}} = \ker Q_B / \operatorname{im} Q_B$. The no-ghost theorem then proves positivity: it shows that this covariant construction has exactly the positive-norm transverse content of light-cone quantization, so the gauge-fixed theory is unitary. The equivalence of the two constructions, proved at the end, is what allows later chapters to switch freely between constraint and cohomology language [10, 28].

First ask what BRST quantization does and how it is related to ordinary gauge fixing. Gauge fixing is the practical normalization step in a gauge theory. The path integral has an infinite overcounting because all points on the same gauge orbit describe the same physical configuration. Choosing a gauge removes that overcounting, and the Faddeev–Popov determinant records the Jacobian of this choice. In the string path integral this determinant is represented by the anticommuting b, c ghosts.

This, however, is not yet the whole story. After a gauge has been chosen, the original gauge symmetry is no longer manifest. BRST symmetry is the cohomological way to keep its physical content. The original fields and the ghosts are combined into a larger system with a fermionic charge Q_B satisfying

$$Q_B^2 = 0.$$

Because this charge is nilpotent, its cohomology is well-defined. Physical states are the BRST-closed states, and two states that differ by a BRST-exact state are the same physical state:

$$Q_B|\psi\rangle = 0, \qquad |\psi\rangle \sim |\psi\rangle + Q_B|\Lambda\rangle.$$

Gauge fixing is not a small special case of BRST. Rather, gauge fixing removes the redundant gauge volume from the path integral, while BRST gives the gauge-fixed theory its invariant meaning. It tells us which states, operators, and amplitudes survive as genuine observables. In degree zero, the BRST cohomology reproduces gauge-invariant functions; in the string Hilbert space, the relevant cohomology gives the physical string spectrum.

For strings this is economical. Old covariant quantization imposes the Virasoro constraints and then removes null descendants. BRST quantization packages both operations into one cohomological statement. The Virasoro constraints become the condition $Q_B|\psi\rangle = 0$, and the removal of spurious or pure-gauge states becomes the equivalence relation $|\psi\rangle \sim |\psi\rangle + Q_B|\Lambda\rangle$. The rest of this chapter explains this statement first in the older language and then in the BRST language, so that the equivalence between the two is completely explicit [1, 3, 10].

2.1 Old covariant quantization

Old covariant quantization works entirely in the matter sector. States are built by acting with the creation operators α^{μ}_{-n} $(n>0)$ on the momentum vacuum $|0;k\rangle$; the reparametrization ghosts b_n, c_n are not part of this construction and enter only in the BRST treatment of Section 2.2, where they play an essential role. The immediate problem is that not all of these matter oscillators create positive-norm states: the timelike modes α^0_{-n} have negative norm,

$$\left\|\alpha^0_{-n}|0\rangle\right\|^2=\langle 0|\alpha^0_n\alpha^0_{-n}|0\rangle=\langle 0|\left[\alpha^0_n,\alpha^0_{-n}\right]|0\rangle=n\,\eta^{00}\langle 0|0\rangle=-n \qquad (n>0). \tag{2.1}$$

The physical Hilbert space must therefore be a constrained subspace of this naive oscillator Fock space, chosen so that such negative-norm states do not appear in physical amplitudes. The constraint can be read directly from gauge independence. After fixing the worldsheet metric to $g_{ab}(\sigma)$, a physical amplitude must be unchanged under an infinitesimal change of gauge slice, $g_{ab}(\sigma)\mapsto g_{ab}(\sigma)+\delta g_{ab}(\sigma)$. The corresponding first variation of the amplitude is[1];

$$\delta\langle f\mid i\rangle=-\frac{1}{4\pi}\int d^2\sigma\sqrt{g(\sigma)}\delta g_{ab}(\sigma)\langle f|T^{ab}(\sigma)|i\rangle$$

Since this variation should vanish for arbitrary $\delta g_{ab}(\sigma)$[2], we have;

$$\langle f|T^{ab}(\sigma)|i\rangle=0\Rightarrow\langle i|T^{ab}(\sigma)|f\rangle=0\quad\forall a,b$$

Not every pair of oscillator states satisfies this condition; it is precisely the condition that selects the physical subspace. Therefore, for physical states $|\psi\rangle$ and $|\psi'\rangle$, one has

$$\langle\psi|T_{ab}(\sigma)\,|\psi'\rangle=0\Rightarrow\langle\psi'|\,T_{ab}(\sigma)|\psi\rangle=0\quad\forall a,b \tag{4.1.1}$$

The condition is equivalent to the following in terms of the stress tensor on the plane;

$$\langle\psi|T(z)\,|\psi'\rangle=\langle\psi|T_{zz}(z)\,|\psi'\rangle=\frac{1}{4}\langle\psi|T_{00}(\sigma)-2iT_{10}(\sigma)-T_{11}(\sigma)\,|\psi'\rangle=0$$
$$\langle\psi|\bar T(\bar z)\,|\psi'\rangle=\langle\psi|\bar T_{\bar z\bar z}(\bar z)\,|\psi'\rangle=\frac{1}{4}\langle\psi|T_{00}(\sigma)+2iT_{10}(\sigma)-T_{11}(\sigma)\,|\psi'\rangle=0$$

where we used the following relations between stress tensor components;

$$T(z)=T_{zz}(z)=\frac{1}{4}\left(T_{00}(\sigma)-2iT_{10}(\sigma)-T_{11}(\sigma)\right) \tag{4.1.2}$$
$$\bar T(\bar z)=T_{\bar z\bar z}(\bar z)=\frac{1}{4}\left(T_{00}(\sigma)+2iT_{10}(\sigma)-T_{11}(\sigma)\right) \tag{4.1.3}$$

Using the mode expansion of $T(z)$ and $\bar T(\bar z)$ gives

$$\langle\psi|T(z)\,|\psi'\rangle=\sum_{m\in\mathbb{Z}}z^{-m-2}\langle\psi|\left(L_m+A\delta_{m,0}\right)|\psi'\rangle=0,$$
$$\langle\psi|\bar T(\bar z)\,|\psi'\rangle=\sum_{m\in\mathbb{Z}}\bar z^{-m-2}\langle\psi|\left(\bar L_m+\bar A\delta_{m,0}\right)|\psi'\rangle=0.$$

where the normal-ordering constants A and $\bar A$ have been included. Since these equations hold for all z, each Laurent coefficient must vanish:

$$\langle\psi|\left(L_m+A\delta_{m,0}\right)|\psi'\rangle=0,$$
$$\langle\psi|\left(\bar L_m+\bar A\delta_{m,0}\right)|\psi'\rangle=0,\qquad\forall m\in\mathbb{Z}.$$

The discussion so far used the full stress tensor. It decomposes into the matter stress tensor T^m_{ab}, which contains the X^μ CFT or another matter CFT in an internal sector, and the ghost

[1] With the stress tensor normalized as $T^{ab}=-(4\pi/\sqrt{g})\,\delta S/\delta g_{ab}$, a first-order change of the metric shifts the action by $\delta S=-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\delta g_{ab}T^{ab}$, and expanding $e^{-S-\delta S}=e^{-S}(1-\delta S+\cdots)$ inside the path integral gives exactly the displayed formula. No boundary terms arise on a closed worldsheet.

[2] Here $\hat g_{ab}$ is a gauge choice, and physical amplitudes may not depend on the choice of gauge slice in any direction. Variations along diff×Weyl directions carry no new information, they reproduce the Ward identities and the anomaly condition of Chapter 1, but the variations transverse to the gauge orbit are equally allowed, and it is those that force the matrix elements of every component of T^{ab} to vanish between physical states.

stress tensor T^g_{ab}:

$$T_{ab} = T^m_{ab} + T^g_{ab}$$

Old covariant quantization imposes the matter-sector constraints

$$\langle\psi|T^m(z)\,|\psi'\rangle = \langle\psi|\,(L^m_n + A\delta_{n,0})\,|\psi'\rangle = 0 \quad \forall n \in \mathbb{Z}$$
$$\langle\psi|\bar{T}^m(\bar{z})\,|\psi'\rangle = \langle\psi|\,\left(\bar{L}^m_n + \bar{A}\delta_{n,0}\right)|\psi'\rangle = 0 \quad \forall n \in \mathbb{Z}$$

If these conditions were imposed for all integer modes on ket states, one would obtain

$$\begin{aligned}(L^m_n + A\delta_{n,0})\,|\psi'\rangle &= 0 \quad \forall n \in \mathbb{Z},\\ \left(\bar{L}^m_n + \bar{A}\delta_{n,0}\right)|\psi'\rangle &= 0 \quad \forall n \in \mathbb{Z}.\end{aligned} \tag{4.1.4}$$

This condition is inconsistent. For $n \neq 0, 1$, Eq. (4.1.4) would imply, for a physical state $|\psi\rangle$,

$$\left[L^m_n, L^m_{-n}\right]|\psi\rangle = L^m_n L^m_{-n}|\psi\rangle - L^m_{-n}L^m_n|\psi\rangle = 0$$

The same commutator is fixed by the Virasoro algebra:

$$\begin{aligned}\left[L^m_n, L^m_{-n}\right]|\psi\rangle &= 2n\,(L^m_0 + A)\,|\psi\rangle + \frac{D}{12}n\left(n^2-1\right)|\psi\rangle\\ &= \frac{D}{12}n\left(n^2-1\right)|\psi\rangle.\end{aligned}$$

For $n \neq 0, 1$ this cannot vanish unless the central term is absent. Hence Eq. (4.1.4) cannot be imposed for every integer mode. The negative modes L_{-n} create Virasoro descendants; they are not annihilation constraints on ket states. The old covariant physical-state conditions are the positive-mode constraints together with the L_0 condition:

$$(L^m_n + A\delta_{n,0})\,|\psi\rangle = 0 \quad n \geq 0. \tag{4.1.5}$$

The case $n = -1$ does not produce the central contradiction in the commutator, but it is still not imposed on ket states. Instead, states generated by L_{-1} and the other negative modes appear as spurious descendants; if such a descendant is also physical, it is a null state and is set to zero in the physical inner product.

Thus every physical state has conformal weight $-A$. The same condition also implies

$$\begin{aligned}0 = \langle\psi'|\,L^m_n|\psi\rangle &= \langle\psi'|\,L^m_n|\psi\rangle^\dagger = \langle\psi|L^{\dagger m}_n\,|\psi'\rangle\\ &= \langle\psi|L^m_{-n}\,|\psi'\rangle \Rightarrow \langle\psi|L^m_{-n}\,|\psi'\rangle = 0 \quad (n > 0).\end{aligned}$$

In the second step the matrix element is replaced by its complex conjugate because it vanishes, and in the last step $L^{\dagger m}_n = L^m_{-n}$ has been used. This adjoint relation follows from

$$L^m_n = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{n-l}.\alpha_l:,$$
$$L^{\dagger m}_n = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{-l}.\alpha_{-n+l}: = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{-n-l}.\alpha_l: = L^m_{-n} \quad (n \neq 0).$$

Here $\alpha_{-l}\cdot\alpha_{-n+l} = \alpha_{-n+l}\cdot\alpha_{-l}$ for $n \neq 0$, and the final equality follows by the relabelling $l \to -l$. Therefore every state of the form

$$|\chi\rangle = \sum_{n=1}^{\infty} L^m_{-n}\,|\chi_n\rangle$$

is orthogonal to all physical states, for arbitrary $|\chi_n\rangle$:

$$\langle\psi \mid \chi\rangle = \sum_{n=1}^{\infty}\langle\psi|L^m_{-n}\,|\chi_n\rangle = \sum_{n=1}^{\infty}\langle\chi_n|\,L^m_{-n}|\psi\rangle^\dagger = 0$$

Such a state is called spurious. A state that is both spurious and physical is a null state. Null states do not change physical inner products:

$$\langle\psi' \mid \psi\rangle + \langle\psi' \mid \chi\rangle = \langle\psi' \mid \psi\rangle$$

where $|\psi\rangle$ and $|\psi'\rangle$ are physical states and $|\chi\rangle$ is null. The state $|\psi\rangle$ therefore has the same physical matrix elements as $|\psi\rangle + |\chi\rangle$, so the two states represent the same physical state[3]. The old covariant Hilbert space is consequently the quotient

$$\mathcal{H}_{\text{OCQ}} = \mathcal{H}_{\text{physical}} \,/\mathcal{H}_{\text{null}}$$

The constraints in Eq. (4.1.5) may now be applied to the lowest open-string states. The relevant oscillator forms are

$$\begin{aligned} L_n^m &= \frac{1}{2}\sum_{l\in\mathbb{Z}} : \alpha^{\mu}_{n-l}\alpha^{\nu}_{l} : \eta_{\mu\nu} \\ &= \frac{1}{2}\sum_{l=n+1}^{\infty} \alpha^{\mu}_{n-l}\alpha^{\nu}_{l}\eta_{\mu\nu} + \frac{1}{2}\sum_{l=-\infty}^{n} \alpha^{\nu}_{l}\alpha^{\mu}_{n-l}\eta_{\mu\nu}, \qquad n \geq 0. \end{aligned} \tag{4.1.6}$$

$$\begin{aligned} L_{-n}^m &= \frac{1}{2}\sum_{l\in\mathbb{Z}} : \alpha^{\mu}_{-n-l}\alpha^{\nu}_{l} : \eta_{\mu\nu} \\ &= \frac{1}{2}\sum_{l=1}^{\infty} \alpha^{\mu}_{-n-l}\alpha^{\nu}_{l}\eta_{\mu\nu} + \frac{1}{2}\sum_{l=1}^{n-1} \alpha^{\mu}_{-n+l}\alpha^{\nu}_{-l}\eta_{\mu\nu} \\ &\quad + \frac{1}{2}\sum_{l=n+1}^{\infty} \alpha^{\mu}_{-l}\alpha^{\nu}_{l-n}\eta_{\mu\nu} + \alpha^{\nu}_{0}\alpha^{\mu}_{-n}\eta_{\mu\nu}, \qquad n > 0. \end{aligned} \tag{4.1.7}$$

For $n \geq 0$, the lightest state $|0;k\rangle$ gives

$$\begin{aligned} L_n^m|0;k\rangle &= \frac{1}{2}\sum_{l=n+1}^{\infty} \alpha^{\mu}_{n-l}\alpha^{\nu}_{l}\eta_{\mu\nu}|0;k\rangle + \frac{1}{2}\sum_{l=-\infty}^{n} \alpha^{\nu}_{l}\alpha^{\mu}_{n-l}\eta_{\mu\nu}|0;k\rangle \\ &= \frac{1}{2}\sum_{l=-\infty}^{n} \alpha^{\nu}_{l}\alpha^{\mu}_{n-l}\eta_{\mu\nu}|0;k\rangle. \end{aligned} \tag{4.1.8}$$

This expression vanishes unless $n = 0$. For $n = 0$,

$$\begin{aligned} L_0^m|0;k\rangle &= \frac{1}{2}\sum_{l=-\infty}^{n} \alpha^{\nu}_{l}\alpha^{\mu}_{-l}\eta_{\mu\nu}|0;k\rangle \\ &= \frac{1}{2}\alpha^{\nu}_{0}\alpha^{\mu}_{0}\eta_{\mu\nu}|0;k\rangle = \alpha' k^2|0;k\rangle. \end{aligned} \tag{4.1.9}$$

where $\alpha^{\mu}_0 = \sqrt{2\alpha'}k^{\mu}$ has been used. Therefore the only non-trivial constraint on $|0;k\rangle$ is

$$\begin{aligned} (L_0^m + A)\,|0;k\rangle &= \big(\alpha' k^2 + A\big)\,|0;k\rangle = 0 \\ &\Rightarrow k^2 = -\frac{A}{\alpha'} \Rightarrow m^2 = \frac{A}{\alpha'}. \end{aligned}$$

where $m^2 = -k^2$ is the mass of the state. There are no spurious states at this level. The next set of states is as follows:

$$e.\alpha_{-1}|0;k\rangle = e_{\beta}\alpha^{\beta}_{-1}|0;k\rangle \tag{4.1.10}$$

Equation (4.1.6) determines the action of the positive Virasoro modes on this state. For the L_0 constraint one obtains

[3]Every observable of the free theory is an inner product among physical states, and interactions preserve the property, because null states also decouple from tree amplitudes, they are total derivatives on moduli space, as shown in Chapter 4. The physical inner product is positive semi-definite on the constrained space, and passing to the quotient by its null directions is the standard construction of a Hilbert space from a semi-definite form.

$$
\begin{aligned}
L_0^m e_\beta \alpha_{-1}^\beta |0;k\rangle &= \frac{1}{2}\left(\alpha_{-1}^\mu \alpha_1^\nu \eta_{\mu\nu} + \alpha_{-1}^\nu \alpha_1^\mu \eta_{\mu\nu} + \alpha_0^\nu \alpha_0^\mu \eta_{\mu\nu}\right) e_\beta \alpha_{-1}^\beta |0;k\rangle \\
&= \alpha_{-1}^\mu \eta_1^{\nu\beta} \eta_{\mu\nu} e_\beta |0;k\rangle + \alpha' k^2 e_\beta \alpha_{-1}^\beta |0;k\rangle \\
&= (e.\alpha_{-1})\,|0;k\rangle + \alpha' k^2\,(e.\alpha_{-1})\,|0;k\rangle \\
&= \left(1+\alpha' k^2\right)(e.\alpha_{-1})\,|0;k\rangle.
\end{aligned}
$$

In the second step the oscillator commutation relation and $\alpha_1^\mu|0;k\rangle = 0$ have been used. Therefore

$$
\begin{aligned}
\left(L_0^m + A\right) e_\beta \alpha_{-1}^\beta |0;k\rangle &= \left(1 + A + \alpha' k^2\right)(e.\alpha_{-1})\,|0;k\rangle = 0 \\
\Rightarrow m^2 &= \frac{1+A}{\alpha'}.
\end{aligned}
$$

For the level-one state in Eq. (4.1.10), the positive-mode constraints with $m > 0$ can act nontrivially only when an annihilation oscillator has mode one. The $m = 0$ case has already produced the mass-shell condition, so the only remaining positive Virasoro constraint is L_1. Only the corresponding term in Eq. (4.1.6) contributes, giving

$$
\begin{aligned}
0 = L_1 e_\beta \alpha_{-1}^\beta |0;k\rangle &= \frac{1}{2}\alpha_1^\nu \alpha_0^\mu \eta_{\mu\nu} e_\beta \alpha_{-1}^\beta |0;k\rangle \\
&= \frac{1}{2}\sqrt{2\alpha'}\,\eta_{\mu\nu} k^\mu \eta^{\nu\beta} e_\beta |0;k\rangle = e.k|0;k\rangle \Rightarrow e.k = 0.
\end{aligned}
$$

At level one there is a single possible spurious state:

$$
L_{-1}^m|0;k\rangle = \sqrt{2\alpha'} k^\nu \alpha_{-1}^\mu \eta_{\mu\nu}|0;k\rangle + \frac{1}{2}\sum_{l=2} \alpha_{-l}^\mu \alpha_{l-1}^\nu \eta_{\mu\nu}|0;k\rangle = \sqrt{2\alpha'} k.\alpha_{-1}|0;k\rangle
$$

where we used Eq. (4.1.7); for $n = 1$, the first and second terms do not contribute when acting on the vacuum. This state is physical, and hence null, precisely when $k^2 = 0$. The intercept is fixed by the following alternatives;

- If $A + 1 > 0$, the state is massive. In the rest frame,

$$
k^\mu = (m, \underbrace{0, \ldots, 0}_{d=D-1})
$$

The little group of this momentum is $\mathrm{SO}(D-1)$, and there are no null states because $k^2 \neq 0$. No interacting extension satisfying unitarity and Lorentz invariance is known for this case.[4] The closed-string representation analysis below rules out this possibility independently by using the $\mathrm{SO}(D-1)$ content at higher levels.

- If $A + 1 = 0$, the state is massless and the spurious state is null. In a convenient frame,

$$
k^\mu = (k, \underbrace{0, \ldots, 0}_{D-2}, k)
$$

The little group of this momentum is $SO(D-2)$.

- If $A + 1 < 0$, then $m^2 < 0$ and thus, we have a multiplet of tachyons. This case is unacceptable.

---

[4]As a free theory the case $A + 1 > 0$ is unitary: the constraints leave $D - 1$ positive-norm polarizations of a massive vector. The obstruction appears at the interacting level. For $a = 1$ the null states of the spectrum decouple from amplitudes and act as gauge symmetries; for $a < 1$ this decoupling structure is absent, and no interacting extension consistent with unitarity and Lorentz invariance has been constructed. The closed-string argument mentioned below (counting states in $SO(D-1)$ representations at higher levels) excludes the case independently.

The light-cone calculation gives the same normal-ordering constant from the transverse phase space. Impose

$$X^+ = x^+ + 2\alpha' p^+ \tau.$$

Then X^+ has no oscillators, and the Virasoro constraints solve X^- in terms of the transverse coordinates. The oscillator algebra follows directly from the symplectic form

$$\Omega = \int_0^{2\pi} d\sigma\, \delta P_\mu(\sigma) \wedge \delta X^\mu(\sigma), \qquad P_\mu = \frac{1}{2\pi\alpha'} \dot{X}_\mu.$$

Terms mixing a zero mode with an oscillator vanish because

$$\int_0^{2\pi} d\sigma\, e^{\pm in\sigma} = 0, \qquad n \neq 0.$$

Thus the oscillator part of Ω is obtained entirely from the transverse fields. With

$$X^i = x^i + 2\alpha' p^i \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha^i_n e^{-in(\tau+\sigma)} + \widetilde{\alpha}^i_n e^{-in(\tau-\sigma)} \right),$$

the $\alpha\alpha$ and $\widetilde{\alpha}\widetilde{\alpha}$ terms survive with $m = -n$. The mixed $\alpha\widetilde{\alpha}$ terms cancel pairwise by antisymmetry of the wedge product. Hence

$$\Omega_{\text{osc}} = i \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \frac{1}{n} \left(\delta\alpha^i_n \wedge \delta\alpha^i_{-n} + \delta\widetilde{\alpha}^i_n \wedge \delta\widetilde{\alpha}^i_{-n} \right).$$

Inverting this two-form gives the Poisson brackets

$$\{\alpha^i_n, \alpha^j_m\}_{\text{P.B.}} = -in\,\delta^{ij}\delta_{n+m,0}, \qquad \{\widetilde{\alpha}^i_n, \widetilde{\alpha}^j_m\}_{\text{P.B.}} = -in\,\delta^{ij}\delta_{n+m,0}.$$

Quantization gives

$$[\alpha^i_n, \alpha^j_m] = n\,\delta^{ij}\delta_{n+m,0}, \qquad [\widetilde{\alpha}^i_n, \widetilde{\alpha}^j_m] = n\,\delta^{ij}\delta_{n+m,0}.$$

For $n > 0$, define $a^i_n = \alpha^i_n/\sqrt{n}$ and $(a^i_n)^\dagger = \alpha^i_{-n}/\sqrt{n}$, with the same definition in the right-moving sector. Each transverse oscillator of mode number n is therefore a harmonic oscillator of frequency n. For one transverse boson the zero-point contribution is

$$E_0^{(1)} = \frac{1}{2} \sum_{n=1}^{\infty} n = \frac{1}{2}\zeta(-1) = -\frac{1}{24}.$$

Thus the transverse light-cone Hamiltonian contains

$$E_0 = (D-2)E_0^{(1)} = -\frac{D-2}{24}.$$

Equivalently, the open-string mass formula can be written as

$$m^2 = \frac{1}{\alpha'}\left(N - \frac{D-2}{24} \right).$$

Comparing this with the old-covariant formula $m^2 = (N+A)/\alpha'$ gives

$$A = -\frac{D-2}{24}.$$

The level-one vector has $N = 1$. Lorentz invariance requires it to be massless, since a massless vector has $D-2$ transverse polarizations, while a massive vector would carry $D-1$. Therefore

$$1 + A = 0 \quad \Longrightarrow \quad 1 - \frac{D-2}{24} = 0 \quad \Longrightarrow \quad D = 26.$$

For the closed string the same zero-point energy appears in both chiral sectors. Level matching sets $N = \widetilde{N}$, and the mass formula becomes

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24} \right).$$

The condition $D = 26$ is therefore the light-cone form of the anomaly and BRST conditions derived below [3].

2.1.1 Critical dimension and the closed-string spectrum

The dimension is fixed before the closed-string spectrum is interpreted: the vanishing of the total central charge $c_{\text{tot}} = c_X + c_{bc} = D - 26$, Eq. (1.16), requires $D = 26$ for the critical bosonic string. For the closed string the physical-state conditions are

$$(L_0 - 1)|\psi\rangle = (\widetilde{L}_0 - 1)|\psi\rangle = 0, \qquad L_n|\psi\rangle = \widetilde{L}_n|\psi\rangle = 0 \quad (n > 0). \tag{2.2}$$

The first excited closed-string state is

$$\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\widetilde{\alpha}^{\nu}_{-1}|0;k\rangle. \tag{2.3}$$

Using the decomposition in Eq. (1.34), this state contains the graviton, the Kalb–Ramond two-form, and the dilaton. The mass formula and level matching are

$$\frac{\alpha' M^2}{4} = N + \widetilde{N} - 2, \qquad N = \widetilde{N}. \tag{2.4}$$

For $N = \widetilde{N} = 1$ one gets $M^2 = 0$, which is the massless closed-string multiplet. The no-ghost theorem states that, after the Virasoro constraints or BRST cohomology are imposed, the physical Hilbert space has the positive-norm transverse content expected from light-cone quantization [10, 28].

The same dimension follows from requiring the light-cone spectrum to assemble into Lorentz representations, in the manner of Minwalla's lectures [3]. Light-cone gauge breaks manifest Lorentz invariance, so at each mass level the transverse states must recombine into a representation of the little group: $SO(D-1)$ for a massive level and $SO(D-2)$ for a massless one. At the first excited level the closed string carries

$$\widetilde{\alpha}^{i}_{-1}\alpha^{j}_{-1}|0;k\rangle, \qquad i, j = 1, \ldots, D-2,$$

which is $(D-2)^2$ states, the vector times vector of $SO(D-2)$. These cannot be assembled into an $SO(D-1)$ representation, so the level must be massless, and Eq. (2.4) at $N = \widetilde{N} = 1$ forces

$$0 = 1 - \frac{D-2}{24} \quad \Longrightarrow \quad D = 26.$$

Once $D = 26$ is imposed, every massive level must instead fill an $SO(D-1)$ representation. At $N = \widetilde{N} = 2$ the transverse states of one chiral sector are

$$\alpha^{i}_{-1}\alpha^{j}_{-1}|0;k\rangle \;\oplus\; \alpha^{i}_{-2}|0;k\rangle,$$

with

$$\tfrac{1}{2}(D-2)(D-1) + (D-2) = \tfrac{1}{2}D(D-1) - 1$$

states, which is exactly the dimension of the symmetric traceless rank-two tensor of $SO(D-1)$. The level is therefore Lorentz invariant, and the same counting works at every higher level. This is the light-cone counterpart of the no-ghost theorem and of the vanishing of the Weyl anomaly at $D = 26$.

2.2 BRST quantization

In a gauge theory the Faddeev–Popov ghosts keep track of infinitesimal gauge motion. The BRST charge Q_B is the quantum operator that generates this motion on the enlarged matter-plus-ghost Hilbert space. Physical states must be annihilated by this charge,

$$Q_B|\psi\rangle = 0. \tag{2.5}$$

A state of the form $Q_B|\Lambda\rangle$ has zero matrix elements with all BRST-closed physical states because $Q_B^2 = 0$ and Q_B can be moved by contour deformation in a BRST-invariant correlator. Therefore such states are identified with zero. The physical state space is

$$Q_B^2 = 0, \qquad \mathcal{H}_{\text{phys}} = \frac{\ker Q_B}{\operatorname{im} Q_B}, \qquad |\psi\rangle \sim |\psi\rangle + Q_B|\Lambda\rangle. \tag{2.6}$$

The derivation below starts from a general gauge-fixing condition and then specializes this construction to the string [10, 28].

Gauge fixing may be described first in a general notation. For a point particle the fields are $X^\mu(\tau)$ and the einbein $e(\tau)$; more generally, the degrees of freedom are field values at each spacetime point, together with their tensor or internal indices. All such labels are collected into a single index i, so the fields are denoted by ϕ_i.

Let $\delta_\alpha\phi_i$ be the variation of ϕ_i under the gauge transformation labelled by α. Since gauge transformations are local, α includes the spacetime point as well as any field index carried by the gauge parameter. An infinitesimal transformation is written as $\epsilon^\alpha \delta_\alpha$, where ϵ^α is a real infinitesimal parameter. Closure of the gauge transformations means that the commutator of two such transformations is another gauge transformation:

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma$$

where the $f^\gamma_{\alpha\beta}$ are structure constants. A gauge-fixing condition is a function of the fields that is set to zero. Since there is one such condition for each local gauge redundancy, the condition also carries a collective index, including spacetime coordinates and field indices:

$$F^A(\phi) = 0$$

where A denotes this collective index. Let the original gauge-invariant action be S_1. Imposing $F^A(\phi) = 0$ for every A requires a Lagrange multiplier $-iB_A$, so the gauge-fixing term gives

$$S_1 - iB_AF^A(\phi) = S_1 + S_2 \text{ where } S_2 = -iB_AF^A(\phi)$$

The summation over A includes integration over spacetime, so the Lagrange-multiplier term is itself an action term. In the string Faddeev–Popov procedure, the fields ϕ_i are $X^\mu(\sigma)$ and $g_{ab}(\sigma)$, while the multipliers B_A play the role of the auxiliary field ζ. The Faddeev–Popov construction gives

$$\int \frac{[d\phi_i]}{V_{\text{gauge}}} \exp\left(-S_1\right) = \int [d\phi_i dB_A db_A dc^\alpha] \exp\left(-S_1 - S_2 - S_3\right)$$

where S_3 is the Faddeev Popov action;

$$S_3 = b_A c^\alpha \delta_\alpha F^A(\phi)$$

This form matches the Faddeev–Popov ghost action obtained for the string:

$$\int d^2\sigma \sqrt{\hat{g}(\sigma)} b_{ab} \left(\hat{P}_1 c\right)^{ab}$$

Recall that $\hat{P}_1$ comes from the variation of the metric. The remaining term in the metric variation has zero contraction with b_{ab} because b_{ab} is traceless, and hence it does not appear in the ghost action. The total action is therefore

$$S_1 + S_2 + S_3 = S_1 - iB_AF^A(\phi) + b_A c^\alpha \delta_\alpha F^A(\phi). \tag{4.2.1}$$

Two facts will be used below. First, Eq. (4.2.1) is invariant under the BRST transformation

$$\begin{aligned} \delta_B \phi_i &= -i\epsilon c^\alpha \delta_\alpha \phi_i, \qquad & \delta_B B_A &= 0, \\ \delta_B b_A &= \epsilon B_A, \qquad & \delta_B c^\alpha &= \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma. \end{aligned} \tag{4.2.2}$$

Because the variation of an anticommuting field can be commuting, as in $\delta_B b_A = \epsilon B_A$, the parameter ϵ is itself anticommuting with b_A and c^α.

Ghost-number symmetry assigns conserved ghost numbers to the anticommuting variables b_A, ϵ, and c^α, while the commuting fields have ghost number zero. The variation of ϕ_i in Eq. (4.2.2) shows that c^α and ϵ carry opposite ghost numbers, whereas the variation of b_A shows that b_A and ϵ carry the same ghost number. Thus c^α has ghost number $+1$, while b_A and ϵ have ghost number -1.

The invariance of Eq. (4.2.1) under Eq. (4.2.2) follows from the following variations:

$$S_1 \to S_1. \tag{4.2.3}$$

because S_1 is gauge invariant and $\delta_B \phi_i$ is a gauge transformation with parameter $-i\epsilon c^\alpha$. Moreover,

$$
\begin{aligned}
S_2 \to S_2 &- iB_A \delta_B F^A(\phi) \\
&= S_2 - iB_A \delta_B \phi_i \, \partial_i F^A(\phi) \\
&= S_2 - iB_A \left(-i\epsilon c^\alpha \delta_\alpha \phi_i\right) \partial_i F^A(\phi) \\
&= S_2 - \epsilon B_A c^\alpha \delta_\alpha \phi_i \partial_i F^A(\phi) = S_2 - \epsilon B_A c^\alpha \delta_\alpha F^A(\phi).
\end{aligned}
\tag{4.2.4}
$$

Finally,

$$
\begin{aligned}
S_3 \to{}& (b_A + \epsilon B_A) \left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right) \left(\delta_\alpha F^A(\phi) + \delta_B \phi_i \, \partial_i \delta_\alpha F^A(\phi)\right) \\
&= (b_A + \epsilon B_A) \left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right) \left(\delta_\alpha F^A(\phi) - i\epsilon c^\beta \delta_\beta \phi_i \, \partial_i \delta_\alpha F^A(\phi)\right) \\
&= (b_A + \epsilon B_A) \left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right) \left(\delta_\alpha F^A(\phi) - i\epsilon c^\beta \delta_\beta \delta_\alpha F^A(\phi)\right) \\
&= S_3 - \frac{i}{2}\epsilon b_A f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha F^A(\phi) + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) - i\epsilon b_A c^\alpha c^\beta \delta_\beta \delta_\alpha F^A(\phi).
\end{aligned}
$$

All terms proportional to ϵ^2 vanish because ϵ is anticommuting. Using

$$
\begin{aligned}
c^\alpha c^\beta \delta_\beta \delta_\alpha &= \frac{1}{2}\left(c^\alpha c^\beta \delta_\beta \delta_\alpha - c^\beta c^\alpha \delta_\beta \delta_\alpha\right) \\
&= -\frac{1}{2}\left(c^\alpha c^\beta \delta_\alpha \delta_\beta - c^\alpha c^\beta \delta_\beta \delta_\alpha\right) \\
&= -\frac{1}{2} c^\alpha c^\beta \left[\delta_\alpha, \delta_\beta\right] = -\frac{1}{2} c^\alpha c^\beta f^\gamma_{\alpha\beta} \delta_\gamma.
\end{aligned}
$$

one obtains

$$
\begin{aligned}
S_3 \to S_3 &- \frac{i}{2}\epsilon b_A f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha F^A(\phi) + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) \\
&+ \frac{i}{2}\epsilon b_A c^\alpha c^\beta f^\gamma_{\alpha\beta} \delta_\gamma F^A(\phi) = S_3 + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi).
\end{aligned}
\tag{4.2.5}
$$

Equations (4.2.4) and (4.2.5) show that $S_2 + S_3$ is invariant. Together with Eq. (4.2.3), this proves that $S_1 + S_2 + S_3$ is BRST invariant.

The BRST variation of $b_A F^A(\phi)$ is

$$
\begin{aligned}
b_A F^A(\phi) \to{}& (b_A + \epsilon B_A) \left(F_A(\phi) + \delta_B \phi_i \, \partial_i F_A(\phi)\right) \\
&= (b_A + \epsilon B_A) \left(F_A(\phi) - i\epsilon c^\alpha \delta_\alpha \phi_i \partial_i F_A(\phi)\right) \\
&= (b_A + \epsilon B_A) \left(F_A(\phi) - i\epsilon c^\alpha \delta_\alpha F_A(\phi)\right) \\
&= b_A F^A + \underbrace{i\epsilon \left(b_A c^\alpha \delta_\alpha F_A(\phi) - iB_A F^A(\phi)\right)}_{\delta_B (b_A F^A)} \\
&= b_A F^A + \underbrace{i\epsilon \left(S_2 + S_3\right)}_{\delta_B (b_A F^A)}.
\end{aligned}
$$

Hence

$$
\delta_B \left(b_A F^A\right) = i\epsilon \left(S_2 + S_3\right). \tag{4.2.6}
$$

Change the gauge-fixing condition by an infinitesimal amount $\epsilon \, \delta F^A(\phi)$. In the path-integral formalism the transition amplitude is represented as

$$
\langle f \mid i \rangle = \int [d\phi_i] \, e^{-S_1} \xrightarrow{\text{gauge fixing}} \int [d\phi_i \, db_A \, dB_A \, dc^\alpha] \, e^{-(S_1+S_2+S_3)}.
$$

where the integrations over b_A, B_A, and c^α arise from gauge fixing and from the Faddeev–Popov determinant. The variation of this amplitude is

$$\begin{aligned}\langle f \mid i\rangle &\to \int [d\phi_i\, db_A\, dB_A\, dc^\alpha]\, e^{-(S_1+S_2+S_3)} \left(1 - \epsilon\delta\left(S_2+S_3\right) + \mathcal{O}\left(\epsilon^2\right)\right)\\ &= \langle f \mid i\rangle - \epsilon\langle f|\delta\left(S_2+S_3\right)|i\rangle + \mathcal{O}\left(\epsilon^2\right).\end{aligned}$$

Here S_1 does not depend on the gauge-fixing condition, so $\delta S_1 = 0$ under the variation of $F^A(\phi)$. Using Eq. (4.2.6),

$$\epsilon\delta\langle f \mid i\rangle = -\epsilon\langle f|\delta\left(S_2+S_3\right)|i\rangle = i\langle f|\delta_B\left(b_A\delta F^A\right)|i\rangle.$$

A symmetry variation generated by a conserved charge is represented by a commutator with that charge when the symmetry parameter is commuting. For an anticommuting parameter the corresponding expression is the graded commutator, which becomes an anticommutator on odd operators.[5] For BRST symmetry this gives

$$\epsilon\delta\langle f \mid i\rangle = i\langle f|\delta_B\left(b_A\delta F^A\right)|i\rangle \propto \epsilon\langle f|\left\{Q_B, b^A\delta F_A\right\}|i\rangle. \tag{4.2.7}$$

where Q_B is the BRST charge. Gauge independence of the amplitude requires

$$\langle f|\left\{Q_B, b^A\delta F_A\right\}|i\rangle = 0 \text{ for arbitrary } \delta F^A(\phi) \tag{4.2.8}$$

Thus physical states must satisfy, for arbitrary δF^A,

$$\begin{aligned}0 &= \langle\psi|\left\{Q_B, b^A\delta F_A\right\}|\psi'\rangle\\ &= \langle\psi|Q_B b^A\delta F_A|\psi'\rangle - \langle\psi|b^A\delta F_A Q_B|\psi'\rangle.\end{aligned}$$

This condition is satisfied if physical states are BRST closed, $Q_B|\psi\rangle = 0$. If $Q_B^\dagger = Q_B$, it also follows that $\langle\psi|Q_B = 0$. The hermiticity of Q_B follows from the reality of the gauge parameter.[6] Since the variation of F^A is arbitrary, choose

$$\delta F^A = -iB_B M^{AB}$$

for some arbitrary constant matrix. We immediately see that δ_α variation for this variation of F^A is zero and thus, S_3 is invariant. However, S_2 changes as follows:

$$S_2 \to -iB_A\left(F^A - iB_B M^{AB}\right) = S_2 - B_A B_B M^{AB}$$

The original action S_1 is unchanged, so the net effect is the addition of one term:

$$S_1+S_2+S_3 \to S_1+S_2+S_3 - B_A B_B M^{AB}$$

and the B_A integration becomes Gaussian instead of a delta-function constraint.

Because the gauge-fixed action in Eq. (4.2.1) is BRST invariant, changing the gauge choice must preserve the conservation of the BRST charge. Equivalently, Q_B must commute with the change in the Hamiltonian. The original Hamiltonian already commutes with Q_B, so only the variation of the Hamiltonian is relevant. For amplitudes between states at different times, the time-evolution factor gives

$$\begin{aligned}\langle f|e^{-iHt}|i\rangle &\to \langle f|e^{-i(H+\delta H)t}|i\rangle = \langle f|e^{-i(H)t}(1-it\delta H)|i\rangle + \mathcal{O}\left((\delta H)^2\right)\\ &= \langle f|e^{-iHt}|i\rangle - it\langle f|e^{-iHt}\delta H|i\rangle + \mathcal{O}\left((\delta H)^2\right)\end{aligned}$$

[5] For any charge Q and parameter ϵ, the variation is $\delta\phi = [\epsilon Q, \phi]$. If ϵ and Q are both Grassmann odd, then moving ϵ out of the commutator flips signs whenever it passes an odd object: $[\epsilon Q, \phi] = \epsilon Q\phi - \phi\,\epsilon Q = \epsilon(Q\phi + \phi Q)$ for odd ϕ, so the graded bracket $\{Q, \phi\}$ appears; for even ϕ one finds $\epsilon[Q, \phi]$. Both cases are summarized by the graded commutator $[\epsilon Q, \phi] = \epsilon\,[Q, \phi\}$.

[6] The BRST transformation is generated by the Noether charge of a symmetry whose parameter is a single anticommuting constant taken to be real. "Real Grassmann" means the parameter is Grassmann-odd, so it anticommutes with the fermionic fields, and at the same time equal to its own complex conjugate. The transformation preserves the reality of every field: ϕ_i and B_A are ordinary real fields, while the ghosts b_A and c^α are real and anticommuting. When a symmetry preserves these reality conditions, its Noether charge is hermitian in the inner product in which the fields are self-adjoint. The ghost zero modes make that inner product degenerate on part of the space, but the degeneracy does not affect the formal hermiticity used here.

Comparison with Eq. (4.2.7), including the same time-evolution factor when the initial and final states are separated in time, shows that δH is proportional to[7]

$$\left\{Q_B, b^A \delta F_A\right\}$$

Therefore the requirement $[Q_B, \delta H] = 0$ implies

$$\left[Q_B, \left\{Q_B, b_A \delta F^A\right\}\right] = 0$$

This gives

$$Q_B^2 b_A \delta F^A - b_A \delta F^A Q_B^2 = \left[Q_B^2, b_A \delta F^A\right] = 0 \tag{4.2.9}$$

for arbitrary δF^A. Equation (4.2.2) fixes the ghost-number assignment: b_A has ghost number -1, the variation of $\langle f|i\rangle$ has ghost number zero, and the BRST charge has ghost number $+1$. Hence Q_B^2 has ghost number $+2$. It cannot be a nonzero constant, since constants carry ghost number zero.[8] Thus Eq. (4.2.9) is satisfied by imposing

$$Q_B^2 = 0 \tag{4.2.10}$$

[9]

2.2.1 Point particle example

For a point particle the coordinate is τ, and the fields are $X^\mu(\tau)$ and $e(\tau)$. The gauge label α in δ_α is therefore replaced by the continuous label τ_1. A convenient basis for gauge transformations is chosen so that the transformation localized at τ_1 acts only at that point:

$$\delta_{\tau_1} \tau = \delta\left(\tau - \tau_1\right)$$

A general transformation of τ is then

$$\delta \tau = \epsilon^{\tau_1} \delta_{\tau_1} \tau = \epsilon\left(\tau_1\right) \delta\left(\tau - \tau_1\right) = \epsilon(\tau) \Rightarrow \tau \rightarrow \tau + \eta(\tau)$$

The point-particle action gives the following field variations under this transformation:

$$\begin{aligned}
\delta X^\mu(\tau) &= -\eta(\tau) \partial_\tau X^\mu(\tau) = -\epsilon^{\tau_1} \delta_{\tau_1} \partial_\tau X^\mu(\tau) \\
&= -\epsilon^{\tau_1} \delta(\tau - \tau_1) \partial_{\tau_1} X^\mu(\tau_1) \Rightarrow \delta_{\tau_1} X^\mu = -\delta(\tau - \tau_1) \partial_{\tau_1} X^\mu(\tau_1), \\
\delta e(\tau) &= -\partial_\tau(\epsilon(\tau) e(\tau)) = -\epsilon^{\tau_1} \partial_\tau\left(\delta_{\tau_1} e(\tau)\right) \\
&= -\epsilon^{\tau_1} \partial_{\tau_1}\left(\delta(\tau - \tau_1) e(\tau_1)\right) \Rightarrow \delta_{\tau_1} e(\tau) = -\partial_{\tau_1}\left(\delta(\tau - \tau_1) e(\tau_1)\right).
\end{aligned}$$

The structure constants of the BRST algebra are obtained from the commutator $[\delta_{\tau_1}, \delta_{\tau_2}]$. Acting on $X^\mu(\tau)$, the commutator is

$$\left[\delta_{\tau_1}, \delta_{\tau_2}\right] X^\mu(\tau) = \delta_{\tau_1} \delta_{\tau_2} X^\mu(\tau) - (1 \leftrightarrow 2)$$

Explicitly,

[7] The argument is correct and can be stated without comparing time evolutions: the entire dependence of the gauge-fixed action on F^A resides in $S_2 + S_3 = \delta_B(b_A F^A)/(i\epsilon)$, so changing the gauge condition changes the action, and hence the Hamiltonian, by a BRST-exact term. This is the general statement that gauge-fixed Hamiltonians for different gauge choices differ by $\{Q_B, \cdot\}$, which is also why physical (BRST-closed) matrix elements are gauge independent.

[8] $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\}$ carries ghost number $+2$, while a c-number is neutral, so Q_B^2 can only be a ghost-number-two *operator* commuting with all $b_A \delta F^A$. In an irreducible realization no such nontrivial operator exists, hence $Q_B^2 = 0$. The hidden assumption is that the gauge algebra closes off shell with field-independent structure constants; when it does not, nilpotency requires the Batalin–Vilkovisky extension mentioned in the next footnote.

[9] For gauge algebras that close only on shell, or whose structure functions depend on the fields, the transformation law above is not nilpotent off shell; the systematic repair is the Batalin–Vilkovisky formalism, in which every field acquires an antifield and $Q_B^2 = 0$ is replaced by the master equation $(S, S) = 0$. The bosonic string in conformal gauge has a closing algebra, so the direct construction used here is sufficient, and the physical-state cohomology defined below is unambiguous.

$$\delta_{\tau_1}\delta_{\tau_2}X^\mu(\tau) = \delta_1\left(\delta\left(\tau-\tau_2\right)\partial_\tau X^\mu(\tau)\right) = \delta\left(\tau-\tau_2\right)\partial_\tau\left(\delta_1 X^\mu(\tau)\right) = \delta\left(\tau-\tau_2\right)\partial_\tau\left(\delta\left(\tau-\tau_1\right)\partial_\tau X^\mu(\tau)\right)$$
$$= \delta\left(\tau-\tau_2\right)\delta\left(\tau-\tau_1\right)\partial_\tau^2 X^\mu(\tau) + \delta\left(\tau-\tau_2\right)\partial_\tau\delta\left(\tau-\tau_1\right)\partial_\tau X^\mu(\tau)$$

The first term cancels against the term with $1 \leftrightarrow 2$. Therefore

$$\left[\delta_{\tau_1}, \delta_{\tau_2}\right] X^\mu(\tau) = \left[\delta\left(\tau-\tau_2\right)\partial_\tau\delta\left(\tau-\tau_1\right) - \delta\left(\tau-\tau_2\right)\partial_\tau\delta\left(\tau-\tau_1\right)\right]\partial_\tau X^\mu(\tau)$$

To express this commutator in the form $f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X^\mu$, with summation over τ_3 understood as integration over τ_3, rewrite it as

$$\left[\delta_{\tau_1}, \delta_{\tau_2}\right] X^\mu(\tau) = \left[\delta\left(\tau-\tau_2\right)\partial_\tau\delta\left(\tau-\tau_1\right) - \delta\left(\tau-\tau_2\right)\partial_\tau\delta\left(\tau-\tau_1\right)\right]\partial_\tau X^\mu(\tau)$$

$$= \int d\tau_3\left[\delta\left(\tau-\tau_3\right)\delta\left(\tau_3-\tau_2\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_1\right) - \delta\left(\tau-\tau_3\right)\delta\left(\tau_3-\tau_1\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_2\right)\right]\partial_\tau X^\mu(\tau)$$

$$= -\int d\tau_3\left[\delta\left(\tau_3-\tau_2\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_1\right) - \delta\left(\tau_3-\tau_1\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_2\right)\right]\delta_{\tau_3}X^\mu(\tau) = \int d\tau_3 f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X^\mu(\tau)$$

Thus

$$f^{\tau_3}_{\tau_1\tau_2} = \delta\left(\tau_3-\tau_1\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_2\right) - \delta\left(\tau_3-\tau_2\right)\partial_{\tau_3}\delta\left(\tau_3-\tau_1\right)$$

The BRST transformations are then

$$\delta_B X^\mu(\tau) = -\int d\tau_1 i\epsilon c^{\tau_1}\delta_{\tau_1}X^\mu(\tau) = i\epsilon\int d\tau_1 c^{\tau_1}\delta\left(\tau-\tau_1\right)\partial_\tau X^\mu(\tau) = i\epsilon c(\tau)\partial_\tau X^\mu(\tau) = i\epsilon c(\tau)\dot{X}^\mu(\tau) \tag{4.2.11}$$

$$\delta_B e(\tau) = -i\epsilon\int d\tau_1 c^{\tau_1}\delta_{\tau_1}e(\tau) = i\epsilon\int d\tau_1 c^{\tau_1}\partial_\tau\left(\delta\left(\tau-\tau_1\right)e(\tau)\right)$$
$$= i\epsilon\partial_\tau\int d\tau_1 c\left(\tau_1\right)\delta\left(\tau-\tau_1\right)e(\tau) = i\epsilon\partial_\tau(c(\tau)e(\tau)) = i\epsilon(ce) \tag{4.2.12}$$

$$\delta_B B(\tau) = 0 \tag{4.2.13}$$

$$\delta_B b(\tau) = \epsilon B(\tau) \tag{4.2.14}$$

$$\delta_B c^{\tau_1} = \delta_B c\left(\tau_1\right) = \frac{i}{2}\epsilon f^{\tau_1}_{\tau_2\tau_3}c^{\tau_2}c^{\tau_3} = \frac{i}{2}\epsilon\int d\tau_2 d\tau_3\left[\delta\left(\tau_1-\tau_2\right)\partial_{\tau_1}\delta\left(\tau_1-\tau_3\right) - \delta\left(\tau_1-\tau_3\right)\partial_\tau\delta\left(\tau_1-\tau_2\right)\right]c\left(\tau_2\right)c\left(\tau_3\right)$$
$$= \frac{i\epsilon}{2}\left[\int d\tau_3\partial_{\tau_1}\delta\left(\tau_1-\tau_3\right)c\left(\tau_1\right)c\left(\tau_3\right) - \int d\tau_2\partial_{\tau_1}\delta\left(\tau_1-\tau_2\right)c\left(\tau_2\right)c\left(\tau_1\right)\right] = -i\epsilon\int d\tau_3\partial_{\tau_1}\delta\left(\tau_1-\tau_3\right)c\left(\tau_3\right)c\left(\tau_1\right)$$
$$= -i\epsilon\partial_{\tau_1}\left[\int d\tau_3\delta\left(\tau_1-\tau_3\right)c\left(\tau_3\right)\right]c\left(\tau_1\right) = i\epsilon c\left(\tau_1\right)\partial_{\tau_1}c\left(\tau_1\right) \Rightarrow \delta_B c(\tau) = i\epsilon c(\tau)\partial_\tau c(\tau) = i\epsilon c(\tau)\dot{c}(\tau) \tag{4.2.15}$$

The anticommutativity of the c ghosts has been used repeatedly, and in the second line the integration variable τ_2 has been renamed as τ_3. For the point particle the gauge-fixing condition is $e(\tau) = 1$, so $F^\tau = 1 - e(\tau)$. The three action terms are

$$S_1 = \frac{1}{2}\int d\tau\left(e^{-1}\dot{X}^\mu\dot{X}_\mu + em^2\right) \tag{4.2.16}$$

$$S_2 = -i\int d\tau B_\tau F^\tau = -i\int d\tau B(\tau)(1-e(\tau)) = i\int d\tau B(\tau)(e(\tau)-1) \tag{4.2.17}$$

$$S_3 = \int d\tau\int d\tau_1 b_\tau c^{\tau_1}\delta_{\tau_1}F^\tau = \int d\tau\int d\tau_1 b(\tau)c\left(\tau_1\right)\delta_{\tau_1}(1-e(\tau)) = \int d\tau\int d\tau_1 b(\tau)c\left(\tau_1\right)\partial_\tau\left(\delta\left(\tau-\tau_1\right)e(\tau)\right)$$
$$= \int d\tau b(\tau)\partial_\tau\left(\int d\tau_1 c\left(\tau_1\right)\delta\left(\tau-\tau_1\right)\right)e(\tau) = \int d\tau b(\tau)\dot{c}(\tau)e(\tau) \tag{4.2.18}$$

The total action is thus,

$$S=\int d\tau\left(\frac{1}{2}e^{-1}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{e}{2}m^{2}-iB(e-1)+b\dot{c}e\right)=\int d\tau\left(\frac{1}{2}e^{-1}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{e}{2}m^{2}-iB(e-1)-e\dot{b}c\right)$$

[10] The equations of motion for B and e are

$$B\text{EOM}: e=1$$

$$e\text{ EOM}:\quad -\frac{1}{2}e^{-2}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{1}{2}m^{2}-iB-\dot{b}\,c=0,$$

$$\begin{aligned}B&=-i\left[\frac{1}{2}e^{-2}\dot{X}^{\mu}\dot{X}_{\mu}-\frac{1}{2}m^{2}+\dot{b}\,c\right]\\&=i\left[-\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{1}{2}m^{2}-\dot{b}\,c\right].\end{aligned}$$

In the last step $e=1$ has been imposed using the B equation of motion. The reduced action is therefore

$$S=\int d\tau\left(\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{1}{2}m^{2}-\dot{b}c\right)\tag{4.2.19}$$

The BRST transformation in Eq. (4.2.14) becomes

$$\delta_{B}b=i\epsilon\left[-\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}+\frac{1}{2}m^{2}-\dot{b}c\right]\tag{4.2.20}$$

with Eqs. (4.2.11) and (4.2.15) unchanged. The variation of the X^{μ} kinetic term is

$$\begin{aligned}\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}\to\frac{1}{2}\left(\partial_{\tau}\left(X^{\mu}+i\epsilon c\dot{X}^{\mu}\right)\partial_{\tau}\left(X_{\mu}+i\epsilon c\dot{X}_{\mu}\right)\right)&=\frac{1}{2}\left(\dot{X}^{\mu}+i\epsilon\dot{c}\dot{X}^{\mu}+i\epsilon c\ddot{X}^{\mu}\right)\left(\dot{X}_{\mu}+i\epsilon\dot{c}\dot{X}_{\mu}+i\epsilon c\ddot{X}_{\mu}\right)\\&=\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}+i\epsilon\dot{c}\dot{X}^{\mu}\dot{X}_{\mu}+i\epsilon c\ddot{X}^{\mu}\dot{X}_{\mu}\end{aligned}$$

The mass term is invariant. The $-\dot{b}c$ term changes as follows:

$$\begin{aligned}-\dot{b}\,c&\to-\partial_{\tau}\left[b-i\epsilon\left(\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu}-\frac{1}{2}m^{2}+\dot{b}\,c\right)\right](c+i\epsilon c\dot{c})\\&=\left(-\dot{b}+i\epsilon\dot{X}^{\mu}\ddot{X}_{\mu}+i\epsilon\dot{b}\,c+i\epsilon\dot{b}\,\dot{c}\right)(c+i\epsilon c\dot{c})\\&=-\dot{b}\,c+i\epsilon c\,\dot{X}^{\mu}\dot{X}_{\mu}.\end{aligned}$$

The point-particle and ghost systems obey[11]

$$[X^{\mu},p^{\nu}]=i\eta^{\mu\nu},\qquad\{b,c\}=1,\quad H=\frac{1}{2}\left(p^{2}+m^{2}\right)$$

where because of the Euclidean signature, $\dot{X}^{\mu}$ is replaced by $-i\dot{X}^{\mu}$ and thus

$$p_{\mu}=\frac{\partial L}{\partial\dot{X}^{\mu}_{l}}=\frac{\partial L}{-i\partial\dot{X}^{\mu}_{e}}=\frac{\partial L}{-i\partial\dot{X}^{\mu}}=i\frac{\partial L}{\partial\dot{X}^{\mu}}=i\dot{X}_{\mu}$$

where $\dot{X}^{\mu}_{l}$ and $\dot{X}^{\mu}_{e}$ are derivatives with respect to Lorentzian and Euclidean time, respectively, with $\tau_{l}=i\tau_{e}$ and $\tau_{e}=\tau$. Noether's theorem then gives the BRST charge, with L denoting the

[10]The last step is integration by parts in τ: $\int d\tau\,b\dot{c}\,e=-\int d\tau\,\dot{b}\,c\,e-\int d\tau\,b\,c\,\dot{e}$, and on the gauge slice enforced by the B equation of motion, $e=1$, the $\dot{e}$ term vanishes; boundary terms are dropped as usual.

[11]The mismatch found above comes from eliminating B and setting $e=1$ before checking the symmetry, while still using a transformation law inherited from the unreduced action. Off shell the BRST transformations close with the auxiliary field B; once B is eliminated the transformation of b is only on-shell nilpotent, and invariance of the reduced action holds up to equations of motion and total derivatives. The safer route constructs the BRST charge before eliminating B, giving $Q_{B}=c\,H$ with $H=\frac{1}{2}(p^{2}+m^{2})$ and $Q_{B}^{2}=0$ because $c^{2}=0$ and H commutes with c — the point-particle analogue of keeping the Nakanishi–Lautrup field until the BRST algebra has been checked. [10].

Lagrangian:

$$Q_B = \frac{\partial L}{-i\partial \dot{X}^\mu}\Delta X^\mu + \frac{\partial L}{-i\partial \dot{b}}\Delta b + \frac{\partial L}{-i\partial \dot{c}}\Delta c = i\dot{X}^\mu\left(ic\dot{X}^\mu\right) + ic\left[i\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right)\right]$$
$$= \frac{c}{2}\left(-\dot{X}^\mu\dot{X}_\mu + m^2\right) = \frac{c}{2}\left(p^2+m^2\right) = cH$$

The derivative with respect to $\dot{b}$ anticommutes with $\dot{b}$. The ghost Hilbert space is the two-state system

$$|k;\uparrow\rangle, \quad |k;\downarrow\rangle$$

where the following is true;

$$p^\mu|k;\uparrow\rangle = p^\mu|k;\downarrow\rangle = k^\mu$$
$$b|k;\uparrow\rangle = |k;\downarrow\rangle, \quad b|k;\downarrow\rangle = 0$$
$$c|k;\uparrow\rangle = 0, \quad c|k;\downarrow\rangle = |k;\uparrow\rangle \tag{4.2.21}$$

This implies

$$Q_B|k;\downarrow\rangle = \frac{c}{2}\left(p^2+m^2\right)|k;\downarrow\rangle = \frac{1}{2}\left(k^2+m^2\right)|k;\uparrow\rangle, \quad Q_B|k;\uparrow\rangle = 0 \quad \text{as } c|k;\uparrow\rangle = 0$$

The BRST-closed states are all $|k;\uparrow\rangle$ states, together with the $|k;\downarrow\rangle$ states satisfying $k^2+m^2 = 0$. Thus the mass-shell condition is imposed on the $|k;\downarrow\rangle$ branch. For $k^2+m^2 \neq 0$, the state $|k;\uparrow\rangle$ is proportional to $Q_B|k;\downarrow\rangle$ and is therefore exact. The closed states that are not exact are consequently

$$|k;\downarrow\rangle, \quad |k;\uparrow\rangle \quad \text{for } k^2+m^2 = 0$$

The $|k;\uparrow\rangle$ branch is then removed by the usual relative-cohomology condition. Since exact states have vanishing amplitudes, any amplitude with an external $|k;\uparrow\rangle$ state vanishes away from $k^2+m^2 = 0$; allowing such a state only on the mass shell would require a term proportional to $\delta(k^2+m^2)$, which is not the form of an ordinary local S-matrix contribution in dimensions greater than two. Thus the physical point-particle spectrum is represented by the $|k;\downarrow\rangle$ branch.[12] This projection can also be obtained by imposing the constraint

$$b|\psi\rangle = 0$$

under which only $|k;\downarrow\rangle$ remains.

2.3 BRST quantization of the string

For the bosonic string the BRST charge combines the matter Virasoro generators with the ghost oscillators. For the open string it can be written schematically as

$$Q_B = \sum_n c_{-n}\left(L_n^X + \frac{1}{2}L_n^{\text{gh}} - a\delta_{n0}\right), \tag{2.7}$$

with the standard additional normal-ordering terms included in L_n^{gh}. Nilpotency gives

$$Q_B^2 = 0 \iff D = 26, \qquad a = 1 \tag{2.8}$$

for the open bosonic string. Thus the BRST calculation reproduces the same critical dimension and intercept that appeared in old covariant quantization [10].

BRST cohomology of the string.

In the open string, an unintegrated physical vertex has the form cV_m, where V_m is a matter primary. The c ghost has ghost number one, so the physical open-string cohomology is taken at

[12] An S-matrix element in $D > 2$ is a distribution with poles at particle masses, produced by long-distance propagation in the LSZ limit. A contribution proportional to $\delta(k^2+m^2)$ has support on a measure-zero set of external momenta and corresponds to no local coupling; amplitudes at generic momenta are blind to such states. Consistency of factorization then forces them out of the spectrum altogether, which is what the $b_0|\psi\rangle = 0$ condition implements.

ghost number one:

$$\mathcal{H}_{\text{phys}}^{\text{open}} = H^1(Q_B) = \frac{\ker Q_B \cap \mathcal{H}_{\text{gh}=1}}{\operatorname{im} Q_B \cap \mathcal{H}_{\text{gh}=1}}, \qquad |\psi\rangle \sim |\psi\rangle + Q_B|\Lambda\rangle. \tag{2.9}$$

The physical meaning is the same as in the open-vector example in Eq. (1.36). A BRST-exact state changes the representative of the vertex operator but not the amplitude. For example, a longitudinal photon polarization gives a total derivative on the boundary, so it is removed from the physical spectrum.

The vertex condition can be checked directly from the BRST current. Up to total derivatives in the current, take

$$j_B = cT_m + \frac{1}{2}{:}cT_g{:} + \frac{3}{2}\partial^2 c.$$

Let $\mathcal{O}$ be a matter primary of holomorphic weight h. Only cT_m contributes to the singular part of $j_B(z)\mathcal{O}(w)$. Expanding $c(z)$ around w gives

$$j_B(z)\mathcal{O}(w) \sim \frac{h\, c\mathcal{O}(w)}{(z-w)^2} + \frac{h\,(\partial c)\mathcal{O}(w) + c\,\partial\mathcal{O}(w)}{z-w}.$$

Hence

$$[Q_B, \mathcal{O}] = h(\partial c)\mathcal{O} + c\,\partial\mathcal{O}.$$

For $h = 1$, this is a total derivative:

$$[Q_B, \mathcal{O}] = \partial(c\mathcal{O}).$$

Therefore an integrated boundary insertion $\int dz\, \mathcal{O}$ is BRST closed up to endpoint terms precisely when the matter operator has weight one. The unintegrated insertion gives the complementary calculation,

$$j_B(z)c\mathcal{O}(w) \sim \frac{(1-h)c\,\partial c\,\mathcal{O}(w)}{z-w}, \qquad \{Q_B, c\mathcal{O}\} = (1-h)c\,\partial c\,\mathcal{O}.$$

Thus $c\mathcal{O}$ is BRST closed exactly at $h = 1$. For the closed string the same argument is applied in both chiral sectors, so $\int d^2z\, V$ and $c\tilde{c}\, V$ are physical when V has weight $(1, 1)$ [3].

Siegel gauge gives a useful check. If

$$b_0|\psi\rangle = 0, \qquad \{Q_B, b_0\} = L_0, \tag{2.10}$$

then a BRST-closed state satisfies

$$0 = b_0 Q_B|\psi\rangle = \{b_0, Q_B\}|\psi\rangle - Q_B b_0|\psi\rangle = L_0|\psi\rangle. \tag{2.11}$$

Thus the mass-shell condition is the zero-mode part of BRST invariance in this gauge. This is the BRST version of the null-state identification used in old covariant quantization [10].

2.4 The no-ghost theorem

The no-ghost theorem states that covariant quantization gives the same positive-norm physical spectrum as light-cone quantization in the critical bosonic string. In BRST language,

$$[\psi] \in H(Q_B), \qquad [\psi] \neq 0 \quad \Rightarrow \quad \langle[\psi]|[\psi]\rangle \geq 0, \tag{2.12}$$

with zero norm only for null representatives that are set to zero in cohomology. The theorem is essential because the covariant Fock space contains time-like oscillators with negative norm before the constraints are imposed [10, 28].

Proof idea and physical content.

The proof tracks where the negative-norm oscillators go. One separates the oscillators into transverse modes and pairs involving longitudinal/time-like oscillators together with the b, c ghosts. These non-transverse variables form BRST quartets. A quartet has the schematic form

$$|u\rangle, \quad Q_B|u\rangle = |v\rangle, \qquad |\tilde{v}\rangle, \quad Q_B|\tilde{v}\rangle = |\tilde{u}\rangle. \tag{2.13}$$

States paired in this way are either BRST-exact or have zero inner product with all physical states, so they do not survive in the cohomology. Technically one filters the BRST charge as

$$Q_B = Q_0 + Q_1 + Q_2 + \cdots, \qquad Q_0^2 = 0, \tag{2.14}$$

and first computes the Q_0 cohomology. The result contains only the transverse oscillators. The remaining terms in Q_B do not reintroduce negative-norm states when $D = 26$ and $a = 1$. Furuuchi and Ohta implement this argument by a similarity transformation, which gives a compact check on the longer no-ghost proof [28].

The quartet argument can be written as a short cohomology calculation. Choose a frame with $k^+ \neq 0$, and grade the nonzero modes by the light-cone number $N_{\rm lc}$, which counts $+$ oscillators against $-$ oscillators. The BRST charge decomposes as

$$Q_B = Q_1 + Q_0 + Q_{-1}, \qquad [N_{\rm lc}, Q_j] = jQ_j.$$

The highest-grade term is

$$Q_1 = -\sqrt{2\alpha'}\, k^+ \sum_{m\neq 0} c_m \alpha^-_{-m}.$$

Nilpotency of Q_B implies $Q_1^2 = 0$. Define

$$R = \frac{1}{\sqrt{2\alpha'}\, k^+} \sum_{m\neq 0} b_m \alpha^+_{-m}, \qquad S = \{Q_1, R\}.$$

Using $\{b_m, c_n\} = \delta_{m+n,0}$ and $[\alpha^+_m, \alpha^-_n] = -m\delta_{m+n,0}$, one obtains

$$S = \sum_{m=1}^{\infty} m \left(N_{b,m} + N_{c,m} + N^+_m + N^-_m\right).$$

Thus S is a non-negative number operator for all non-transverse matter oscillators and all nonzero ghost oscillators. Since $[Q_1, S] = 0$, the Q_1-cohomology can be computed at fixed S. If $Q_1|\psi\rangle = 0$ and $S|\psi\rangle = s|\psi\rangle$ with $s > 0$, then

$$|\psi\rangle = \frac{1}{s} S|\psi\rangle = \frac{1}{s} Q_1 R|\psi\rangle.$$

Hence every closed state with $s > 0$ is exact. Only the kernel of S survives:

$$N_{b,m} = N_{c,m} = N^+_m = N^-_m = 0 \qquad (m \geq 1).$$

The surviving representatives contain only transverse oscillators,

$$\left(\prod\nolimits_a \alpha^{i_a}_{-n_a}\right)|0;k\rangle, \qquad i = 1, \ldots, D-2. \tag{2.15}$$

Their norms are positive because

$$\left\|\alpha^i_{-n}|0;k\rangle\right\|^2 = \langle 0;k|\alpha^i_n \alpha^i_{-n}|0;k\rangle = n\,\delta^{ii} > 0.$$

The remaining pieces $Q_0 + Q_{-1}$ do not change this conclusion on a fixed L_0 eigenspace; they move representatives inside the same filtered cohomology class. Thus the BRST cohomology has the transverse light-cone spectrum, provided $D = 26$ and $a = 1$ so that $Q_B^2 = 0$ and the mass-shell condition is compatible with the transverse representatives [3, 28].

2.4.1 BRST-OCQ equivalence

The equivalence with old covariant quantization can be summarized as the statement that the Virasoro constraints plus the identification by spurious states are the same physical equivalence relation as BRST cohomology:

$$\frac{\{|\psi\rangle : (L_0 - a)|\psi\rangle = 0,\ L_n|\psi\rangle = 0\ (n > 0)\}}{\{L_{-n}|\chi_n\rangle :\ n > 0\}} \simeq H(Q_B). \tag{2.16}$$

After the earlier OCQ calculation: OCQ imposes constraints by hand and removes null descendants, while BRST makes the same removal into a cohomological gauge equivalence. [10].

Chapter 3

The string S-matrix and moduli space

This chapter converts the gauge-fixed path integral into a prescription for scattering amplitudes. Gauge fixing does not remove every freedom in the worldsheet metric. On a given topology a finite number of metric deformations, the moduli, cannot be reached by any diffeomorphism or Weyl rescaling, and a finite number of gauge transformations, the conformal Killing vectors, survive the fixing and must not be counted twice. The construction proceeds by decomposing an arbitrary metric variation into gauge directions and moduli directions, counting each by the Riemann–Roch theorem, and observing that the Faddeev–Popov procedure supplies one b-ghost insertion for every modulus and one c-ghost insertion for every conformal Killing vector. This gives the amplitude formula of string perturbation theory, an integral over the moduli space of punctured Riemann surfaces with BRST-invariant vertex operators for the external states. Gauge independence is then established: BRST-exact insertions decouple up to boundary terms on moduli space, and those boundary terms are identified with the factorization poles required by unitarity. Thus a condition imposed on the worldsheet becomes a statement about the spacetime S-matrix [10, 17].

What does "moduli space" mean physically? In this chapter the immediate moduli are worldsheet moduli: they describe inequivalent complex structures and puncture positions of the Riemann surface on which the path integral is evaluated. But the same word appears in compactification with a closely related meaning. There a modulus is a scalar parameter whose constant value changes the background without costing potential energy. It may set the radius of a circle, the shape of a torus, a Wilson line, or a Kähler or complex-structure parameter of a compact space.

The moduli space is the geometric space whose points label all such valid and distinct choices. It is therefore a map of the theory's physical possibilities: each point represents a background or configuration, and moving in moduli space changes quantities such as sizes, shapes, and radii of the extra dimensions. These points are not gauge copies of one another; they correspond to different values of physical scalar fields.

The simplest example is compactification on a circle. If one internal direction obeys

$$X^d \sim X^d + 2\pi R$$

and the metric contains

$$ds^2 = \cdots + e^{2\beta\phi}(dX^d)^2,$$

then the physical circumference is proportional to $2\pi R e^{\beta\phi}$. The scalar ϕ therefore measures the size of the compact dimension. If no potential is generated for ϕ, every constant value is an allowed vacuum, so these values form a one-dimensional moduli space. More generally, when the lower-dimensional action contains a kinetic term

$$-\frac{1}{2}G_{ij}(\phi)\,\partial_\mu\phi^i\partial^\mu\phi^j,$$

the coefficient G_{ij} is interpreted as the metric on moduli space. For one radius modulus this metric has one component; for many shape and size moduli it gives the line element on the full space of compactification backgrounds.

3.1 The circle and the torus

The torus has one complex metric modulus τ, appearing in the metric as

$$ds^2 = \left|d\sigma^1 + \tau d\sigma^2\right|^2$$

This modulus remains after choosing reparametrizations that preserve the periodicities of $\sigma^{1,2}$. The large reparametrizations are the $SL(2,\mathbb{Z})/\mathbb{Z}_2$ transformations

$$\begin{pmatrix}\sigma^{1'}\\ \sigma^{2'}\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sigma^{1}\\ \sigma^{2}\end{pmatrix}, \quad ad-bc=1 \quad a,b,c,d\in\mathbb{Z}$$

The metric and periodicities are also invariant under rigid translations,

$$\sigma^a \to \sigma^a + \xi^a$$

The residual symmetry group left after choosing the metric is the conformal Killing group (CKG). One vertex-operator position must be fixed for each independent generator of this group.

3.2 Moduli and Riemann surfaces

Metric deformations not generated by diffeomorphisms and Weyl transformations are represented by variations δg_{ab} orthogonal to the diff $\times$ Weyl variations

$$-2\left(P_1\delta\sigma\right)_{ab} + (2\delta\omega - \nabla\cdot\delta\sigma)g_{ab} \tag{5.2.1}$$

where the operator P_1 acts on $\delta\sigma^a$ as

$$\left(P_1\delta\sigma\right)_{ab} = \frac{1}{2}\left(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - g_{ab}\nabla_c\delta\sigma^c\right)$$

Orthogonality of δg_{ab} to Eq. (5.2.1) gives

$$\int d^2\sigma\sqrt{g}\delta g_{ab}\left(-2\left(P_1\delta\sigma\right)^{ab} + (2\delta\omega-\nabla\cdot\delta\sigma)g^{ab}\right) = 0$$

$$\Rightarrow \int d^2\sigma\sqrt{g}\left(-2\delta g_{ab}\left(P_1\delta\sigma\right)^{ab} + g^{ab}\delta g_{ab}(2\delta\omega-\nabla\cdot\delta\sigma)\right) = 0$$

Integrating by parts gives

$$\delta g_{ab}\left(P_1\delta\sigma\right)^{ab} = \delta g_{ab}\nabla^a\delta\sigma^b - \frac{g^{ab}\delta g_{ab}}{2}\nabla\cdot\delta\sigma = \nabla^a\left(\delta g_{ab}\delta\sigma^b\right) - \left(\nabla^a\delta g_{ab}\right)\delta\sigma^b - \frac{g^{ab}\delta g_{ab}}{2}\nabla\cdot\delta\sigma$$
$$= \left(P_1^T\delta g\right)_a\delta\sigma^a - \frac{g^{ab}\delta g_{ab}}{2}\nabla\cdot\delta\sigma + \text{ total derivative}$$

Therefore[1]

$$\int d^2\sigma\sqrt{g}\left(\left(P_1^T\delta g\right)_a\delta\sigma^a - g^{ab}\delta g_{ab}\delta\omega\right) = 0$$

Since $\delta\sigma^a$ and $\delta\omega$ are arbitrary, the coefficients must vanish:

$$\left(P_1^T\delta g\right)_a = 0, \quad g^{ab}\delta g_{ab} = 0$$

Thus δg_{ab} is traceless, and the number of independent solutions of the first equation gives the number of moduli.

Conformal Killing vectors occur when a diffeomorphism plus Weyl transformation produces no metric variation, i.e. when Eq. (5.2.1) vanishes. Contracting with g^{ab} gives

$$\delta\omega = \frac{1}{2}\nabla\cdot\delta\sigma$$

[1] If the trace part and the $\delta\omega$ variation are eliminated first, only $(P_1^T\delta g)_a = 0$ survives; if instead $\delta\omega$ is kept as an independent variation, its coefficient $g^{ab}\delta g_{ab}$ appears alongside. Setting the coefficients of the independent variations $\delta\sigma^a$ and $\delta\omega$ separately to zero reproduces the two conditions below in either scheme.

The remaining condition is $(P_1\delta g)_a = 0$, the conformal Killing equation. Around conformal gauge,

$$g_{ab}(z,\bar{z}) = \frac{1}{2}e^{\omega(z,\bar{z})}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the conformal Killing equation becomes[2]

$$\partial\delta g_{\bar{z}\bar{z}} = \bar{\partial}\delta g_{zz} = \partial\delta\bar{z} = \bar{\partial}\delta z = 0$$

Thus, the number of moduli (denoted as μ) is $\dim\left(\ker P_1^T\right)$ and the number of CKVs (denoted as κ) is $\dim(\ker P_1)$. The Riemann-Roch theorem (which can be seen as an index theorem) states that;

$$\dim\left(\ker P_1^T\right) - \dim(\ker P_1) = -3\chi \Rightarrow \mu - \kappa = -3\chi \tag{5.2.2}$$

For $\chi < 0$ there are no CKVs. A Weyl transformation can bring the Ricci curvature to a constant. The old and new curvatures obey

$$\sqrt{g'}R' = \sqrt{g}\left(R - 2\nabla^2\omega\right) \Rightarrow 2\nabla^2\omega = R - e^{2\omega}R'$$

which can be solved for constant R'.[3] Moreover, due to Gauss-Bonnet theorem (for no boundaries);

$$\int_M d^2\sigma\sqrt{g}R = 2\pi\chi(M)$$

the sign of a constant R is the sign of χ. The identity needed below is

$$\left(P_1^T P_1 u\right)_a = -\frac{1}{2}\left(\nabla^b\nabla_a u_b + \nabla^b\nabla_b u_a - \nabla^b\left(g_{ab}\nabla_c u^c\right)\right) = -\frac{1}{2}\nabla^2 u_a - \frac{1}{2}\left(\nabla^b\nabla_a u_b - \nabla^b\left(g_{ab}\nabla_c u^c\right)\right)$$

$$= -\frac{1}{2}\nabla^2 u_a + \frac{1}{2}\left(\nabla_a\nabla_c - \nabla_c\nabla_a\right)u^c = -\frac{1}{2}\nabla^2 u_a - \frac{1}{4}R u_a \tag{5.2.3}$$

Using the facts that $(P_1\delta\sigma)_{ab}$ is traceless and symmetric, we can derive the following identity as well;

$$(P_1\delta\sigma)_{ab}(P_1\delta\sigma)^{ab} = \nabla_a\delta\sigma_b\nabla^b\delta\sigma^a = \text{ total derivative } -\delta\sigma_b\nabla_a\nabla^b\delta\sigma^a = \text{ total derivative } -\delta\sigma_b\left(P_1^T P_1\delta\sigma\right)^b$$

where the last step uses the fact that $\left(P_1^T\sigma\right)^{ab}$ is traceless and symmetric. Using Eq. (5.2.3), one obtains

$$\int d^2\sigma\sqrt{g}\,(P_1\delta\sigma)_{ab}(P_1\delta\sigma)^{ab} = \sqrt{g}\int d^2\sigma\delta\sigma_a\left(P_1^T P_1\delta\sigma\right)^a = \int d^2\sigma\sqrt{g}\left(-\delta\sigma^a\frac{1}{2}\nabla^2\delta\sigma_a - \frac{1}{4}R\delta\sigma^a\delta\sigma_a\right)$$

$$= \int d^2\sigma\sqrt{g}\left(\frac{1}{2}\nabla_b\delta\sigma^a\nabla^b\delta\sigma_a - \frac{1}{4}R\delta\sigma^a\delta\sigma_a\right)$$

where surface terms are dropped in the last line. If $\chi < 0$, then $R < 0$ and the integral in the last line is positive definite. Hence $(P_1\delta\sigma)_{ab}$ cannot vanish, and there are no CKVs for negative Euler characteristic.

Riemann surfaces in the S-matrix.

A string Feynman diagram is a two-dimensional worldsheet with prescribed external insertions. In an ordinary point-particle diagram each internal line carries a Schwinger proper time; in string theory the corresponding parameters are the moduli of the Riemann surface. After fixing

[2] In conformal gauge the only nonvanishing metric components are $g_{z\bar{z}}$, so a traceless symmetric deformation has pure components δg_{zz}, $\delta g_{\bar{z}\bar{z}}$. The gauge variation gives $(P_1\delta\sigma)_{zz} = \nabla_z\delta\sigma_z = g_{z\bar{z}}\partial_z\delta\sigma^{\bar{z}}$, and similarly for the conjugate. A conformal Killing vector is one producing no traceless deformation, $\partial_{\bar{z}}\,\delta z = \partial_z\,\delta\bar{z} = 0$: holomorphic vector fields. Likewise $\ker P_1^T$ consists of δg_{zz} with $\partial_{\bar{z}}\delta g_{zz} = 0$: holomorphic quadratic differentials.

[3] The argument is correct; the displayed equation with constant R' is the Liouville equation, and the existence of a solution on any compact surface, with the sign of R' dictated by χ through Gauss–Bonnet, is the content of the uniformization theorem: positive constant curvature on the sphere, flat metrics on the torus, negative constant curvature for genus two and higher.

diffeomorphisms and Weyl rescalings, a general metric variation decomposes as

$$\delta g_{ab} = \sum_{r=1}^{\mu} \delta m^r\, \psi^{(r)}_{ab} + (P_1\delta\sigma)_{ab} + 2\delta\omega\, g_{ab}, \qquad \psi^{(r)} \in \ker P_1^T. \tag{3.1}$$

The last two terms are changes produced by a diffeomorphism and a Weyl rescaling. The first term cannot be removed by these gauge transformations and therefore labels a true deformation of the complex structure. The operator P_1 already appeared in the Faddeev–Popov determinant: its kernel counts conformal Killing vectors, while $\ker P_1^T$ counts moduli. For a closed orientable surface of genus h, the complex dimension of moduli space is 0 for the sphere, 1 for the torus, and $3h-3$ for $h \geq 2$ [10, 17].

3.3 The measure for moduli

The expression for the S matrix with n momenta $k_1, \ldots, k_n$ and internal states $j_1, \ldots, j_n$ is given by as follows:

$$S_{j_1,\ldots j_n}(k_1,\ldots,k_n) = \sum_{\text{comp. top.}} \int_F \frac{d^\mu t}{n_R} \int [d\phi db dc] e^{-S_m - S_g - \lambda\chi}$$
$$\times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_k \hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)}\mathcal{V}_{j_i}(k_i,\sigma_i) \tag{5.3.1}$$

where F is the moduli space, n_R is the order of discrete symmetries, f contains fixed coordinates and the inner product is defined as follows:

$$(b, \partial_k \hat{g}) = \int d^2\sigma \sqrt{g} b_{ab} \partial_k \hat{g}^{ab}$$

[4]

Determinant form of the measure.

The determinant form follows by performing the Gaussian decomposition of metric fluctuations. The component of δg_{ab} along $P_1\delta\sigma$ is a gauge direction and contributes a Faddeev–Popov determinant, while the components along $\psi^{(r)}_{ab}$ remain as integrations over the moduli m^r. Omitting zero modes, the schematic result is

$$\frac{[dg]}{\text{Diff} \times \text{Weyl}} \longrightarrow d^\mu m\, \frac{\left(\det{}' P_1^\dagger P_1\right)^{1/2}}{\text{Vol(CKG)}} \prod_{r=1}^{\mu} (b, \partial_{m^r}\hat{g}). \tag{3.2}$$

The prime means that conformal Killing zero modes are omitted. The insertions $(b, \partial_{m^r}\hat{g})$ are the ghost representation of the Jacobian for moving along moduli space. If vertex positions are fixed using conformal Killing transformations, the missing ghost zero modes are supplied by the corresponding $c\tilde{c}$ insertions at the fixed vertices [10].

Counting moduli and conformal Killing vectors.

The Riemann–Roch theorem supplies the index count used in Eq. (3.1). For the operator P_1 acting on vector fields one has

$$\text{ind}\, P_1 = \dim\ker P_1 - \dim\ker P_1^T = 3\chi. \tag{3.3}$$

With the notation $\kappa = \dim\ker P_1$ and $\mu = \dim\ker P_1^T$, this gives $\mu - \kappa = -3\chi$. On the sphere, $\chi = 2$ and there are six real conformal Killing parameters; this is why three complex insertion points may be fixed. On the torus, $\chi = 0$ and one complex modulus remains. For genus $h \geq 2$ there are no conformal Killing vectors and $3h-3$ complex moduli must be integrated. This counting is needed before the amplitude measure can be written correctly [10, 17].

[4]The determinant form of this measure, and the path-integral proof of the Riemann–Roch count used above, are assembled in the two boxed paragraphs that follow; the index computation itself is the standard heat-kernel evaluation of $\dim\ker P_1 - \dim\ker P_1^T$ and we do not repeat it beyond the statement in Eq. (3.3).

3.4 Moduli-space measure and gauge independence

The moduli-space measure is the part of the amplitude that remains after the gauge directions have been removed. If μ_r is the Beltrami differential representing a change of complex structure, the corresponding ghost insertion is

$$(b, \mu_r) = \int_\Sigma d^2 z \; (b_{zz}\, \mu^z_{r\bar z} + b_{\bar z\bar z}\, \bar\mu^{\bar z}_{rz}) \,. \tag{3.4}$$

The same insertion can be obtained from transition functions. Cover the surface by coordinate patches U_m, with holomorphic overlap maps $z_m = f_{mn}(z_n;t)$. A small change of the modulus t^r can be represented inside each patch by a vector field,

$$z_m \longrightarrow z_m + \delta t^r\, v^z_{r,m}(z_m, \bar z_m), \qquad \bar z_m \longrightarrow \bar z_m + \delta t^r\, v^{\bar z}_{r,m}(z_m, \bar z_m).$$

The induced Beltrami differential is locally

$$\mu^z_{r\bar z} = \partial_{\bar z_m} v^z_{r,m}, \qquad \bar\mu^{\bar z}_{rz} = \partial_{z_m} v^{\bar z}_{r,m}.$$

Using $\bar\partial b_{zz} = 0$ away from operator insertions, the holomorphic part of the pairing becomes a boundary term:

$$\int_{U_m} d^2 z_m\, b^{(m)}_{zz}\, \partial_{\bar z_m} v^z_{r,m} = \oint_{\partial U_m} \frac{dz_m}{2\pi i}\, b^{(m)}_{zz} v^z_{r,m}.$$

On an overlap $U_m \cap U_n$, the two boundary contours have opposite orientations. Their difference is the change of the holomorphic transition map,

$$\delta_r f^z_{mn} = v^z_{r,m} - \frac{\partial f_{mn}}{\partial z_n}\, v^z_{r,n}.$$

Thus

$$(b, \mu_r) = \sum_{m<n} \oint_{C_{mn}} \frac{dz_m}{2\pi i}\, b^{(m)}_{zz}\, \delta_r f^z_{mn} + \text{ antiholomorphic part.}$$

This formula is useful because a puncture can be treated as another modulus. If a new local coordinate y_i is centered at the i-th puncture,

$$y_i = z - z_i, \qquad \frac{\partial y_i}{\partial z_i} = -1.$$

The boundary of the excised puncture disc has the opposite orientation, so the modulus z_i supplies

$$(b, \mu_{z_i}) = \oint_{z_i} \frac{dz}{2\pi i}\, b(z).$$

The OPE $b(z)c(z_i) \sim (z - z_i)^{-1}$ then gives

$$\oint_{z_i} \frac{dz}{2\pi i}\, b(z)c(z_i) = 1.$$

Consequently

$$d^2 z_i\, (b, \mu_{z_i})(\bar b, \bar\mu_{\bar z_i})\, c(z_i)\bar c(\bar z_i)\, \mathcal{V}_i(z_i, \bar z_i) = d^2 z_i\, \mathcal{V}_i(z_i, \bar z_i).$$

This is the local calculation behind the usual exchange between an integrated vertex and a fixed unintegrated vertex. The positions of punctures and the ordinary complex-structure moduli are therefore treated by the same b-ghost measure [3, 17].

This is the CFT form of differentiating the fiducial metric with respect to a modulus. Near a degeneration, for example when a tube of modulus q becomes long, the measure contains $d^2 q/|q|^2$ together with the insertion of a complete set of intermediate states. The boundary of moduli space therefore gives the physical pole structure of the amplitude.

Gauge invariance appears as the decoupling of null or BRST-exact insertions. If an external vertex is shifted by

$$V \longrightarrow V + \{Q_B, \Lambda\}, \tag{3.5}$$

then the variation of an amplitude of BRST-closed external states is

$$\delta\mathcal{A} = \int_{\mathcal{M}} d\mu \left\langle \{Q_B, \Lambda\} \prod_i V_i \right\rangle. \tag{3.6}$$

Moving Q_B through the correlator gives zero in the interior of moduli space because the other vertices are BRST closed. Boundary terms arise only from degeneration regions, where they are interpreted as factorization contributions. Thus Ward identities and unitarity are controlled by the same BRST condition.

For a genus-h closed orientable surface with n punctures, the corresponding measure has the form

$$\mathcal{A}_{h,n} = g_s^{2h-2+n} \int_{\mathcal{M}_{h,n}} \left\langle \prod_{r=1}^{3h-3+n} (b, \mu_r) \prod_{i=1}^{n} V_i \right\rangle . \tag{3.7}$$

For the sphere one must additionally divide by $PSL(2, \mathbb{C})$; equivalently, three positions are fixed and three $c\tilde{c}$ insertions absorb the ghost zero modes. The tree-level formulas below are special cases of this expression [10, 17].

Chapter 4

Tree-level amplitudes

This chapter applies the formalism of the previous one to explicit amplitudes. We evaluate the genus-zero amplitudes on the three tree-level surfaces, the sphere for closed strings, the disk for open strings, the projective plane for unoriented strings, and the method is the same in each case: solve for the scalar Green function on the surface, saturate the ghost zero modes by fixing the positions allowed by the conformal Killing group, and reduce the correlator to elementary integrals. The four-tachyon disk amplitude is the Veneziano formula, whose infinite sequence of poles displays the entire string spectrum inside a single integral, and whose three degeneration regions give the s-, t-, and u-channels of a single amplitude, the content of worldsheet duality. Factorizing on the tachyon pole fixes the disk normalization in terms of the couplings; this is the first place where unitarity is used to compute . Chan–Paton factors then show how nonabelian gauge interactions enter through the open string, and the closed-string analogue, the Virasoro–Shapiro amplitude, closes the chapter [10, 17].

4.1 Riemann surfaces

At genus zero the general formula in Eq. (3.7) simplifies because there are no genuine complex-structure moduli on the sphere. What remains is the treatment of punctures and conformal Killing transformations. The sphere has $PSL(2,\mathbb{C})$ symmetry, so three closed-string insertion points can be fixed. The disk has $PSL(2,\mathbb{R})$ symmetry, so three real boundary positions can be fixed. The projective plane implements the unoriented closed-string projection through a crosscap identification. These three surfaces are the elementary tree-level backgrounds used in the rest of the chapter.

4.1.1 The sphere

The sphere is the basic closed-string tree diagram. Its conformal Killing group is $PSL(2,\mathbb{C})$, so three complex insertion points may be fixed, for example

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3 = \infty. \tag{4.1}$$

The ghost correlator then supplies the Faddeev-Popov Jacobian,

$$\langle c(z_1)c(z_2)c(z_3)\rangle = z_{12}z_{13}z_{23}, \tag{4.2}$$

with an identical antiholomorphic factor for closed strings. Therefore three unintegrated closed-string vertices appear at tree level; the configuration and its cross-ratio are shown in Figure 4.1. [17].

4.1.2 The disc

The disc is the tree surface for open strings and for closed-string insertions in the presence of a boundary. Its conformal Killing group is $PSL(2,\mathbb{R})$, so three real boundary positions may be fixed. The doubling trick turns the upper half plane into a holomorphic calculation with image points:

$$\langle X^\mu(z)X^\nu(w)\rangle_D = -\frac{\alpha'}{2}\eta^{\mu\nu}\left[\ln|z-w|^2 + \ln|z-\bar{w}|^2\right], \tag{4.3}$$

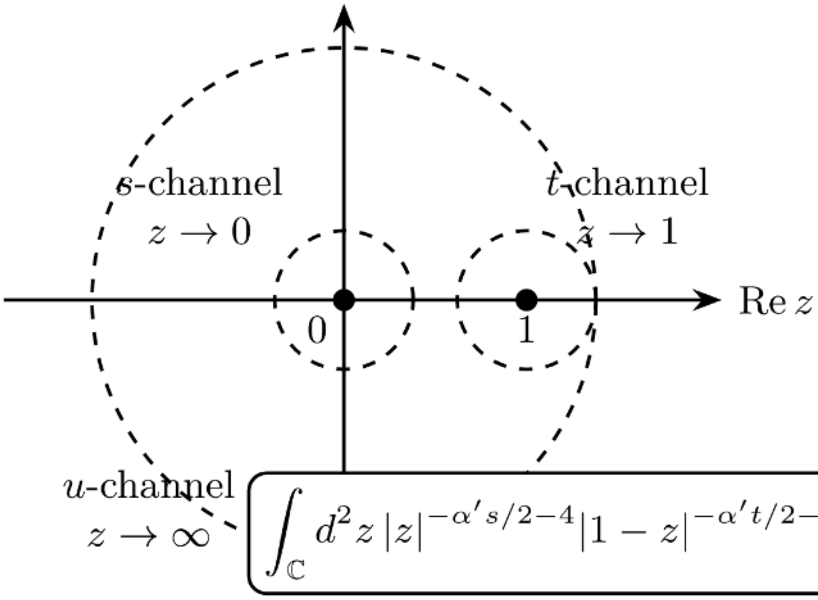


Figure 4.1: On the sphere, the projective symmetry $SL(2,\mathbb{C})$ fixes three punctures. In a four-point amplitude the only remaining invariant is the cross-ratio, which is usually denoted by the single complex coordinate z.

for Neumann boundary conditions. The image-charge term in the disc Green function follows from this relation. [17].

4.1.3 The projective plane

The projective plane RP^2 is the tree-level unoriented closed-string surface. It can be represented as a sphere identified by the antipodal antiholomorphic involution

$$z \sim -\frac{1}{\bar{z}}. \tag{4.4}$$

Consequently, correlation functions may be computed by adding an image insertion at the involuted point. The closed-string analogue of the disc doubling trick, but with a crosscap in place of a boundary. The same surface appears again in the unoriented-string and orientifold discussions. [10].

4.2 Scalar expectation values

To derive the differential equation for the Green functions, start from the generating functional

$$Z[J] = \left\langle \exp\left(i \int d^2\sigma J_\mu(\sigma) X^\mu(\sigma) \right) \right\rangle \tag{6.2.1}$$

Consider the Helmholtz eigenvalue problem

$$\left(\partial^2 + \omega_I^2\right) X_I(\sigma) = 0$$

whose solutions form a complete set. The field X^μ may therefore be expanded as

$$X^\mu = \sum_I x_I^\mu X_I$$

Because the Helmholtz equation is linear, the eigenfunctions can be rescaled. Choose the normalization

$$\int d^2\sqrt{g} d^2\sigma X_I X_{I'} = \delta_{II'}$$

Orthogonality follows because the Helmholtz problem is a Sturm–Liouville problem. Equation (6.2.1) then gives

$$Z[J] = \prod_{\mu,I} \int dx_I^\mu \exp\left(ix_I^\mu \int d^2\sigma J_\mu(\sigma) X_I^\mu(\sigma)\right) \exp\left(-\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \sum_{I'} \omega_{I'}^2 x_I^\mu x_{\mu I'} X_I X_{I'}\right)$$

where the sums over I and μ have been written as a product over modes. Orthogonality of the X_I gives

$$Z[J] = \prod_{\mu,I} \int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + ix_I^\mu J_{\mu I}\right)$$

where

$$J_{\mu I} = \int d^2\sigma J_\mu(\sigma) X_I(\sigma)$$

The zero mode is constant, and orthonormality gives

$$X_0 = \left(\int d^2\sigma \sqrt{g}\right)^{-1/2}$$

The Helmholtz equation gives $\omega_0^2 = 0$, so all dx_I^μ integrals in $Z[J]$ are Gaussian except the zero-mode integrals dx_0^μ. The zero-mode integrations give factors of $2\pi\delta(J_{0\mu})$, and the product over μ gives $(2\pi)^d \delta^{(d)}(J_0)$. For the nonzero modes, use

$$\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

which gives

$$\int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + ix_I^\mu J_{\mu I}\right) = \sqrt{\frac{4\pi\alpha'}{w_I^2}} \exp\left(-\pi\alpha' \frac{J_{I\mu} J_{I\mu}}{w_I^2}\right)$$

Taking the product over μ gives

$$\prod_\mu \int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + ix_I^\mu J_{\mu I}\right) = \left(\frac{4\pi\alpha'}{w_I^2}\right)^{d/2} \exp\left(-\pi\alpha' \frac{J_I . J_I}{w_I^2}\right)$$

The $\mu = 0$ contribution has the opposite sign in the quadratic term. Rotating $x_I^0 \to ix_I^0$ gives the convergent Gaussian and contributes an additional factor of i to $Z[J]$. The result is

$$Z[J] = i(2\pi)^d \delta^{(d)}(J_0) \prod_{I \neq 0} \left(\frac{4\pi\alpha'}{w_I^2}\right)^{d/2} \exp\left(-\pi\alpha' \frac{J_I \cdot J_I}{w_I^2}\right)$$

Use the identity

$$-\pi\alpha' \frac{J_I \cdot J_I}{w_I^2} = -\frac{1}{2} \int d^2\sigma d^2\sigma' \left[J(\sigma) J(\sigma') \times \frac{2\pi\alpha'}{w_I^2} X_I(\sigma) X_I(\sigma')\right]$$

The eigenvalues of $-\nabla^2$ are ω_I^2, so the product over nonzero ω_I^2 gives the determinant of $-\nabla^2$ with the zero mode omitted. Therefore

$$Z[J] = i(2\pi)^d \delta^{(d)}(J_0) \prod_{I \neq 0} \left(\det \frac{-\nabla^2}{4\pi\alpha'}\right)^{d/2} \exp\left(-\frac{1}{2} \int d\sigma d\sigma' J(\sigma) J(\sigma') G(\sigma, \sigma')\right)$$

where

$$G(\sigma, \sigma') = \sum_{I \neq 0} \frac{2\pi\alpha'}{\omega_I^2} X_I(\sigma) X_I(\sigma')$$

This Green's function satisfies the following differential equation;

$$\nabla^2 G(\sigma, \sigma') = -2\pi\alpha' \sum_{I \neq 0} X_I(\sigma) X_I(\sigma')$$

To put this equation in a more useful form, expand the σ dependence of $\delta(\sigma - \sigma')$ in the complete basis X_I, with the σ' dependence carried by the coefficients:

$$\delta(\sigma-\sigma') = \sum_I x_I(\sigma')X^I(\sigma) \Rightarrow \int \sqrt{g(\sigma)}X^{I'}(\sigma)\delta(\sigma-\sigma')\,d^2\sigma = \sum_I x_I(\sigma')\int d^2\sigma\sqrt{g(\sigma)}X^I(\sigma)X^{I'}(\sigma)$$

$$\Rightarrow x_{I'}(\sigma') = \sqrt{g(\sigma')}X^{I'}(\sigma') \Rightarrow x_I(\sigma') = \sqrt{g(\sigma')}X^I(\sigma')$$

$$\Rightarrow \delta(\sigma-\sigma') = \sum_I \sqrt{g(\sigma')}X^I(\sigma')X^I(\sigma) \Rightarrow (g(\sigma'))^{-1/2}\delta(\sigma-\sigma') = \left(X^0\right)^2 + \sum_{I\neq 0} X^I(\sigma')X^I(\sigma)$$

$$\Rightarrow \sum_{I\neq 0} X^I(\sigma)X^I(\sigma') = (g(\sigma))^{-1/2}\delta(\sigma-\sigma') - \left(X^0\right)^2$$

Here the delta function is even and X_0 is constant. Hence

$$-\frac{1}{2\pi\alpha'}\nabla^2 G(\sigma,\sigma') = (g(\sigma))^{-1/2}\delta(\sigma-\sigma') - \left(X^0\right)^2 \tag{6.2.2}$$

On the sphere the metric is known explicitly, and Eq. (6.2.2) is solved by[1]

$$G(z_1,\bar z_1,z_2,\bar z_2) = -\frac{\alpha'}{2}\ln|z_{12}|^2 + f(z_1,\bar z_1) + f(z_2,\bar z_2) \tag{6.2.3}$$

with

$$f(z,\bar z) = \frac{\alpha' X_0^2}{4}\int d^2z' e^{2\omega(z',\bar z')}\ln|z-z'| + k$$

where k is a constant determined by zero-mode orthogonality. Consider the expectation value

$$A^n_{S^2}(k,\sigma) = \langle :\exp(ik_1X(\sigma_1)) : \ldots : \exp(ik_nX(\sigma_n)) :\rangle$$

$$= \langle :\exp(ik_1X(\sigma_1)+\ldots+ik_nX(\sigma_n)) :\rangle = \left\langle \exp\left[i\left(\sum_j\int d^2\sigma\delta(\sigma-\sigma_j)k_j\right)\right]\cdot X(\sigma_j)\right\rangle$$

This corresponds to the source

$$J^\mu(\sigma) = \sum_j k_j^\mu\delta^{(2)}(\sigma-\sigma_j) \Rightarrow J_I^\mu = \int d^2\sigma\sum_j k_j^\mu\delta^{(2)}(\sigma-\sigma_j)X_I(\sigma) = \sum_j k_j^\mu X_I(\sigma_j)$$

$$\Rightarrow J_0^\mu = X_0\sum_j k_j^\mu \Rightarrow \delta^{(d)}(J_0) = X_0^{-d}\delta^{(d)}\left(\sum_j k_j^\mu\right)$$

The exponent then becomes

$$= -\int d^2\sigma d^2\sigma' J(\sigma)\cdot J(\sigma')G(\sigma,\sigma') = -\int d^2\sigma d^2\sigma'\sum_{jl}k_j\cdot k_lG(\sigma,\sigma')\delta(\sigma-\sigma_j)\delta(\sigma-\sigma_l)G(\sigma,\sigma')$$

$$= -\frac{1}{2}\sum_{j,l}k_j\cdot k_lG(\sigma_j,\sigma_l) = -\sum_{j<l}k_j\cdot k_lG(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j):$$

The required amplitude is therefore

$$A^n_{S^2}(k,\sigma) = iC^X_{S^2}(2\pi)^d\delta^{(d)}\left(\sum_j k_j\right)\exp\left(-\sum_{j<l}k_j.k_lG(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j):\right)$$

where

[1] Here $\nabla^2 = 4e^{-2\omega}\partial\bar\partial$ in conformal gauge, and $\partial\bar\partial\ln|z_{12}|^2 = 2\pi\delta^2(z_1-z_2)$ reproduces the delta function, while $\partial\bar\partial$ acting on the f terms produces the constant $-X_0^2$ piece, since f is built from the area form. The additive constant k is fixed by demanding orthogonality to the zero mode, $\int d^2\sigma\sqrt{g}\,G(\sigma,\sigma') = 0$, which is the condition under which G inverts $-\nabla^2$ on the space orthogonal to constants.

$$C^X_{S^2} = X_0^{-d}\left(\det\frac{-\nabla^2}{4\pi\alpha'}\right)^{-d/2}_{S^2}$$

We next define the normal ordered $:G(\sigma,\sigma'):$ operator. It can be shown that[2];

$$:G(\sigma,\sigma') := G(\sigma,\sigma') + \frac{\alpha'}{2}\ln d^2(\sigma,\sigma')$$

and by using 6.2.3, we get[3];

$$:G(\sigma,\sigma) := 2f(\sigma) + \alpha'\omega(\sigma) \Rightarrow: G(z=z',\bar z=\bar z') := 2f(z,\bar z) + \alpha'\omega(z,\bar z)$$

Using momentum conservation, the exponent simplifies as follows:

$$\begin{aligned}
&-\sum_{j<l} k_j.k_l\, G(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j):\\
&= \frac{\alpha'}{2}\sum_{j<l} k_j.k_l \ln|\sigma_j-\sigma_l|^2 - \sum_{j<l} k_j.k_l f(\sigma_j) - \sum_{l<j} k_j.k_l f(\sigma_l)\\
&\quad - \sum_j k_j^2 f(\sigma_j) - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j)\\
&= \frac{\alpha'}{2}\sum_{j<l} k_j.k_l \ln|\sigma_j-\sigma_l|^2 - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j)\\
&\quad - \sum_j\left(\sum_{l<j} + \sum_{l>j} + \sum_{l=j}\right) k_l.k_j f(\sigma_j)\\
&= \sum_{j<l} \ln|\sigma_j-\sigma_l|^{\alpha' k_j.k_l} - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j)\\
&= \sum_{j<l} \ln|z_{jl}|^{\alpha' k_j.k_l} - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j).
\end{aligned}$$

In the second line the indices j and l have been interchanged in one of the $k\cdot k\, f(\sigma)$ terms. The amplitude becomes

$$A^n_{S^2}(k,\sigma) = iC^X_{S^2}(2\pi)^d\delta^{(d)}\left(\sum_j k_j\right)\exp\left(-\frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j)\right)\prod_{j<l}|z_{jl}|^{\alpha' k_j k_l} \tag{6.2.4}$$

The result can be generalized to the following;

$$\begin{aligned}
&\left\langle \prod_{j=1}^n : e^{ik_jX(z_j,\bar z_j)} : \prod_{k=1}^p \partial X^{\mu_k}(z_k') \prod_{l=1}^q \partial X^{\mu_l}(z_l'')\right\rangle_{S^2} = iC^X_{S^2}(2\pi)^d\delta^{(d)}\left(\sum_j k_j\right)\prod_{j<l}|z_{jl}|^{\alpha' k_j k_l}\\
&\qquad\qquad \times\left\langle \prod_{j=1}^p \partial X^{\mu_j}(z_j')\prod_{k=1}^q \bar\partial X^{\mu_j}(\bar z_j'')\right\rangle\\
&= iC^X_{S^2}(2\pi)^d\delta^{(d)}\left(\sum_j k_j\right)\prod_{j<l}|z_{jl}|^{\alpha' k_j k_l}\times\left\langle\prod_{j=1}^p\left[v^{\mu_j}(z_j') + q^{\mu_j}(z_j')\right]\prod_{k=1}^q\left[\tilde v^{\mu_j}(\bar z_j') + \tilde q^{\mu_j}(\bar z_j')\right]\right\rangle
\end{aligned} \tag{6.2.5}$$

[2]This is the same renormalization introduced for vertex operators in Chapter 1: normal ordering subtracts the coincident-point logarithm, and using the geodesic distance $d(\sigma,\sigma')$ in the subtraction makes it covariant, at the cost of the Weyl factor appearing explicitly in $:G:$ below.

[3]Near coincidence $G \to -\frac{\alpha'}{2}\ln|z_{12}|^2 + 2f(z,\bar z)$, while the geodesic distance in conformal gauge is $d^2 \simeq e^{2\omega(z)}|z_{12}|^2$. Adding $\frac{\alpha'}{2}\ln d^2 = \frac{\alpha'}{2}\ln|z_{12}|^2 + \alpha'\omega$ cancels the logarithm and leaves the displayed finite result.

where $v^{\mu_j}\left(z_j'\right)$ and $\tilde{v}^{\mu_j}\left(z_j'\right)$ are as follows:

$$v^\mu(z) = -i\frac{\alpha'}{2}\sum_{i=1}^n \frac{k_i^\mu}{z-z_i}, \quad \tilde{v}^\mu(\bar{z}) = -i\frac{\alpha'}{2}\sum_{i=1}^n \frac{k_i^\mu}{\bar{z}-\bar{z}_i}$$

and $q^\mu(z) = \partial X^\mu(z) - v^\mu(z), \tilde{q}^\mu(\bar{z}) = \bar{\partial} X^\mu(\bar{z}) - \tilde{v}^\mu(\bar{z})$. [4]

4.2.1 The disc

The Green function for the disc is obtained by the method of images:

$$G\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right) = -\frac{\alpha'}{2}\ln\left|z_1 - z_2\right|^2 - \frac{\alpha'}{2}\ln\left|z_1 - \bar{z}_2\right|^2 \tag{6.2.6}$$

The second term is the image charge term.[5] The resulting Gaussian expectation value is

$$A^n_{D^2}(k,\sigma) = iC^X_{D^2}(2\pi)^d\delta^{(d)}\left(\sum_j k_j\right)\prod_{j<l}\left|z_j - z_l\right|^{\alpha' k_j k_l}\left|z_j - \bar{z}_l\right|^{\alpha' k_j k_l}\prod_i\left|z_i - \bar{z}_i\right|^{\alpha' k_i^2/2}. \tag{6.2.7}$$

4.3 The projective plane

The projective plane RP^2 is relevant because it is the tree-level surface for an unoriented closed-string process with one crosscap. It is obtained by the antiholomorphic involution

$$z \sim -\frac{1}{\bar{z}}. \tag{4.5}$$

The Green function is obtained by the method of images. Schematically,

$$G_{RP^2}(z,w) = -\frac{\alpha'}{2}\ln|z-w|^2 - \frac{\alpha'}{2}\ln|1+z\bar{w}|^2 + \text{zero-mode and normal-ordering terms.} \tag{4.6}$$

The second logarithm is the image contribution produced by the crosscap. The amplitude therefore probes how closed-string states couple to an orientifold plane. This is the tree-level version of the crosscap contribution that later appears in the Klein-bottle and Möbius-strip one-loop amplitudes [10, 352].

4.4 The bc CFT on genus-zero surfaces

The ghost correlators computed in this section are controlled entirely by zero modes. The $\lambda = 2$ ghost system has action $S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c$, so on a curved surface the saddle-point equations are $\bar{\partial}c = \bar{\partial}b = 0$: zero modes of c are globally holomorphic vector fields (conformal Killing vectors), while zero modes of b are holomorphic quadratic differentials (moduli directions). The Riemann–Roch count of Chapter 1 therefore fixes the correlators before any computation: the sphere has three c zero modes $(1, z, z^2)$ and no b zero modes, so the first nonvanishing holomorphic correlator is $\langle c(z_1)c(z_2)c(z_3)\rangle$, and every Grassmann integral with fewer c insertions vanishes identically. The disc and projective plane inherit these statements through the doubling trick and the involution $z \sim -1/\bar{z}$ respectively, with the real form $PSL(2,\mathbb{R})$ leaving three real fixable positions. This zero-mode is exactly what converts the formal division by the conformal Killing group into the ghost factors that appear in the Veneziano computation below [10, 17].

The same selection rule follows from the ghost current. For a first-order bc system with b of

[4] The holomorphicity method used here is the same one used for ghost correlators: once the singular parts of the OPE are fixed, the remaining correlator is a globally defined holomorphic function of the insertion points, with prescribed behavior at coincident points and at infinity. On the sphere this fixes the answer up to the overall normalization already absorbed into the surface constant.

[5] On the upper half-plane, Neumann boundary conditions require the normal derivative of the Green function to vanish on the real axis. Placing an image source at the reflected point $\bar{z}_2$ with the same sign gives $\partial_n \ln|z - z_2|^2 + \partial_n \ln|z-\bar{z}_2|^2 = 0$ at $z = \bar{z}$, so the boundary term in the variation of X^μ vanishes. A Dirichlet boundary condition would instead use the opposite sign for the image charge.

weight λ and c of weight $1-\lambda$, define

$$j_{\text{gh}}(z) =: cb : (z).$$

Its OPEs with the elementary fields are

$$j_{\text{gh}}(z)c(w) \sim \frac{c(w)}{z-w}, \qquad j_{\text{gh}}(z)b(w) \sim -\frac{b(w)}{z-w}.$$

Thus c has ghost number $+1$, while b has ghost number -1. The stress tensor

$$T_{bc} = (1-\lambda) : (\partial b)c : -\lambda : b\partial c :$$

gives

$$T_{bc}(z)j_{\text{gh}}(w) \sim \frac{1-2\lambda}{(z-w)^3} + \frac{j_{\text{gh}}(w)}{(z-w)^2} + \frac{\partial j_{\text{gh}}(w)}{z-w}.$$

The third-order pole shows that the ghost current is not a primary current when $\lambda \neq \frac{1}{2}$. On a genus-g surface this becomes a background ghost charge,

$$\int_\Sigma d^2\sigma\, \sqrt{g}\, \nabla_a j^a_{\text{gh}} = 2\pi(1-2\lambda)(1-g).$$

Therefore a holomorphic correlator can be nonzero only if the inserted ghost numbers satisfy

$$\sum_r q_r = (2\lambda-1)(1-g).$$

For reparametrization ghosts, $\lambda = 2$, so

$$\sum_r q_r = 3(1-g).$$

This gives three c-ghost insertions on the sphere, zero net ghost number on the torus, and $3g-3$ b-ghost insertions for a closed surface of genus $g \geq 2$ [3].

4.4.1 The sphere

On the sphere there are no moduli, so no b-ghost insertion is required. The six real conformal Killing vectors require three holomorphic and three antiholomorphic c insertions, and the minimal nonzero ghost correlator is

Justification: on the sphere the holomorphic bc system has three c-zero modes, corresponding to the three holomorphic conformal Killing vectors $1, z, z^2$. A path integral over Grassmann zero modes vanishes unless every zero mode is soaked up. Therefore

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{S^2} \neq 0, \qquad \langle c(z_1)c(z_2)\rangle_{S^2} = 0, \tag{4.7}$$

and similarly for the antiholomorphic sector. Thus the minimal nonzero closed-string ghost correlator has three c's and three $\tilde{c}$'s. This is the ghost path-integral version of fixing three complex positions by $PSL(2,\mathbb{C})$. [10, 17].

The same result follows in the operator formalism and fixes the polynomial dependence without a separate ansatz. For the $\lambda = 2$ system,

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}, \qquad b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}, \qquad \{c_n, b_m\} = \delta_{n+m,0}.$$

The state created by the identity obeys

$$c_n|1\rangle = 0 \quad (n \geq 2), \qquad b_n|1\rangle = 0 \quad (n \geq -1).$$

Since $c(z)|1\rangle = c_1|1\rangle + O(z)$, the state $c(0)|1\rangle$ is annihilated by b_n for $n \geq 0$ and by c_n for $n \geq 1$. It is the lower ghost vacuum, $|\downarrow\rangle = c_1|1\rangle$. Acting once with the zero mode gives

$$|\uparrow\rangle = c_0|\downarrow\rangle = c_0c_1|1\rangle, \qquad b_0|\uparrow\rangle = |\downarrow\rangle, \qquad c_0|\uparrow\rangle = 0.$$

The sphere bra is normalized by

$$\langle 1|c_{-1}c_0c_1|1\rangle = 1.$$

Keeping only the three zero-mode terms in $c(z) = c_1 + zc_0 + z^2c_{-1} + \cdots$, one obtains

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{S^2} = \det\begin{pmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{pmatrix} = z_{12}z_{23}z_{31}.$$

Thus the three fixed insertion points are not a convention-dependent decoration of the amplitude;

they are the three ghost zero modes required by the bc Fock space. The radial-quantization conventions behind this residue calculation are developed in Notes I [3, 4].

$$\langle c(z_1)\, c(z_2)\, c(z_3)\, \tilde{c}(z_4)\, \tilde{c}(z_5)\, \tilde{c}(z_6)\rangle$$

This correlator is the determinant $\det C^a_{0j}$ from the c-zero-mode part of the Faddeev–Popov ghost action. The C^a_{0j} are conformal Killing vectors; on the sphere,

$$C^z_{01} = 1, \quad C^z_{02} = z, \quad C^z_{03} = z^2, \quad C^z_{04} = C^z_{05} = C^z_{06} = 0,$$
$$C^{\bar z}_{01} = C^{\bar z}_{02} = C^{\bar z}_{03} = 0, \quad C^{\bar z}_{04} = 1, \quad C^{\bar z}_{05} = \bar z, \quad C^{\bar z}_{06} = \bar z^2.$$

Since this basis is not orthonormal, its Gram determinant contributes a Jacobian. This gives an overall constant $C^g_{S^2}$ multiplying the $(C^{z_i}_{0j}, C^{\bar z_i}_{0j})$ matrix.[6] The determinant is

$$C^g_{S^2} \det \begin{pmatrix} C^{z_1}_{01} & C^{z_2}_{01} & C^{z_3}_{01} & C^{\bar z_4}_{01} & C^{\bar z_5}_{01} & C^{\bar z_6}_{01} \\ C^{z_1}_{02} & C^{z_2}_{02} & C^{z_3}_{02} & C^{\bar z_4}_{02} & C^{\bar z_5}_{02} & C^{\bar z_6}_{02} \\ C^{z_1}_{03} & C^{z_2}_{03} & C^{z_3}_{03} & C^{\bar z_4}_{03} & C^{\bar z_5}_{03} & C^{\bar z_6}_{03} \\ C^{z_1}_{04} & C^{z_2}_{04} & C^{z_3}_{04} & C^{\bar z_4}_{04} & C^{\bar z_5}_{04} & C^{\bar z_6}_{04} \\ C^{z_1}_{05} & C^{z_2}_{05} & C^{z_3}_{05} & C^{\bar z_4}_{05} & C^{\bar z_5}_{05} & C^{\bar z_6}_{05} \\ C^{z_1}_{06} & C^{z_2}_{06} & C^{z_3}_{06} & C^{\bar z_4}_{06} & C^{\bar z_5}_{06} & C^{\bar z_6}_{06} \end{pmatrix} = C^g_{S^2} \det \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 \\ z_1^2 & z_2^2 & z_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \bar z_4 & \bar z_5 & \bar z_6 \\ 0 & 0 & 0 & \bar z_4^2 & \bar z_5^2 & \bar z_6^2 \end{pmatrix}$$
$$= C^g_{S^2} z_{12} z_{23} z_{31} \bar z_{45} \bar z_{56} \bar z_{64} \tag{6.4.1}$$

We can deduce a result with b insertions as well. We should have additional insertions for c ghosts if we have b insertions because we want them to contract with one another. The result is as follows[7];

$$\left\langle \prod_{i=1}^{p+3} c(z_i) \prod_{j=1}^{p} b(z'_i)\,.\ \text{anti} \right\rangle = C^g_{S^2} \frac{z_{p+1,p+2} z_{p+1,p+3} z_{p+2,p+3}}{(z_1 - z'_1)\ldots(z_p - z'_p)}.\ \text{anti}\ \pm\ \text{(permutations)} \tag{6.4.2}$$

The overall sign will not be needed below.[8]

4.4.2 The disc

The doubling trick gives us the following for the disc;

$$\langle c(z_1)\, c(z_2)\, c(z_3)\rangle_{D^2} = C^g_{D^2} z_{12} z_{23} z_{31}$$
$$\langle c(z_1)\, c(z_2)\, \tilde c(\bar z_3)\rangle_{D^2} = \langle c(z_1)\, c(z_2)\, c(z'_3)\rangle_{D^2} = C^g_{D^2} z_{12}(z_2 - \bar z_3)(\bar z_3 - z_1)$$

4.4.3 The projective plane

Similarly, for the projective plane, we can deduce the following;

$$\langle c(z_1)\, c(z_2)\, c(z_3)\rangle_{RP_2} = C^g_{RP_2} z_{12} z_{23} z_{31}$$
$$\langle c(z_1)\, c(z_2)\, \tilde c(\bar z_3)\rangle_{RP_2} = \langle c(z_1)\, c(z_2)\, c(z'_3)\rangle_{RP_2} = C^g_{RP_2} z_{12}(1 + z_1 \bar z_3)(1 + z_2 \bar z_3)$$

[6] The Jacobian is $(\det\langle C_j, C_k\rangle)^{-1/2}$, the Gram determinant of the six conformal Killing vectors in the inner product defined by the round metric. Its numerical value is never needed separately: it enters only through the combination $C_{S^2} = e^{-2\lambda} C^X_{S^2} C^g_{S^2}$, which is fixed once and for all by unitarity of the four-point amplitude, Eq. (6.5.4).

[7] Saturate the three zero modes with any three c's and contract every b against one of the remaining c's using $\langle b(z)c(w)\rangle \sim (z-w)^{-1}$; summing over the ways of pairing gives the permutations indicated. The same answer follows in one line from bosonization of the ghost system.

[8] The holomorphicity argument fixing these correlators runs as follows: a bc correlator is a meromorphic function of each insertion point, with poles only at b–c coincidences dictated by the OPE and zeros forced by the c zero-mode structure; a meromorphic function on the sphere with prescribed poles and zeros is fixed up to a constant, which is $C^g_{S^2}$. This is the standard shortcut to Eq. (6.4.1)–(6.4.2).

4.5 The Veneziano amplitude

The four-tachyon amplitude is the prototype for crossing symmetry, a single worldsheet reproducing the s-, t-, and u-channel exchanges at once (Figure 4.2).

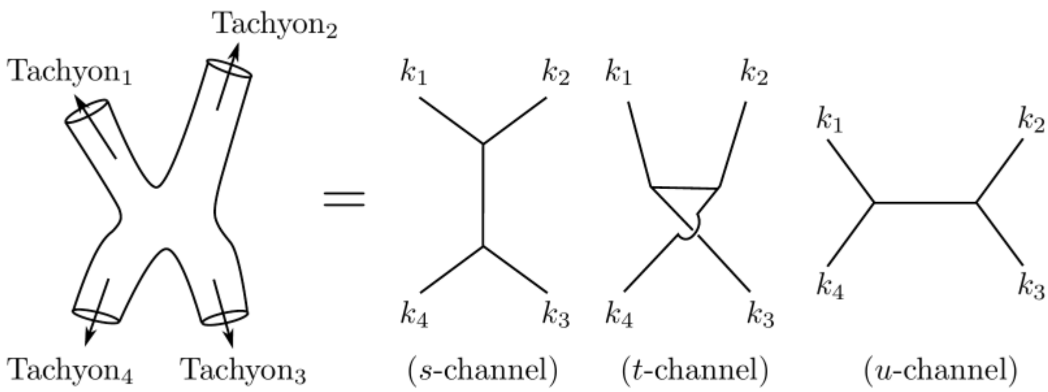


Figure 4.2: The four-tachyon amplitude on the Riemann sphere. The single worldsheet, with four legs running to infinity, reproduces the s-, t-, and u-channel exchanges at once, because topologically equivalent worldsheets are not distinguished in the path integral. This worldsheet duality is why the Veneziano amplitude carries poles in every channel. From Ref. [17].

We calculate the tachyon three function now. The vertex operator for the tachyon is as follows:

$$g_0 \int_{\mathbb{R}} ds : e^{ik\cdot X(s)} :$$

where g_0 is the string coupling. We use the upper half plane, with y as the boundary coordinate. Three boundary points can be fixed by the three real conformal Killing vectors of the disc. The group $\mathrm{PSL}(2,\mathbf{R})$ preserves the cyclic ordering of boundary insertions, so the amplitude includes the two inequivalent cyclic orderings. Since all three insertion points are fixed by conformal Killing symmetry, no modulus remains and no b-ghost insertion is required. The three-point S matrix is therefore

$$S_{D^2}(k_1,k_2,k_3) = e^{-\lambda} g_0^3 \left\langle {}_*^* c^1 e^{ik_1\cdot X}(y_1)_{**}^{**} c^1 e^{ik_2\cdot X}(y_2)_{**}^{**} c^1 e^{ik_3\cdot X}(y_3)_*^* \right\rangle_{D_2} + (k_2 \leftrightarrow k_3)$$

where the normal ordering is the boundary normal ordering. Using the boundary expectation value for the disc and (6.4.1), we get;

$$S_{D^2}(k_1,k_2,k_3) = iC_{D_2}(2\pi)^{26}\delta^{26}\left(\sum_j k_j\right) g_0^3 |y_{12}|^{1+2\alpha' k_1\cdot k_2} |y_{23}|^{1+2\alpha' k_2\cdot k_3} |y_{31}|^{1+2\alpha' k_1\cdot k_3}$$

$$+ iC_{D_2}(2\pi)^{26}\delta^{26}\left(\sum_j k_j\right) g_0^3 |y_{12}|^{1+2\alpha' k_1\cdot k_3} |y_{23}|^{1+2\alpha' k_2\cdot k_3} |y_{31}|^{1+2\alpha' k_1\cdot k_2}$$

where $C_{D_2} = e^{-\lambda C^X_{D_2} C^g_{D_2}}$. Using the on-shell tachyon condition,

$$k_1^2 = k_2^2 = k_3^2 = \frac{1}{\alpha'}$$

because we have tachyons here. Moreover, due to the momentum conservation is as follows:

$$k_1 + k_2 + k_3 = 0$$

Using these two conditions, we derive the following result;

$$(k_1+k_2)^2 = k_3^2 = \frac{1}{\alpha'} \text{ and } (k_1+k_2)^2 = k_1^2 + k_2^2 + 2k_1\cdot k_2 = \frac{2}{\alpha'} + 2k_1\cdot k_2 \Rightarrow 2\alpha' k_1\cdot k_2 + 1 = 0$$

and similar results follow for $2\alpha' k_2.k_3 + 1$ and $2\alpha' k_1.k_3 + 1$. Therefore, the S matrix becomes;

$$S_{D^2}(k_1, k_2, k_3) = 2iC_{D_2} g_0^3 (2\pi)^{26} \delta^{26}\left(\sum_j k_j\right) \tag{6.5.1}$$

We calculate the four point tachyon function now. Three points can be fixed and hence, we will have to integrate over the fourth coordinate. We have to sum over all the orderings but we can do that by just interchanging k_1 and k_2. We get the following;

$$\begin{aligned} S_{D^2}(k_1, k_2, k_3, k_4) &= ig_0^4 C_{D^2} (2\pi)^{26} \delta^{(26)}\left(\sum_j k_j\right) |y_{12}y_{23}y_{31}| \int dy_4 \prod_{j<l} |y_{jl}|^{2\alpha' k_j \cdot k_l} + (k_2 \leftrightarrow k_3) \\ &= ig_0^4 C_{D^2} (2\pi)^{26} \delta^{(26)}\left(\sum_j k_j\right) |y_{12}y_{23}y_{31}| \int dy_4 \prod_{j<l} |y_{jl}|^{2\alpha' k_j \cdot k_l} + (k_2 \leftrightarrow k_3) \end{aligned}$$

where we have the following;

$$\begin{aligned} 2\alpha' k_1 \cdot k_2 &= -2 - 2\alpha' s, 2\alpha' k_2 \cdot k_3 = -2 - 2\alpha' u, 2\alpha' k_1 \cdot k_3 = -2 - 2\alpha' t \\ 2\alpha' k_2 \cdot k_4 &= -2 - 2\alpha' t, 2\alpha' k_3 \cdot k_4 = -2 - 2\alpha' s, 2\alpha' k_1 \cdot k_4 = -2 - 2\alpha' u \end{aligned}$$

where

$$\begin{aligned} s &= (k_1 + k_2)^2 = (k_3 + k_4)^2, & t &= (k_1 + k_3)^2 = (k_2 + k_4)^2, \\ u &= (k_1 + k_4)^2 = (k_2 + k_3)^2, & s + t + u &= -\frac{4}{\alpha'}, \qquad s = -\frac{2}{\alpha'}. \end{aligned}$$

and thus, the factors dependent on y s that appear in the amplitude are as follows:

$$\begin{aligned} &|y_{12}|^{-1-\alpha' s} |y_{13}|^{-1-\alpha' t} |y_{14}|^{-2-\alpha' u} |y_{23}|^{-1-\alpha' u} |y_{24}|^{-2-\alpha' t} |y_{34}|^{-2-\alpha' s} \\ &= y_3^{-1-\alpha' t} y_4^{-2-\alpha' u} (1 - y_3)^{-1-\alpha' u} (1 - y_4)^{-2-\alpha' t} (y_3 - y_4)^{-2-\alpha' s} \end{aligned}$$

where we set $y_1 = 0$ and $y_2 = 1$. If we then send $y_3 \to \infty$, the exponent of y_3 is $-4-\alpha'(s+t+u)$, which vanishes by the on-shell relation. This choice removes all remaining y_3-dependent factors, and the amplitude becomes

$$ig_0^4 C_{D^2} (2\pi)^{26} \delta^{(26)}\left(\sum_j k_j\right) |y_{12}y_{23}y_{31}| \int dy_4 \left[|y_4|^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' t} + (t \to s)\right]$$

where the indicated replacement implements the interchange of k_2 and k_3. We split the integral as follows:

$$\begin{aligned} &\int_{-\infty}^0 dy_4 |y_4|^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' t} + \int_0^1 dy_4 y_4^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' t} + \int_1^\infty dy_4 y_4^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' t} \\ +&\int_{-\infty}^0 dy_4 |y_4|^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' s} + \int_0^1 dy_4 y_4^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' s} + \int_1^\infty dy_4 y_4^{-2-\alpha' u} (1 - y_4)^{-2-\alpha' s} \end{aligned}$$

We can show using Möbius transformations[9] that the first and fifth integrals give $I(u, s)$, the third and sixth integrals give $I(s, t)$ and the second and fourth integrals give $I(t, u)$ where $I(s, t)$ is given as follows:

$$I(s, t) = \int_0^1 dy y^{-2-\alpha' s} (1 - y)^{-2-\alpha' t}$$

So, the four-point tachyon function becomes;

[9] The transformations $y \to 1 - y$, $y \to 1/y$, and $y \to y/(y - 1)$ generate the permutations of the three fixed points $\{0, 1, \infty\}$. Applying $y_4 \to 1/y_4$ maps $(1, \infty)$ to $(0, 1)$, and $y_4 \to y_4/(y_4 - 1)$ maps $(-\infty, 0)$ to $(0, 1)$; tracking the exponents through the Jacobians permutes the Mandelstam labels exactly as stated.

$$S_{D_2}(k_1,\dots k_4) = 2ig_0^4 C_{D^2}(2\pi)^{26}\delta^{(26)}\left(\sum_j k_j\right)[I(s,t)+I(t,u)+I(u,s)] \tag{6.5.2}$$

This integral will converge only if $-2-\alpha' s > -1 \Rightarrow \alpha' s < -1$. At $\alpha' s + 1 = 0$, we can probe the behavior of this integral by looking at the behavior of the leading term as follows:

$$I(s,t) = \int_0^1 dy \left[y^{-2-\alpha' s} + \mathcal{O}\left(y^{-1-\alpha' s}\right)\right] = \frac{1}{\alpha' s + 1} + \mathcal{O}\left(1/\left(\alpha' s\right)\right)$$

so there is a simple pole at $s = -1/\alpha'$, the tachyon pole. With the standard small imaginary part, the principal-value formula for a simple pole is

$$\frac{1}{x - x_0 \mp i\epsilon} = \mathcal{P}\frac{1}{x-x_0} \pm i\pi\delta\left(x - x_0\right) \Rightarrow \frac{1}{\alpha' s + 1 \mp i\epsilon} = \mathcal{P}\frac{1}{\alpha' s+1} \pm i\pi\delta\left(\alpha' s + 1\right)$$

Using unitarity, we can relate the four-point function to three-point functions (as required by factorization)) as follows:

$$S_{D_2}(k_1,\dots k_4) = i\int \frac{d^{26}k}{(2\pi)^{26}}\frac{S_{D_2}(k_1,k_2,k)\,S_{D_2}(-k,k_3,k_4)}{-k^2+1/\alpha+i\epsilon} \tag{6.5.3}$$

Using Eqs. (6.5.1) and (6.5.2), and collecting the terms singular in $\alpha' s + 1$ from $I(s,t)$ and $I(u,s)$, one obtains

$$C_{D_2}\alpha' g_0^2 = 1 \Rightarrow C_{D_2} = \frac{1}{\alpha' g_0^2} \tag{6.5.4}$$

The expressions of determinants on different topologies (using renormalization) agree with the results from unitarity (with the determinant normalization fixed by factorization). Using unitarity and the definition of the beta function, we write the four-point function as follows:

$$\begin{aligned} S_{D_2}(k_1,\dots k_4) = {} & \frac{2ig_0^2}{\alpha'}(2\pi)^{26}\delta^{(26)}\left(\sum_j k_j\right) \\ & \times \left[B\left(-\alpha_0(s),-\alpha_0(t)\right) + B\left(-\alpha_0(s),-\alpha_0(u)\right)B\left(-\alpha_0(t),-\alpha_0(u)\right)\right] \end{aligned} \tag{6.5.5}$$

where

$$\alpha_0(x) = 1 + \alpha' x, B(a,b) = \int_0^1 dy y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The Mandelstam variables in terms of center of mass energy E and mass m are easily given as follows (we take k_3^0, k_4^0 negative[10]);

$$s = E^2, t = \left(4m^2 - E^2\right)\sin^2\frac{\theta}{2} u = \left(4m^2-E^2\right)\cos^2\frac{\theta}{2}$$

where θ is the angle between particle 1 and particle 3. It means that the Regge limit;
Regge limit: $s \to \infty, t$ is fixed
just sends the energy to infinity and the hard scattering limit

$$\text{Hard scattering limit: } s \to \infty, t/s \text{ is fixed } \Rightarrow \frac{t}{s} \to -\sin^2\frac{\theta}{2}$$

sends energy to infinity at a fixed angle. Using Stirling approximation, (just using the x^x factor from the approximation of $\Gamma(x+1)$), we see that in the large s limit, we have;

$$\begin{aligned} B\left(-\alpha_0(s),-\alpha_0(t)\right) &= \frac{\Gamma\left(-\alpha' s - 1\right)\Gamma\left(-\alpha_0(t)\right)}{\Gamma\left(-\alpha' s - \alpha' t - 2\right)} \\ &\sim \frac{s^{-\alpha' s - 1}}{s^{-\alpha' s - \alpha' t - 2}}\Gamma\left(-\alpha_0(t)\right) \end{aligned}$$

[10] This is the all-incoming convention: momenta are directed into the diagram, so physical outgoing particles carry negative energy components, and $s = (k_1+k_2)^2$, $t = (k_1+k_3)^2$, $u = (k_1+k_4)^2$ obey $s+t+u = \sum_i m_i^2$ without extra signs.

$$= s^{\alpha' t+1}\Gamma\left(-\alpha_0(t)\right) = s^{\alpha_0(t)}\Gamma\left(-\alpha_0(t)\right). \qquad (6.5.6)$$

The Regge behavior follows from Eq. (6.5.6). In the hard-scattering limit, Stirling's approximation applied symmetrically to the three beta-function channels gives

$$S_{D^2}\left(k_1,\ldots k_4\right) \sim e^{-\alpha' s f(\theta)}, \qquad f(\theta) = -\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}.$$

[11]

4.6 Chan–Paton factors and gauge interactions

In 26 dimensions, closed string theory is the unique closed theory.[12] Additional endpoint degrees of freedom for open strings are called Chan–Paton degrees of freedom. A state then becomes

$$|N, k; ij\rangle, i, j = 1, \ldots, N$$

Then, we can define a basis as follows:

$$|N, k, a\rangle = \sum_{ij} |N, k, ij\rangle \lambda^a_{ij}$$

where λ^a_{ij} 's are hermitian matrices with the following normalization;

$$\operatorname{Tr}\left(\lambda^a\lambda^b\right) = \delta^{ab}$$

These Chan–Paton indices are inert under Poincare transformations and do not modify the worldsheet stress tensor. They are also conserved along a boundary segment between adjacent vertex insertions, so amplitudes acquire traces of Chan–Paton factors. The construction is compatible with unitarity because the completeness of the matrices λ^a implies[13]

$$\operatorname{Tr}\left(A\lambda^a\right)\operatorname{Tr}\left(B\lambda^a\right) = \operatorname{Tr}(AB)$$

Using the completeness relation for the Chan–Paton generators, normalized so that $\operatorname{Tr}(\lambda^a\lambda^b) = \delta^{ab}$, this identity gives the factorization condition in Eq. (6.5.3). The same trace algebra is the standard Chan–Paton derivation of nonabelian open-string gauge factors [10, 352].

4.6.1 Gauge interactions

Chan-Paton factors turn the massless open-string vector into a nonabelian gauge boson. A color-ordered tree amplitude has the schematic form

$$\mathcal{A}_n^{\text{open}} = g_o^{n-2}\sum_\rho \operatorname{Tr}(\lambda^{a_{\rho(1)}}\cdots\lambda^{a_{\rho(n)}})A(\rho(1),\ldots,\rho(n)). \qquad (4.8)$$

The trace is forced by endpoint index conservation along the boundary, while the low-energy limit of the vector amplitude gives Yang–Mills interactions. The first massive corrections are stringy, but the leading gauge transformation is fixed by Eq. (1.35). [17].

4.6.2 The unoriented string

The unoriented string requires Ω which does the following

[11] The derivation keeps t/s fixed and uses $t \simeq -s\sin^2(\theta/2)$, $u \simeq -s\cos^2(\theta/2)$, with $s+t+u$ fixed by the external masses. The logarithms from $\Gamma(-\alpha' s)$, $\Gamma(-\alpha' t)$, and $\Gamma(-\alpha' u)$ combine so that the powers of s cancel except for the exponential shown. This is the Gross–Mende saddle behavior: fixed-angle string scattering is exponentially softer than field-theory hard scattering.

[12] Any closed-string sector contains a massless symmetric tensor state, and consistency of its interactions forces it to couple universally to energy–momentum, i.e. to be a graviton. Two independent closed-string sectors would give two massless spin-two states, and no consistent theory of two interacting gravitons with independent couplings exists. So the closed bosonic string cannot be doubled or replaced, and the freedom found in this section resides entirely in the open-string (Chan–Paton) sector.

[13] Completeness of the $U(n)$ generators normalized by $\operatorname{Tr}(\lambda^a\lambda^b) = \delta^{ab}$ reads $\sum_a \lambda^a_{ij}\lambda^a_{kl} = \delta_{il}\delta_{jk}$; contracting with $A_{ji}B_{lk}$ gives $\operatorname{Tr}(A\lambda^a)\operatorname{Tr}(B\lambda^a) = \operatorname{Tr}(AB)$, which is exactly the factorization of a one-loop trace into two tree traces required by unitarity.

$$\Omega\alpha_n^\mu\Omega^{-1} = (-1)^n\alpha_n^\mu \text{ (Open strings)}$$

$$\Omega\alpha_n^\mu\Omega^{-1} = \tilde{\alpha}_n^\mu, \Omega\tilde{\alpha}_n^\mu\Omega^{-1} = \alpha_n^\mu \text{ (Closed strings)}$$

Ω does the following to the states;

$$\Omega|N,k\rangle = (-1)^N|N,k\rangle = (-1)^{1+\alpha' m^2}|N,k\rangle = \omega_N|N,k\rangle$$

If we include the Chan Paton indices, we get the following;

$$\Omega|N,k;ij\rangle = \omega_N|N,k;ji\rangle$$

The interchange of Chan–Paton indices occurs because worldsheet parity interchanges the string endpoints. Since orientation reversal is a projection on the Hilbert space and must be compatible with sewing of open-string worldsheets, it must act involutively on physical states: $\Omega^2 = 1$. This gives the following possibilities:

$\alpha' m^2$	N	λ_{ij}^a	Gauge
Even	Odd	antisymmetric	$SO(N)$
Odd	Even	symmetric	Traceless sym.+singleton

[14] The more general worldsheet parity is given as follows:

$$\text{Special Case}: \Omega|N,k;ij\rangle = \omega_N|N,k;ji\rangle = \omega_N\mathbb{I}_{jj'}\,|N,k;j'i'\rangle\,\mathbb{I}_{i'i}$$

$$\text{General case}: \Omega_\gamma|N,k;ij\rangle = \omega_N\gamma_{jj'}\,|N,k;j'i'\rangle\,\gamma_{i'i}^{-1}, \quad \gamma\in U(N)$$

For the generalized parity action, the same projection condition requires $\Omega_\gamma^2 = 1$. Acting twice on a Chan–Paton state gives

$$\Omega_\gamma^2|N;k;ij\rangle = \gamma_{jj'}\gamma_{i'i}^{-1}\gamma_{i'i''}\gamma_{j''j'}^{-1}\,|N;k;i''j''\rangle = \left(\left(\gamma^T\right)^{-1}\gamma\right)_{ii''}|N;k;i''j''\rangle\left(\left(\gamma^T\right)^{-1}\gamma\right)^{-1}_{j''j}$$

Imposing $\Omega_\gamma^2 = 1$ gives two possibilities, distinguished below by the symmetry of γ:

$$\left(\gamma^T\right)^{-1}\gamma = \pm\mathbb{I} \Rightarrow \gamma^T = \pm\gamma$$

Now, to interpret these two possibilities, we get do a $U(N)$ rotation of the Chan Paton factors as follows:

$$|N;k;ij\rangle' = U_{ii'}^{-1}\,|N;k;i'j'\rangle\,U_{j'j}U \in U(N)$$

For the transformed states, we have the following worldhseet parity transformation;

$$\begin{aligned}
\Omega\,|N;k;ij\rangle' &= \gamma'_{jj'}\,|N;k;j'i'\rangle'\,(\gamma')_{i'i}^{-1}\\
&= \left(\gamma' U^{-1}\right)_{jj'}|N'k;j'i'\rangle\left(U\gamma'^{-1}\right)_{i'i},\\
\Rightarrow\quad \Omega U_{jj'}^T\,|N;k;j'i'\rangle\left(U^T\right)_{i'i}^{-1} &= \left(\gamma' U^{-1}\right)_{jj'}|N'k;j'i'\rangle\left(U\gamma'^{-1}\right)_{i'i},\\
\Rightarrow\quad \Omega\,|N;k;ij\rangle &= \left(\left(U^T\right)^{-1}\gamma' U^{-1}\right)_{jj'}|N'k;j'i'\rangle\left(U\gamma'^{-1}U^T\right)_{i'i}\\
&= \left(\left(U^T\right)^{-1}\gamma' U^{-1}\right)_{jj'}|N'k;j'i'\rangle\left(\left(U^T\right)^{-1}\gamma' U^{-1}\right)^{-1}_{i'i}.
\end{aligned}$$

So, we see that γ matrices can be written as follows:

$$\gamma = \left(U^T\right)^{-1}\gamma' U^{-1} \Rightarrow \gamma' = U^T\gamma U$$

This transformation can be used to set $\gamma' = 1$, equivalently to bring γ to the identity by a unitary congruence when the symmetric projection is chosen. For $\gamma' = 1$, the equation above

[14]The wording in the table means the following: when λ^a is antisymmetric, the massless vectors form the adjoint of $SO(n)$; when symmetric, the states at that level fill the symmetric tensor of $SO(n)$, which decomposes into the traceless symmetric part plus a singlet. For the symplectic projection introduced below, the roles are exchanged, the adjoint of $Sp(n/2)$ is the symmetric representation, which is why both projections define consistent unoriented theories.

constrains only the symmetric part of γ. Writing the symmetric and antisymmetric parts as γ_s and γ_a, one has

$$U^T\gamma_s U + U^T\gamma_a U = 1 \Rightarrow U^T\gamma_s U - U^T\gamma_a U = 1 \Rightarrow U^T\gamma_s U = 1, \quad U^T\gamma_a U = 0$$

So, if γ is symmetric, then we can always solve for γ and thus, set $\gamma' = 1$. It gives the trivial worldsheet parity. For γ being anti symmetric, we can set γ' equal to the following[15];

$$\gamma' = M = i \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}, \quad n = 2k$$

[16]

4.7 Closed string tree amplitudes

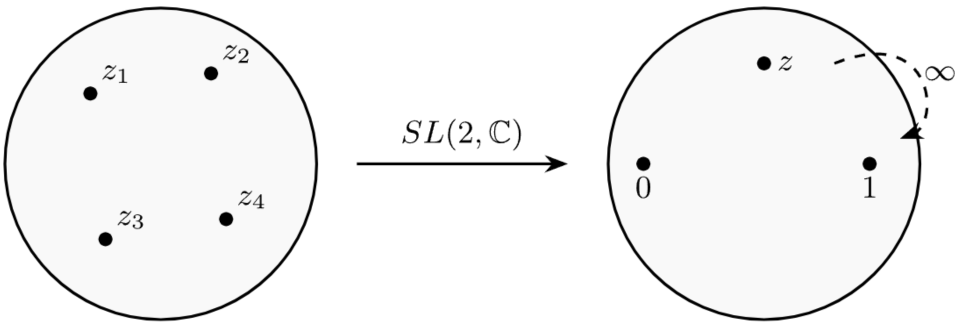
The closed-string four-tachyon amplitude is obtained from the sphere correlator after fixing three insertion points by $PSL(2,\mathbb{C})$. Taking the insertions at 0, 1, ∞, and z, the remaining integral has the schematic form

$$\mathcal{A}_4^{\text{closed}} = C_{S^2} g_c^2 \int_{\mathbb{C}} d^2z\, |z|^{-\alpha' s/2-4} |1-z|^{-\alpha' t/2-4}, \tag{4.9}$$

where C_{S^2} depends on the normalization of the sphere path integral and of the external tachyon vertices. Performing the complex beta-function integral gives the Virasoro–Shapiro amplitude

$$\mathcal{A}_4^{\text{closed}} = C_{S^2} g_c^2 \frac{\Gamma\left(-1-\frac{\alpha' s}{4}\right)\Gamma\left(-1-\frac{\alpha' t}{4}\right)\Gamma\left(-1-\frac{\alpha' u}{4}\right)}{\Gamma\left(2+\frac{\alpha' s}{4}\right)\Gamma\left(2+\frac{\alpha' t}{4}\right)\Gamma\left(2+\frac{\alpha' u}{4}\right)}, \qquad s+t+u = -\frac{16}{\alpha'}. \tag{4.10}$$

The constant C_{S^2} is not fixed by conformal invariance alone; it is fixed by the chosen normalization of the path integral or by matching factorization residues. The amplitude is symmetric in s, t, u and has an infinite sequence of poles corresponding to on-shell closed-string intermediate states, one channel of which is drawn in Figure 4.3 [10, 17].



$$\chi = \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad \longrightarrow \quad z$$

Figure 4.3: The same four-punctured sphere contains the three degeneration limits of the closed-string four-point function: $z \to 0$, $z \to 1$, and $z \to \infty$. These limits give the s-, t-, and u-channel poles of one crossing-symmetric integral.

Factorization of tree amplitudes.

The tree-level consistency check is factorization at each physical pole. Near a pole in a channel with mass M_N, the amplitude must separate into two lower-point amplitudes joined by a

[15]Any nondegenerate antisymmetric matrix can be brought to this canonical block form by a congruence $\gamma \to U^T\gamma U$, this is the antisymmetric analogue of diagonalizing a quadratic form, sometimes called the symplectic Gram–Schmidt or Darboux construction, and nondegeneracy forces the dimension $n = 2k$ to be even. The factor of i is a phase convention making γ unitary.

[16]The remaining material of the source section, the action of Ω_γ on the Chan–Paton wavefunctions and the resulting $SO(n)$ and $Sp(k)$ spectra at each mass level, is used and completed in the unoriented one-loop computation of Chapter 5 and in the type-I construction of Chapter 8, so it is not duplicated here.

propagator:

$$\mathcal{A}_n \xrightarrow{s \to M_N^2} \sum_{\Psi_N \in \mathcal{H}_{\rm phys}} \mathcal{A}_L(\cdots, \Psi_N) \frac{1}{s - M_N^2} \mathcal{A}_R(\Psi_N, \cdots). \tag{4.11}$$

The sum is over the physical Hilbert space, not over the full oscillator Fock space. The no-ghost theorem is used here in a precise way: negative-norm states cannot appear as independent states in the pole residues [17, 28].

Closed strings on D_2 and RP^2.

The disk and the projective plane show how closed strings couple to boundaries and crosscaps. In the closed-string channel their amplitudes are written as overlaps with a boundary state or a crosscap state,

$$\mathcal{A}_{D_2} \sim \langle B|V_1 \cdots V_n|0\rangle, \qquad \mathcal{A}_{RP^2} \sim \langle C|V_1 \cdots V_n|0\rangle. \tag{4.12}$$

The boundary state imposes Neumann or Dirichlet gluing conditions on left- and right-moving oscillators. The crosscap state imposes the analogous gluing condition after worldsheet orientation reversal. Tadpoles are read from the one-point functions of massless closed-string states on these surfaces. Their cancellation is the tree-channel form of the one-loop cancellation among cylinder, Möbius-strip, and Klein-bottle diagrams [10, 352].

General tree-level formula.

For n closed-string tachyons on the sphere, the Koba–Nielsen form is

$$\mathcal{A}_n^{S^2} = C_{S^2} g_c^{n-2} \int \frac{\prod_i d^2 z_i}{\mathrm{Vol}\, PSL(2,\mathbb{C})} \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}. \tag{4.13}$$

Fixing three insertion points is the practical implementation of the division by the conformal Killing group. For higher external states, derivative insertions such as ∂X^μ and polarization tensors are inserted in the same correlator and evaluated by Wick contractions [10, 17].

4.7.1 Möbius invariance

Möbius invariance is the explicit form of conformal gauge independence on the sphere. Under

$$z \mapsto \frac{az+b}{cz+d}, \qquad ad - bc = 1, \tag{4.14}$$

a primary operator transforms as

$$\mathcal{V}(z,\bar{z}) \mapsto \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{V}(z',\bar{z}'). \tag{4.15}$$

Integrated closed-string vertices are invariant only for $h = \bar{h} = 1$, which is why the mass-shell and transversality conditions are already encoded in conformal invariance. [17].

4.7.2 Path integrals and matrix elements

The path-integral amplitude and the operator matrix element are two languages for the same punctured surface. Cutting the sphere around an insertion produces a state, and sewing two cuts inserts a complete set:

$$1_{\rm phys} = \sum_{\Psi \in \mathcal{H}_{\rm phys}} |\Psi\rangle\langle\Psi|. \tag{4.16}$$

Factorization follows from the degeneration of moduli space. A long cylinder carries a physical intermediate state, so the path-integral derivation and the operator proof of unitarity give the same pole structure. [17].

4.7.3 Operator calculations

Operator calculations reproduce the same amplitudes by radial ordering and OPEs. The basic contraction

$$X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z-w|^2 \tag{4.17}$$

immediately gives the Koba-Nielsen factor in Eq. (4.13). Derivative insertions, such as ∂X^μ, are then handled by Wick contraction with the exponentials. Therefore the long path-integral expression collapses into a practical vertex-operator calculus. [17].

BPZ inner product in operator calculations.

The inner product used in the operator derivation is the BPZ pairing of radial quantization. If $\mathcal{O}_A$ creates $|A\rangle$ at the origin, the dual state is obtained by the conformal inversion $I(z) = 1/z$:

$$\langle A|B\rangle = \langle (I \circ \mathcal{O}_A)(0)\, \mathcal{O}_B(0)\rangle\,. \tag{4.18}$$

This formula is used when a degenerating surface is cut into two pieces and a complete set of states is inserted along the cut. The ghost number of the BPZ pairing determines which combinations of matter and ghost operators give nonzero amplitudes [10, 17].

4.8 Feynman diagrams, worldsheets, and the tree-level picture

The tree amplitudes computed in this chapter have a simple geometric lesson behind them. In ordinary quantum field theory a Feynman diagram is a graph: it has one-dimensional lines and special points where the lines meet. In string theory the graph is replaced by a smooth two-dimensional surface. This replacement is not only a change of drawing. It changes how interactions, ultraviolet limits, and intermediate states are encoded.

Let us first recall the field-theory side. A scalar propagator can be written with a Schwinger proper time,

$$\frac{1}{p^2+m^2} = \int_0^\infty dt\, \exp\bigl[-t(p^2+m^2)\bigr]\,. \tag{4.19}$$

For a fixed Feynman graph, one assigns such a parameter t_i to every internal line. The graph may then be viewed as a singular one-dimensional geometry: the internal edges have lengths t_i, and the interaction vertices are the singular points at which the edges are glued. Up to reparametrizations along each edge, the lengths t_i are the invariant metric of the graph.

Momentum conservation at a vertex can also be written geometrically. For example,

$$\int d^dx\, \exp\left(i\sum_i p_i\cdot x\right) = (2\pi)^d\delta^{(d)}\left(\sum_i p_i\right), \tag{4.20}$$

so a vertex position x enforces the statement that the momenta flowing into that point add to zero. In this language a Feynman graph carries two kinds of input, a one-dimensional metric on the graph, reduced to the proper times t_i, and a map X from the graph into spacetime.

The position-space propagator makes this picture even clearer:

$$G(x,y) = \int_0^\infty dt \int \frac{d^dp}{(2\pi)^d}\exp\bigl[ip\cdot(x-y) - t(p^2+m^2)\bigr]\,. \tag{4.21}$$

At fixed t, this is the kernel for a particle to travel from y to x in proper time t. Equivalently,

$$G(x,y;t) = \int_{X(0)=y}^{X(t)=x} \mathcal{D}X(t')\, \exp\left[-\int_0^t dt'\left(\dot{X}^2+m^2\right)\right]. \tag{4.22}$$

Thus ordinary perturbation theory already has a worldline interpretation: it integrates over maps from a one-dimensional object into spacetime, together with the metric lengths of that object.

The two ends of the Schwinger integral have different meanings. The large-t region gives the physical pole of the propagator,

$$\int_\Lambda^\infty dt\, e^{-t(p^2+m^2)} \sim \frac{1}{p^2+m^2}. \tag{4.23}$$

This region must remain, because it is the long-distance propagation of an intermediate particle. The small-t region is different. It is the short-distance region of the graph and is the source of ultraviolet singularities in local quantum field theory. A hard lower cutoff on t would remove that region, but it would also interfere with the locality structure of the theory.

String theory keeps the long-distance factorization region but changes the short-distance one.

Instead of integrating over singular one-dimensional graphs, it integrates over smooth two-dimensional worldsheets. The analog of the field-theory input is now

$$\text{metric on } \Sigma \quad \text{and} \quad X : \Sigma \to M, \tag{4.24}$$

where Σ is the worldsheet and M is target spacetime. The one-dimensional original of this object is the einbein form of the point-particle action,

$$S_{\text{pp}} = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - e\, m^2\right), \tag{4.25}$$

in which the worldline metric $e(\tau)$ trades the awkward square root of $\int d\tau\, m\sqrt{-\dot{X}^2}$ for a polynomial action that, unlike the square-root form, still makes sense for $m = 0$. Varying e gives $e^2 = -\dot{X}^2/m^2$, and substituting it back recovers the square-root action, so the two forms agree on shell; in this language a point particle is a worldline coupled to one-dimensional gravity, with e the analogue of $\sqrt{h}$. The Polyakov action is the two-dimensional version of this worldline action:

$$I_{2d}[h, X] = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\, \sqrt{h}\, h^{\alpha\beta}\partial_{\alpha}X^I\partial_{\beta}X^J G_{IJ}(X). \tag{4.26}$$

The important new point is Weyl invariance. A two-dimensional metric has three local components, while diffeomorphisms remove only two local functions. The Weyl equivalence

$$h_{\alpha\beta} \sim e^{2\omega(\sigma)}h_{\alpha\beta} \tag{4.27}$$

removes the remaining local scale. After quotienting by diffeomorphisms and Weyl transformations, the metric integral is reduced to a finite-dimensional integral over moduli. These moduli play the role that Schwinger parameters play in field theory, but now they are moduli of Riemann surfaces.

This is why the tree-level amplitudes above have no separate choice of interaction vertex. A field-theory graph has special vertices where particles meet. A string worldsheet is smooth there. One may draw a pair of pants and call it a splitting interaction, but a different worldsheet time slicing changes where that splitting appears to happen. The interaction is not an extra local rule inserted at a point. It is part of the topology of the surface.

At one loop this same geometry explains the ultraviolet improvement most sharply. The field-theory loop integrates over the whole proper-time range $0 < t < \infty$. The endpoint $t \to 0$ is the ultraviolet region. For the closed string, the analogous surface is a torus. A rectangular torus with sides s and s' may be interpreted either as a string of circumference s propagating for time s', or as a string of circumference s' propagating for time s. Modular invariance identifies these two descriptions. In the simplified rectangular picture, the ratio

$$t = \frac{s}{s'} \tag{4.28}$$

need not be integrated over both $t < 1$ and $t > 1$, because the two regions are equivalent. The fundamental region keeps the long-tube limits, which are infrared and give factorization on physical states, but removes an independent short-proper-time ultraviolet boundary.

The same idea is already visible in the tree amplitudes of this chapter. The Veneziano amplitude is a single worldsheet integral, yet its degeneration limits reproduce the different field-theory channels. What field theory draws as several diagrams is encoded in string theory by different boundary regions of one moduli space. The poles are still present because long tubes represent physical intermediate strings. What is absent is a separate local short-distance vertex where arbitrary ultraviolet behavior can be inserted.

Finally, this geometric reformulation also explains why gravity is unavoidable. Varying the target-space metric changes the worldsheet action by

$$\delta I = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\, \sqrt{h}\, h^{\alpha\beta}\partial_{\alpha}X^I\partial_{\beta}X^J \delta G_{IJ}(X). \tag{4.29}$$

The integrand is a worldsheet vertex operator. By the state–operator correspondence, such an insertion corresponds to an external string state. Therefore a fluctuation of the spacetime metric is one of the string states: the graviton is not added by hand, but appears in the spectrum of the closed string. This completes the geometric message of the tree-level calculation: smooth worldsheets replace singular Feynman graphs, moduli replace Schwinger parameters, degeneration replaces factorization channels, and the spacetime metric itself becomes a string field.

Chapter 5

One-loop amplitudes, modular invariance, and tadpoles

One-loop amplitudes provide the first quantum consistency tests of string theory, and this chapter carries them out. The surfaces of Euler number zero, torus, cylinder, Möbius strip, Klein bottle, each contribute a vacuum amplitude, and each answers a different question. On the torus, the modular group $SL(2,\mathbb{Z})$ is a residual gauge symmetry, so the modulus must be integrated over a fundamental domain; the region responsible for ultraviolet divergences in field theory is absent, and finiteness follows from gauge invariance. We develop the theta-function technology needed to make this explicit and verify modular invariance of the full partition function. The open and unoriented surfaces can be described two ways, as an open-string loop or a closed-string exchange between boundary and crosscap states, and their residual divergence is infrared: a massless tadpole, or equivalently a one-point function of the massless closed-string fields, showing that flat space with the wrong gauge group does not solve the equations of motion. Summing cylinder, Möbius strip, and Klein bottle, the tadpole cancels precisely for the Chan–Paton group $SO(8192)$; this is the first example in which a worldsheet consistency condition selects a gauge group, and it prepares the $SO(32)$ computation for the superstring [10, 352].

5.1 Riemann surfaces

The Euler number for genus g surfaces with b boundaries and c crosscaps is as follows:

$$\chi = 2 - 2g - b - c$$

If we want $\chi = 0$, then the following possibilities occur;

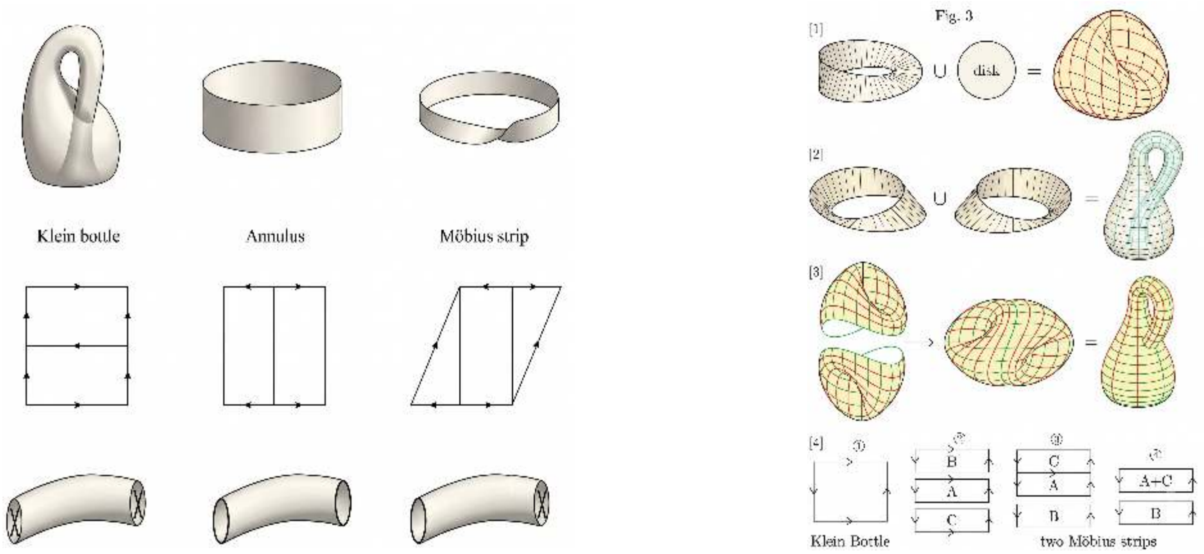
g	b	c	Surface	CKVs
1	0	0	Torus	2
0	2	0	Cylinder	1
0	1	1	Möbius Strip	1
0	0	2	Klein bottle	1

where we also mentioned the number of CKVs.[1]

5.1.1 The torus

The torus is the oriented closed-string one-loop surface, shown together with the other one-loop surfaces in Figure 5.1. Its modulus is τ in the upper half-plane, with the modular group

[1]For a genus-g closed oriented surface the conformal Killing vectors are holomorphic vector fields: there are three complex generators on the sphere, one complex translation on the torus, and none for $g \geq 2$. Boundaries and crosscaps impose reality or involution conditions on the doubled surface; the disk retains the three real generators of $PSL(2,\mathbb{R})$, while the cylinder, Möbius strip, and Klein bottle retain one real translation. These are precisely the zero modes that must be divided out or fixed in the one-loop measures.



(a) (b)

Figure 5.1: The open and unoriented one-loop surfaces, from edge identifications and from cut-and-glue pictures. (a) The annulus has two boundaries, the Möbius strip has one boundary and one crosscap, and the Klein bottle has two crosscaps. (b) Disks cap boundaries and Möbius strips generate crosscaps, so different decompositions can describe the same closed unoriented surface.

identifying equivalent descriptions:

$$\tau \sim \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{5.1}$$

The vacuum amplitude is therefore an integral over the fundamental domain, instead of whole upper half-plane. Modular invariance is the invariance of the gauge-fixed one-loop path integral under large diffeomorphisms of the torus, and the resulting fundamental domain is shown in Figure 5.2. [10].

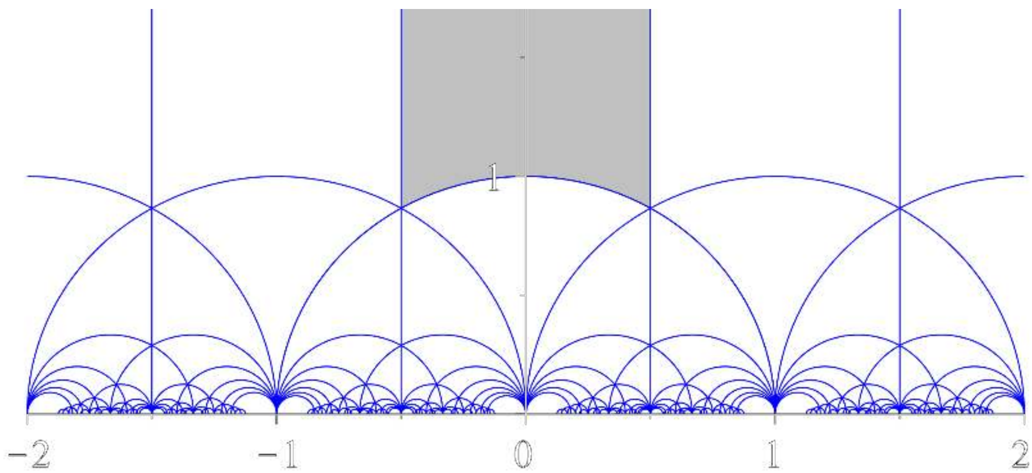


Figure 5.2: The moduli space of the torus is the fundamental domain of $SL(2, \mathbb{Z})$ (shaded, $|\mathrm{Re}\,\tau| < \frac{1}{2}$, $|\tau| > 1$), whose images tessellate the upper-half τ-plane. The one-loop vacuum amplitude is integrated over this single domain, and the images accumulate on the real axis, which is the region kept out of the string integral that would otherwise produce the field-theory ultraviolet divergence.

5.1.2 The cylinder

The cylinder can be described in two equivalent ways, as an open-string one-loop trace or a closed-string tree exchange. The modular parameter transformation $t \mapsto 1/t$ gives the equality already emphasized in Eq. (1.1). Physically, this means that a UV region in the open-string channel is an IR exchange of closed-string states:

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{open}}} e^{-2\pi t(L_0-a)} \quad\longleftrightarrow\quad \langle B|e^{-\pi\ell(L_0+\tilde{L}_0-2a)}|B\rangle, \qquad \ell = \frac{1}{t}. \tag{5.2}$$

It provides a direct diagnostic for tadpoles and D-brane tensions. [10, 40].

5.1.3 The Klein bottle

The Klein bottle is the unoriented closed-string one-loop surface. In the operator trace it inserts worldsheet parity:

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{dt}{t}\, \mathrm{Tr}_{\mathcal{H}_{\mathrm{closed}}}\left(\Omega\, e^{-2\pi t(L_0+\tilde{L}_0-2a)}\right). \tag{5.3}$$

After a channel transformation it becomes closed-string exchange between crosscaps. Therefore the Klein bottle must be discussed together with RP^2: both encode the same orientifold and crosscap structure at different orders in perturbation theory. [10].

5.2 CFT on the torus

Putting a CFT on the torus converts its entire spectrum into a single function. With modulus $\tau = \tau_1 + i\tau_2$, propagation around the time cycle inserts $e^{2\pi i\tau_1 P - 2\pi\tau_2 H}$, where $P = L_0 - \bar{L}_0$ generates the spatial translation and $H = L_0 + \bar{L}_0 - (c+\bar{c})/24$ contains the Casimir shift from mapping the cylinder to the plane. The trace over the Hilbert space is therefore

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\, q^{L_0-c/24}\,\bar{q}^{\bar{L}_0-\bar{c}/24}, \qquad q = e^{2\pi i\tau}, \tag{5.4}$$

and every statement of this section is a statement about this trace: the scalar contributes $(\sqrt{\tau_2}|\eta|^2)^{-1}$ per dimension, the ghosts contribute through their zero-mode-saturated trace, and modular invariance of the combination is the nontrivial requirement. Two consequences derived below deserve emphasis, because they constrain *all* CFTs and not just the string: invariance under $\tau \to \tau+1$ forces $h - \bar{h} \in \mathbb{Z}$ sector by sector and $c + \bar{c} \in 24\mathbb{Z}$ for the vacuum, and invariance under $\tau \to -1/\tau$ ties the high-temperature density of states to the central charge, the germ of Cardy's formula used for black-hole entropy in Chapter 13 [10, 359].

5.2.1 Scalar correlators

The differential equation in (6.2.2) can be written as follows:

$$\frac{1}{\alpha'}\nabla^2 G(\sigma,\sigma') = -2\pi(g(\sigma))^{-1/2}\delta^{(2)}(\sigma-\sigma') + 2\pi\left(X_0\right)^2$$

Now, we know that;

$$X_0^2 = \left(\int d^2\sigma\sqrt{g}\right)^{-1} = \frac{1}{\text{Area}}$$

So, we get the following

$$\frac{1}{\alpha'}\nabla^2 G(\sigma,\sigma') = -2\pi(g(\sigma))^{-1/2}\delta^{(2)}(\sigma-\sigma') + \frac{2\pi}{\text{Area}}$$

We now write this equation in terms of $w, \bar{w}$ coordinates. We see that;

$$w = \sigma^1 + i\sigma^2, \bar{w} = \sigma^1 - i\sigma^2 \Rightarrow \partial_1 = \partial + \bar{\partial}, \partial_2 = i(\partial - \bar{\partial}) \Rightarrow \nabla^2 = \partial_1^2 + \partial_2^2 = 4\partial\bar{\partial}$$

Moreover, we see that;

$$\int d^2\sigma\,\delta(\sigma-\sigma') = \frac{1}{2}\int d^2w\,\delta(\sigma-\sigma') = \int d^2w\,\delta(w-w'),$$
$$\Rightarrow \quad \delta(w-w') = \frac{1}{2}\delta(\sigma-\sigma'), \qquad \delta(\sigma-\sigma') = 2\delta(w-w').$$

Lastly, the metric on the torus covering space ($w,\bar{w}$) is flat. Using these results, we get;

$$\frac{2}{\alpha'}\partial\bar{\partial}G\left(w,\bar{w},w',\bar{w}'\right) = -2\pi\delta^{(2)}\left(w-w'\right)+\frac{\pi}{\text{Area}}$$

In the covering space the torus has length 2π and height $2\pi\tau_2$, and hence area $4\pi^2\tau_2$. Therefore

$$\frac{2}{\alpha'}\partial\bar{\partial}G\left(w,\bar{w},w',\bar{w}'\right) = -2\pi\delta^{(2)}\left(w-w'\right)+\frac{1}{4\pi\tau_2}$$

The corresponding scalar Green function is

$$G\left(w,w',\bar{w},\bar{w}'\right) = -\frac{\alpha'}{2}\ln\left|\vartheta_1\left(\frac{w-w'}{2}\middle|\,\tau\right)\right|^2+\frac{\alpha'}{4\pi\tau_2}\left(\operatorname{Im}\left(w-w'\right)\right)^2+k\left(\tau,\bar{\tau}\right)$$

After the self-contractions are removed by normal ordering, the torus expectation value is

$$\left\langle\prod_{i=1}^{n}:e^{ik_iX(z_i,\bar{z}_i)}:\right\rangle_{T^2} = iC_{T^2}^X(\tau)(2\pi)^d\delta^d\left(\sum_j k_j\right)\times\prod_{i<j}\left|\frac{2\pi}{\partial_\nu\vartheta_1(0\mid\tau)}\vartheta_1\left(\frac{w_{ij}}{2\pi}\right)\exp\left[-\frac{\left(\operatorname{Im}w_{ij}\right)^2}{4\pi\tau_2}\right]\right|^{\alpha'k_i.k_j}$$

5.2.2 The scalar partition function

We now compute the partition function as follows:

$$\begin{aligned}
Z(\tau) &= \operatorname{Tr}\left[\exp\left(2\pi i\tau_1P-2\pi\tau_2H\right)\right] \text{ where } P=L_0-\bar{L}_0, H=L_0+\bar{L}_0-\frac{1}{24}(c+\tilde{c})\\
&= \operatorname{Tr}\left(e^{2\pi i(\tau_1+i\tau_2)L_0}e^{-2\pi i(\tau_1-i\tau_2)\tilde{L}_0}\right)e^{2\pi\tau_2\frac{(c+\tilde{c})}{24}}\\
&= \operatorname{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right)(q\bar{q})^{-\frac{(c+\tilde{c})}{48}} \text{ where } q=e^{2\pi i\tau}, \bar{q}=e^{-2\pi i\bar{\tau}}
\end{aligned}$$

Now, for bosonic string, the c and $\tilde{c}$ can be the number of dimensions d and thus, we get;

$$Z(\tau) = \operatorname{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right)(q\tilde{q})^{-d/24}$$

To calculate this trace, we need arbitrary states in the theory. They are labelled by the occupation numbers of the modes α_{-n}^{μ} and $\tilde{\alpha}_{-n}^{\mu}$ which we will call $N_{\mu n}$ and $\tilde{N}_{\mu n}$. The levels of the state thus then become;

$$N=\sum_{n,\mu}nN_{\mu n}, \tilde{N}=\sum_{n,\mu}n\tilde{N}_{\mu n}$$

For these plane modes, the zero-mode constant is absent from L_0, and one has

$$\begin{aligned}
L_0 &= \frac{\alpha'k^2}{4}+N = \frac{\alpha'k^2}{4}+\sum_{n,\mu}nN_{\mu n} \Rightarrow q^{L_0} = e^{\pi i\tau\alpha'k^2/2}\prod_{\mu n}q^{nN_{\mu n}}\\
\tilde{L}_0 &= \frac{\alpha'k^2}{4}+\tilde{N} = \frac{\alpha'k^2}{4}+\sum_{n,\mu}n\tilde{N}_{\mu n} \Rightarrow \tilde{q}^{L_0} = e^{-\pi i\tilde{\tau}\alpha'k^2/2}\prod_{\mu n}\bar{q}^{n\tilde{N}_{\mu n}}\\
&\Rightarrow q^{L_0}\bar{q}^{\tilde{L}_0} = e^{\pi i\tilde{\alpha}'k^2(\tau-\bar{\tau})/2}\prod_{\mu n}q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}} = e^{-\pi\alpha'k^2\tau_2}\prod_{\mu n}q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}}
\end{aligned}$$

The expressions of q^{L_0} and $\bar{q}^{\bar{L}_0}$ above in terms of the occupation numbers are meant between states with definite occupation numbers, i.e. eigenstates of the occupation-number operators. This is exactly the situation in the trace, so these expressions may be used there. To take the trace, we sum over all possible occupancy numbers and integrate over the center-of-mass momentum. To keep the integration measure dimensionless, we multiply the $d^d k$ measure with the spacetime volume V_d. So, we get;

$$Z(\tau) = V_d (q\bar{q})^{-d/24} \int \frac{d^d k}{(2\pi)^d} e^{-\pi\alpha' k^2 \tau_2} \prod_{\mu,n} \sum_{N_{\mu n}, \tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}} \bar{q}^{n\tilde{N}_{\mu n}}$$

Now, we derive the following results;

$$\sum_{N=0}^{\infty} q^{nN} = 1 + q^n + \cdots = \frac{1}{1-q^n},$$

$$\Rightarrow \prod_{\mu,n} \sum_{N_{\mu n}, \tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}} \bar{q}^{n\tilde{N}_{\mu n}} = \prod_{\mu} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \frac{1}{1-\bar{q}^n} \right)$$

$$= \left(\prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2} \right)^d .$$

$$= (q\bar{q})^{d/24} \frac{1}{|\eta(\tau)|^{2d}}$$

Moreover,

$$\int \frac{d^d k}{(2\pi)^d} e^{-\pi\alpha' k^2 \tau_2} = \prod_{j=0}^{d-1} \left(\int_{-\infty}^{\infty} \frac{dk^j}{2\pi} e^{-(\pi\alpha'\tau_2)k_j^2} \right)$$

In the above integrals, $j = 0$ integral is a problem because k_0^2 is negative. For this, we do the Wick rotation $k^0 \to ik^d$ and thus, k^0 integral will give a factor of i. So, we get;

$$\int \frac{d^d k}{(2\pi)^d} e^{-\pi\alpha' k^2 \tau_2} = i \prod_{j=1}^{d} \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} = i \left(\frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \right)^d$$

So, the partition function becomes;

$$Z(\tau) = iV_d \left(Z_X(\tau) \right)^d, \quad Z_X(\tau) = \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{|\eta(\tau)|^2}$$

For one free scalar on the torus, the oscillator determinant and zero mode give

$$Z_X(\tau,\bar{\tau}) = \frac{1}{\sqrt{\tau_2}\,|\eta(\tau)|^2}, \qquad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \quad q = e^{2\pi i\tau}. \tag{5.5}$$

The factor $|\eta|^{-2}$ counts left- and right-moving oscillators, while $\tau_2^{-1/2}$ comes from integrating the non-compact zero mode. This form also explains why modular transformations work, since the eta-function has a modular weight and the zero-mode factor supplies the compensating power. In compactification this zero-mode piece is replaced by a momentum-winding lattice sum, so the scalar partition function is the prototype for the Narain partition function. [10].

The single factor $\tau_2^{-1/2}|\eta|^{-2}$ is the Euclidean harmonic-oscillator determinant repeated mode by mode. For one oscillator the kernel over Euclidean time β requires

$$\det\left(-\partial_\tau^2 + \omega^2\right) = \prod_{n=1}^{\infty} \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) \tag{5.6}$$

over periodic fluctuations. The product diverges, and the standard treatment splits it as $\prod_n (4\pi^2 n^2/\beta^2) \cdot \prod_n \left(1 + \beta^2\omega^2/4\pi^2 n^2\right)$, whose second factor converges to $\sinh(\beta\omega/2)/(\beta\omega/2)$ by Euler's product for sinh, while the first is regularized with $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$, leaving $Z(\beta) = \left[2\sinh(\beta\omega/2)\right]^{-1} = \sum_{n\geq 0} e^{-\beta\omega(n+1/2)}$, so the regularization keeps the full oscillator spectrum, zero-point energy included. Each worldsheet oscillator contributes such a factor, the $q^{-c/24}$ prefactor of the η-function is the zeta-regularized zero-point sum $\sum_n n \to \zeta(-1) = -\frac{1}{12}$, and modular invariance is the statement that the regularization was done the same way in every sector [8, 10].

5.2.3 The bc CFT

For the bc ghost system with b having the weight of 2, the L_0 mode (as derived in chapter 2), is given as follows:

$$L_0 = -\sum_{n=-\infty}^{\infty} n : b_n c_{-n} : -1 = -\sum_{n=1}^{\infty} n c_{-n} b_n + \sum_{n=1}^{\infty} n b_{-n} c_n - 1 = \sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) - 1$$

Moreover, recall that we have the following anti-commutation relation;

$$\{b_n, c_m\} = \delta_{m+n,0}$$

We can see that $L_0 + 1$ is nothing but the number operator (or the level N operator). We verify it as follows:

$$\sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) b_{-m} |0\rangle = \sum_{n=1}^{\infty} n b_{-n} c_n b_{-m} |0\rangle = m b_{-m} |0\rangle$$

$$\sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) c_{-m} |0\rangle = -\sum_{n=1}^{\infty} n c_{-n} b_n c_{-m} |0\rangle = m c_{-m} |0\rangle$$

Each term in $L_0 + 1$ gives the level generated by a different ghost field. Denote the occupancy numbers of b_{-m} and c_{-m} by B_m and C_m, respectively; each can only be 0 or 1 because of anticommutativity.

$$q^{L_0} \left| \{B_m\}, \{C_n\} \right\rangle = q^{mB_m + nC_n} q^{-1} \left| \{B_m\}, \{C_n\} \right\rangle$$

So, we see that sandwiching between states of definite B_m and C_n quantum numbers will give us factors like

$$q^{mB_m + nC_n - 1} \text{ and } \bar{q}^{m\bar{B}_m + n\bar{C}_n - 1}$$

Now, summing over all states, we get;

$$\operatorname{Tr}\left(q^{L_0} \bar{q}^{\bar{L}_0} \right) = (q\bar{q})^{-1} \prod_{m=1}^{\infty} \sum_{B_m, \tilde{B}_m = 0}^{1} q^{mB_m} \bar{q}^{m\tilde{B}_m} \prod_{m=0}^{\infty} \sum_{C_m, \tilde{C}_m = 0}^{1} q^{mC_m} \bar{q}^{m\tilde{C}_m}$$

$$= (q\bar{q})^{-1} \prod_{m=1} (1 + q^m)(1 + \bar{q}^m) \prod_{m=0} (1 + q^m)(1 + \bar{q}^m) = 4 \prod_{m=1}^{\infty} |1 + q^m|^4$$

For c, the product starts at $m = 0$ because c_0 is a creation operator, whereas b_0 is not. For the bc ghost system, $c = \tilde{c} = -26$, and hence the partition function becomes

$$Z(\tau) = 4(q\bar{q})^{13/12} (q\bar{q})^{-1} \prod_{m=1}^{\infty} |1 + q^m|^4 = 4(q\bar{q})^{1/12} \prod_{m=1}^{\infty} |1 + q^m|^4$$

We see that if we had calculated

$$(q\bar{q})^{-(c+\tilde{c})/48} \operatorname{Tr}\left[(-1)^F q^{L_0} \bar{q}^{\tilde{L}_0} \right]$$

then due to the c_0 zero modes, we will get a factor of

$$\left(1 - q^0\right)\left(1 - \bar{q}^0\right) = 0$$

and thus, this quantity vanishes. The zero-mode and Grassmann-trace reason is given immediately below. Since the torus has one metric modulus and one complex CKV, the torus expectation value contains one b insertion and one c insertion. The simplest ghost correlation function is therefore

Reason: the physical ghost functional integral is a Grassmann trace, so it contains the statistics sign $(-1)^F$. Without insertions, the $c_0, \tilde{c}_0$ zero modes are unsaturated and the trace vanishes, exactly as the factor $(1 - q^0)(1 - \bar{q}^0)$ indicates. In an amplitude the zero modes are not left unsaturated: the torus has one complex conformal Killing vector and one complex modulus, so the required insertions are one $c\tilde{c}$ and one $b\tilde{b}$. Equivalently,

$$\operatorname{Tr}\Big[(-1)^F b_0\tilde{b}_0 c_0\tilde{c}_0\, q^{L_0}\bar{q}^{\bar{L}_0}\Big] \neq 0, \qquad \operatorname{Tr}\Big[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0}\Big] = 0. \tag{5.7}$$

The anticommutativity is essential: a Grassmann zero-mode integral is nonzero only for the coefficient of the product of all zero modes. [10].

$$\Big\langle c\,(w_1)\, b\,(w_2)\, \tilde{c}\,(w_3)\, \tilde{b}\,(w_4)\Big\rangle$$

Only the zero-mode part contributes. It projects onto states with occupancy numbers $C_0 = \tilde{C}_0 = 1$, equivalently the state $|\uparrow\uparrow\rangle$. Hence

$$\operatorname{Tr}\left((-1)^F c_0 b_0 \tilde{c}_0 \tilde{b}_0 q^{L_0}\bar{q}^{\bar{L}_0}\right) = (q\bar{q})^{1/12}\prod_{m=1}^{\infty}(1-q^m)(1-\bar{q}^m)\prod_{m=1}^{\infty}(1-q^m)(1-\bar{q}^m) = (q\bar{q})^{1/12}\prod_{m=1}^{\infty}|1-q^m|^4$$

5.2.4 General CFTs

For a general CFT, the eigenvalue of L_0 is the conformal weight. The general partition function is therefore

$$Z(\tau) = \sum_i q^{h_i - c/24}\bar{q}^{\bar{h}_i - \bar{c}/24}(-1)^{F_i}$$

Now, let $\tau = il$ and then, we have;

$$Z(il) = \sum_i e^{-2\pi l\left(h_i + \bar{h}_i - (c+\bar{c})/24\right)}(-1)^{F_i}$$

Now, we have;

$$\tau \to \tau + 1 \Rightarrow il \to il + 1 \Rightarrow l \to l - i$$

and thus, this transformation gives the following phase in the partition function;

$$\exp\left[2\pi i\left(h_i + \bar{h}_i - (c+\bar{c})/24\right)\right]$$

It shows that to retain modular invariance, we should have;

$$h_i + \bar{h}_i - \frac{c+\bar{c}}{24} \in \mathbb{Z}\forall i$$

However, for the unity operator, we have $h_i = \bar{h}_i = 0$ and thus, $c + \bar{c}$ should be a multiple of 24 .

For the argument that follows, we assume that all the conformal weights are non-negative (i.e. the CFT is unitary). Now, if $l \to 0$, then the exponential in the partition function becomes unity and the partition function is determined by the density of states. Now, usually density of states increases with increasing weight (this is an underlying assumption for this argument) and thus, the partition function is dominated by the density of states for high h_i. Now, we also have the following constraint;

$$Z(il) = Z(\tau) = Z\left(-\frac{1}{\tau}\right) = Z\left(\frac{i}{l}\right)$$

Thus, we have;

$$Z(il) = Z\left(\frac{i}{l}\right) = \sum_i \exp\left[-\frac{2\pi}{l}\left(h_i + \bar{h}_i - (c+\bar{c})/24\right)\right](-1)^{F_i}$$

Now, if $l \to 0$, then the exponential in $Z(i/l)$ is dominated by lowest $h_i + \bar{h}_i$ which is 0 for the unity operator. So, we have;

$$\lim_{l\to 0} Z\left(\frac{i}{l}\right) \sim \exp\left[\frac{\pi(c+\bar{c})}{12l}\right]$$

So, we can say that;

$$\lim_{l\to 0} Z(il) \sim \exp\left[\frac{\pi(c+\bar{c})}{12l}\right]$$

5.2.5 Theta Functions

The definition of theta function is as follows:

$$\vartheta(\nu,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n\nu\right)$$

We can readily derive the following;

$$\begin{aligned}\vartheta(\nu+1,\tau) &= \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n\nu + 2\pi i n\right)\\ &= \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n\nu\right) = \vartheta(\nu,\tau).\end{aligned}$$

Moreover, we can derive the following;

$$\begin{aligned}\vartheta(\nu+\tau,\tau) &= \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n\nu + 2\pi i n\tau\right)\\ &= \sum_{n=-\infty}^{\infty} \exp\left(\pi i(n^2+2n)\tau + 2\pi i n\nu\right)\\ &= e^{-\pi i\tau - 2\pi i\nu} \sum_{m=-\infty}^{\infty} \exp\left(\pi i m^2\tau + 2\pi i m\nu\right)\\ &= e^{-\pi i\tau-2\pi i\nu}\vartheta(\nu,\tau), \qquad m = n+1.\end{aligned}$$

The transformation under modular transformations is as follows:

$$\vartheta(\nu,\tau+1) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + \pi i n^2 + 2\pi i n\nu\right)$$

(the proof follows from the two modular generators). The other modular transformation is as follows:

$$\vartheta\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) = \sum_n \exp\left(-\frac{\pi i n^2}{\tau} + \frac{2\pi i n\nu}{\tau}\right) = \sqrt{-i\tau}\exp\left[i\pi\tau\left(k+\frac{\nu}{\tau}\right)^2\right]$$

$$= \sqrt{-i\tau}\exp\left(\frac{i\pi\nu^2}{\tau}\right)\sum_k \exp\left[i\pi\tau k^2 + 2\pi i\nu k\right] = \sqrt{-i\tau}\exp\left(\frac{i\pi\nu^2}{\tau}\right)\vartheta(\nu,\tau)$$

where we used the Poisson resummation formula. The zero of ϑ is as follows:

$$\vartheta\left(\nu=\frac{\tau+1}{2},\tau\right) = \sum_n \exp\left(\pi i\tau\left(n^2+n\right)+\pi i n\right) = \sum_{n\text{ even}} \exp\left(\pi i\tau\left(n^2+n\right)\right) - \sum_{n\text{ odd}} \exp\left(\pi i\tau\left(n^2+n\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(\pi i\tau\left(4n^2+2n\right)\right) - \sum_{n=-\infty}^{\infty} \exp\left(\pi i\tau\left((2n-1)^2+2n-1\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(2\pi i\tau\left(2n^2+n\right)\right) - \sum_{n=-\infty}^{\infty} \exp\left(2\pi i\tau\left(2n^2-n\right)\right) = 2i\sum_{n=-\infty}^{\infty} \exp\left(4\pi i\tau n^2\right)\sin(2\pi i\tau n) = 0$$

The infinite product representation is easily derived using Jacobi triple product identity;

$$\prod_{m=1}^{\infty}(1-q^m)\left(1+q^{m-1/2}z\right)\left(1+q^{m-1/2}z^{-1}\right)=\sum_{n\in\mathbb{Z}}q^{n^2/2}z^n \tag{7.2.1}$$

Setting $q=e^{2\pi i\tau}$ and $z=e^{2\pi i\nu}$, the right hand side of 7.2.1 becomes $\vartheta(\nu,\tau)$. Using the characteristics definition of theta functions;

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](\nu,\tau)=\sum_{n=-\infty}^{\infty}\exp\left(\pi i(n+a)^2\tau+2\pi i(n+a)(\nu+b)\right)$$

This gives the following theta functions in the notation used below:

$$\vartheta\left[\begin{array}{c}0\\0\end{array}\right](\nu,\tau)=\vartheta_{00}(\nu,\tau)=\vartheta_3(\nu\mid\tau)=\vartheta(\nu,\tau)=\sum_{n\in\mathbb{Z}}q^{n^2/2}z^n$$

$$\vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](\nu,\tau)=\vartheta_{01}(\nu,\tau)=\vartheta_4(\nu\mid\tau)=\sum_{n\in\mathbb{Z}}\exp\left(\pi in^2\tau+2\pi in\left(\nu+\frac{1}{2}\right)\right)=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}z^n$$

$$\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](\nu,\tau)=\vartheta_{10}(\nu,\tau)=\vartheta_2(\nu\mid\tau)=\sum_{n\in\mathbb{Z}}\exp\left(\pi i\left(n+\frac{1}{2}\right)^2\tau+2\pi i\left(n+\frac{1}{2}\right)\nu\right)$$

$$=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}z^{n+\frac{1}{2}}=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}z^{n-\frac{1}{2}}$$

where in the last step, we shifted the index from n to $n-1$. Lastly, we have;

$$\vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](\nu,\tau)=\vartheta_{11}(\nu,\tau)=-\vartheta_1(\nu\mid\tau)=\sum_{n\in\mathbb{Z}}\exp\left(\pi i\left(n+\frac{1}{2}\right)^2\tau+2\pi i\left(n+\frac{1}{2}\right)\left(\nu+\frac{1}{2}\right)\right)$$

$$=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n+\frac{1}{2}\right)}z^{n+\frac{1}{2}}e^{\pi\left(n+\frac{1}{2}\right)}=i\sum_{n\in\mathbb{Z}}(-1)^nq^{\frac{1}{2}\left(n+\frac{1}{2}\right)}z^{n+\frac{1}{2}}=-i\sum_{n\in\mathbb{Z}}(-1)^nq^{\frac{1}{2}\left(n-\frac{1}{2}\right)}z^{n-\frac{1}{2}}$$

where we again shifted the index. Now, we derive the modular transformations of these theta functions as follows:

$$\vartheta_3(\nu\mid\tau+1)=\sum_{n\in\mathbb{Z}}q^{n^2/2}e^{\pi in^2}z^n=\sum_{n\in\mathbb{Z}}q^{n^2/2}e^{\pi in}z^n=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}z^n=\vartheta_4(\nu\mid\tau)$$

where we used the fact that n^2 is odd iff n is odd and n^2 is even iff n is even and thus, we have $e^{\pi in^2}=e^{\pi in}$. Continuing, we have;

$$\vartheta_4(\nu\mid\tau+1)=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}e^{\pi in^2}z^n=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}e^{\pi in}z^n=\sum_{n\in\mathbb{Z}}q^{n^2/2}z^n=\vartheta_3(\nu\mid\tau)$$

$$\vartheta_2(\nu\mid\tau+1)=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}e^{\pi i\left(n-\frac{1}{2}\right)^2}z^{n-1/2}=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}e^{\pi i\left(n^2-n\right)+\pi i/4}z^{n-1/2}$$

$$=e^{\pi i/4}\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}z^{n-1/2}=e^{\pi i/4}\vartheta_2(\nu\mid\tau)$$

$$\vartheta_1(\nu\mid\tau+1)=i\sum_{n\in\mathbb{Z}}(-1)^nq^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}e^{\pi i\left(n-\frac{1}{2}\right)^2}z^{n-1/2}=i\sum_{n\in\mathbb{Z}}(-1)^nq^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}e^{\pi i\left(n^2-n\right)+\pi i/4}z^{n-1/2}$$

$$=ie^{\pi i/4}\sum_{n\in\mathbb{Z}}(-1)^nq^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}z^{n-1/2}=e^{\pi i/4}\vartheta_1(\nu\mid\tau)$$

The S transformations are done as follows:

$$\vartheta_3\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right)=\sum_{n\in\mathbb{Z}}e^{-\pi in^2/\tau+2\pi in\nu/\tau}$$

$$
\begin{aligned}
&= \sqrt{-i\tau}\sum_{n\in\mathbb{Z}} \exp\left[-i\pi\tau\left(k+\frac{\nu}{\tau}\right)^2\right] \\
&= \sqrt{-i\tau}e^{i\pi\nu^2/\tau}\sum_{k\in\mathbb{Z}} q^{k^2/2}z^k \\
&= \sqrt{-i\tau}e^{i\pi\nu^2/\tau}\vartheta_3(\nu \mid \tau)
\end{aligned}
$$

where we used the Poisson resummation formula. The other theta functions transform by the same calculation, with the characteristics shifted by the S and T actions on the two torus cycles. The Jacobi identity of the theta functions is as follows:

$$\vartheta_3^4(0 \mid \tau) = \vartheta_2^4(0 \mid \tau) + \vartheta_4^4(0 \mid \tau) \tag{7.2.2}$$

This identity follows from the product representations below: the ratio $\vartheta_3^4/(\vartheta_2^4+\vartheta_4^4)$ is an elliptic function without poles and with value 1 at the cusp. From the product representation, it can also be seen that $\vartheta_1(0 \mid \tau) = 0$ as follows:

$$\vartheta_1(\nu \mid \tau) \propto \sin(\pi\nu) \Rightarrow \vartheta_1(0 \mid \tau) = 0 \tag{7.2.3}$$

We again used the fact that $n^2 - n$ is always even (i.e. for all $n \in \mathbb{Z}$). The modular transformation of the eta function is calculated now. We have the following;

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \Rightarrow \eta(\tau+1) = q^{1/24}e^{\pi i/12}\prod_{n=1}^{\infty}\left(1-q^n e^{2\pi i n}\right) = e^{\pi i/12}\eta(\tau)$$

To derive the S transformation, we first do another calculation. We will need the product representations of the theta functions; these follow from the Jacobi triple product applied to the series definitions in (7.2.1). We proceed as follows:

$$
\begin{aligned}
\frac{\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{\eta^3(\tau)} &= \frac{2e^{\pi i\tau/4}\prod_{n=1}^{\infty}(1-q^m)^3(1+q^m)^2\left(1-q^{m-\frac{1}{2}}\right)^2\left(1+q^{m-\frac{1}{2}}\right)^2}{e^{\pi i\tau/4}\prod_{n=1}^{\infty}(1-q^n)^3} \\
\Rightarrow \sqrt{\frac{\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{2\eta^3(\tau)}} &= \prod_{n=1}^{\infty}(1+q^m)\left(1-q^{2m-1}\right)
\end{aligned}
$$

Now, we derive an identity to proceed;

$$
\begin{aligned}
\left(1-q^{2m-1}\right)\left(1-q^{2m}\right) &= 1-q^{2m}-q^{2m-1}+q^{4m-1} = 1-\left(e^{2\pi i(2m)}\right)^{\tau}-\left(e^{2(2m-1)\pi i}\right)^{\tau}+\left(e^{2(4m-1)\pi i}\right)^{\tau} \\
&= 1-(1)^{\tau}-(-1)^{\tau}+(-1)^{\tau} = 1-(1)^{\tau} = 1-\left(e^{2\pi i m}\right)^{\tau} = 1-q^m \Rightarrow 1-q^{2m-1} = \frac{1-q^m}{1-q^{2m}}
\end{aligned}
$$

So, we get the following;

$$\sqrt{\frac{\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{2\eta^3(\tau)}} = \prod_{n=1}^{\infty}(1+q^m)\frac{1-q^m}{1-q^{2m}} = \prod_{n=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m}} = 1$$

Thus, we readily have the following;

$$
\begin{aligned}
\eta(\tau) &= \sqrt[3]{\frac{\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{2}} \\
\Rightarrow \eta\left(-\frac{1}{\tau}\right) &= \sqrt[3]{\frac{(-i\tau)^{3/2}\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{2}} = \sqrt{-i\tau}\sqrt[3]{\frac{\vartheta_3(0 \mid \tau)\vartheta_4(0 \mid \tau)\vartheta_2(0 \mid \tau)}{2}} = \sqrt{-i\tau}\quad\eta(\tau)
\end{aligned}
$$

This implies the following;

$$\bar{\eta}\left(-\frac{1}{\bar{\tau}}\right) = \sqrt{i\bar{\tau}}\bar{\eta}(\bar{\tau}) \Rightarrow |\eta(\tau)|^2 = \frac{1}{|\tau|}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2$$

We also note that;

$$\tau_1 + i\tau_2 = \tau \to -\frac{1}{\tau} = -\frac{1}{\tau_1 + i\tau_2} = -\frac{\tau_1}{|\tau|^2} + i\frac{\tau_2}{|\tau|^2} \Rightarrow \tau_2 \to \frac{\tau_2}{|\tau|^2} \Rightarrow \sqrt{\tau_2} \to \frac{\sqrt{\tau_2}}{|\tau|} \Rightarrow \sqrt{\tau_2} = |\tau|\sqrt{\tau_2'} \tag{7.2.4}$$

where τ_2' is the transformed τ_2. It means that $\sqrt{\tau_2}|\eta(\tau)|^2$ is a modular invariant quantity (note that in T transformation, τ_2 is invariant).

5.3 The torus amplitude

Since the torus metric is invariant under the transformation

$$\sigma^a \to -\sigma^a \Rightarrow w \to -w, \bar{w} \to -\bar{w}$$

The torus has a symmetry group of $\mathbb{Z}_2$ and thus, we will have a factor of 2 in the denominator of the moduli integration. Only one CKV is there for the torus and therefore, we insert one $c\tilde{c}$ insertion. Because of one complex modulus, we have one $B\tilde{B}$ insertion where B is given as follows:

$$B = \frac{1}{4\pi}\left(b, \partial_\tau g\right) = \frac{1}{4\pi}\int d^2w b_{ww}\partial_\tau g_{ww} = 2\pi i b_{ww}(0)$$

The last equality follows by evaluating the Beltrami differential for the torus modulus in the flat coordinate and using translation invariance to place the resulting b_{ww} insertion at the origin. The torus amplitude becomes

$$\begin{aligned} S(1,\ldots,n)_{T^2} &= \frac{1}{2}\int d\tau d\bar{\tau}\left\langle B\tilde{B}c\tilde{c}\mathcal{V}\left(w_1,\bar{w}_1\right)\prod_{i=2}^{n}\int dw_i d\bar{w}_i \mathcal{V}_i\left(w_i,\bar{w}_i\right)\right\rangle_{T^2} \\ &= 2\pi^2\int d\tau d\bar{\tau}\left\langle b(0)\tilde{b}(0)c\tilde{c}\mathcal{V}\left(w_1,\bar{w}_1\right)\prod_{i=2}^{n}\int dw_i d\bar{w}_i \mathcal{V}_i\left(w_i,\bar{w}_i\right)\right\rangle_{T^2} \end{aligned}$$

The position w_1 can be averaged over the torus because translation invariance leaves the correlator independent of the chosen origin; the corresponding ghost zero mode is still saturated by the $c\tilde{c}$ insertion. Thus

$$\begin{aligned} \int \frac{d\sigma_1 d\sigma_2}{\text{Area of torus}} c\left(w_1\right)\tilde{c}\left(\bar{w}_1\right)\mathcal{V}\left(w_1,\bar{w}_1\right) &= \int \frac{dw_1 d\bar{w}_1}{2\text{ Area of torus}} c\left(w_1\right)\tilde{c}\left(\bar{w}_1\right)\mathcal{V}\left(w_1,\bar{w}_1\right) \\ &= c(0)\tilde{c}(0)\int \frac{dw_1 d\bar{w}_1}{2(2\pi)^2\tau_2}\mathcal{V}\left(w_1,\bar{w}_1\right) \end{aligned}$$

So, torus amplitude becomes;

$$S_{T^2}(1,\ldots,n) = \int \frac{d\tau d\bar{\tau}}{4\tau_2}\left\langle b(0)\tilde{b}(0)c(0)\tilde{c}(0)\prod_{i=1}^{n}\int dw_i d\bar{w}_i \mathcal{V}_i\left(w_i,\bar{w}_i\right)\right\rangle_{T^2}$$

We first calculate the amplitude without any vertex operators and include the matter partition function for 26 dimensions, we get;

$$\begin{aligned} Z_{T^2} &= \int \frac{d\tau d\bar{\tau}}{4\tau_2}\langle b(0)\tilde{b}(0)c(0)\tilde{c}(0)\rangle_{T^2} iV_{26}\left(Z_X\right)^{26} = iV_{26}\int \frac{d\tau d\bar{\tau}}{4\tau_2}\left(4\pi\alpha'\tau_2\right)^{-13}|\eta(\tau)|^{4-2\times 26} \\ &= iV_{26}\int \frac{d\tau d\bar{\tau}}{4\tau_2}\left(4\pi\alpha'\tau_2\right)^{-13}|\eta(\tau)|^{-48} \end{aligned}$$

For a general CFT, we can write down the partition function similarly by integrating over the COM momentum (with d non-compact dimensions and where $d \geq 2$ so that ghosts can remove the oscillators corresponding to these directions) as follows:

$$Z_{T^2} = V_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \int \frac{d^d k}{(2\pi)^d} e^{-\pi\tau_2\alpha' k^2} \sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{\bar{h}_i-1} = iV_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi\alpha'\tau_2)^{-d/2} \sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{\bar{h}_i-1} \tag{7.3.1}$$

This is the same structure as the torus partition function. The momenta appear in the integrand because $\mathcal{H}_\perp$ does not include non-compact momenta, ghosts, or the two directions $\mu = 0, 1$ removed by the constraints. The corresponding field-theory result for particle paths with circle topology is

$$Z_{S_1}(m^2) = V_d \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{dl}{2l} e^{-(k^2+m^2)l/2} = V_d \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{dl}{2l} e^{-k^2 l/2} e^{-m^2 l/2}$$
$$= iV_d \int_0^\infty \frac{dl}{2l} \left(\int \frac{dk_j}{2\pi} e^{-k_j^2 l/2} \right)^d e^{-m^2 l/2} = iV_d \int_0^\infty \frac{dl}{2l} (2\pi l)^{-d/2} e^{-m^2 l/2}$$

where the factor of i comes from Wick rotating $k^0 \to ik^0$ so that the k^0 integral converges. The variables k_j are Euclidean integration variables, while k^μ denote the non-compact spacetime momenta.[2] Summing this partition function over the string mass spectrum gives

$$m^2 = \frac{2}{\alpha'}(h + \bar{h} - 2) \Rightarrow \frac{m^2 l}{2} = \frac{l}{\alpha'}(h + \bar{h} - 2)$$

Notice that the formula this time has h instead of the oscillator level N.[3] However, we also have the level matching condition i.e. $h = \tilde{h}$. We can impose this condition as follows:

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i(h-\bar{h})\theta} = \delta(h - \bar{h}) = \delta_{h,\bar{h}}$$

The last step makes sense only if $h - \bar{h}$ is an integer but we saw before that due to modular invariance, this is the case and thus, there is no problem in the last step. So, we get;

$$\sum_{i\in\mathcal{H}_\perp} Z_{S^1}(m_i^2) = iV_d \int_0^\infty \frac{dl}{2l} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (2\pi l)^{-d/2} \sum_{i\in\mathcal{H}_\perp} e^{-l\left(h_i+\bar{h}_i-2\right)/\alpha' + i\left(h_i-\bar{h}_i\right)\theta}$$

Now we change the variables. The new variables are as follows:

$$\tau = \frac{\theta}{2\pi} + \frac{il}{2\pi\alpha'}, \bar{\tau} = \frac{\theta}{2\pi} - \frac{il}{2\pi\alpha'} \Rightarrow \theta = \pi(\tau + \bar{\tau}), \quad l = -i(\tau - \bar{\tau})\pi\alpha' \tag{7.3.2}$$

The real and imaginary parts of τ lie in the ranges

$$R: -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad 0 \leq \tau_2 < \infty$$

We have called this region R. The Jacobian is calculated as follows:

$$d\theta dl = \begin{vmatrix} \partial\theta/\partial\tau & \partial\theta/\partial\bar{\tau} \\ \partial l/\partial\tau & \partial l/\partial\bar{\tau} \end{vmatrix} d\tau d\bar{\tau} = 2\pi^2\alpha' d\tau d\bar{\tau}$$

Moreover, we can also see that $l = 2\pi\alpha'\tau_2$. Lastly, we see that;

$$-\frac{l}{\alpha'}(h_i + \bar{h}_i - 2) + i(h_i - \bar{h}_i)\theta$$
$$= \frac{i(\tau - \bar{\tau})\pi\alpha'}{\alpha'}(h_i + \bar{h}_i - 2) + i\pi(\tau + \bar{\tau})(h_i - \bar{h}_i)$$
$$= 2\pi i(h_i - 1)\tau - 2\pi i(\bar{h}_i - 1)\bar{\tau},$$

[2]The formula is the Schwinger proper-time representation of a one-loop vacuum graph. The factor $dl/(2l)$ is the residual translation volume of the circular worldline, and the Gaussian momentum integral gives $(2\pi l)^{-d/2}$. The string torus formula replaces the single particle mass m^2 by the complete tower of worldsheet states and then imposes level matching.

[3]For a general CFT, h is the eigenvalue of L_0 after all oscillator, momentum, winding, and internal contributions have been included. In a free oscillator theory h reduces to the level plus the appropriate zero-mode contribution and intercept. Therefore replacing N by h is not a change of principle; it is the CFT-invariant way to write the same mass formula.

and hence

$$e^{-l(h_i+\bar{h}_i-2)/\alpha'+i(h_i-\bar{h}_i)\theta}=e^{2\pi i(h_i-1)\tau-2\pi i(\bar{h}_i-1)\bar{\tau}}=q^{h_i-1}\bar{q}^{\bar{h}_i-1}.$$

In these new variables, one obtains

$$\sum_{i\in\mathcal{H}_\perp} Z_{S^1}\left(m_i^2\right)=iV_d\int_R\frac{d\tau d\bar{\tau}}{4\tau_2}\left(4\pi^2\alpha'\tau_2\right)^{-d/2}\sum_{i\in\mathcal{H}_\perp}q^{h_i-1}\bar{q}^{\bar{h}_i-1} \tag{7.3.3}$$

The only difference between this partition function and the actual string partition function is the domain of τ integration. To study divergences, first note that

$$Z_{S_1}\left(m^2\right)\sim\int_0^\infty\frac{dl}{l^{d/2+1}}=-\frac{2}{d}\left|\frac{1}{l^{d/2}}\right|_0^\infty=\infty\quad(d\geq 2)$$

The divergence comes from the endpoint $l=0$. Summing over the string spectrum gives a sum of divergent terms, since all terms in Eq. (7.3.3) have the same sign. From Eq. (7.3.2), $l=0$ means $\tau=\bar{\tau}$, so τ lies on the real axis. This axis is absent from F_0, and therefore the ultraviolet divergent region is absent.
It remains to examine the $\tau_2\to\infty$ region of Eq. (7.3.1). Use the variables

$$\tau=\tau_1+i\tau_2,\quad \bar{\tau}=\tau_1-i\tau_2\Rightarrow d\tau d\bar{\tau}=2d\tau_1 d\tau_2$$

We also do the following manipulation;

$$q^{h_i-1}\bar{q}^{\bar{h}_i-1}=e^{2\pi i\left(h_i-\bar{h}_i\right)\tau_1}e^{-2\pi\left(h_i+\bar{h}_i-2\right)\tau_2}$$

Now, using these variables, we can write 7.3.1 for $d=26$ as follows:

$$iV_{26}\int\frac{d\tau_1 d\tau_2}{2\tau_2}\left(4\pi\alpha'\tau_2\right)^{-13}\sum_{i\in\mathcal{H}_\perp}e^{2\pi i\left(h_i-\bar{h}_i\right)\tau_1}e^{-2\pi\left(h_i+\bar{h}_i-2\right)\tau_2}=iV_{26}\int_{\tau_{2\text{lower}}}^\infty\frac{d\tau_2}{2\tau_2}\left(4\pi\alpha'\tau_2\right)^{-13}\sum_{i\in\mathcal{H}_\perp}e^{-4\pi(h_i-1)\tau_2}$$

where the τ_1 integration just ensures that $h_i=\bar{h}_i$. Now, we can sum over different h_i (which is just level i.e. N). For $h_i=0$, we have only one state i.e. the tachyon. For $h_i=1$, we have 24^2 states because the h_i state contains the following creation operators on the vacuum;

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1},\quad i,j\in\{1,\ldots,24\}$$

So, the expansion of 7.3.1 is as follows:

$$iV_{26}\int_{\tau_{2\text{lower}}}^\infty\frac{d\tau_2}{2\tau_2}\left(4\pi\alpha'\tau_2\right)^{-13}\left(e^{4\pi\tau_2}+(24)^2+\ldots\right)$$

The first tachyon term diverges. Using the fact that;

$$m_i^2=\frac{2}{\alpha'}\left(h_i+\bar{h}_i-2\right)=\frac{4}{\alpha'}\left(h_i-1\right)\Rightarrow 4\left(h_i-1\right)=\alpha' m_i^2$$

(where the level matching condition is used i.e. $h_i=\bar{h}_i$) we can write down the partition function in the form above for $d\neq 26$ as well. It is as follows:

$$iV_d\int_{\tau_{2\text{lower}}}^\infty\frac{d\tau_2}{2\tau_2}\left(4\pi\alpha'\tau_2\right)^{-d/2}\sum_{i\in\mathcal{H}_\perp}\exp\left(-\pi\alpha' m_i^2\tau_2\right)$$

5.3.1 Physics of the vacuum amplitude

Since $Z_{S_1}\left(m^2\right)$ is the vacuum partition function for a path of circle topology, the total vacuum amplitude sums over any number of disconnected such paths, with identical configurations counted once. Thus

$$Z_{\text{vac}}\left(m^2\right)=1+Z_{S^1}\left(m^2\right)+\frac{1}{2!}\left(Z_{S^1}\left(m^2\right)\right)^2+\ldots=\exp\left[Z_{S^1}\left(m^2\right)\right]$$

where $1/n!$ avoids overcounting identical components. The vacuum amplitude also has the representation

$$Z_{\text{vac}}\left(m^2\right)=\langle 0|e^{-iHT}|0\rangle=\langle 0|e^{-i\rho_0 VT}|0\rangle=\langle 0|e^{-i\rho_0 V_d}|0\rangle=e^{-i\rho_0 V_d}$$

where ρ_0 is the vacuum energy density, which must be constant in a Lorentz-invariant vacuum, V is the spatial volume, T is time, and V_d is the spacetime volume. Comparing the two expressions for the vacuum amplitude gives

$$Z_{S^1}\left(m^2\right) = -i\rho_0 V_d \Rightarrow \rho_0 = \frac{i}{V_d} Z_{S_1}\left(m^2\right)$$

The lower limit $\tau_{2\text{lower}}$ is a proper-time infrared cutoff in the particle representation and a modulus-space cutoff in the string representation. In field theory one would also have to impose an ultraviolet cutoff at small proper time, but for the torus amplitude the modular fundamental domain removes the region $\tau_2 \to 0$ as a separate integration region. The remaining large-τ_2 divergence is therefore an infrared divergence from tachyonic or massless physical states, not a short-distance ultraviolet divergence. The generalization to particles with different spins is as follows:

$$\rho_0 = \frac{i}{V_d} \sum_j (-1)^{\mathbf{F}_j} Z_{S_1}\left(m_j^2\right) \tag{7.3.4}$$

5.4 Open and unoriented one loop graphs

The annulus, Möbius strip, and Klein bottle must be discussed together because each has both a loop-channel and a transverse-channel interpretation. The annulus already shows the essential relation,

$$\int_0^\infty \frac{dt}{2t} \operatorname{Tr}_{\mathcal{H}_{\text{open}}}\left(e^{-2\pi t(L_0-a)}\right) = \int_0^\infty d\ell\, \langle B|e^{-\pi\ell(L_0+\tilde{L}_0-2a)}|B\rangle, \qquad \ell = \frac{1}{t}. \tag{5.8}$$

The Möbius amplitude inserts the orientation reversal Ω in the open-string trace, and the Klein bottle is the corresponding closed-string unoriented surface. The transverse massless pole is the diagnostic, since uncancelled NS-NS or R-R tadpoles mean that the assumed background is not a valid solution. [10, 352].

5.4.1 The cylinder

The cylinder partition function is relevant for the open strings. The cylinder has one modulus that we call t. The height of the cylinder is $2\pi t$ and the circumference is π. So, we have to calculate;

$$Z_{C_2} = \frac{1}{2}\cdot\frac{1}{2}(2\pi)^2 \int_0^\infty \frac{dt}{2\pi^2 t} \operatorname{Tr}_0'\left[e^{-2\pi t L_0}\right] = \int_0^\infty \frac{dt}{2t} \operatorname{Tr}_0'\left[e^{-2\pi t L_0}\right]$$

where the first half comes due to the Jacobian of $d\sigma^1 d\sigma^2 \to dw d\bar{w}$ transformation, the second one due to the $\mathbb{Z}_2$ symmetry of the cylinder, $(2\pi)^2$ from the B insertions (although they aren't here, they were used in torus case to have $b(0)$ in the expectation value instead of B) and the $2\pi^2 t$ comes from the area of the cylinder. L_0 is the open string Hamiltonian and it is given as follows:

$$L_0 = \alpha' p^2 + N$$

Now, we sum over all possible states and integrate over the momentum to get;

$$Z_{C_2} = V_d n^2 \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t\alpha' k^2} \int_0^\infty \frac{dt}{2t} \prod_{\mu,n} \sum_{N_{\mu n}} e^{-2\pi t n N_{\mu n}}$$

$$= iV_d n^2 \left(8\pi^2\alpha' t\right)^{-d/2} \int_0^\infty \frac{dt}{2t} \prod_{\mu,n} \sum_{N_{\mu n},\mathcal{H}^\perp} e^{-2\pi t n N_{\mu n}} = in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \left[\prod_{n=1}^\infty \frac{1}{1-e^{-2\pi t n}}\right]$$

$$= in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \left[\prod_{n=1}^\infty \frac{1}{1-e^{2i\pi(it)n}}\right] = in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \eta(it)^{-(d-2)} \tag{7.4.1}$$

where in the second line $\mathcal{H}^{\perp}$ denotes the trace over transverse excitations. The factor n^2 comes from the Chan–Paton degrees of freedom, and $d-2$ appears because the ghosts remove two non-compact oscillator directions. Setting $d=26$ gives

$$Z_{C_2} = iV_{26}n^2 \int_0^\infty \frac{dt}{2t}\left(8\pi^2\alpha' t\right)^{-13}\eta(it)^{-24}$$

The limiting regions of the cylinder modulus have the usual open- and closed-string interpretations. The limit $t\to\infty$ is the open-string infrared region, analogous to the $\tau_2\to\infty$ limit on the torus. The limit $t\to 0$ is studied by a modular transformation to the closed-string channel. Defining $s=\pi/t$ gives

$$\eta\left(\frac{i}{t}\right) = \sqrt{t}\eta(it) \Rightarrow \eta\left(\frac{is}{\pi}\right) = \sqrt{\frac{\pi}{s}}\eta\left(\frac{i\pi}{s}\right)$$

Then, the partition function becomes;

$$\begin{aligned}
-iV_{26}\int_0^\infty \frac{d(\pi/s)}{2(\pi/s)}\left(\frac{8\pi^3\alpha'}{s}\right)^{-13}\eta(i\pi/s)^{-24} &= -iV_{26}\int_0^\infty \frac{d(\pi/s)}{2(\pi/s)}\left(\frac{8\pi^3\alpha'}{s}\right)^{-13}\left(\frac{\pi}{s}\right)^{12}\eta(is/\pi)^{-24}\\
&= \frac{iV_{26}n^2}{2\pi\left(8\pi^2\alpha'\right)^{13}}\int_0^\infty ds\eta(is/\pi)
\end{aligned}$$

Using the expansion of $\eta(\tau)$ i.e.;

$$\eta(\tau) = q^{1/24}\left(1-q-q^2+\ldots\right)$$

we have;

$$\begin{aligned}
\eta(\tau)^{-24} &= q^{-1}(1-q+\ldots)^{-24} = e^{-2\pi i\tau}(1+24q+\ldots),\\
\Rightarrow\quad \eta(is/\pi)^{-24} &= e^{2s}\left(1+24e^{-2s}+\ldots\right) = e^{2s}+24+\mathcal{O}\left(e^{-2s}\right).
\end{aligned} \tag{7.4.2}$$

So, we have divergence from the massless states as well. The massless divergent part is as follows:

$$2\frac{24iV_{26}n^2}{4\pi\left(8\pi^2\alpha'\right)^{13}}\int_0^\infty ds \tag{7.4.3}$$

The divergence in Eq. (7.4.3) is a closed-string infrared divergence in the transverse exchange. In spacetime language it is sourced by the disk one-point function of massless NS–NS fields, mainly the dilaton and graviton, and if the total tadpole from all one-loop surfaces does not cancel, the assumed flat background is not a solution of the effective field equations. Boundary states make this precise,

$$\mathcal{A}_{\text{cyl}} = \langle B|\,\Delta\,|B\rangle, \tag{5.9}$$

so the large-cylinder limit isolates the overlap of $|B\rangle$ with massless closed-string states. Tadpole cancellation then means the net massless source from boundary and crosscap states vanishes. [10, 352].

5.4.2 The Klein bottle

The Klein bottle partition becomes;

$$\int_0^\infty \frac{dt}{4t}\operatorname{Tr}'_c\left[\Omega\exp\left(-2\pi t\left(L_0+\tilde{L}_0\right)\right)\right]$$

where one factor of $1/2$ comes from the residual automorphism of K_2, and another factor of $1/2$ comes from the orientifold projection operator, as explained below. Using the expression for L_0 and $\bar{L}_0$ and noticing that the momentum integration is going to be like the torus case, we get;

The two factors of $1/2$ have different origins and should not be merged. One factor comes from the orientifold projection operator $P_\Omega = (1+\Omega)/2$ in the closed-string trace. The Klein-bottle term is the piece with the Ω insertion. The other factor is the residual automorphism of the

Klein bottle, equivalently the overcounting of the fundamental region of the modular parameter. Thus the schematic vacuum contribution is

$$\mathcal{K} \sim \frac{1}{2}_{\text{proj}} \frac{1}{2}_{\text{aut}} \int_0^\infty \frac{dt}{t} \operatorname{Tr}_{\mathcal{H}_{\text{closed}}} \left[\Omega \, e^{-2\pi t (L_0 + \bar{L}_0 - 2)} \right]. \tag{5.10}$$

The prime on the trace indicates the usual omission/saturation of zero modes. This is why the coefficient is not an arbitrary normalization: it encodes both Hilbert-space projection and worldsheet automorphism. [10, 352].

$$iV_d \int_0^\infty \frac{dt}{4t} (4\pi a l' t)^{-d/2} \sum_{i \in \mathcal{H}_\perp} \Omega_i \exp\left[-2\pi t \left(h_i + \bar{h}_i - 2\right)\right]$$

We see that we have to sum over the states with a definite value of Ω_i i.e. the diagonal states of Ω. Since Ω exchanges left and right movers, only left-right paired states contribute to the trace with an Ω insertion: schematically $\langle L, R|\Omega|L, R\rangle = \langle L, R|R, L\rangle$, which vanishes unless the two oscillator configurations match. Thus, we do not sum over two independent occupation-number sets as in the torus case, but over one paired set. The oscillator argument is doubled, giving $\eta(2it)$ instead of $\eta(it)$. So, we would have;

Diagonal-state trace: the Ω-inserted trace is a trace over the closed-string Hilbert space with an operator that interchanges the two chiral sectors. A basis state contributes only if it has nonzero overlap with its exchanged state. This is the closed-string analogue of why an insertion of a permutation operator traces only states invariant under that permutation. [10, 352].

$$Z_{K_2} = iV_{26} \int_0^\infty \frac{dt}{4t} (4\pi\alpha' t)^{-13} \eta(2it)^{-24}$$

Include the rise of cross-cap here. For the modular transformations, we do the variable change to $s = \pi/2t$ and then =, we get;

$$Z_{K_2} = \frac{iV_{26}}{4} \int_0^\infty \frac{ds}{s} (2\pi\alpha')^{-13} s^{13} \eta\left(\frac{i\pi}{s}\right)^{-24} \pi^{-13} = \frac{iV_{26}}{4\pi\,(2\pi\alpha')^{13}} \int_0^\infty ds \eta(is/\pi)^{-24}$$
$$= \frac{iV_{26} 2^{26}}{4\pi\,(8\pi\alpha')^{13}} \int_0^\infty ds \eta(is/\pi)^{-24}$$

where we used the fact;

$$\eta(i\pi/s) = \sqrt{\frac{s}{\pi}} \eta(is/\pi)$$

We can extract the massless divergent part from this partition function just like the cylinder case by expanding $\eta(is/\pi)$ and we saw from the cylinder case that the relevant term from this expansion is 24 and thus, the massless divergent part of Z_{K_2} is;

$$\frac{24 iV_{26} 2^{26}}{4\pi\,(8\pi\alpha')^{13}} \int_0^\infty ds \tag{7.4.4}$$

5.4.3 The Möbius strip

The Möbius strip partition function has the same open-string trace structure as the cylinder, with an additional factor of 1/2 from the orientifold projection. The required partition function is

$$Z_{M_2} = iV_{26} \int_0^\infty \frac{dt}{4t} \left(8\pi^2\alpha' t\right)^{-13} \sum_{i \in \mathcal{H}_o^\perp} \Omega_i e^{-2\pi t (h_i - 1)} = iV_{26} \int_0^\infty \frac{dt}{4t} \left(8\pi^2\alpha' t\right)^{-13} e^{2\pi t} \sum_{i \in \mathcal{H}_o^\perp} \Omega_i e^{-2\pi t N_i}$$

where N_i is the level of the i-th state and $\mathcal{H}_o^\perp$ is the transverse Hilbert space for open strings. Recall that

$$\Omega \alpha_n^\mu \Omega^{-1} = (-1)^n \alpha_n^\mu$$

and thus, we can find the trace as follows:

$$e^{2\pi t}\prod_{\mu,N}\sum_{N_{\mu n}=0}^{\infty}(-1)^{nN_{\mu n}}e^{-2\pi tnN_{\mu n}}=e^{2\pi t}\prod_{\mu,N}\frac{1}{1-(-1)^n e^{-2\pi tn}}=e^{2\pi t}\prod_{n=1}^{\infty}\left[1-(-1)^n e^{-2\pi tn}\right]^{-24}$$

where $N_{\mu n}$ is the occupation number. This product can be written in terms of $\vartheta_{00}(0,2it)$ as

$$e^{2\pi t}\prod_{n=1}^{\infty}\left[1-(-1)^n e^{-2\pi tn}\right]^{-24}=\vartheta_{00}(0,2it)^{-12}\eta^{-12}(2it)$$

A digression here. All the states are constrained to have $\Omega^2=1$ but as we saw in Chapter 4, the total eigenvalue of $\Omega|N,ij\rangle$ (where i,j are $SO(n)$ indices) which we call ω, is made up of two parts;

$$\omega=\omega_N s_a,\quad \omega_N=(-1)^N$$

where s_a is $+1$ when λ^a_{ij} is symmetric and -1 when λ^a_{ij} is anti-symmetric. So, at even mass levels, we should have λ^a_{ij} anti-symmetric and vice versa. However, in the calculation of the partition function above, we included ω_N part of ω only. We can now include s_a part as well. The symmetric matrices contribute with $s_a=1$ and vice versa. For the $Sp(k)=Sp(n/2)$ case (where $n=2k$ and where $Sp(k)$ contains $2k\times 2k$ matrices), this flips and we have the symmetric matrices contributing with $s_a=-1$ and vice versa. All of this is discussed in Chapter 4.
For $n\times n$ matrices, we have $n(n+1)/2$ symmetric matrices and $n(n-1)/2$ anti-symmetric states. So, the Chan Paton part of the partition function gives us the following factors;

$$SO(n):\frac{n(n+1)}{2}-\frac{n(n-1)}{2}=n$$
$$Sp(k):-\frac{n(n+1)}{2}+\frac{n(n-1)}{2}=-n$$

So, the total partition function becomes;

$$Z_{M_2}=\pm inV_{26}\int_0^{\infty}\frac{dt}{4t}\left(8\pi^2\alpha' t\right)^{-13}\vartheta_{00}(0,2it)^{-12}\eta(2it)^{-12}$$

where the upper sign is for $SO(n)$ and lower sign is for $Sp(k)$. We now do the modular transformation of this partition function. We set $s=\pi/4t$ and then, we get;

$$Z_{M_2}=\pm\frac{inV_{26}}{4\left(2\pi^3\alpha'\right)^{13}}\int_0^{\infty}ds s^{12}\vartheta_{00}\left(0,\frac{i\pi}{2s}\right)^{-12}\eta\left(\frac{i\pi}{2s}\right)^{-12}$$

Using the modular properties of $\vartheta_{00}(\nu,\tau)$ and $\eta(\tau)$ by setting $\nu=0$ and $\tau=2is/\pi$, we get the following;

$$\vartheta_{00}\left(0,\frac{i\pi}{2s}\right)=\sqrt{\frac{2s}{\pi}}\vartheta_{00}\left(0,\frac{2is}{\pi}\right),\quad \eta\left(\frac{i\pi}{2s}\right)=\sqrt{\frac{2s}{\pi}}\eta\left(\frac{2is}{\pi}\right)$$

and thus, the partition function becomes;

$$Z_{M_2}=\pm\tfrac{inV_{26}}{4(2\pi^3\alpha')^{13}}\tfrac{\pi^{12}}{2^{12}}\int_0^{\infty}ds\vartheta_{00}\left(0,\tfrac{2is}{\pi}\right)^{-12}\eta\left(\tfrac{2is}{\pi}\right)^{-12}=\pm 2\tfrac{inV_{26}2^{13}}{4\pi(8\pi^2\alpha')^{13}}\int_0^{\infty}ds\vartheta_{00}\left(0,\tfrac{2is}{\pi}\right)^{-12}\eta\left(\tfrac{2is}{\pi}\right)^{-12}$$

We now calculate the massless divergent part of this partition function. Using the expansion for η, we have;

$$\eta(\tau)=q^{1/24}(1-q-\ldots)=e^{\pi i\tau/12}\left(1-e^{2\pi i\tau}-\ldots\right)\Rightarrow\eta(2is/\pi)=e^{-s/6}\left(1-e^{-4s}-\ldots\right)$$
$$\Rightarrow\eta(2is/\pi)^{-12}=e^{2s}\left(1-e^{-4s}-\ldots\right)^{-12}=e^{2s}+12e^{-2s}-\ldots$$

Moreover, from the expansion of $\vartheta_{00}(0,\tau)$, we have;

$$\vartheta_{00}(0,\tau)=\sum_{n\in\mathbb{Z}}q^{n^2/2}=\sum_{n\in\mathbb{Z}}e^{\pi in^2\tau}=1+2\sum_{n=1}^{\infty}e^{\pi in^2\tau}=1+2e^{\pi i\tau}+2e^{8i\pi\tau}+\ldots$$

$$\Rightarrow\vartheta_{00}(0,2is/\pi)=1+2e^{-2s}+\ldots\Rightarrow\vartheta_{00}(0,2is/\pi)=\left(1+2e^{-2s}+\ldots\right)^{-12}=1-24e^{-2s}+\ldots$$

Therefore, we have;

$$\vartheta_{00}(0, 2is/\pi)^{-12}\eta(2is/\pi)^{-12} = \left(e^{2s} + 12e^{-2s} - \dots\right)\left(1 - 24e^{-2s} + \dots\right) = e^{2s} - 24 + \dots$$

So, the massless divergent part from Z_{M_2} is as follows:

$$\mp 2\frac{24inV_{26}2^{13}}{4\pi\left(8\pi^2\alpha'\right)^{13}}\int_0^\infty ds \tag{7.4.5}$$

Now, we can add all the massless divergent terms (which are also known as tadpole divergences) from C_2, K_2 and M_2 (recall that there is no massless divergent term from T^2). For the cylinder case, there will be an additional factor of 1/2 in (7.4.3) because of the projection operator. Adding (7.4.4), (7.4.5) and half of (7.4.3), we get;

$$\frac{24iV_{26}}{4\pi\left(8\pi^2\alpha'\right)^{13}}\Big(\underbrace{n^2}_{\text{cylinder}}\ \underbrace{\mp 2.2^{13}n}_{\text{Möbius strip}} + \underbrace{2^{26}}_{\text{Klein bottle}}\Big)\int_0^\infty ds = \frac{24iV_{26}}{4\pi\left(8\pi^2\alpha'\right)^{13}}\left(2^{13}\mp n\right)^2\int_0^\infty ds$$

This term does not vanish for the plus sign, which is the lower sign associated with $\mathrm{Sp}(k) = \mathrm{Sp}(n/2)$. The symplectic choice therefore has an uncanceled tadpole. For the minus sign, the divergence cancels for $n = 2^{13} = 8192$. The upper sign is the $SO(n)$ choice, so the divergence vanishes only for $SO(8192)$.

Chapter 6

Higher-order amplitudes and all-order consistency

This chapter establishes that string perturbation theory is well defined beyond the orders that can be computed explicitly. The method controls amplitudes through the geometry of moduli space. A genus-g surface with n punctures has complex dimension $3g-3+n$, and the physically important limits occur at its boundary. We introduce the plumbing-fixture construction, where a degenerating surface is formed by joining two smaller surfaces with $zw = q$. The integral over q becomes a Schwinger-parameter representation of a spacetime propagator with a complete set of physical states flowing through it. Unitarity is therefore recovered, order by order, from worldsheet geometry. The same analysis classifies the possible divergences of string theory: tachyon instabilities, infrared massless exchange, and tadpoles. Each has a distinct physical meaning and remedy. We then describe the structures needed beyond this controlled boundary analysis. These include Schottky and Fuchsian presentations of higher-genus surfaces, string field theory as the off-shell completion of sewing, the divergent large-order behaviour of the genus expansion that foreshadows D-instantons, and the high-energy and Hagedorn regimes where the spectrum as a whole controls the physics [10, 17].

All-order amplitude formula. A direct way to read this chapter is to start from one gauge-fixed formula and then study its boundary limits. For a closed oriented genus-g surface with n punctures,

$$\mathcal{A}_{g,n} = g_s^{2g-2+n} \int_{\mathcal{M}_{g,n}} \left\langle \prod_{r=1}^{3g-3+n} (b,\mu_r)(\bar{b},\bar{\mu}_r) \prod_{i=1}^{n} V_i \right\rangle_{\Sigma_g} . \tag{6.1}$$

Here μ_r are Beltrami differentials, tangent to moduli space, and

$$(b,\mu_r) = \frac{1}{2\pi} \int_{\Sigma_g} d^2z\, b_{zz}\, \mu^{z}_{\bar{z},r}. \tag{6.2}$$

The power of g_s follows from the Euler characteristic weight $g_s^{-\chi} = g_s^{2g-2}$, together with one coupling factor per external closed-string vertex. The formula is complete only after the conformal Killing vectors are treated separately: on the sphere three points are fixed and three $c\tilde{c}$ insertions appear; on the torus one translation zero mode must be fixed or averaged; for $g \geq 2$ there are no conformal Killing vectors. This is the same Faddeev–Popov logic used in Chapters 5–7, now written in a way that makes the moduli explicit [10, 17, 121].

From a plumbing fixture to a propagator. Cut two small discs with local coordinates z and w, then sew the annuli by

$$z\,w = q, \qquad q = e^{-t+i\theta}, \qquad |q| < 1. \tag{6.3}$$

Large t is a long thin cylinder. Evolution along this cylinder is

$$\int_0^\infty dt\, e^{-t(L_0+\bar{L}_0-2)} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(L_0-\bar{L}_0)}. \tag{6.4}$$

The θ-integral imposes level matching $L_0 - \bar{L}_0 = 0$, while the t-integral is the Schwinger parameter of the spacetime propagator. This proves the factorization claim directly. A degeneration boundary of moduli space becomes propagation of a complete set of CFT states. After BRST reduction that complete set is the physical string Hilbert space, so the no-ghost theorem is what makes the pole residue a positive-norm unitarity sum.

Four-point pole and factorization. In the $z \to 0$ degeneration of a four-point sphere correlator,

$$V_1(z,\bar{z})V_2(0) \sim \sum_I C_{12I} z^{h_I-h_1-h_2} \bar{z}^{\bar{h}_I-\bar{h}_1-\bar{h}_2} V_I(0). \tag{6.5}$$

For physical closed-string external vertices $h_i = \bar{h}_i = 1$, the local integral contains

$$\int_{|z|<\epsilon} d^2z\, z^{h_I-2}\bar{z}^{\bar{h}_I-2}. \tag{6.6}$$

With $z = \rho e^{i\theta}$, the angular integral gives $\delta_{h_I-\bar{h}_I}$, and the radial integral becomes the first factor in Eq. (6.4). Therefore near an s-channel pole one obtains

$$\mathcal{A}_4(1,2,3,4) \underset{s\to M_N^2}{\sim} \sum_{I\in\mathcal{H}_{\text{phys}}(N)} \mathcal{A}_3(1,2,I)\,\frac{i}{s-M_N^2+i\epsilon}\,\mathcal{A}_3(I,3,4). \tag{6.7}$$

This is the precise statement behind "worldsheet duality": the same punctured sphere has $z \to 0$, $z \to 1$, and $z \to \infty$ boundary regions, giving the s-, t-, and u-channel expansions of one integral.

Classification of degeneration divergences. A degeneration integral

$$\int_{|q|<\epsilon} \frac{d^2q}{|q|^2}\, q^{h-1}\bar{q}^{\bar{h}-1} \tag{6.8}$$

must be interpreted case by case. A tachyon gives a power divergence and means the perturbative vacuum is unstable. A massless on-shell state gives an infrared effect from long-range propagation. A massless tadpole is stronger: it is a one-point source, so the assumed background is not a solution of the spacetime equations. This is why type-I tadpole cancellation is not just a partition-function trick; it is the worldsheet version of setting the massless source term in the effective field equations to zero [10, 352].

High-energy and thermal regimes. For fixed-angle hard scattering, Stirling's approximation to the Veneziano beta function gives

$$\mathcal{A}(s,t) \sim \exp[-\alpha' s\, f(\theta)], \qquad f(\theta) = -\sin^2\frac{\theta}{2}\log\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\log\cos^2\frac{\theta}{2}. \tag{6.9}$$

For Regge scattering, $s \to \infty$ at fixed t, the ratio of gamma functions gives $\mathcal{A} \sim s^{\alpha(t)}$, with linear trajectory $\alpha(t) = \alpha(0) + \alpha' t$. At finite temperature, Euclidean time is compact with $R_0 = \beta/(2\pi)$, so closed strings also have thermal winding:

$$p^0_{L,R} = \frac{n}{R_0} \pm \frac{wR_0}{\alpha'}. \tag{6.10}$$

For the closed bosonic string the $n = 0, w = 1, N = \tilde{N} = 0$ thermal winding mode has

$$M^2_{\text{th}}(\beta) = -\frac{4}{\alpha'} + \frac{\beta^2}{4\pi^2\alpha'^2}, \qquad \beta_H = 4\pi\sqrt{\alpha'}. \tag{6.11}$$

This is the thermal-scalar derivation of the Hagedorn bound: above $T_H = 1/\beta_H$, the naive canonical ensemble around the zero-temperature vacuum is no longer stable [14, 45, 48].

Noncritical strings and the Liouville mode. If the matter CFT does not cancel the bc-ghost anomaly, the Weyl factor cannot simply be discarded. It becomes the Liouville field φ, with

$$c_{\text{matter}} + c_{\text{L}} + c_{bc} = 0, \qquad c_{\text{L}} = 1 + 6Q^2, \qquad c_{bc} = -26. \tag{6.12}$$

A matter primary $\mathcal{O}_\Delta$ is dressed to a physical integrated vertex by

$$V = \int d^2z\, e^{2a\varphi}\mathcal{O}_\Delta, \qquad a(Q-a) + \Delta = 1. \tag{6.13}$$

The last equation is just the $(1,1)$ condition for the dressed operator. Thus "away from $D = 26$" is not an uncontrolled off-shell deformation; conformal invariance is restored by adding the Liouville degree of freedom [10, 42].

6.0.1 General tree level amplitudes

The same formula applies to tree and loop amplitudes once the genus is specified. For a genus-g closed surface with n punctures,

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{r=1}^{3g-3+n} (b,\mu_r)(\bar{b},\bar{\mu}_r) \prod_{i=1}^{n} V_i \right\rangle_g . \tag{6.14}$$

The Beltrami differential μ_r is a tangent vector to moduli space. The insertion (b, μ_r) is the ghost factor associated with that tangent direction. For $g = 0$, the formula is supplemented by fixing the conformal Killing group; for $g \geq 2$, the dimension $3g - 3 + n$ gives the number of complex moduli directly.

6.0.2 Three point amplitudes

On the sphere the conformal Killing group $PSL(2, \mathbb{C})$ has enough freedom to fix three insertion points. A closed-string three-point amplitude can therefore be written as

$$\mathcal{A}_3 = \langle c\tilde{c}V_1(z_1)\, c\tilde{c}V_2(z_2)\, c\tilde{c}V_3(z_3)\rangle_{S^2} . \tag{6.15}$$

The three $c\tilde{c}$ factors absorb the ghost zero modes associated with the fixed conformal Killing transformations. For tachyon vertices this reduces to the momentum-conserving delta function times the chosen normalization constant. For massless vertices, Wick contractions of ∂X and $\bar{\partial}X$ with the exponentials give the polarization-dependent cubic couplings.

6.0.3 Four point amplitudes and world-sheet duality

The four-point sphere amplitude has one complex modulus after three insertion points are fixed. For closed-string tachyons,

$$\mathcal{A}_4 = C_{S^2} g_c^2 \int_{\mathbb{C}} d^2z\, |z|^{-\alpha' s/2-4}\, |1-z|^{-\alpha' t/2-4}. \tag{6.16}$$

The regions $z \to 0$, $z \to 1$, and $z \to \infty$ are the s-, t-, and u-channel degeneration limits of the same integral. A single worldsheet integral therefore contains the three channel expansions that would be drawn as separate diagrams in point-particle perturbation theory.

6.0.4 Unitarity of the four point amplitude

The pole expansion of the four-point amplitude gives the unitarity test. Near an s-channel pole,

$$\mathcal{A}_4(1,2,3,4) \underset{s\to M_N^2}{\sim} \sum_{I\in\mathcal{H}_{\text{phys}}(N)} \mathcal{A}_3(1,2,I)\, \frac{i}{s - M_N^2 + i\epsilon}\, \mathcal{A}_3(I,3,4). \tag{6.17}$$

The residue is a sum over physical string states at level N. The no-ghost theorem is therefore used in loop and tree amplitudes in the same way: it guarantees that the intermediate states appearing in residues have a positive-norm physical interpretation [17, 28].

6.1 Higher genus Riemann surfaces

For a closed orientable surface of genus $g \geq 2$ with n punctures, the complex dimension of moduli space is

$$\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3g - 3 + n. \tag{6.18}$$

The amplitude takes the same form as Eq. (6.14),

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{r=1}^{3g-3+n} (b,\mu_r)(\bar{b},\bar{\mu}_r) \prod_{i=1}^{n} V_i \right\rangle_g . \tag{6.19}$$

The b-ghost insertions are the Faddeev–Popov measure for the moduli directions. When a higher-genus surface approaches a degeneration boundary, the sewing parameter of the long tube becomes the Schwinger parameter of an intermediate string propagator. This is the geometric origin of loop factorization [10, 17].

6.1.1 Schottky groups

A Schottky description presents a genus-g Riemann surface as a space obtained from a domain in the Riemann sphere by a freely generated subgroup of $PSL(2,\mathbb{C})$. Each generator is a Möbius map

$$L_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, \qquad a_i d_i - b_i c_i = 1, \qquad i = 1, \ldots, g. \tag{6.20}$$

The multipliers and fixed points of the L_i give coordinates on a patch of moduli space. This parametrization is useful for amplitudes because degeneration limits become transparent: a multiplier going to zero is a long thin tube, so the Schottky parameter is a direct sewing parameter for the factorization channel. The same sewing parameter appears in the pole formula Eq. (6.17): near the boundary of moduli space, the surface develops a long tube and the amplitude factorizes through physical string states. [16].

6.1.2 Fuchsian groups

A Fuchsian description realizes a Riemann surface as a space obtained from the upper half-plane by a discrete subgroup of $PSL(2,\mathbb{R})$. Locally the action is

$$z \longmapsto \gamma(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1. \tag{6.21}$$

The example is useful because it replaces a drawn surface by a precise uniformization. In the amplitude chapters the same idea appears physically: changing the description of the surface must not change the factorization channels or the BRST-invariant content of the amplitude. [10, 17].

6.1.3 The period matrix

A genus-g Riemann surface is controlled by its periods. Choosing canonical cycles A_i, B_i and holomorphic one-forms ω_i normalized by $\oint_{A_i} \omega_j = \delta_{ij}$, the period matrix is

$$\Omega_{ij} = \oint_{B_i} \omega_j, \qquad \Omega^T = \Omega, \qquad \operatorname{Im}\Omega > 0. \tag{6.22}$$

Theta functions and lattice sums are functions of Ω. At higher genus, modular covariance means invariance under the mapping-class group acting on the period matrix. It is a condition on the amplitude itself, because theta functions and lattice sums depend directly on Ω. [10, 17, 58, 359].

6.2 Sewing and cutting worldsheets

Sewing is the constructive converse of degeneration, and it explains *why* the master formula

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{r=1}^{3g-3+n} (b, \mu_r)(\bar{b}, \bar{\mu}_r) \prod_{i=1}^{n} V_i \right\rangle_g \tag{6.23}$$

holds at every genus once it holds for the three-punctured sphere. Any surface in $\mathcal{M}_{g,n}$ can be assembled by gluing pairs of punctures on lower surfaces with the plumbing map $zw = q$ of Eq. (6.3); the gluing supplies exactly one complex modulus q per seam and one $(b,\mu)(\bar{b},\bar{\mu})$ pair for it, so the counting $3g - 3 + n$ is reproduced seam by seam, a pair-of-pants decomposition has $2g - 2 + n$ vertices and $3g - 3 + n$ seams, matching the ghost insertions one to one. The nontrivial content is that the sewn measure agrees with the intrinsic Weil–Petersson-type measure on $\mathcal{M}_{g,n}$ near the boundary and continues it to the interior; BRST invariance guarantees that different decompositions of the same surface give the same amplitude, which is the worldsheet version of the field-theory statement that different Feynman-diagram channels come from one action [10, 17].

6.2.1 A graphical decomposition

No picture is needed here: the decomposition is a pair-of-pants statement. A genus-g surface with n punctures can be cut along

$$3g-3+n \tag{6.24}$$

nonintersecting closed curves into $2g-2+n$ three-holed spheres. Each cut curve carries a length/twist or, in the conformal sewing language, a complex parameter q. The string interpretation is that each internal curve is an intermediate closed-string propagator, so the decomposition is a geometric version of inserting a complete set of states. The diagram is replaced by combinatorial input, the topology is encoded by the number of cuts and by the sewing parameters. [10].

6.3 Sewing and cutting CFTs

Cutting a CFT correlator along a circle turns it into a state in the Hilbert space; sewing two circles inserts the inverse BPZ metric and integrates over the gluing parameter. In formulas the local operation is

$$\mathbf{1}_{\mathcal{H}} = \sum_{\alpha,\beta} |\alpha\rangle\, G^{\alpha\beta}\, \langle\beta|, \qquad \int_{|q|<1} \frac{d^2q}{|q|^2}\, q^{L_0}\bar{q}^{\bar{L}_0}. \tag{6.25}$$

The second factor is the closed-string propagator written in worldsheet language. Therefore factorization, modular integration, and BRST cohomology have to be discussed together: a BRST-exact state inserted across the cut must decouple, while a physical state gives a genuine pole or threshold. [10, 40].

6.3.1 General amplitudes

At the level of the full CFT, cutting and sewing upgrade Eq. (6.23) into a set of consistency conditions on the theory itself, and it is worth recording what they demand. Writing the amplitude once more,

$$\mathcal{A}_{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{r=1}^{3g-3+n} (b,\mu_r)(\bar{b},\bar{\mu}_r) \prod_{i=1}^{n} V_i \right\rangle_g, \tag{6.26}$$

the requirement that every way of cutting the surface gives the same answer reduces, after unwinding the sewing definitions, to exactly three local conditions on the matter CFT: associativity of the operator product expansion (equality of s- and t-channel cuttings of the four-punctured sphere), modular covariance of one-point functions on the torus, and the existence of a single nondegenerate BPZ pairing used at every seam. These are Sonoda's sewing constraints, and a CFT satisfying them defines consistent string amplitudes to *all* orders, there is no separate consistency condition at genus fifty. This is the precise sense in which perturbative string theory is determined by tree-level and one-loop information, and it is why the modular-invariance checks of Chapters 7 and 10 carry so much weight [10, 17].

6.3.2 String divergences

String divergences come from boundaries of moduli space, not from arbitrary short-distance regions of a fixed Feynman graph. Near a separating degeneration the local integral has the form

$$\int_{|q|<\epsilon} \frac{d^2q}{|q|^2}\, q^{h-1}\bar{q}^{\bar{h}-1}, \tag{6.27}$$

where $(h,\bar{h})$ are the weights of the state propagating through the tube. Tachyons make this integral power divergent, massless states give infrared logarithms or tadpoles, and massive states are harmless. The same diagnostic used earlier for the cylinder, Klein bottle, and Möbius strip applies here: a one-loop divergence in one channel becomes tree-level exchange of a physical closed-string state in the dual channel [40].

Equation (6.8) separates tachyon divergences, massless infrared exchange, and massless tadpoles as different physical statements. The last case is the one that becomes a background equation of motion and explains why type-I tadpole cancellation is a consistency condition, not a decorative partition-function identity.

The same consistency conditions can be imposed in singular limits of the sewing problem. They are the ones used above,

$$Q_B^2 = 0, \qquad Z(\tau+1) = Z(\tau), \qquad Z(-1/\tau) = Z(\tau),$$

$$\mathcal{A}_{g,n} \xrightarrow[q\to 0]{} \sum_{\phi\in\mathcal{H}_{\text{phys}}} \mathcal{A}^{\phi}_{g_1,n_1+1} \frac{dq\, d\bar{q}}{q\bar{q}}\, q^{h_\phi}\bar{q}^{\bar{h}_\phi} \mathcal{A}^{\phi}_{g_2,n_2+1}.$$

Supercosets, AdS_3 integrability, the RNS and Berkovits–Vafa–Witten formalisms, and nonrelativistic string limits are settings where they continue to hold [225]. Under a critical electric or R–R scaling the finite energy is a brane tension minus a divergent rest energy,

$$E_{\text{finite}} = \lim_{\epsilon\to 0}\big(T_{\text{brane}}(\epsilon) - q\, C(\epsilon)\big),$$

and the surviving light sector is tensionless, Carrollian, ambitwistor, Matrix-theoretic, or spin-matrix-like, according to the scaling [222, 224, 226, 230, 241].

Ambitwistor strings give a chiral degeneration of the sewing formula. In the tensile theory a separating tube contributes

$$\int \frac{d^2q}{|q|^2}\, q^{h-1}\bar{q}^{\bar{h}-1},$$

so all physical levels with $h = \bar{h}$ can propagate. In the ambitwistor limit the worldsheet constraint is instead $P^2(z) = 0$. The degeneration localizes on the scattering equations,

$$\text{Res}_{q=0}\, \frac{dq}{q}\, \bar{\delta}\big(P^2\big) \quad\Longrightarrow\quad k_I^2 = 0$$

for the internal momentum k_I. Ohmori's analysis of bosonic and RNS ambitwistor worldsheet moduli spaces verifies factorization on this massless pole [221]. A recent BRST-cohomology computation reaches the same conclusion algebraically: the spectrum is massless, but it contains an additional vector and does not give the Hilbert space of an ordinary unitary Maxwell theory [242]. Factorization therefore survives the chiral limit, while the physical state space changes.

6.4 String Field Theory

String field theory is the off-shell completion of the sewing construction. In Witten's cubic open string field theory the string field Ψ is a ghost-number-one state and the action is

$$S_{\text{open}} = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right), \tag{6.28}$$

with gauge transformation

$$\delta\Psi = Q_B\Lambda + \Psi * \Lambda - \Lambda * \Psi. \tag{6.29}$$

The plus sign between the kinetic and cubic terms is essential; without it this is not the standard cubic open-string field theory action. It makes precise what was only implicit in the Polyakov amplitude: off shell one must choose local coordinates around punctures and an allowed sewing prescription. The BRST charge, BPZ inner product, and star product are therefore the algebraic version of cutting and gluing worldsheets. [22, 215].

In string field theory an off-shell deformation is measured by the failure of BRST invariance. If the worldsheet CFT is perturbed by a state Φ, the conformal condition reads

$$Q_B\Phi + \frac{1}{2}[\Phi, \Phi] + \cdots = 0.$$

Away from the conformal locus the left-hand side is not set to zero by changing k^2 alone; the BV vertices and the decomposition of moduli space must also change [240]. Pure-spinor calculations give the corresponding on-shell test. Tree-level massive three-point vertices are fixed by

$$Q_B V_i = 0, \qquad \delta_{\text{Mobius}}\langle V_1 V_2 V_3\rangle = 0,$$

and closed-string two-point prescriptions require nonstandard ghost-number assignments because the closed-string cohomology is more constrained than the tensor product of two open-string cohomologies [232, 235]. The same operator Q_B therefore controls physical states, vertices, and off-shell deformations.

On the moduli space of classical open-SFT solutions, Choi constructs a two-form connection whose curvature is built from the star product and the open-string integration map [268]. In schematic form,

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge_* \mathcal{A}, \qquad \mathcal{A} \sim \int_{\text{open}} \Psi^{-1} * d\Psi,$$

and in suitable boundary-condition moduli spaces this curvature is identified with the closed-string B-field. The star product in Eq. (6.28) therefore transports background fields as well as defining the cubic vertex. For AdS amplitudes, the flat-space monodromy relation

$$\sum_\sigma e^{i\pi\alpha' k\cdot k_\sigma}\, \mathcal{I}_{\text{open}}(\sigma) = 0$$

is replaced by a relation among curved-space worldsheet integrals dressed by multiple polylogarithms; the corresponding closed-string integrals obey a KLT-type factorization with single-valued analogues [288]. Open/closed factorization persists, now with the AdS worldsheet integral as the kernel in place of the flat Veneziano one.

Hull's 2025 superstring-field-theory action reduces to Sen's superstring field theory in an appropriate limit and reproduces the same physical superstring amplitudes, but introduces a shift symmetry that compensates background changes by shifting the string fields [301]. This gives a version of background independence inside the doubled-field structure of superstring field theory: the physical interacting sector remains coupled as before, while the additional sector acquires self-interactions yet stays decoupled from physical amplitudes. It sharpens the distinction made here between a worldsheet on-shell prescription and an off-shell field theory over the space of backgrounds.

Maccaferri, Ruffino, and Vosmera construct gauge-invariant free open and closed SFT kinetic terms on target spaces with boundary by promoting boundary gauge parameters to dynamical boundary modes and writing the corresponding BV master action [318]. Kaja, Maccaferri, Portugal, and Vosmera then use this boundary formulation to define gauge-invariant observables analogous to Brown–York charges; the observables arise from boundary tadpoles and are associated with isometries of the SFT gauge group around a chosen background [348]. In an off-shell SFT with boundary, the same boundary term that gives a tadpole, an equation of motion, can instead become a gauge-invariant observable measuring how the background is sourced.

6.5 Large order behaviour

Large-order behavior is compared with the genus expansion introduced in the Polyakov path integral. The perturbative free energy has the formal structure

$$F(g_s) = \sum_{h=0}^{\infty} g_s^{2h-2} F_h. \tag{6.30}$$

The sum over genera is not expected to be an ordinary convergent expansion. Its asymptotic nature is physically meaningful because nonperturbative objects, especially D-branes, appear as effects invisible at any fixed genus. [10, 38, 351, 352].

Large-order behavior is not only a genus-expansion issue. Kervyn and Stieberger analyze tree-level n-point amplitudes in the $\alpha' \to \infty$ regime using saddle points, finite-difference equations, Mellin–Barnes representations, twisted intersection theory, and Lefschetz thimbles [333]. The resulting asymptotic $1/\alpha'$ series is organized by Bernoulli numbers, not the multiple zeta values of the low-energy $\alpha' \to 0$ expansion, and resurgence packages the analytic continuation between kinematic regions. Thus the high-energy softness in Eq. (6.32) has a nonperturbative transseries refinement, not just a leading saddle exponent.

6.6 High energy and high temperature

Thermal string theory is obtained by compactifying Euclidean time on a circle of radius $R_0 = \beta/(2\pi)$. The thermal zero modes have

$$p^0_{L,R} = \frac{n}{R_0} \pm \frac{wR_0}{\alpha'}. \tag{6.31}$$

This differs from ordinary finite-temperature field theory because strings can wind around the Euclidean-time circle. The Hagedorn behavior is then visible in the same momentum-winding spectrum that appears for spatial circle compactification. [10, 10, 350].

6.6.1 Hard scattering

Fixed-angle hard scattering sends $s, t \to \infty$ with t/s held fixed. Unlike field-theory power counting, string amplitudes are exponentially soft in this regime,

$$\mathcal{A}(s,t) \sim \exp[-\alpha' s\, f(\theta)], \tag{6.32}$$

for a positive angular function f. This behavior is a direct signal of finite string length and is contrasted with the Regge limit, where the exchanged trajectory is the main organizing idea. [10, 17].

6.6.2 Regge scattering

The Regge limit keeps t fixed while $s \to \infty$. String amplitudes then organize themselves by trajectories,

$$\alpha(s) = \alpha(0) + \alpha' s. \tag{6.33}$$

The physical conclusion is that high-energy exchange is not dominated by a single particle. It is dominated by an entire tower of string states lying on the same trajectory. Worldsheet gauge invariance and conformal invariance thereby package infinitely many spacetime particles into one analytic object. [10, 17].

The same tower can be generated algorithmically. Basile and Markou construct closed-string trajectories from an open-string seed, dressing the left- and right-moving sectors by a symplectic-algebra "tiling" procedure; Howe duality produces the candidates before the physical constraints are imposed [341]. The closed-string spectrum is therefore not the unrestricted product $\mathcal{H}_{\text{open}} \otimes \mathcal{H}_{\text{open}}$. It is the constrained subspace

$$\mathcal{H}_{\text{closed}} = \{\, |\psi\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R : (L_0 - \bar{L}_0)|\psi\rangle = 0, \quad Q_B|\psi\rangle = 0 \,\}/Q_B\mathcal{H}.$$

Level matching and BRST cohomology remain the decisive steps after the trajectory candidates have been generated.

6.6.3 High temperature and the Hagedorn point

The thermal spectrum of Eq. (6.31) contains one state that field theory has no analogue of: the mode winding once around Euclidean time with no momentum and no oscillators. Its mass follows from the closed-string formula with $n = 0$, $w = 1$, $N = \widetilde{N} = 0$,

$$M^2_{\text{th}}(\beta) = -\frac{4}{\alpha'} + \left(\frac{\beta}{2\pi\alpha'}\right)^2, \tag{6.34}$$

so this "thermal scalar" is heavy at low temperature and becomes massless at $\beta_H = 4\pi\sqrt{\alpha'}$, tachyonic beyond. Since a winding tachyon signals an instability of the assumed background, the canonical ensemble around the flat vacuum ceases to exist above $T_H = 1/\beta_H$. The same conclusion follows from state counting: the single-string density of states grows as $\rho(M) \sim M^{-a}e^{\beta_H M}$, so the thermal sum $\int dM\, \rho(M)e^{-\beta M}$ diverges for $\beta < \beta_H$, the exponential growth of the spectrum, visible already in the η^{-24} expansion of Chapter 5, and the winding-mode instability are two descriptions of one phenomenon. Whether T_H marks a limiting temperature or a phase transition to different degrees of freedom is a question perturbation theory cannot settle [14, 48].

In the Rindler construction the tensionless limit is obtained from increasingly accelerated worldsheets: as the worldsheet horizon is approached, the effective tension goes to zero and a null string appears [222]. The Virasoro-sandwich formulation imposes physicality as a matrix element condition,

$$\langle \phi | L_n | \psi \rangle = 0, \qquad \langle \phi | \bar{L}_n | \psi \rangle = 0,$$

between physical states, and it also produces parity and time-reversal sectors relevant for accelerated worldsheets [226]. Edge-mode calculations near the Minkowski–Rindler horizon sum over the string tower and keep the one-loop expression modular invariant [233]. Hagedorn growth, Virasoro constraints, and horizon thermality are therefore linked by the same worldsheet state sum.

The open-string Hagedorn phase enters brane-antibrane cosmology through the same divergence as in Eq. (6.34). For open strings on an unstable brane system,

$$Z_{\text{open}}(\beta) \sim \int_0^\infty dE\, \rho_{\text{open}}(E) e^{-\beta E}, \qquad \rho_{\text{open}}(E) \sim E^{-a} e^{\beta_H E}.$$

The integral ceases to converge for $\beta \le \beta_H$. In perturbatively stabilized brane-antibrane inflation, tachyon condensation releases energy into open-string degrees of freedom before a conventional radiation fluid is guaranteed to exist [343]. The instability is therefore a change in the relevant degrees of freedom: the thermal scalar or open-string tachyon signals that the expansion around the original brane background has ended.

A recent review identifies the tensionless limit with a Carrollian limit on the worldsheet, in which the two Virasoro algebras are replaced by the two-dimensional conformal Carroll (equivalently BMS_3) algebra [267]. High-energy scattering can then be formulated intrinsically in the induced-vacuum null-string theory: the ultra-high-energy tensile string approaches a null worldsheet, and amplitudes are organized by the Carrollian symmetry instead of the fixed-tension oscillator expansion [277]. Carrollian worldsheets admit two Weyl-type scalings, so a consistent null-string action should gauge the extra Carroll-Weyl symmetry; a Hamiltonian analysis then enlarges the BMS_3 constraint algebra by a weight-one operator [283, 284]. Finite-lifetime worldsheets provide another route, where shrinking a causal diamond to the birth of the string produces a global ultra-local Carrollian phase [279]. Thus the $T \to 0$ limit is not just the formal removal of α'^{-1}: it changes the gauge symmetry, the constraint algebra, and the admissible worldsheet phase space.

6.7 Low temperature and noncritical strings

At low temperature, $\beta \gg \beta_H$, the thermal winding state in Eq. (6.11) is massive and the canonical expansion is controlled by the ordinary low-lying string spectrum. Noncritical strings answer a different question: what should be done when the matter CFT does not cancel the Weyl anomaly? The correct treatment keeps the Weyl factor as the Liouville mode instead of treating the gauge-fixed path integral as if it were still conformal [10, 42].

6.7.1 Non critical strings

Noncritical strings are what happens when the Weyl anomaly is not cancelled by the matter CFT alone. Instead of discarding the Weyl factor, one keeps the Liouville mode φ and demands total central charge balance:

$$c_{\text{matter}} + c_{\text{L}} + c_{bc} = 0, \qquad c_{\text{L}} = 1 + 6Q^2, \qquad c_{bc} = -26. \tag{6.35}$$

A matter primary $\mathcal{O}_\Delta$ is then dressed to a physical integrated vertex by

$$V = \int d^2 z\, e^{2a\varphi} \mathcal{O}_\Delta, \qquad a(Q - a) + \Delta = 1. \tag{6.36}$$

It gives a controlled meaning to the phrase “off shell” or “away from $D = 26$”: the extra Liouville field restores conformal invariance instead of leaving the Weyl factor as an unaccounted gauge mode. The construction is governed by the anomaly coefficient in Eq. (1.16). [10].

Complex Liouville strings use the same central-charge cancellation as Eq. (6.35). The matter sector consists of two conjugate Liouville CFTs,

$$c_1 = 13 + i\lambda, \qquad c_2 = 13 - i\lambda, \qquad c_{\text{matter}} = c_1 + c_2 = 26.$$

Including the bc ghosts gives $c_{\text{matter}} + c_{bc} = 26 - 26 = 0$, so the Weyl anomaly cancels although the individual CFTs are complex [227]. The same central-charge underlies recent $c = 1$ matrix-integral descriptions, where amplitudes are computed through intersection numbers on moduli space, and super-Virasoro minimal strings, where spacelike and timelike super-Liouville sectors distinguish the 0A and 0B choices [237, 239]. Liouville dressing therefore remains the explicit mechanism that restores conformal invariance when the matter central charge is not critical by itself.

The $c = 1$ matrix model gives a quantitative form of the Liouville discussion. In the double-scaled description, eigenvalue fermions occupy a phase-space droplet, and the closed-string collective field is the fluctuation of its density. Time-dependent Liouville walls correspond to time-dependent deformations of this droplet. Finite-N corrections and the infrared wall smooth singular behavior that appears in the strict double-scaling limit [263]. In the adjoint sector, the matrix quantum mechanics produces a Regge-like scaling

$$\Delta^2 \sim \frac{n}{\alpha'},$$

which is interpreted on the string side as an excitation of a folded open string [254]. Liouville line defects add a localized cosmological constant; weak coupling shifts open-channel spectra, while strong coupling glues hyperbolic surfaces across the defect [253]. Crosscap amplitudes in complex Liouville strings also match an $SL(2, \mathbb{Z})$ family of semiclassical dS_3 observer saddles [260]. In each case matrix eigenvalues, defects, and crosscaps give computable deformations of the same Liouville path integral.

Minimal superstrings give an explicit test of the open/closed degeneration rules. Barman, Kaushik, Mahajan, Murdia, and Sen compute disk and annulus contributions induced by ZZ instanton boundary conditions in type-0 minimal superstrings, where open-string-channel degenerations must be regulated using open-closed string field theory and picture-changing choices have to be specified [342]. This is a compact noncritical-string example of the general sewing principle used above: boundary instantons generate closed-string effects, but the divergent corners of moduli space still have to be assigned by a consistent SFT prescription.

Chapter 7

Toroidal compactification and T-duality

This chapter asks what a compact dimension means to a string, and finds that the answer differs from the point-particle case. A field on a circle has quantized momenta n/R and nothing more; a closed string in addition winds, $X(\sigma+2\pi) = X(\sigma) + 2\pi wR$, and the two quantum numbers enter the mass formula symmetrically. The spectrum, and, we argue, the full conformal field theory, is invariant under $R \leftrightarrow \alpha'/R$ with $n \leftrightarrow w$. We develop this in stages: the circle, several dimensions with the Narain lattice (where modular invariance forces the momentum lattice to be even and self-dual, the arithmetic later responsible for the heterotic gauge groups), orbifolds with their obligatory twisted sectors, and open strings, where T-duality converts a Wilson line into the position of a hypersurface on which string endpoints live. That hypersurface is a dynamical object: the D-brane. We derive its Dirac–Born–Infeld action from gauge invariance of the worldsheet couplings and compute its tension from the cylinder amplitude, completing the identification [10, 350].

7.1 Toroidal compactification in field theory

We consider a $D = d+1$ dimensional theory with one dimension compactified $x_d \sim x_d + 2\pi R$. The five-dimensional metric is G^D_{MN} and we adopt a particular convention for the metric as follows:

$$ds^2 = G^D_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\mu + G_{dd}\left(dx^d + A_\mu dx^\mu\right)^2 \tag{8.1.1}$$

Comparing the two-line elements, we have the following correspondences;

$$G_{dd} = G^D_{dd}, G_{\mu d} = \frac{G^D_{\mu d}}{A_\mu}, G_{\mu\nu} = G^D_{\mu\nu} - G^D_{dd} A_\mu A_\nu$$

All the fields depend on non-compact coordinates only and thus, the metric 8.1.1 is invariant in x^d translations and it is also manifestly invariant in $x'^\mu\left(x^\nu\right)$ reparametrizations. The expanded form of the metric is as follows:

$$\left(G_{\mu\nu} + G_{dd} A_\mu A_\nu\right) dx^\mu dx^\nu + G_{dd} dx^d dx^d + 2G_{dd} A_\mu dx^d dx^\mu$$

This metric is invariant under the reparametrizations of the form

$$x^d \to x^d + \lambda\left(x^\mu\right)$$

if A_μ 's transform as follows.[1]

$$A_\mu \to A_\mu - \partial_\mu \lambda$$

If we include x^d dependence, then the fourier series for a scalar field $\phi\left(x^M\right)$ is written as follows:

$$\phi\left(x^M\right) = \sum_{n\in\mathbb{Z}} \phi_n\left(x^\mu\right) e^{inx^d/R}$$

[1] Substitute $dx^d \to dx^d + \partial_\mu\lambda\, dx^\mu$ into $dx^d + A_\mu dx^\mu$. The metric keeps the same Kaluza–Klein form provided the one-form $A = A_\mu dx^\mu$ transforms as $A \to A - d\lambda$, or $A_\mu \to A_\mu - \partial_\mu\lambda$ in components. Thus a higher-dimensional coordinate transformation becomes an abelian gauge transformation in the lower-dimensional theory.

and then, the D dimensional wave equation becomes;

$$\partial_M \partial^M \phi\left(x^M\right) = 0 \Rightarrow \phi\left(x^M\right) = \sum_{n\in\mathbb{Z}} \partial_\mu \partial^\mu \phi_n\left(x^\mu\right) - \frac{n^2 G^{dd}}{R^2}\phi_n\left(x^\mu\right) e^{inx^d/R} = 0$$

$$\Rightarrow \partial_\mu \partial^\mu \phi_n - \frac{n^2 G^{dd}}{R^2}\phi_n = 0 \Rightarrow \partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2}\phi_n = 0$$

where in the last step $G_{dd} = 1$ has been set for simplicity. Comparing this equation with the Klein–Gordon equation shows that ϕ_n has mass squared n^2/R^2. This is the Kaluza–Klein tower. The massless sector contains the lower-dimensional metric, the Kaluza–Klein gauge field, and the radius modulus; their couplings are obtained by reducing the higher-dimensional Einstein–Hilbert action and integrating over the circle.

7.2 Toroidal compactification in CFT

In CFT language a compact spatial direction is described by a compact boson,

$$X \sim X + 2\pi R. \tag{7.1}$$

The closed-string boundary condition allows winding,

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi w R, \qquad w \in \mathbb{Z}, \tag{7.2}$$

and single-valuedness of the wavefunction on the circle gives quantized center-of-mass momentum $p = n/R$. Splitting the solution into left- and right-moving pieces gives

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}. \tag{7.3}$$

A compact-boson vertex operator therefore has the form

$$V_{p_L,p_R}(z,\bar{z}) =: \exp(ip_L X_L(z) + ip_R X_R(\bar{z})) :, \tag{7.4}$$

with conformal weights

$$h = \frac{\alpha' p_L^2}{4} + N, \qquad \bar{h} = \frac{\alpha' p_R^2}{4} + \widetilde{N}. \tag{7.5}$$

The spin $h - \bar{h}$ must be an integer. This gives the level-matching condition

$$N - \widetilde{N} + nw = 0. \tag{7.6}$$

The one-loop partition function sums over the full momentum-winding lattice,

$$Z_R(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{\alpha'}{4}p_L^2} \bar{q}^{\frac{\alpha'}{4}p_R^2}. \tag{7.7}$$

Under $R \mapsto \alpha'/R$ together with $n \leftrightarrow w$, the pair (p_L, p_R) is mapped to $(p_L, -p_R)$ and the partition function is unchanged. Locality of the compact-boson vertex operators also imposes integrality of the lattice pairing: the phase obtained by taking one vertex around another is trivial only when

$$l \circ l' \in \mathbb{Z}, \qquad l \circ l \in 2\mathbb{Z}. \tag{7.8}$$

This is the CFT origin of the even self-dual Narain lattice in toroidal compactification [10, 350].

7.3 Closed strings and T-duality

For closed strings on a circle, the mass formula is

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2), \qquad N - \widetilde{N} + nw = 0. \tag{7.9}$$

It is invariant under

$$R \longleftrightarrow \frac{\alpha'}{R}, \qquad n \longleftrightarrow w, \qquad p_L \longrightarrow p_L, \qquad p_R \longrightarrow -p_R. \tag{7.10}$$

The physical meaning is simple: the energy of momentum modes grows as the circle becomes small, while the energy of winding modes decreases. The full string spectrum contains both, so the two descriptions are equivalent.

At the self-dual radius $R = \sqrt{\alpha'}$, additional states become massless. For example, the states with $n = w = \pm 1$ and one side at the appropriate oscillator level can satisfy $M^2 = 0$. These states supply extra gauge bosons, enhancing the abelian currents of the circle compactification to a nonabelian current algebra. Moving away from the self-dual radius gives these states a mass proportional to the displacement of the radius modulus; in the lower-dimensional spacetime description this is the Higgsing of the enhanced gauge symmetry.

For a d-torus the pair (n_m, w^m) forms a lattice vector, and T-duality becomes the arithmetic group preserving the Narain inner product,

$$\begin{pmatrix} n \\ w \end{pmatrix} \longrightarrow \Omega \begin{pmatrix} n \\ w \end{pmatrix}, \qquad \Omega^T \eta \Omega = \eta, \qquad \Omega \in O(d, d; \mathbb{Z}). \tag{7.11}$$

The circle duality is the simplest element of this group. Other elements include large diffeomorphisms of the torus and integer shifts of the B-field [10, 350].

Maharana's worldsheet review extends Eq. (7.11): when the background is independent of the compact coordinates, the compact-coordinate equations of motion and the NSR superfield equations can be written in $O(d, d)$-covariant form, and massive vertex operators in simple compactifications organize into T-duality multiplets [349]. For a circle, the basic relation is

$$\partial_\alpha \widetilde{X} = \epsilon_{\alpha\beta} \partial^\beta X \quad \Longrightarrow \quad X_L + X_R \leftrightarrow X_L - X_R.$$

Hence p_L is fixed and p_R changes sign. Decoupling-limit work applies the same identity in singular regimes, relating critical IIA D0-brane limits to BFSS/DLCQ Matrix theory, tensionless and ambitwistor strings, Carrollian strings, and spin-matrix limits of AdS/CFT [224]. The elementary $R \leftrightarrow \alpha'/R$ transformation is the local form of this broader duality relation.

Non-supersymmetric duality-web constructions test how far this seed extends. In wedge-singularity backgrounds the internal sector is a wedge sum, for example $S^1 \vee S^1$, and worldsheet effects of the wedge combine with D0-brane emergence arguments to reproduce the expected type-0 tachyon mass in a scaling limit [259]. The related type-0A tachyon-condensation analysis tracks doubled R–R fields and D-brane charges as one branch of the wedge shrinks; a Gauss-law estimate makes unscreened collapsing-branch R–R charge infinitely costly as the circle size goes to zero, while an effective higher-form Stueckelberg screening channel distinguishes flux screening from localized charge discharge [261]. These results are not part of the standard supersymmetric T-duality derivation, but they show how the same momentum/winding, D0-brane, and R–R charge can constrain singular non-supersymmetric limits.

7.4 Compactification of several dimensions

If we compactify k dimensions, we have;

$$X^m \sim X^m + 2\pi R, 26 - k \leq m \leq 25$$

7.4.1 The string spectrum

Then the internal metric is G_{mn} and we have a B field as B_{mn}. We also have $A_{\mu m}$ bosons (from G_{MN}) and $B_{\mu m}$ bosons (from B_{MN}). We do the analysis for zero modes. We take the zero modes as follows:

$$X^m\left(\sigma^1, \sigma^2\right) = x^m\left(\sigma^2\right) + w^m \sigma^1$$

The worldsheet lagrangian is as follows:

$$L = \frac{1}{2\pi}\left(g^{ab} G_{mn} + i\epsilon^{ab} B_{mn}\right) \partial_a X^m \partial_b X^n$$

$$= \frac{1}{2\alpha'}\left(g^{22} G_{mn} \partial_2 X^m \partial_2 X^n + g^{11} G_{mn} \partial_1 X^m \partial_1 X^n + i\epsilon^{12} G_{mn} \partial_1 X^m \partial_2 X^n + i\epsilon^{21} G_{mn} \partial_2 X^m \partial_1 X^n\right)$$

$$= \frac{1}{2\alpha'} G_{mn}\left(\dot{x}^m \dot{x}^n + w^n w^m R^2\right) - \frac{i}{\alpha'} B_{mn} w^m \dot{x}^n R$$

The conjugate momentum p_m is given as;

$$p_m = i\frac{\partial L}{\partial \dot{x}^m} = \frac{1}{\alpha'}v_m - \frac{1}{\alpha'}B_{mn}w^m R \Rightarrow v_m = \alpha'\frac{n_m}{R} - B_{mn}w^n R$$

where $v_m = i\dot{x}_m$. Define p_{mL} and p_{mR} as follows:

$$p_{mL} = \frac{n_m}{R} + (\delta_{mn} - B_{mn})\frac{w^n R}{\alpha'} = \frac{v_{Lm}}{\alpha'}, \quad p_{mR} = \frac{n_m}{R} - (\delta_{mn} + B_{mn})\frac{w^n R}{\alpha'} = \frac{v_{Rm}}{\alpha'}$$

where

$$v_L^m = v^m + w^m R, v_R^m = v^m - w^m R$$

It gives the following zero Virasoro operators;

$$L_0 = \frac{\alpha' p_L^2}{4} + N, \quad \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \tilde{N}$$

The zero-mode Hamiltonian follows by substituting the zero-mode expansion into the world-sheet action and performing the Legendre transform:

$$H = \frac{1}{2\alpha'}G_{mn}\left(v^m v^n + w^m w^n R^2\right) = \frac{1}{4\alpha'}G_{mn}\left(v_L^m v_L^n + v_R^m v_R^n\right)$$

Indeed, $v_L^m = v^m + Rw^m$ and $v_R^m = v^m - Rw^m$, so the cross terms cancel in $G_{mn}(v_L^m v_L^n + v_R^m v_R^n)$ and the remaining terms give $2G_{mn}(v^m v^n + R^2 w^m w^n)$. Therefore, the mass in uncompactified dimensions is

$$m^2 = \frac{1}{2\alpha'^2}G_{mn}\left(v_L^m v_L^n + v_R^m v_R^n\right) + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

Moreover, from the $L_0, \tilde{L}_0$ modes above, we can compute the difference constraint as follows:

$$L_0 - \tilde{L}_0 = 0 \Rightarrow G_{mn}\left(v_L^m v_L^n - v_R^m v_R^n\right) + 4\alpha'(N - \tilde{N}) = 0 \Rightarrow n^m w_m + N - \tilde{N} = 0$$

The same quantities must also define a modular-invariant torus partition function. This is the Narain-lattice viewpoint: the left- and right-moving zero modes form a vector (k_L, k_R) in an even self-dual lattice of signature (k, k), and the oscillator factors multiply the lattice theta function in the one-loop trace. Define the alternative momenta k_{rL} and k_{rR} and coordinates X_L^r, X_R^r as follows:

$$G_{mn} = e_m^r e_n^r, \quad X^r = e_m^r X^m, k_{rL} = e_r^m \frac{v_{mL}}{\alpha'}, k_{rR} = e_r^m \frac{v_{mR}}{\alpha'}$$

then the mass expression and the difference constraint become;

$$m^2 = \frac{1}{2}\left(k_{rL}k_{rL} + k_{rR}k_{rR}\right) + \frac{2}{\alpha'}(N + \tilde{N} - 2), \alpha'\left(k_{rL}k_{rL} - k_{rR}k_{rR}\right) + 4(N - \tilde{N}) = 0 \qquad (8.4.1)$$

7.4.2 Narain compactification

For Narain compactifications, define dimensionless momenta as follows:

$$l_{L,R} = \sqrt{\frac{\alpha'}{2}}k_{L,R}$$

For single valuedness of the OPE;

$$: e^{ik_L X_L(z) + ik_R X_R(\bar{z})} :: e^{ik'_L X_L(0) + ik'_R X_R(0)} :\sim z^{\alpha' k_L k'_L/2}\bar{z}^{\alpha' k_R k'_R/2} : e^{i(k+k')_L X_L(0) + i(k+k')_R X_R(0)} :$$

$$= z^{l_L l'_L}\bar{z}^{l_R l'_R} : e^{i(k+k')_L X_L(z) + i(k+k')_R X_R(z)} :$$

we need the following phase to be unity;

$$\exp\left[2\pi i\left(l_L l'_L - l_R l'_R\right)\right] = 1 \Rightarrow l_L l'_L - l_R l'_R \in \mathbb{Z} \Rightarrow l \circ l' \in \mathbb{Z}$$

where a new dot product has been defined with signature (k, k). So, the lattice of l 's i.e. Γ is in its self dual lattice Γ^* i.e.

$$\Gamma \subset \Gamma^*$$

The partition function for a CFT (with $c = \bar{c}$) on the torus is given as follows:

$$\sum_{|\psi\rangle} \langle\psi| q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} |\psi\rangle = \sum_{|\psi\rangle} \langle\psi| e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)} e^{-2\pi i \bar{\tau}\left(\tilde{L}_0 - \frac{c}{24}\right)} |\psi\rangle$$

and it acquires the following phase under $\tau \to \tau + 1$;

$$2\pi i \left(L_0 - \tilde{L}_0\right)$$

and thus, for single valuedness, we need $L_0 - \tilde{L}_0 \in \mathbb{Z}$. Using the expressions above, we get;

$$L_0 - \tilde{L}_0 = \frac{1}{2}\left(l_{Lr} l_{Lr} - l_{Rr} l_{Rr}\right) + N - \tilde{N} \in \mathbb{Z} \Rightarrow l_{Lr} l_{Lr} - l_{Rr} l_{Rr} \in 2\mathbb{Z} \Rightarrow l \circ l \in 2\mathbb{Z}$$

From this condition, the single-valuedness of the OPE can be derived;

$$(l + l') \circ (l + l') - l \circ l - l' \circ l' \in 2\mathbb{Z} \Rightarrow l \circ l' \in \mathbb{Z}$$

The partition function for the mth direction, with possible vacua $|0; l_{rL}, l_{rR}\rangle$, is as follows:

$$Z_m = \frac{1}{|\eta(\tau)|^2} \sum_{l_L^m, l_R^m} q^{\frac{1}{2} l_L^{m2}} \bar{q}^{\frac{1}{2} l_R^{m2}} = \frac{1}{|\eta(\tau)|^2} \sum_{l_L^m, l_R^m} \exp\left(\pi i \tau l_L^{m2} - \pi i \bar{\tau} l_R^{m2}\right)$$

which means that the partition function for all m directions is as follows:

$$Z_\Gamma = \frac{1}{|\eta(\tau)|^{2k}} \sum_{l \in \Gamma} \exp\left(\pi i \tau l_L^2 - \pi i \bar{\tau} l_R^2\right)$$

The S transformation is as follows:

$$\begin{aligned} Z_\Gamma &= \frac{1}{|\eta(\tau)|^{2k}} \sum_{l' \in \Gamma} \int d^{2k} l \, \delta(l - l') \exp\left(\pi i \tau l_L^2 - \pi i \bar{\tau} l_R^2\right) \\ &= V_\Gamma^{-1} \frac{1}{|\eta(\tau)|^{2k}} \sum_{l'' \in \Gamma} \int d^{2k} l \, \exp\left(2\pi i \, l'' \circ l + \pi i \tau l_L^2 - \pi i \bar{\tau} l_R^2\right). \end{aligned}$$

where we used the Poisson summation identity on the lattice;

$$\sum_{l' \in \Gamma} \delta\left(l - l'\right) = V_\Gamma^{-1} \sum_{l'' \in \Gamma^*} \exp\left(2\pi i l'' \circ l\right)$$

This identity is the Fourier series of the periodic delta-comb on the vector space $\Gamma \otimes \mathbb{R}$. The dual lattice Γ^* consists of those vectors whose $O(k,k)$ inner product with every vector of Γ is integral; the normalization V_Γ^{-1} is fixed by integrating both sides over one fundamental cell.
Using Poisson resummation formula and the modular transformation of η function, we get;

$$Z_\Gamma(\tau) = V_\Gamma^{-1} Z_{\Gamma^*}(-1/\tau)$$

Then, $\Gamma = \Gamma^*$ is sufficient to ensure modular invariance. It is also necessary for a purely lattice CFT with the same oscillator factor: the S transformation maps the theta function of Γ to the theta function of Γ^*, and equality for all τ requires the two spectra, with their multiplicities, to coincide. Thus Γ has to be an even, self-dual lattice. If $\Lambda \in O(k,k,\mathbb{R})$, then Γ is still an even, self-dual lattice because the modular invariance conditions depend on the $O(k,k)$ product only. But $O(k,k,\mathbb{R})$ is not a symmetry because the expression for mass and the difference constraint aren't invariant under it. They are invariant only in $O(k,\mathbb{R}) \times O(k,\mathbb{R})$. Thus, the group that generates inequivalent theories (e.g. theories on lattices of different radii) is as follows:

$$\frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R}) \times O(k,\mathbb{R})}$$

and the dimension of this group is;

$$\frac{2k(2k-1)}{2} - 2 \times \frac{k(k-1)}{2} = k(2k-1) - k(k-1) = k^2$$

which is the same as the DOF from G_{mn} and B_{mn}. We can start from a lattice (known as Γ_0) with all compact dimensions compact and all radii being self dual radii. If we denote the group that maps Γ_0 to itself as $O(k,k,\mathbb{Z})$, then the following lattices are the same;

$$\Lambda'\Lambda\Lambda''\Gamma_0 = \Lambda\Gamma, \quad \Lambda \in O(k,k,\mathbb{R}), \Lambda'' \in O(k,k,\mathbb{Z}), \Lambda' \in O(k,\mathbb{R}) \times O(k,\mathbb{R})$$

Recalling the expressions for l_L^r and l_R^r;

$$l_L^r = \frac{n^r}{R} + (\delta^{rs} - B^{rs}) \frac{\omega_s R}{\alpha'}, l_R^r = \frac{n^r}{R} - (\delta^{rs} + B^{rs}) \frac{\omega_s R}{\alpha'}$$

we see that Z_Γ is invariant if we do the following T-duality transformation together with the exchange of momentum and winding numbers, $n^r \leftrightarrow w_r$:

$$R \longleftrightarrow \frac{\alpha'}{R}$$

We see that if x^m is periodic with period $2\pi R$ then so is the combination;

$$x^m = L_n^m x^n, L_n^m \in \mathbb{Z}$$

but if $\det L \neq 1$, then the volume of the k torus changes and hence, we should have $\det L = 1$. Thus, $L \in SL(2,\mathbb{Z})$. Another component in the T duality transformations is the following;

$$b_{mn} \to b_{mn} + N_{mn} \Rightarrow B_{mn} \to B_{mn} + \frac{\alpha' N_{mn}}{R^2}, \quad N_{mn} \in \mathbb{Z}$$

This transformation does the following;

$v_m^L = \frac{n_m}{R} + (\delta_{mn} - B_{mn}) w^n R \to \frac{(n_m - N_{mn} w^n)}{R} + (\delta_{mn} - B_{mn}) w^n R = \frac{n'_m}{R} + (\delta_{mn} - B_{mn}) w^n R, n'_m = n_m - N_{mn} w^n \in \mathbb{Z}$

and similarly, for v_m^R. So, we see that the partition function doesn't change because we are still summing over integer n^m and w^m.

A explicit example is a two-torus with a B-field shift. The integer shift

$$B_{mn} \longrightarrow B_{mn} + N_{mn}, \qquad N_{mn} \in \mathbb{Z}, \tag{7.12}$$

is compensated by the momentum relabeling

$$n_m \longrightarrow n_m - N_{mn} w^n. \tag{7.13}$$

Since the partition function sums over all integer n_m and w^m, this is a true duality of the spectrum. The example is useful because it shows how $O(d,d;\mathbb{Z})$ contains more than $R \leftrightarrow \alpha'/R$: it also contains large gauge transformations of the B-field. [79].

7.5 Orbifolds

Orbifolds are introduced here because toroidal compactification gives smooth compact directions, while many string compactifications require controlled fixed points. In field theory a quotient by a reflection would usually produce a singular space. In string theory the worldsheet prescription also adds twisted sectors, and these sectors contain states localized near the fixed points. The orbifold construction is therefore a direct test of whether a singular-looking target space can still define a consistent conformal field theory. This material is used below in the discussion of twisted sectors, modular-invariant partition functions, and the relation between orbifold and torus CFTs [10, 58, 86].

Consider a $\mathbb{Z}_2$ orbifold with action

$$X^{25} \to -X^{25} \tag{8.5.1}$$

which sends all the operators appearing in X^{25} to their negatives. A compact space is made by an additional identification;

$$X^{25} \to -X^{25}, X^{25} \sim X^{25} + 2\pi R m, m \in \mathbb{Z} \tag{8.5.2}$$

This produces a new sector, the twisted sector, defined by

$$X^{25}(\sigma + 2\pi) \sim -X^{25}(\sigma) \tag{8.5.3}$$

Since X^{25} is antiperiodic, it has the expansion

$$X^{25}(z,\bar{z}) = i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{1}{m+1/2}\left(\frac{\alpha_m^{25}}{z^{m+1/2}}+\frac{\alpha_m^{25}}{\bar{z}^{m+1/2}}\right) \tag{8.5.4}$$

For a compact orbifold, there is another twisted sector:

$$X^{25}(\sigma+2\pi)\sim 2\pi R - X^{25}(\sigma) \tag{8.5.5}$$

For this sector, we have the following expansion;

$$X^{25}(z,\bar{z}) = \pi R + i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{1}{m+1/2}\left(\frac{\alpha_m^{25}}{z^{m+1/2}}+\frac{\tilde{\alpha}_m^{25}}{\bar{z}^{m+1/2}}\right) \tag{8.5.6}$$

The allowed massless states are obtained by imposing the orbifold projection. For generic massless states, $n=w=0$, and invariance under the orbifold action requires an even number of creation operators in the 25 direction. This condition removes the Kaluza–Klein gauge bosons and the gauge bosons descending from the antisymmetric tensor. The zero-point energy in the twisted sector, with 23 periodic bosons and one antiperiodic boson, is

$$a_{\text{twisted}} = -\frac{23}{24}+\frac{1}{48} = -\frac{15}{16} \tag{8.5.7}$$

Then, the mass-shell condition becomes as follows:

$$\frac{\alpha' p^2}{4}+N+a_{\text{twisted}} = \frac{\alpha' p^2}{4}+N-\frac{15}{16} = 0 \Rightarrow m^2 = \frac{4}{\alpha'}\left(N-\frac{15}{16}\right) \tag{8.5.8}$$

Thus, the lowest mass state is tachyonic ($N=0$). The level matching condition requires $N=\tilde{N}$ and due to the anti periodicity, X^{25} oscillators make half integral contributions to the number operator. Thus, the next excited state involves $\alpha_{-1/2}^{25}\tilde{\alpha}_{-1/2}^{25}$ and hence, the this state is also tachyonic. There is no way to set $N=15/16$ and hence, there is no massless state.
Twisted-state vertex operators are constrained by the orbifold projection and are localized at the fixed points. Tree-level amplitudes with only untwisted external states receive contributions only from untwisted intermediate states; this is the inheritance principle. The untwisted-sector partition function is fixed by the projected trace:

$$\operatorname{Tr}\left(\frac{1+r}{2}q^{L_0-\frac{1}{24}}\bar{q}^{\bar{L}_0-\frac{1}{24}}\right) = \frac{1}{2}Z_{\text{tor}}(R,\tau)+\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \tag{8.5.9}$$

where r in the orbifold action and ϑ_2 is given as follows:

$$\vartheta_2(\tau) = 2\eta(\tau)q^{\frac{1}{12}}\prod_{r\geq 1}(1+q^r)^2 = \sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau) = \vartheta_{10} \tag{8.5.10}$$

where ϑ_{10} denotes the theta function with characteristic $(\frac{1}{2},0)$. The last expression follows from the Jacobi triple product identity:

$$q^{-\frac{1}{24}}\prod_{r\geq 1}\left(1+q^{r+\frac{1}{2}}w\right)\left(1+q^{r+\frac{1}{2}}w^{-1}\right) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}q^{\frac{n^2}{2}}w^n \tag{8.5.11}$$

The partition function in 8.5.9 is not modular invariant because it is not invariant in S transformations. We will have to add the twisted sector. The partition function for the twisted sector is as follows:

$$\operatorname{Tr}\left(\frac{1+r}{2}q^{L_0+\frac{1}{48}}\bar{q}^{\bar{L}_0+\frac{1}{48}}\right) = \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|+\left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| \tag{8.5.12}$$

The factor of 1/48 comes from the twisted-sector ground-state energy. For one real boson reflected by $X\mapsto -X$, the oscillators are half-integer moded, and zeta-function regularization gives a twist-field weight $h=\bar{h}=1/16$. Since the trace is written with $q^{L_0-c/24}$ and $c=1$, the exponent becomes $h-c/24 = 1/16-1/24 = 1/48$. Moreover, the functions used are defined as follows:

$$\vartheta_3(\tau) = q^{-\frac{1}{24}}\eta(\tau)\prod_{r=0}^{\infty}\left(1+q^{r+\frac{1}{2}}\right)^2 = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \vartheta_{00} \tag{8.5.13}$$

$$\vartheta_4(\tau) = q^{-\frac{1}{24}}\eta(\tau)\prod_{r=0}^{\infty}\left(1-q^{r+\frac{1}{2}}\right)^2 = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}} = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau) = \vartheta_{01} \tag{8.5.14}$$

These functions are related to each other by modular transformations as follows:

$$\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \overset{T}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \overset{S}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| \overset{T}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| \overset{S}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| \tag{8.5.15}$$

The general formula for ϑ functions is as follows:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2} e^{2\pi i(n+a)(z+b)} \tag{8.5.16}$$

for which a product representation follows from the Jacobi triple product identity;

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \eta(\tau)e^{2\pi i a(z+b)}q^{\frac{a^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^{n+a-\frac{1}{2}}e^{2\pi i(z+b)}\right)\left(1+q^{n-a-\frac{1}{2}}e^{-2\pi i(z+b)}\right) \tag{8.5.17}$$

7.5.1 Twisting

A closed string on an orbifold need not return to the same point in the covering space after one trip around the string. It only needs to return to a point identified by the orbifold group. Thus a sector labelled by $g \in G$ obeys

$$X(\sigma + 2\pi) = g \cdot X(\sigma). \tag{7.14}$$

For the reflection $X \mapsto -X$, a twisted string satisfies $X(\sigma+2\pi) = -X(\sigma)$ in the covering theory, so its center is pinned to a fixed point of the reflection. Quantization in this sector gives oscillators with fractional moding and a shifted ground-state conformal weight. The orbifold Hilbert space is the sum of the untwisted and twisted sectors followed by projection onto invariant states,

$$\mathcal{H}_{\text{orb}} = \bigoplus_{[g]} \mathcal{H}_g^{C(g)}. \tag{7.15}$$

The twisted states are the degrees of freedom that make the orbifold CFT well defined at fixed points where the geometric quotient is singular [10, 58].

7.5.2 $c = 1$ CFTs and the orbifold branch

The moduli space of compact $c = 1$ CFTs contains the torus theories and the orbifold theories. These meet at a point because their partition functions agree there:

$$Z_{\text{orb}}(\sqrt{\alpha}, \tau) = Z_{\text{tor}}(2\sqrt{\alpha}, \tau)$$

i.e. the partition function of the torus theory at twice the self-dual radius is the same as the partition function of the orbifold theory at the self-dual radius. The same organizing principle applies to the finite subgroups used in higher-dimensional orbifolds. Cyclic groups Z_k and dihedral groups D_k give the elementary rotation and reflection orbifolds, while the tetrahedral, octahedral, and icosahedral groups T, O, I are the finite rotation groups of the sphere. In every case modular invariance requires the torus sum over commuting twists,

$$Z_{\text{orb}} = \frac{1}{|G|}\sum_{gh=hg} Z\begin{bmatrix}g\\h\end{bmatrix},$$

so the invariant projection and the twisted Hilbert spaces are not independent choices.

7.6 Open strings

Start with the pure gauge configuration $A_{25}\left(x^M\right)=0$, which is gauge equivalent to the following constant connection. The gauge coupling is kept outside the connection and appears in the Wilson line through the charge q:

$$A_{25}\left(x^M\right)=-i\exp\left(\frac{i\theta x^{25}}{2\pi R}\right)\partial_{25}\exp\left(\frac{-i\theta x^{25}}{2\pi R}\right)=-\frac{\theta}{2\pi R}$$

where θ is constant. Now, the Wilson line for this potential is as follows:

$$W=\exp\left(iq\oint dx^{25}A_{25}\right)=\exp\left(-\frac{iq\theta}{2\pi R}\oint dx^{25}\right)=\exp\left(-\frac{iq\theta}{2\pi R}(2\pi R)\right)=\exp(-iq\theta) \quad (8.6.1)$$

where q is the coupling constant. Now, suppose that we have a point particle coupled to an electromagnetic field. The Euclidean worldline action is obtained from the Lorentzian action by $t=-i\tau$, so the minimal coupling term acquires the factor $-iqA_M\dot{X}^M$:

$$S=\int d\tau\left(\frac{1}{2}\dot{X}^M\dot{X}_M+\frac{m^2}{2}-iqA_M\dot{X}^M\right)$$

The conjugate momentum p_{25} is as follows:

$$p^{25}=\frac{\partial L}{\partial\left(\partial X^{25}/\partial\tau_{\text{Minkowski}}\right)}=-\frac{\partial L}{i\partial\left(\partial X^{25}/\partial\tau_{\text{Euclidean}}\right)}=i\left(\dot{X}^{25}-iqA^{25}\right)=i\dot{X}^{25}+qA^{25}=v^{25}-\frac{q\theta}{2\pi R}$$

where $v^{25}=i\dot{X}^{25}$. Now, due to the periodicity, $p^{25}=l/R, l\in\mathbb{Z}$. and thus, we have;

$$v^{25}=\frac{l}{R}+\frac{q\theta}{2\pi R}$$

The Hamiltonian is derived by solving the momentum relation for v^{25} and substituting into the mass-shell constraint:

$$H=\frac{1}{2}\left(p^\mu p_\mu+v_{25}^2+m^2\right)$$

Since this hamiltonian annihilates the physical states, the mass-shell condition for the mass that is visible in non-compact dimensions becomes;

$$-p^\mu p_\mu=m^2+v_{25}^2=m^2+\left(\frac{l}{R}+\frac{q\theta}{2\pi R}\right)^2$$

we see that if $R\to\infty$, the mass receives no correction. For the nonabelian gauge theory, an open string carries $U(n)$ Chan–Paton indices at its endpoints. If $A_{25}^aT^a$ is constant, with T^a generators of $U(n)$, then hermiticity allows a constant unitary gauge transformation to diagonalize $A_{25}^aT^a$:

$$A_{25}^aT^a=-\frac{1}{2\pi R}\operatorname{diag}\left(\theta_1,\ldots,\theta_n\right)$$

For a general Chan paton state;

$$|N;k;a\rangle=\sum_{ij}|N;k;ij\rangle\lambda_{ij}^a$$

the gauge field A_{25} couples as follows:

$$|N;k;a\rangle=\sum_{ij}|N;k;ij\rangle\left[A_{25},\lambda^a\right]_{ij}$$

The required commutator is as follows:

$$\left[A_{25},\lambda^a\right]_{ik}=\frac{1}{2\pi R}\sum_j\left(\theta_i\delta_{ij}\lambda_{jk}-\theta_j\lambda_{ij}\delta_{jk}\right)=\left(\theta_i-\theta_k\right)\lambda_{ik}\Rightarrow\left[A_{25},\lambda^a\right]_{ij}=\left(\theta_i-\theta_j\right)\lambda_{ij}$$

Thus the state $|N;k;ij\rangle$ has charge $+1$ under $U(1)_i$ and charge -1 under $U(1)_j$. The corresponding expression for v_{25} is

$$v_{25} = \frac{l}{R} + \frac{\theta_i - \theta_j}{2\pi R}$$

Thus, mass of the string state $|N; k; ij\rangle$ is as follows:

$$m^2 = \frac{1}{\alpha'}(N-1) + \frac{(2\pi l + \theta_i - \theta_j)^2}{4\pi^2 R^2}$$

The lowest non-tachyonic states have the following mass;

$$m_{ij}^2 = \frac{(\theta_i - \theta_j)^2}{4\pi^2 R^2}, \quad l = 0, N = 1 \tag{8.6.2}$$

These states are massive if all the θ 's are different. If θ 's are equal in s sets of size r_i, then the massless states form the following representation;

$$U(n) \to U(r_1) \times \ldots \times U(r_s)$$

7.6.1 T-duality

For open strings, T-duality maps Wilson lines into D-brane positions. If a constant gauge field around the original circle has eigenvalues θ_i, an open string with Chan–Paton labels i, j sees the difference $\theta_i - \theta_j$ in its momentum quantization. On the dual circle of radius $R' = \alpha'/R$, the same quantity is interpreted as the separation of two branes,

$$y_i = \frac{\theta_i}{2\pi} R', \qquad m_{ij}^2 = \frac{|y_i - y_j|^2}{(2\pi\alpha')^2} + \frac{N-1}{\alpha'}. \tag{7.16}$$

The mass of the stretched string is therefore the energy of a string segment of length $|y_i - y_j|$ plus its oscillator contribution. Coincident eigenvalues give coincident branes and restore the corresponding nonabelian gauge symmetry [10, 350].

In decoupling limits this same open-string mass estimate diagnoses which degrees of freedom survive. If a background R–R field cancels a D0-brane rest energy, the surviving light objects are D0-branes and the off-diagonal open strings that become the BFSS matrix degrees of freedom; matching the fundamental-string worldsheet through T-duality then explains why nodal/tensionless worldsheets appear in the dual description [224]. The equation $m_{ij} \sim |y_i - y_j|/(2\pi\alpha')$ is therefore more than a D-brane kinematics formula: after a critical scaling it becomes the rule for which matrix entries remain dynamical and which strings decouple.

7.7 D-branes

The gauge field in the compact direction determines the position of the D-brane in the dual compact direction, and hence the shape of the D-brane. T-dualizing a tangent direction reduces the dimension of the D-brane, while T-dualizing a transverse direction increases it. Coincident r branes give the following massless states:

$$\alpha_{-1}^{\mu}|k; ij\rangle i, j \in \{1, \ldots, r\}$$

and thus, we have r^2 massless vectors and a $U(r)$ symmetry. We also have r^2 massless scalars as follows:

$$\alpha_{-1}^{25}|k; ij\rangle i, j \in \{1, \ldots, r\}$$

7.7.1 The D-brane action

We now construct the low-energy D-brane action. This action is called the DBI action. For a p-brane, we introduce the coordinates $\xi^0, \ldots \xi^p$ on the brane worldvolume. The fields to consider on the brane are the embedding fields $X^\mu(\xi)$ and the gauge fields $A_a(\xi)$ (recall that gauge fields live on the endpoints of the strings). Now, we define the induced metric $G_{ab}(\xi)$ and induced $B_{ab}(\xi)$ field as follows:

$$G_{ab}(\xi) = \frac{\partial X^\mu}{\partial X^a}\frac{\partial X^\nu}{\partial X^b}G_{\mu\nu}(X(\xi)), B_{ab}(\xi) = \frac{\partial X^\mu}{\partial X^a}\frac{\partial X^\nu}{\partial X^b}B_{\mu\nu}(X(\xi))$$

Now, if we look at the Nambu–Goto action, the corresponding generalization of the integrand is $\sqrt{-\det(G_{ab}+B_{ab})}$ with an overall factor of $-T_p$, the brane tension. In the Euclidean Polyakov form the antisymmetric B-coupling carries the factor of i already displayed below; in the Lorentzian DBI action the determinant is real and the same information is carried by the pullback of B. However, this is not the whole story. The open-string tree-level action must carry the normalization g_o^{-2}, and $g_o^{-2} \sim e^{-\phi}$ in the Lagrangian. Now, we prove that B_{ab} should appear only with F_{ab}. In string worldsheet action, we have the following combination;

$$\frac{i}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + i\int_{\partial M} dX^\mu A_\mu$$

where $\delta A_\mu = \partial_\mu \lambda$ is a gauge symmetry of this action. The gauge transformation of $B_{\mu\nu}$ is as follows:

$$B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$$

Under this transformation, the action changes as follows:

$$\frac{i}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu\left(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu\right) = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu \partial_\mu\zeta_\nu$$

$$= \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left(\partial_1 X^\mu \partial_2 X^\nu - \partial_2 X^\mu \partial_1 X^\nu\right)\partial_\mu \zeta_\nu = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_1 X^\mu\partial_2 X^\nu\left(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu\right)$$

We now use the following identities;

$$\partial_1\zeta_\mu = \partial_\nu\zeta_\mu\partial_1 X^\nu, \quad \partial_2\zeta_\mu = \partial_\nu\zeta_\mu\partial_2 X^\nu$$

which imply;

$$\partial_1 X^\mu\partial_2 X^\nu\left(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu\right) = \partial_2 X^\nu\partial_1\zeta_\nu - \partial_1 X^\mu \partial_2\zeta_\mu$$

and thus, the variation of the action becomes;

$$-\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left(\partial_1 X^\mu\partial_2\zeta_\mu - \partial_2 X^\mu\partial_1\zeta_\mu\right) \tag{8.7.1}$$

The first term becomes;

$$-\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_1\left(X^\mu\partial_2\zeta_\mu\right) + \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_1\partial_2\zeta_\mu$$

$$= -\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}\left(X^\mu\partial_2\zeta_\mu\right)\Big|_{\text{boundary}} + \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_1\partial_2\zeta_\mu \tag{8.7.2}$$

The second term in 8.7.1 becomes;

$$\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_2 X^\mu\partial_1\zeta_\mu = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_2\left(X^\mu\partial_1\zeta_\mu\right) - \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_2\partial_1\zeta_\mu$$

$$= -\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_2\partial_1\zeta_\mu \tag{8.7.3}$$

where we used the fact that A_μ variation vanishes at starting and ending times. We also see that (8.7.3) cancels the second term in 8.7.2). So, the variation in the B term becomes;

$$-\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}\left(X^\mu\partial_2\zeta_\mu\right)\Big|_{\text{boundary}} = -\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}X^\mu\partial_2\zeta_\mu$$

where we have excluded the boundary label now because it is understood. This term can be canceled if A_μ also transforms as $\delta A_\mu = -\zeta_\mu/2\pi\alpha'$. This will give the following variation;

$$i\int_{\partial M} dX^\mu \delta A_\mu = i\iint_{\partial M} d\sigma^2\sqrt{g}\partial_2 X^\mu \delta A_\mu = i\int_{\partial M} d\sigma^2\sqrt{g}\partial_2\left(X^\mu\delta A_\mu\right) - \int d\sigma^2\sqrt{g}X^\mu\partial_2\delta A_\mu$$
$$= \frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}X^\mu\partial_2\zeta_\mu$$

where we used the fact that δA_μ vanishes at the initial and final times (i.e. σ^2). This exactly cancels the variation of the B action. Now, the variation of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is as follows:

$$\delta F_{\mu\nu} = \partial_\mu\left(-\frac{\zeta_\nu}{2\pi\alpha'}\right) - \partial_\nu\left(-\frac{\zeta_\mu}{2\pi\alpha'}\right) = -\frac{1}{2\pi\alpha'}\left(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu\right) = -\frac{1}{2\pi\alpha'}\delta B_{\mu\nu} \Rightarrow \delta\left(B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right) = 0$$

Thus, the only gauge invariant quantity is $B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}$ and the induced field on the D-brane is $B_{ab} + 2\pi\alpha' F_{ab}$. So, this combination should appear in the action. The DBI action is thus as follows:

$$S_{\text{DBI}} = -T_p\int d^{p+1}\xi e^{-\phi}\left[-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)\right]^{1/2} \tag{8.7.4}$$

For n separated branes, we have n copies of the action. For n coincident branes, open strings can begin and end on any pair of branes, so their Chan–Paton labels make the low-energy fields matrix-valued. The gauge field becomes a $U(n)$ gauge field and the transverse positions become adjoint scalars X^i. The leading nonabelian potential has the schematic form

$$V(X) \sim \frac{1}{g_{\text{YM}}^2}\sum_{i<j}\text{Tr}\,[X^i, X^j]^2.$$

Its flat directions are the commuting matrices, which can be diagonalized simultaneously; their eigenvalues are the positions of the separated branes. Noncommuting matrices describe genuinely coincident-brane degrees of freedom instead of ordinary positions in a classical transverse space.

The DBI determinant in Eq. (8.7.4) can be checked against open string field theory and the open-string symplectic form. For a constant electric field $E = 2\pi\alpha' F_{01}$, the DBI Lagrangian density is

$$\mathcal{L}_{\text{DBI}} = -T_p\sqrt{1-E^2}, \qquad D = \frac{\partial\mathcal{L}_{\text{DBI}}}{\partial E} = \frac{T_p E}{\sqrt{1-E^2}},$$

so the Hamiltonian density is

$$\mathcal{H}_{\text{DBI}} = DE - \mathcal{L}_{\text{DBI}} = \sqrt{T_p^2 + D^2}.$$

The open-SFT symplectic computation of the same electric-flux energy agrees with this expression after the Ellwood invariant is extended to nonpolynomial open SFT [332]. For a constant B-field the endpoint brackets follow from the boundary part of the covariant symplectic current,

$$\{X^i(\tau,0), X^j(\tau,0)\} = \theta^{ij}, \qquad \theta^{ij} = -(2\pi\alpha')^2\left[(g+2\pi\alpha' B)^{-1}B(g-2\pi\alpha' B)^{-1}\right]^{ij},$$

which is the Seiberg–Witten noncommutativity parameter. In the tensionless limit the bulk term degenerates and the remaining phase space can be supported entirely at the boundary [334]. The three standard descriptions of branes–open-string endpoints, BPS supergravity sources, and gauge-invariant worldvolume actions–are reviewed together in [335].

7.7.2 D-brane tension

We first derive the D-brane tension recursion relation. Since T_p does not depend on fields, we can derive the relation for static D-branes and then use it for the full DBI action. Consider a static D-brane with $B_{ab} = 0$, $F_{ab} = 0$, constant dilaton, and flat induced metric. Then, the action becomes;

$$S = -T_p\int d^{p+1}\xi e^{-\phi} = -T_p e^{-\phi}\int d^{p+1}\xi$$

Now, the action for a static brane has no kinetic energy, and the only potential energy is the mass of the brane. So, action is negative of the mass of the brane. Therefore, the mass per unit volume of the Dp-brane is $T_p e^{-\phi}$. Now, consider the case when Dp-brane is wrapped around a p-torus. The volume of that torus is;

$$\prod_{i=1}^{p}(2\pi R_i)$$

and thus, the mass of the Dp-brane is;

$$T_p e^{-\phi}\prod_{i=1}^{p}(2\pi R_i)$$

Now, let T duality act on p-th direction of the torus. We saw in Section 7.3 that ϕ changes as follows under T-duality;

$$e^{\phi}\to e^{-\phi'}=\frac{R}{\sqrt{\alpha'}}e^{-\phi}\Rightarrow e^{-\phi}=\frac{\sqrt{\alpha'}}{R}e^{-\phi'}$$

Under T-duality, mass is invariant and thus, the mass of the brane is still given as;

$$T_p\frac{\sqrt{\alpha'}}{R_p}e^{-\phi'}\prod_{i=1}^{p}(2\pi R_i)=T_p\sqrt{\alpha'}e^{-\phi'}\prod_{i=1}^{p-1}(2\pi R_i)$$

But, this mass can also be written down as follows (because after T -duality, we have a D$(p-1)$ brane);

$$T_{p-1}e^{-\phi'}\prod_{i=1}^{p-1}(2\pi R_i)$$

and thus, we get;

$$2\pi\sqrt{\alpha'}T_p=T_{p-1}\Rightarrow T_p=\frac{T_{p-1}}{2\pi\sqrt{\alpha'}} \tag{8.7.5}$$

To obtain the expression for the D-brane tension, calculate the cylinder partition function of an open string stretched between two D-branes. It is like the partition function calculated in Chapter 5, but now the string's endpoints can move only on the D-branes, so the momentum integration is over $p+1$ directions and gives V_{p+1} instead of V_{26}. Moreover, there is an additional contribution to the conformal weight h_i for the i-th state because the string is stretched between two D-branes. Assume that the two D-branes are at $X^m=0$ and $X^m=y^m$ for $m=p+1,\ldots,25$. Now, we know that the expansion for X_L^m and X_R^m is as follows:

$$X_L^m=x_L^m+\alpha' p_L^m\sigma^+ +\ldots, X_R^m=x_R^m+\alpha' p_R^m\sigma^- +\ldots$$
$$\Rightarrow X^m(\tau,\sigma)=x_L^m+x_R^m+\alpha'\left(p_L^m+p_R^m\right)\tau+\alpha'\left(p_L^m-p_R^m\right)\sigma+\ldots$$

Imposing $X^m(0,\tau)=0$ and $X^m(\pi,\tau)=y^m$ gives us the following;

$$p_L^m+p_R^m=0,\quad \alpha'\left(p_L^m-p_R^m\right)=y^m\Rightarrow p_L^m=-p_R^m=\frac{y^m}{2\alpha'\pi}$$

Now, we can deduce that;

$$L_0=\alpha' p_L^2+N=N+\frac{y^2}{4\pi^2\alpha'}$$

and thus, the additional contribution from stretching between the branes is $y^2/4\pi^2\alpha$. Now, looking at 7.4.1, we see that we have an additional factor;

$$\int_0^\infty\frac{dt}{2t}\operatorname{Tr}_0'\left[e^{-2\pi tL_0}\right]=\int_0^\infty\frac{dt}{2t}\operatorname{Tr}_0'\left[e^{-2\pi t\left(L_0^{\text{old}}+y^2/4\pi^2\alpha'\right)}\right]=\int_0^\infty\frac{dt}{2t}e^{-y^2t/2\pi\alpha'}\operatorname{Tr}_0'\left[e^{-2\pi tL_0^{\text{old}}}\right]$$

So, the cylinder partition function is as follows:

$$\mathcal{A}=iV_{p+1}\int_0^\infty\frac{dt}{t}\left(8\pi^2\alpha' t\right)^{-(p+1)/2}\exp\left(-ty^2/2\pi\alpha'\right)\eta(it)^{-24}$$

$$= \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{(p+3)/2}} \exp\left(-ty^2/2\pi\alpha'\right) \eta(it)^{-24}$$

Now, we need to extract the large-separation force, so we expand the integrand in the closed-string channel. The modular property of the eta function gives;

$$\eta(it) = \frac{1}{\sqrt{t}}\eta(i/t) \Rightarrow \eta(it)^{-24} = t^{12}\eta(i/t)^{-24} = t^{12}\left(e^{2\pi/t} + 24 + \dots\right)$$

using this expansion, the amplitude becomes;

$$\mathcal{A} = \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt t^{(21-p)/2} \exp\left(-ty^2/2\pi\alpha'\right)\left(e^{2\pi/t} + 24 + \dots\right)$$

We now need to evaluate the second term in this partition function. It is done as follows:

$$\frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt t^{(21-p)/2} \exp\left(-ty^2/2\pi\alpha'\right) = \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \left(\frac{2\pi\alpha'}{y^2}\right)^{(23-p)/2} \int_0^\infty d\mu \mu^{(21-p)/2} e^{-\mu}$$

$$= \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \left(\frac{2\pi\alpha'}{y^2}\right)^{(23-p)/2} \Gamma\left(\frac{23-p}{2}\right) = iV_{p+1}\frac{24}{2^{12}}\left(4\pi\alpha'\right)^{11-p} \pi^{(p-23)/2}\Gamma\left(\frac{23-p}{2}\right)|y|^{p-23}$$

where we did the variable change $\mu = ty^2/2\pi\alpha'$. Now, we use the result that Green's function for the Laplacian $-\nabla^2$ in d transverse Euclidean dimensions is

$$G_d(|y|) = \frac{1}{4\pi^{d/2}|y|^{d-2}}\Gamma\left(\frac{d-2}{2}\right) \tag{8.7.6}$$

This follows by Fourier transforming $1/k^2$:

$$G_d(r) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot y}}{k^2} = \int_0^\infty ds \int \frac{d^dk}{(2\pi)^d} e^{-sk^2+ik\cdot y} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty ds\, s^{-d/2} e^{-r^2/4s},$$

and the change of variable $u = r^2/(4s)$ gives Eq. (8.7.6).
and then, setting $d = 25 - p$, we get;

$$G_{25-p}(|y|) = \frac{1}{4\pi^{(25-p)/2}y^{23-p}}\Gamma\left(\frac{23-p}{2}\right) \Rightarrow 4\pi G_{25-p}(|y|) = \pi^{(p-23)/2}\Gamma\left(\frac{23-p}{2}\right)|y|^{p-23}$$

Therefore, we have;

$$\mathcal{A} = iV_{p+1}\frac{24\pi}{2^{10}}\left(4\pi\alpha'\right)^{11-p} G_{25-p}(|y|)$$

We can convert this Green's function to the momentum-space picture for massless exchange. Since $G_{25-p}(y)$ is the Fourier transform of $1/k_\perp^2$, the pole part of the amplitude is

$$G_{25-p}(|y|) = \frac{1}{k_\perp^2} \Rightarrow \mathcal{A} = iV_{p+1}\frac{24\pi}{2^{10}k_\perp^2}\left(4\pi\alpha'\right)^{11-p} \tag{8.7.7}$$

Now, we do the analogous field-theory calculation because it gives the same amplitude in terms of T_p. We write the spacetime action for the strings with $D = 26$ in the Einstein frame. The antisymmetric tensor is omitted in this one-particle exchange because the static parallel brane with $B_{ab} = F_{ab} = 0$ has no linear source for $B_{\mu\nu}$: the trace of an antisymmetric fluctuation in the DBI determinant vanishes at first order. The relevant terms are as follows:

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}}\left(\tilde{R} - \frac{1}{6}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\right)$$

where recall that $\tilde{\phi}$ has a zero vev because $\phi = \tilde{\phi} + \phi_0$ where $\phi_0 = \langle\phi\rangle$. With the background $B_{ab} = F_{ab} = 0$, the relevant terms from the D-brane action are

$$S = -T_p\int d^{p+1}\xi e^{-\phi}\sqrt{-\det G_{ab}} = -T_pe^{-\phi_0}\int d^{p+1}\xi e^{-\tilde{\phi}}\sqrt{-\det G_{ab}} = -\tau_p\int d^{p+1}\xi e^{-\tilde{\phi}}\sqrt{-\det G_{ab}}$$

where $\tau_p = T_pe^{-\phi_0}$ Now, we recall that the Einstein frame metric $\tilde{G}_{\mu\nu}$ and $G_{\mu\nu}$ are related as follows:

$$\tilde{G}_{\mu\nu} = e^{-4\tilde{\phi}/(D-2)} G_{\mu\nu} = e^{-\tilde{\phi}/6} G_{\mu\nu} \Rightarrow G_{\mu\nu} = e^{\tilde{\phi}/6} \tilde{G}_{\mu\nu} \Rightarrow \tilde{G}_{ab} = e^{\tilde{\phi}/6} G_{ab} \Rightarrow \det \tilde{G} = e^{(p+1)\tilde{\phi}/6} \det G$$
$$\Rightarrow \sqrt{-\det \tilde{G}} = e^{(p+1)\tilde{\phi}/12} \sqrt{-\det G}$$

Therefore, the D-brane action becomes;

$$S = -\tau_p \int d^{p+1}\xi e^{(p-11)\tilde{\phi}/12} \sqrt{-\tilde{G}}$$

In the linear approximation;

$$\tilde{G}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

while raising the indices by the flat metric and using de Donder gauge for the graviton, the quadratic spacetime action becomes

$$S = -\frac{1}{8\kappa^2} \int d^{26}X \left(\partial_\mu h_{\nu\lambda} \partial^\mu h^{\nu\lambda} - \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{2}{3} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right) \tag{8.7.8}$$

where $h = h^\mu_\mu$. The D-brane field action becomes;

$$\begin{aligned} S_p &= -\tau_p \int d^{p+1}\xi \exp\left(\tfrac{p-11}{12}\tilde{\phi}\right) [-\det(\eta_{ab} + h_{ab})] = -\tau_p \int d^{p+1}\xi \left(1 + \tfrac{p-11}{12}\tilde{\phi}\right)\left(1 - \tfrac{1}{2}h_{aa}\right) \\ &= -\tau_p \int d^{p+1}\xi - \tau_p \int d^{p+1}\xi \left[\tfrac{p-11}{12}\tilde{\phi} - \tfrac{h_{aa}}{2}\right] + \mathcal{O}(h\tilde{\phi}) \to -\tau_p \int d^{p+1}\xi \left[\tfrac{p-11}{12}\tilde{\phi} - \tfrac{h_{aa}}{2}\right] + \mathcal{O}(h\tilde{\phi}) \end{aligned} \tag{8.7.9}$$

where we dropped the constant term, which is the brane vacuum energy and does not enter the one-particle exchange amplitude. The term displayed in square brackets is precisely the linear source for h_{ab} and $\tilde{\phi}$; the omitted $\mathcal{O}(h\tilde{\phi})$ and higher terms contain at least two fluctuating fields and do not contribute to the tree-level exchange considered here. We keep h_{aa} as the trace along the brane worldvolume, while $h = h^\mu{}_\mu$ denotes the full spacetime trace. From (8.7.8), we can read off the propagators. Look at the kinetic terms for ϕ. We can define a new dilaton as;

$$\phi' = \frac{i}{\kappa\sqrt{6}} \tilde{\phi}$$

and then, the kinetic term for ϕ' becomes;

$$-\frac{1}{12\kappa^2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} = \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi'$$

i.e the canonical kinetic term. The propagator for ϕ' is well known from QFT and thus, we have;

$$\langle \phi' \phi' \rangle = \frac{i}{k^2} \Rightarrow -\frac{1}{6\kappa^2} \langle \tilde{\phi}\tilde{\phi} \rangle = \frac{i}{k^2} \Rightarrow \langle \tilde{\phi}\tilde{\phi} \rangle = -\frac{6i\kappa^2}{k^2}$$

We see that for an arbitrary number of dimensions, we can define a dilaton as follows:

$$\phi' = \frac{2i}{\kappa\sqrt{D-2}} \tilde{\phi}$$

and ϕ' will still have the canonical kinetic term. So, for arbitrary number of dimensions, we have the following result.

$$\langle \phi' \phi' \rangle = \frac{i}{k^2} \Rightarrow -\frac{4}{(D-2)\kappa^2} \langle \tilde{\phi}\tilde{\phi} \rangle = \frac{i}{k^2} \Rightarrow \langle \tilde{\phi}\tilde{\phi} \rangle = -\frac{(D-2)i\kappa^2}{4k^2} \tag{8.7.10}$$

The de Donder-gauge graviton propagator following from the quadratic Einstein-frame action is

$$\langle h_{\mu\nu} h_{\alpha\beta} \rangle = -\frac{2i\kappa^2}{k^2} \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2} \eta_{\mu\nu}\eta_{\alpha\beta} \right) \tag{8.7.11}$$

Contracting the dilaton source $\tau_p(p-11)\tilde{\phi}/12$ and the graviton source $-\tau_p h_{aa}/2$ with the propagators gives the field-theory exchange graph

$$\mathcal{A} = \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[6\left(\frac{p-11}{12}\right)^2 + \frac{1}{2}\left(2(p+1) - \frac{1}{12}(p+1)^2\right)\right]$$

$$= \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[\frac{1}{24}(p-11)^2 + p + 1 - \frac{1}{24}(p+1)^2\right] = \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[\frac{1}{24}\left(p^2 - 22p + 121\right) + p + 1 - \frac{1}{24}\left(p^2 + 2p + 1\right)\right]$$

$$= \frac{6i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} \qquad (8.7.12)$$

Now, comparing 8.7.7 and 8.7.12, we get;

$$\frac{6i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} = iV_{p+1}\frac{24\pi}{2^{10}k_\perp^2}\left(4\pi^2\alpha'\right)^{11-p} \Rightarrow \tau_p^2 = \frac{\pi^2}{256\kappa^2}\left(4\pi^2\alpha'\right)^{11-p} \Rightarrow \tau_p = \frac{\pi}{16\kappa}\left(4\pi^2\alpha'\right)^{(11-p)/2}$$

$$\Rightarrow T_p e^{-\phi_0} = \frac{\pi}{16\kappa}\left(4\pi^2\alpha'\right)^{(11-p)/2} \Rightarrow T_p = e^{\phi_0}\frac{\pi}{16\kappa}\left(4\pi^2\alpha'\right)^{(11-p)/2} \qquad (8.7.13)$$

Now, we can see that 8.7.13 satisfies 8.7.5 as follows:

$$T_{p-1} = e^{\phi_0}\frac{\pi}{16\kappa}\left(4\pi^2\alpha'\right)^{(11-(p-1))/2} = e^{\phi_0}\frac{\pi}{16\kappa}\left(4\pi^2\alpha'\right)^{(11-p)/2}\left(4\pi^2\alpha'\right)^{1/2} = T_p 2\pi\sqrt{\alpha'} \Rightarrow T_p = \frac{T_{p-1}}{2\pi\sqrt{\alpha'}}$$

The coupling relation used here is the same Euler-characteristic counting derived at the start of the volume. A disk has $\chi = 1$, so an open-string tree amplitude carries one power of g_s^{-1} before external-vertex normalizations; a sphere has $\chi = 2$, so a closed-string tree amplitude carries g_s^{-2}. With canonical external states this gives $g_o^2 \propto g_s$ and $g_c \propto g_s$, hence $g_o^2 \propto g_c$. The proportionality constants are convention-dependent and are fixed by the same factorization comparison that fixes T_p above.

7.8 T-duality of the unoriented string

T-duality changes the action of worldsheet parity on the dual coordinate. If $X^9 = X_L^9 + X_R^9$ and T-duality sends $X_R^9 \mapsto -X_R^9$, then the dual coordinate is $X'^9 = X_L^9 - X_R^9$. Conjugating the orientation reversal Ω by T-duality gives

$$T_9\,\Omega\,T_9^{-1} = \Omega\mathcal{R}_9, \qquad \mathcal{R}_9 : X'^9 \longmapsto -X'^9. \qquad (7.17)$$

The dual projection contains a spacetime reflection. Its fixed loci are orientifold planes. For type I on a circle this gives the type-I$'$ interval with fixed planes at $X'^9 = 0$ and $X'^9 = \pi R'$. Wilson lines in Eq. (7.16) become D-brane positions on this interval, and tadpole cancellation becomes the cancellation of D-brane charge against orientifold-plane charge [10, 167, 352].

7.8.1 Open strings

For open strings, the boundary condition is part of the worldsheet definition of the theory. Varying the Polyakov action gives a boundary term

$$\delta X_\mu\,\partial_\sigma X^\mu\big|_{\partial\Sigma} = 0. \qquad (7.18)$$

Thus Neumann directions obey $\partial_\sigma X^\mu = 0$, while Dirichlet directions obey $\delta X^\mu = 0$ at the boundary. This local equation is the worldsheet origin of the D-brane interpretation: an open-string endpoint is a boundary condition of the worldsheet CFT. Under T-duality, Neumann and Dirichlet conditions are exchanged, so brane positions and Wilson lines become dual descriptions of the same worldsheet input. [10, 38, 350–352].

Chapter 8

Type II and type I superstrings

The bosonic string has a tachyon and no spacetime fermions, so it is not a phenomenologically viable final theory; this chapter constructs the string that removes both problems. We add worldsheet fermions ψ^μ, which enlarges the constraint algebra to the $(1,1)$ superconformal algebra and lowers the critical dimension to ten. The fermions admit two periodicities, Neveu–Schwarz and Ramond, and the entire structure of the ten-dimensional theories follows from asking which combinations of sectors can coexist. We impose only the worldsheet consistency conditions of the Introduction: mutual locality of vertex operators, closure of their operator products, and modular invariance of the one-loop trace. The complete list of solutions is highly restricted, type IIA, type IIB, and the tachyonic 0A and 0B theories, and the GSO projection is the name of the selection these conditions enforce. We verify modular invariance explicitly through the theta-function identity that makes the partition function vanish, the worldsheet signature of spacetime supersymmetry, and close with the unoriented projection to type I, whose Ramond–Ramond tadpole fixes the gauge group to $SO(32)$ [10, 367].

8.1 The superconformal algebra

To include spacetime fermions through worldsheet fields, we introduce a two-dimensional analogue of the Dirac equation:

$$\rho^\alpha \partial_\alpha \psi^\mu = 0 \tag{10.1.1}$$

where $\alpha \in \{\tau, \sigma\}$ and ρ^α are two dimensional Dirac matrices. Now, the Dirac matrices are supposed to satisfy;

$$\left\{\rho^\alpha, \rho^\beta\right\} = 2\eta^{\alpha\beta} \tag{10.1.2}$$

which means that $\left(\rho^0\right)^2 = -1, \left(\rho^0\right)^2 = 1$ and $\{\rho^0, \rho^1\} = 0$. This is satisfied if we take;

$$\rho^0 = -i\sigma^2, \rho^1 = \sigma^1 \tag{10.1.3}$$

Following the construction of the chirality matrix in ten dimensions [2], the chiral matrix ρ takes the form

$$\rho^{0+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \rho^{0-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho = 2S_0 = 2\left(\rho^{0+}\rho^{0-} - \frac{1}{2}\mathbb{I}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the matrices in 10.1.3 are real, it is a Majorana representation and we can take the spinors to be real. We define $\bar{\psi}$ (for Majorana spinor) as follows:

$$\bar{\psi} = \psi^\dagger \rho^0 = \psi^T \rho^0 \tag{10.1.4}$$

The first equality is true for any spinor.
We can now start with the following action;

$$S = S_{\text{bosonic}} + S_{\text{fermionic}} = \frac{1}{2\pi\alpha'} \int d^2 z \partial X \bar{\partial} X + \frac{1}{4\pi} \int d^2 z \left(\psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \bar{\partial} \tilde{\psi}_\mu\right)$$

$$= \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X\bar{\partial}X + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right) \tag{10.1.5}$$

The OPE's are as follows:

$$X^\mu(z)X^\nu(0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2 \tag{10.1.6}$$

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}, \tilde{\psi}^\mu(z)\tilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{z} \tag{10.1.7}$$

which implies that;

$$\{\psi_0^\mu, \psi_0^\nu\} = \left\{\tilde{\psi}_0^\mu, \tilde{\psi}_0^\nu\right\} = \eta^{\mu\nu} \tag{10.1.8}$$

Now, we calculate the energy momentum tensor for 10.1.5. We do remind ourselves that;

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2z\bar{z}} \Rightarrow \sqrt{|g|} = \frac{1}{2z\bar{z}} \tag{10.1.9}$$

which implies that;

$$\partial^z = g^{z\bar{z}}\partial_{\bar{z}} = 2z\bar{z}\bar{\partial}, \partial^{\bar{z}} = g^{\bar{z}z}\partial_z = 2z\bar{z}\partial \tag{10.1.10}$$

The mixed components of the stress tensor are

$$-\frac{1}{2\pi}T_z^{\bar{z}} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\bar{z}}X^\mu\right)}\partial_z X^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_{\bar{z}}\psi^\mu\right)}\partial_z\psi^\mu - g_{zz}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\left(\bar{\partial}X^\mu\right)}\partial X^\mu + \frac{\partial\mathcal{L}}{\partial\left(\bar{\partial}\psi^\mu\right)}\partial\psi^\mu + \frac{\partial\mathcal{L}}{\partial\left(\bar{\partial}^z\tilde{\psi}^\mu\right)}\partial\tilde{\psi}^\mu$$

$$\Rightarrow T_z^{\bar{z}} = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \tag{10.1.11}$$

$$-\frac{1}{2\pi}T_{\bar{z}}^z = \frac{\partial\mathcal{L}}{\partial\left(\partial_z X^\mu\right)}\partial_{\bar{z}}X^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_z\tilde{\psi}^\mu\right)}\partial_{\bar{z}}\tilde{\psi}^\mu - g_{\bar{z}\bar{z}}\mathcal{L} = \frac{\partial\mathcal{L}}{2\partial\left(\partial X^\mu\right)}\bar{\partial}X^\mu + \frac{\partial\mathcal{L}}{2\partial\left(\partial\tilde{\psi}^\mu\right)}\bar{\partial}\tilde{\psi}^\mu$$

$$\Rightarrow T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'}\bar{\partial}X^\mu\bar{\partial}X_\mu - \frac{1}{2}\tilde{\psi}^\mu\partial\tilde{\psi}_\mu \tag{10.1.12}$$

It can be shown that ${T^z}_z = {T^{\bar{z}}}_{\bar{z}} = 0$. We define the stress tensor as follows:

$$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu, \tilde{T}_B(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial}X^\mu\bar{\partial}X_\mu - \frac{1}{2}\tilde{\psi}^\mu\partial\tilde{\psi}_\mu \tag{10.1.13}$$

We denote the holomorphic and antiholomorphic stress-tensor components by T_B and $\bar{T}_B(z)$, following the standard convention [2], and use these names throughout. The $(1,1)$ superconformal transformations are

$$\delta X^\mu = \sqrt{\frac{\alpha'}{2}}\left(\epsilon(z)\psi^\mu(z) + \epsilon^*(z)\tilde{\psi}^\mu(z)\right)$$

$$\delta\psi^\mu = -\sqrt{\frac{2}{\alpha'}}\epsilon(z)\partial X^\mu(z)$$

$$\delta\tilde{\psi}^\mu = -\sqrt{\frac{2}{\alpha'}}\epsilon^*(\bar{z})\bar{\partial}X^\mu(\bar{z}) \tag{10.1.14}$$

It can be checked that the action 10.1.5 is invariant under these transformations. We can calculate the quantity that is conserved in these symmetries by using the Noetherian method.

$$\delta S = \frac{1}{2\pi}\int d^2z\left[\partial\epsilon^*\left(\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu\bar{\partial}X_\mu\right) + \bar{\partial}\epsilon\left(\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu\right)\right]$$

and we define the conserved quantities known as the worldsheet supercurrents [2]:

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu(z), \tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu\bar{\partial}X_\mu(\bar{z}) \tag{10.1.15}$$

Using 10.1.7, we can check that T_B and T_F close an operator algebra (same is the case for anti-holomorphic counterparts). We get;

$$T_B(z)T_B(0) \sim \frac{(3D/2)}{2z^4} + \frac{2T_B(0)}{z^2} + \frac{\partial T_B(0)}{z}$$
$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{\partial T_B(0)}{z}$$
$$T_F(z)T_F(0) \sim \frac{D}{z^3} + \frac{2T_B(0)}{z} \tag{10.1.16}$$

which gives us a central charge of $c = 3D/2$ for the theory. It also shows that T_F is a primary field of conformal weight $\left(\frac{3}{2}, 0\right)$. Written in terms of the central charge, we have;

$$T_B(z)T_B(0) \sim \frac{c}{2z^4} + \frac{2T_B(0)}{z^2} + \frac{\partial T_B(0)}{z}$$
$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{\partial T_B(0)}{z}$$
$$T_F(z)T_F(0) \sim \frac{2c}{3z^3} + \frac{2T_B(0)}{z} \tag{10.1.17}$$

This operator algebra is called the $(1,1)$ superconformal algebra. Two standard examples of superconformal field theories illustrate it. The first is the superconformal generalization of the bc ghost system,

$$S_{bc} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c, \quad h_b = \lambda, \quad h_c = 1-\lambda \tag{10.1.18}$$

with the generalization including the fermionic, commuting ghosts named β and γ. The SCFT action is then given as follows:

$$S_{bc\beta\gamma} = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma), h_\beta = \lambda - \frac{1}{2}, h_\gamma = \frac{3}{2} - \lambda \tag{10.1.19}$$

8.2 NS and R sectors

If we start with the theory on the cylinder for the closed string (with coordinates $w = \sigma + i\tau, \bar{w} = \sigma - i\tau \Rightarrow z = e^{-iw}, \bar{z} = e^{i\bar{w}}$ with $0 \le \sigma \le 2\pi, -\infty \le \tau \le \infty$), then we have the following cases;

$$\psi^\mu(w+2\pi) = \begin{cases} \psi^\mu(w) & \text{(Ramond)} \\ -\psi^\mu(w) & \text{(Neveu Schwarz)} \end{cases} \tag{10.2.1}$$

$$\Rightarrow \quad \psi^\mu(w+2\pi) = e^{2\pi i\nu}\psi^\mu(w), \qquad \begin{cases} \nu = 0 & \text{(Ramond),} \\ \nu = \frac{1}{2} & \text{(Neveu Schwarz).} \end{cases}$$

Similarly,

$$\tilde{\psi}^\mu(w+2\pi) = \begin{cases} \tilde{\psi}^\mu(w) & \text{(Ramond)} \\ -\tilde{\psi}^\mu(w) & \text{(Neveu Schwarz)} \end{cases} \tag{10.2.2}$$

$$\Rightarrow \quad \tilde{\psi}^\mu(w+2\pi) = e^{2\pi i\tilde{\nu}}\tilde{\psi}^\mu(w), \qquad \begin{cases} \tilde{\nu} = 0 & \text{(Ramond),} \\ \tilde{\nu} = \frac{1}{2} & \text{(Neveu Schwarz).} \end{cases}$$

We will take $X^\mu(w)$ to be periodic. Note that in this choice, $T_B(w)$ is periodic in any of the choices in 10.2.1 but $T_F(w)$ has the same periodicity as $\psi^\mu(w)$ (the same thing goes for the anti-holomorphic sector). For the open string, we take $0 \le \sigma \le \pi$. The holomorphic fermion field on the cylinder can be expanded as follows:

$$\psi^\mu(w) = i^{-1/2}\sum_{r\in\mathbb{Z}+\nu}\psi^\mu_{(c)r}e^{irw} \tag{10.2.3}$$

where the (c) subscript indicates that these are cylinder modes. Now, since the free fermion is a primary field with conformal weight $1/2$, we can find the fermion field on the plane as follows:

$$\psi^\mu(z) = \left(\frac{\partial z}{\partial w}\right)^{-1/2} \psi^\mu(w) = \frac{(-i)^{-1/2}}{z^{1/2}}\psi^\mu(w) = \frac{(-i)^{-1/2}}{z^{1/2}} \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_{(c)r} e^{irw} = \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_{(p)r} z^{-r-\frac{1}{2}} \tag{10.2.4}$$

where we defined $\psi^\mu_{(p)r} = (-i)^{-1/2}\psi^\mu_{(c)r}$ as the modes on the plane. We will drop the $(\ p)$ subscript from now on. Please note that on the plane, Ramond fermions are anti-periodic and vice versa (which is the opposite of the cylinder story). Similar expression is obtained for the anti-holomorphic fermionic field;

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \tilde{\psi}^\mu_r \bar{z}^{-r-\frac{1}{2}} \tag{10.2.5}$$

The modes expansions of the bosonic currents are already known to us i.e.

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{\alpha^\mu_m}{z^{m+1}}, \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}} \tag{10.2.6}$$

Using the following identities;

$$\begin{aligned} \oint dz\, [\partial X^\mu(z), \partial X^\nu(w)] &= \oint_{C(w)} dz \mathcal{R}\, [\partial X^\mu(z)\partial X^\nu(w)] \\ \oint dz\, \{\psi^\mu(z), \psi^\nu(w)\} &= \oint_{C(w)} dz \mathcal{R}\, [\psi^\mu(z)\psi^\nu(w)] \end{aligned} \tag{10.2.7}$$

where $\mathcal{R}$ means radial ordering (which gives the OPE of the operators of the in this operator), and the OPE of ∂X^μ with itself and OPE of ψ^μ with itself gives us the following (anti) commutators;

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n}, \{\psi^\mu_r, \psi^\nu_s\} = \left\{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\right\} = \eta^{\mu\nu}\delta_{r+s} \tag{10.2.8}$$

Using the mode expansions of T_B and T_F

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}, T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+\frac{3}{2}}} \tag{10.2.9}$$

we can derive the following algebra;

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}\left(4r^2-1\right)\delta_{r+s} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \end{aligned} \tag{10.2.10}$$

called the RNS algebra or the Ramond algebra for r, s integers and Neveu-Schwarz algebra for r, s half integers.

8.2.1 NS and R spectrum

The NS spectrum is easy to construct. $|0\rangle_{\mathrm{NS}}$ is the ground state (and it is unique) such that;

$$\psi^\mu_r |0\rangle_{\mathrm{NS}} = 0 \text{ if } r > 0 \tag{10.2.11}$$

The R sector has more than one ground state. If $|0\rangle_{\mathrm{R}}$ is a ground state, then so is $\psi^\mu_0|0\rangle_{\mathrm{R}}$ and due to (10.1.8), these states form a representation of the Clifford algebra. Seeing the similarity in 21.1.1 and 10.2.8, we identity the following;

$$\Gamma^\mu \sim \sqrt{2}\psi^\mu_0$$

Therefore $e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$ will take Ramond ground states to Ramond ground states and hence, $|0\rangle_R$ can be thought of having a spinor index and as living in a representation of Clifford algebra. Since a basis for spinors is given in 21.1.6 and a member of this basis is represented by the k-tuple $\mathbf{s}$, we can choose this basis for the Ramond ground states. So, the basis for R-ground states is as follows:

$$|s_0,\ldots,s_4\rangle = |\mathbf{s}\rangle \quad s_a = \pm\frac{1}{2} \tag{10.2.12}$$

Because of 21.1.9, we see that half integral values of s_a mean that R -ground states are spinors. We now construct an operator called the fermion number operator denoted as F. Since Γ^μ is identified as the zero mode of ψ^μ, and since Γ anti commutes with Γ^μ, we can work out a quantity that anticommutes with the full ψ^μ. This quantity is $e^{\pi iF}$ where F is the fermion number operator (which is defined only mod2) with the following desired property;

$$F|\psi\rangle = f|\psi\rangle \Rightarrow F\left(\left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right)|\psi\rangle\right) = (f\pm 1)\left(\left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right)|\psi\rangle\right) = \left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right)F|\psi\rangle \pm \left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right)|\psi\rangle$$
$$\Rightarrow \left[F, \left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right)\right] = \pm\left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right) \tag{10.2.13}$$

where $|\psi\rangle$ is an arbitrary state and f is just a number. So, ψ^μ increases F by one. Using (10.2.13), we get;

$$e^{\pi iF}|\psi\rangle = e^{\pi if}|\psi\rangle \Rightarrow e^{\pi iF}\psi^\mu|\psi\rangle = e^{\pi i(f+1)}\psi^\mu|\psi\rangle = -\psi^\mu e^{\pi if}|\psi\rangle = -\psi^\mu e^{\pi iF}|\psi\rangle \Rightarrow \left\{e^{\pi iF}, \psi^\mu\right\} = 0$$

So, if we can construct an F that satisfies (10.2.13), then $e^{\pi iF}$ satisfies our requirements. Before constructing F, we extend $\Sigma^{\alpha\beta}$ and S_a as well (like we extended Γ^μ and Γ). Here are the extensions;

$$\Sigma^{\alpha\beta} = -\frac{i}{4}\left[\Gamma^\alpha, \Gamma^\beta\right] = -\frac{i}{2}\left[\psi_0^\alpha, \psi_0^\beta\right] \to -\frac{i}{2}\sum_{r\in\mathbb{Z}+\nu}\left[\psi_r^\alpha, \psi_{-r}^\beta\right] = -i\sum_{r\in\mathbb{Z}+\nu}\psi_r^\alpha\psi_{-r}^\beta$$
$$S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1} \Rightarrow S_0 = \sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_{-r}^1, \quad S_a = -i\sum_{r\in\mathbb{Z}+\nu}\psi_r^{2a}\psi_{-r}^{2a+1}, \quad (a\neq 0) \tag{10.2.14}$$

Now, the claim is that F is given as follows:

$$F = S_0 + S_1 + S_2 + S_3 + S_4 \tag{10.2.15}$$

We now derive some anti-commutation relations. For $a = 1,2,3,4$, we have the following two anti-commutation relations;

$$S_a\psi_s^{2a} = -i\sum_{r\in\mathbb{Z}+\nu}\psi_r^{2a}\psi_{-r}^{2a+1}\psi_s^{2a} = i\psi_s^{2a+1} - i\psi_s^{2a}\sum_{r\in\mathbb{Z}+\nu}\psi_r^{2a}\psi_r^{2a+1}$$
$$= i\psi_s^{2a+1} + \psi_s^{2a}S_a \Rightarrow \left[S_a, \psi_s^{2a}\right] = \left[F, \psi_s^{2a}\right] = i\psi_s^{2a+1}$$
$$S_a\psi_s^{2a+1} = -i\sum_{r\in\mathbb{Z}+\nu}\psi_r^{2a}\psi_{-r}^{2a+1}\psi_s^{2a+1} = -i\psi_s^{2a} - i\psi_s^{2a+1}\sum_{r\in\mathbb{Z}+\nu}\psi_r^{2a}\psi_r^{2a+1}$$
$$= -i\psi_s^{2a} + \psi_s^{2a+1}S_a \Rightarrow \left[S_a, \psi_s^{2a+1}\right] = \left[F, \psi_s^{2a+1}\right] = -i\psi_s^{2a}$$

The above two relations give the following;

$$\left[F, \psi_s^{2a} \pm i\psi_s^{2a+1}\right] = i\psi_s^{2a+1} \pm i\left(-i\psi_s^{2a}\right) = \pm\left(\psi_s^{2a} \pm i\psi_s^{2a+1}\right) \tag{10.2.16}$$

which is consistent with (10.2.13). Moreover, for the remaining spinors, we have the following;

$$S_0\psi_s^0 = \sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_{-r}^1\psi_s^0 = -\sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_s^0\psi_{-r}^1 = -\sum_{r\in\mathbb{Z}+\nu}\left(-\delta_{r+s} - \psi_s^0\psi_r^0\right)\psi_{-r}^1$$
$$= \psi_s^1 + \psi_s^0\sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_{-r}^1 = \psi_s^1 + \psi_s^0 S_0 \Rightarrow \left[S_0, \psi_s^0\right] = \psi_s^1$$
$$S_0\psi_s^1 = \sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_{-r}^1\psi_s^1 = \sum_{r\in\mathbb{Z}+\nu}\psi_r^0\left(\delta_{-r+s} - \psi_s^1\psi_{-r}^1\right)$$
$$= \psi_s^0 + \psi_s^1\sum_{r\in\mathbb{Z}+\nu}\psi_r^0\psi_{-r}^1 = \psi_s^0 + \psi_s^1 S_0 \Rightarrow \left[S_0, \psi_s^1\right] = \psi_s^0$$

Therefore, we get;

$$\left[F, \psi_r^0 \pm i\psi_r^1\right] = \psi_r^1 \pm i\psi_r^0 = \pm i\left(\psi_r^0 \mp i\psi_r^1\right)$$

This commutator differs from Eq. (10.2.13) because the fermion-number convention used here includes a Lorentz-generator normalization. A convention often used for the NS sector is

$$F = \sum_{r>0}\sum_{a=2}^{8} \psi_{-r}^a \psi_r^a - 1 \tag{10.2.17}$$

This implies the following (for $b > 2$);

$$\left[F, \psi_s^b\right] = \sum_{r>0}\sum_{a=2}^{8}\left[\psi_{-r}^a\psi_r^a, \psi_s^b\right] = \sum_{r>0}\sum_{a=2}^{8}\left(\psi_{-r}^a\psi_r^a\psi_s^b - \psi_s^b\psi_{-r}^a\psi_r^a\right) = \sum_{r>0}\sum_{a=2}^{8}\eta^{ab}\left(\delta_{r+s}\psi_{-r}^a - \delta_{s-r}\psi_r^a\right)$$
$$= \sum_{r>0}\left(\delta_{r+s}\psi_{-r}^b - \delta_{s-r}\psi_r^b\right)$$

Now, we have the following two cases;

$$\left[F, \psi_s^b\right] = \sum_{r>0}\left(\delta_{r+s}\psi_{-r}^b - \delta_{s-r}\psi_r^b\right) = -\psi_s^b \quad (s > 0)$$
$$\left[F, \psi_s^b\right] = \sum_{r>0}\left(\delta_{r+s}\psi_{-r}^b - \delta_{s-r}\psi_r^b\right) = \psi_s^b \quad (s < 0)$$

which implies the following;

$$\left[F, \psi_{\pm s}^b\right] = \mp\psi_{\pm s}^b \quad (s > 0)$$

and this in turn implies the following;

$$F|\psi\rangle = f|\psi\rangle \Rightarrow F\left(\psi_{\pm s}^b|\psi\rangle\right) = \left(\psi_{\pm s}^b F \mp \psi_{\pm s}^b\right)|\psi\rangle = (f \mp 1)\psi_{\pm s}^b|\psi\rangle$$

So, ψ_s^b decreases F by one and ψ_{-s}^b increases F by one.

8.2.2 Closed string spectra

In the closed string case, the NS-NS vacuum is $|0\rangle_{\text{NS}} \otimes |0\rangle_{\text{NS}}$ and thus, the vacuum has integer spin. In the R-R sector, the vacuum is $|\mathbf{s}, \mathbf{s}'\rangle = |\mathbf{s}\rangle \otimes |\mathbf{s}'\rangle$. This vacuum lies in $\mathbf{32}_{\text{Dirac}} \times \mathbf{32}_{\text{Dirac}}$ representation and using 21.1.35, we can break it down as follows:

$$\mathbf{32}_{\text{Dirac}} \times \mathbf{32}_{\text{Dirac}} = [0]^2 + [1]^2 + [2]^2 + [3]^2 + [4]^2 + [5]$$

If the left and right vacua have a definite chirality, equivalently a definite value of $e^{\pi i F}$, then the left and right Ramond ground states have definite fermion numbers F and $\widetilde{F}$. For different choices of $F, \widetilde{F}$, the breakdown of R–R vacua is as follows, using Eq. (21.1.37):

F	$\tilde{F}$	$e^{\pi i F}$	$e^{\pi i \tilde{F}}$	SO(1, 9) rep
0	0	1	1	$[1] + [3] + [5]_+$
0	1	1	-1	$[0] + [2] + [4]$
1	0	-1	1	$[0] + [2] + [4]$
1	1	-1	-1	$[1] + [3] + [5]_-$

The NS–R and R–NS vacua are spacetime fermions because they contain one spacetime spinor index.

8.3 Vertex operators and bosonization

Using (10.2.4), we see that in the NS sector (with $\nu = 1/2$), the fermion is single-valued. Moreover, 10.1.7, we see that $\psi\psi$ OPE is single value. Hence, the products and derivatives of ψ in the NS sector must be in the NS sector. We now calculate the state corresponding to the k-th derivative of ψ in the NS sector.

$$\partial^k \psi^\mu(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu \frac{(-r-\frac{1}{2})!}{(-r-k-\frac{1}{2})!} z^{-r-k-\frac{1}{2}}$$

So, the state corresponding to $\partial^k\psi^\mu$ is as follows:

$$\left|\partial^k\psi^\mu\right\rangle = \lim_{z\to 0}\partial^k\psi^\mu(z)|0\rangle = \lim_{z\to 0}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{(-r-\frac{1}{2})!}{(-r-k-\frac{1}{2})!} z^{-r-k-\frac{1}{2}}\psi_r^\mu|0\rangle$$

Now, $\psi_r^\mu|0\rangle$ have to be zero for $r+k>1/2$ for all k. The tightest bound comes from $k=0$ and thus;

$$\psi_r^\mu|0\rangle = 0, \quad r = \frac{1}{2}, \frac{3}{2}, \ldots$$

So, we get;

$$\left|\partial^k\psi^\mu\right\rangle = k!\psi^\mu_{-k-\frac{1}{2}}|0\rangle$$

So, we see that for every derivative, we have one state in the NS sector.
For the Ramond vertex operators, we need bosonization. Let $H(z)$ be a holomorphic field (with $\alpha'=2$). The OPE's can be calculated using the following;

$$\mathcal{V}_{k_L,k_R}(z_1,\bar{z}_1)\,\mathcal{V}_{k_L',k_R'}(z_2,\bar{z}_2) \sim z_{12}^{\alpha' k_L k_L'/2}\bar{z}_{12}^{\alpha' k_R k_R'/2}\mathcal{V}_{k_L+k_L',k_R+k_R}(z_2,\bar{z}_2)$$
$$\text{where } \mathcal{V}_{k_L,k_R}(z,\bar{z}) = e^{ik_L X_L(z)+ik_R X_R(\bar{z})} \tag{10.3.1}$$

The OPE's are as follows:

$$e^{iH(z)}e^{-iH(z)} \sim \frac{1}{z}, e^{iH(z)}e^{iH(z)} \sim \mathcal{O}(z)e^{-iH(z)}e^{-iH(z)} \sim \mathcal{O}(z) \tag{10.3.2}$$

Now, consider two Majorana-Weyl fermions;

$$\psi(z) = \frac{1}{\sqrt{2}}\left(\psi^1(z)+i\psi^2(z)\right), \quad \bar{\psi}(z) = \frac{1}{\sqrt{2}}\left(\psi^1(z)-i\psi^2(z)\right) \tag{10.3.3}$$

and the OPE's for fermions is calculated using 10.1.7 as follows:

$$\psi(z)\bar{\psi}(z) \sim \frac{1}{2}\left(\psi^1(z)\psi^1(z)+\psi^2(z)\psi^2(z)\right) = \frac{1}{z}, \quad \psi(z)\psi(z)\sim(O)(z), \quad \bar{\psi}(z)\bar{\psi}(z)\sim(O)(z) \tag{10.3.4}$$

We identify $\psi \sim e^{iH(z)}$ and $\bar{\psi}(z)\sim e^{-iH(z)}$. This identification extends to all operators built from these elementary fields. Expanding the exponential OPEs around the midpoint gives

$$e^{iH(z)}e^{-iH(z)} \sim \frac{1}{2z} + i\partial H(0) + 2zT_B^H(0) + \mathcal{O}\left(z^2\right)$$
$$\psi(z)\psi(-z) \sim \frac{1}{2z} + \psi\bar{\psi}(0) + 2zT_B^\psi(0) + \mathcal{O}\left(z^2\right)$$

So, the stress tensors match and we have $\psi\bar{\psi}\sim i\partial H$. The fermionic sign is reproduced by the monodromy of the exponential operators together with the usual cocycle factors: exchanging two unit-charge exponentials gives the phase $e^{i\pi}=-1$, and the cocycles ensure the same anti-commutation for different complex fermions. Now, consider the general periodicity condition for 10.3.3) (as compared to (10.2.1);

$$\psi(w+2\pi) = e^{2\pi i\nu}\psi(w), \quad \nu\in\mathbb{R} \tag{10.3.5}$$

and define the ground state in the sector with twist ν by the annihilation conditions

$$\psi_{n+\nu}|0\rangle_\nu = \bar{\psi}_{n-\nu+1}|0\rangle_\nu = 0 \tag{10.3.6}$$

It gives the Laurent series of $\psi(z)$ and $\bar{\psi}(z)$ as follows:

$$\psi(z)=\sum_{n=1}^{\infty}\frac{\psi_{\nu-n}}{z^{\nu-n+1/2}}\sim\mathcal{O}\left(z^{1/2-\nu}\right)\Rightarrow\psi(z)\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{1/2-\nu}\right)$$
$$\bar\psi(z)=\sum_{n=0}^{\infty}\frac{\psi_{-\nu-n}}{z^{-\nu-n+1/2}}\sim\mathcal{O}\left(z^{\nu-1/2}\right)\Rightarrow\bar\psi(z)\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{\nu-1/2}\right)$$

where $\mathcal{A}_\nu$ is the corresponding operator for $|0\rangle_\nu$. Using the identifications for $\psi(z)$ and $\bar\psi(z)$ from before, we deduce the following;

$$\psi(z)\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{1/2-\nu}\right)\Rightarrow e^{iH(z)}\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{1/2-\nu}\right)\sim e^{iH(z)}e^{i(1/2-\nu)H(z)}\Rightarrow\mathcal{A}_\nu(z)\sim e^{i(1/2-\nu)H(z)}$$
$$\bar\psi(z)\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{\nu-1/2}\right)\Rightarrow e^{-iH(z)}\mathcal{A}_\nu(0)\sim\mathcal{O}\left(z^{\nu-1/2}\right)\sim e^{-iH(z)}e^{i(1/2-\nu)H(z)}\Rightarrow\mathcal{A}_\nu(z)\sim e^{i(1/2-\nu)H(z)}$$

So, we get a consistent expression for $\mathcal{A}_\nu$. The weight of $\mathcal{A}_\nu$ is easily seen to be as follows:

$$h\left(\mathcal{A}_\nu\right)=\frac{1}{2}\left(\nu-\frac{1}{2}\right)^2$$

The same weight can be derived directly from fermionic modes. We see that 10.3.5 is unchanged if $\nu\to\nu+1$, but the state satisfying the annihilation conditions changes. It is enough to take $0\le\nu\le1$, because ν is defined modulo one by the boundary condition, and different representatives are related by spectral flow. Now, we show that $|0\rangle_{\nu+1}=\bar\psi_{-\nu}|0\rangle_\nu$ by checking that this state satisfies 10.3.6 with $\nu\to\nu+1$.

$$\psi_{n+\nu+1}|0\rangle_{\nu+1}=\psi_{n+\nu+1}\bar\psi_{-\nu}|0\rangle_\nu=-\bar\psi_{-\nu}\psi_{n+\nu+1}|0\rangle_\nu=0$$
$$\bar\psi_{n-\nu}|0\rangle_{\nu+1}=\bar\psi_{n-\nu}\bar\psi_{-\nu}|0\rangle_\nu=-\bar\psi_{-\nu}\bar\psi_{n-\nu}|0\rangle_\nu=\begin{cases}0 & \text{if } n\neq0\,(|0\rangle_\nu \text{ condition })\\ 0 & \text{if } n=0\left(\bar\psi_{-\nu}\bar\psi_{-\nu}=0\right)\end{cases}$$

This phenomenon is called spectral flow. For the Ramond case, we have $\nu=0$ but the state $|0\rangle_0$ will flow into $|0\rangle_1$. The corresponding operators for these states are as follows:

$$|0\rangle_0\sim e^{iH(z)/2},\quad |0\rangle_1\sim e^{-iH(z)/2}$$

We relabel these states as follows:

$$|s\rangle\sim e^{isH(z)},\quad s=\frac{1}{2}$$

This resembles the ground state of the Ramond sector. However, for Ramond sector, we need five different s values and thus, five different H 's. These are dual to five different complex fermions. They are constructed as follows:

$$\frac{1}{\sqrt2}\left(\pm\psi^0(z)+\psi^1(z)\right)\sim e^{\pm iH^0}$$
$$\frac{1}{\sqrt2}\left(\pm\psi^{2a}(z)+\psi^{2a+1}(z)\right)\sim e^{\pm iH^a},\quad a=1,2,3,4$$

Thus finally, we can come up with the vertex operator for the Ramond ground state (denoted as $\Theta_{\mathbf{s}}$) as follows:

$$|\mathbf{s}\rangle\sim\Theta_{\mathbf{s}}=\exp\left[i\sum_a s_aH^a\right]$$

Θ_{s} is also called a spin field. (Bosonization of the bc ghost system and equivalence to linear dilaton)

8.4 The superconformal ghosts

The action of the superconfromal ghost action is as follows:

$$S = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma)$$

and the stress tensor for this theory is as follows (see 2.5.5);

$$T_B(z) =: \partial bc : (z) - \lambda\partial : bc : (z) + : \partial\beta\gamma : (z) - \left(\lambda - \frac{1}{2}\right)\partial : \beta\gamma : (z) \tag{10.4.1}$$

If we choose $\lambda = 2$, we get;

$$T_B(z) =: \partial bc : (z) - 2\partial : bc : (z) + : \partial\beta\gamma : (z) - \frac{3}{2}\partial : \beta\gamma : (z) \tag{10.4.2}$$

The expression for T_F is fixed by requiring that its OPE with b, c, β, γ generate the superconformal transformations with weights $2, -1, 3/2, -1/2$:

$$T_F(z) =: \partial\beta c : (z) + \frac{3}{2} : \beta\partial c : (z) - 2 : b\gamma : (z) \tag{10.4.3}$$

The anomaly cancellation can be checked before fixing $\lambda = 2$. The fermionic first-order pair has

$$h(b) = \lambda, \qquad h(c) = 1 - \lambda, \qquad c_{bc} = 1 - 3(2\lambda - 1)^2.$$

The commuting superghost pair has

$$h(\beta) = \lambda - \frac{1}{2}, \qquad h(\gamma) = \frac{3}{2} - \lambda.$$

For a commuting first-order system with β of weight h_β,

$$c_{\beta\gamma} = 3(2h_\beta - 1)^2 - 1 = 3(2\lambda - 2)^2 - 1 = 12(\lambda - 1)^2 - 1.$$

Hence the total ghost central charge in the superconformal ghost multiplet is

$$c_{\text{gh}}(\lambda) = c_{bc} + c_{\beta\gamma} = \left[1 - 3(2\lambda - 1)^2\right] + \left[12(\lambda - 1)^2 - 1\right] = 9 - 12\lambda.$$

For the RNS string, $\lambda = 2$, so $c_{\text{gh}} = -15$. The matter system (X^μ, ψ^μ) has

$$c_{\text{matter}} = D + \frac{D}{2} = \frac{3D}{2}.$$

The vanishing of the total Weyl anomaly gives

$$\frac{3D}{2} - 15 = 0, \qquad D = 10.$$

Equivalently, the central term in the ghost supercurrent OPE is

$$T_F^G(z)T_F^G(0) \sim \frac{2c_{\text{gh}}}{3z^3} + \frac{2T_B^G(0)}{z},$$

so the same number controls the anomaly of the local superconformal symmetry [3].

So, the conformal weights of the fields are as follows:

$$h(b) = 2, h(c) = -1; h(\beta) = \frac{3}{2}; h(\gamma) = -\frac{1}{2} \tag{10.4.4}$$

and the mode expansions are thus as follows (using the fact that the periodicities of these ghosts should be the same as T_F);

$$b(z) = \sum_{m\in\mathbb{Z}} \frac{b_m}{z^{m+2}}, c(z) = \sum_{m\in\mathbb{Z}} \frac{c_m}{z^{m-1}}, \beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+3/2}}, \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}}, \tag{10.4.5}$$

Now we define the ground state as follows:

$$\begin{gathered}\beta_r|0\rangle_{\text{NS}} = 0, r \geq \frac{1}{2}, \quad \gamma_r|0\rangle_{\text{NS}} = 0, r \geq \frac{1}{2} \\ \beta_r|0\rangle_{\text{R}} = 0, r \geq 0, \quad \gamma_r|0\rangle_{\text{R}} = 0, r \geq 1 \\ b_m|0\rangle_{\text{NS,R}} = 0, m \geq 0, \quad c_m|0\rangle_{\text{NS,R}} = 0, m \geq 1\end{gathered} \tag{10.4.6}$$

In the NS sector, β_r with positive r are annihilation operators, and their conjugates γ_{-r} are creation operators. The R sector has the additional zero-mode choice: we take β_0 to be an annihilation operator, so its conjugate γ_0 is a creation operator. The same convention is used for the b_m, c_m system, with b_0 chosen as an annihilation operator.

8.4.1 Vertex operators

Denote the state corresponding to the unit operator by $|1\rangle$. It belongs to the NS sector because the unit operator has no branch cut. This state satisfies the following conditions, following from Eq. (10.4.4):

$$\beta_r|1\rangle = 0, \left(r > -\frac{3}{2}\right), \gamma_r|1\rangle = 0, \left(r > \frac{1}{2}\right), b_m|1\rangle = 0, (m > -2), c_m|1\rangle = 0, (m > 1) \tag{10.4.7}$$

$|1\rangle$ is not $|0\rangle_{\mathrm{NS}}$. We can show that $|0\rangle_{\mathrm{NS}} = c_1|1\rangle$, which satisfies all the properties of $|0\rangle_{\mathrm{NS}}$ listed in Eq. (10.4.6). The vertex operator for $|0\rangle_{\mathrm{NS}}$ in terms of γ is denoted by $\delta(\gamma(z))$:

$$|0\rangle_{\mathrm{NS}} = \delta(\gamma(0))|1\rangle$$

. We now derive the following results;

$$\begin{aligned}\gamma(z)\delta(\gamma(0))|1\rangle &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{NS}} = \sum_{r<\frac{1}{2}} \frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{NS}} = \gamma_{-\frac{1}{2}} z|0\rangle_{\mathrm{NS}} + \ldots \\ &\sim \mathcal{O}(z)|0\rangle_{\mathrm{NS}} \Rightarrow \gamma(z)\delta(\gamma(0)) \sim \mathcal{O}(z) \end{aligned} \tag{10.4.8}$$

$$\begin{aligned}\beta(z)\delta(\gamma(0))|1\rangle &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{NS}} = \sum_{r<\frac{1}{2}} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{NS}} = \frac{1}{z}\beta_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} + \ldots \\ &\sim \mathcal{O}\left(z^{-1}\right)|0\rangle_{\mathrm{NS}} \Rightarrow \beta(z)\delta(\gamma(0)) \sim \mathcal{O}\left(z^{-1}\right) \end{aligned} \tag{10.4.9}$$

Now we will bosonize the $\beta\gamma$ system to look it like the bosonisation of bc system. We consider the following OPE;

$$\beta(z)\gamma(z)\beta(0)\gamma(0) = \beta(z)\gamma(0)\gamma(z)\beta(0) \sim -\frac{1}{z^2} \tag{10.4.10}$$

This matches the OPE of $\partial\phi$ with itself where ϕ is the chiral, free boson. So, we have the following correspondence;

$$\beta\gamma(z) \sim \partial\phi(z) \tag{10.4.11}$$

Moreover, we can get another correspondence. Firstly, we get the following results;

$$\beta(z)\gamma(z)\beta(0) \sim \frac{\beta(0)}{z} \Rightarrow \partial\phi(z)\beta(0) \sim \frac{\beta(0)}{z}, \beta(z)\gamma(z)\gamma(0) \sim -\frac{\gamma(0)}{z} \Rightarrow \partial\phi(z)\gamma(0) \sim -\frac{\gamma(0)}{z}$$

and then, we consider the following OPEs

$$\partial\phi(z)e^{-\phi(0)} \sim \frac{e^{-\phi(0)}}{z}, \partial\phi(z)e^{\phi(0)} \sim -\frac{e^{\phi(0)}}{z}$$

Comparing the OPEs in the above two equations, we get the following;

$$\beta(z) \sim e^{-\phi(z)}, \gamma(z) \sim e^{\phi(z)} \tag{10.4.12}$$

This will cause a problem. Consider the following OPEs;

$$e^{\pm\phi(z)}e^{\pm\phi(0)} \sim \mathcal{O}\left(z^{-1}\right), e^{\phi(z)}e^{-\phi(0)} \sim \mathcal{O}(z) \tag{10.4.13}$$

this implies that we will have the following OPEs;

$$\beta(z)\beta(0) \sim \mathcal{O}\left(z^{-1}\right), \gamma(z)\gamma(0) \sim \mathcal{O}\left(z^{-1}\right), \beta(z)\gamma(0) \sim \mathcal{O}(z)$$

but these are the wrong OPEs. We can fix this by adding additional fields which are non-singular with respect to ϕ in order to retain the OPE in 10.4.10. The new bosonization is as follows:

$$\beta(z) \sim e^{-\phi(z)}\partial\xi(z), \gamma(z) \sim e^{\phi(z)}\eta(z) \tag{10.4.14}$$

and we need to impose the $\beta\gamma$ OPEs to get the OPEs between $\partial\xi$ and η as follows:

$$\beta(z)\gamma(0) \sim e^{-\phi(z)}\partial\xi(z)e^{\phi(0)}\eta(0) = e^{-\phi(z)}e^{\phi(0)}\partial\xi(z)\eta(0) \sim \mathcal{O}\left(z^{-1}\right) \Rightarrow \xi(z)\eta(0) \sim \mathcal{O}\left(z^{-1}\right)$$
$$\beta(z)\beta(0) \sim e^{-\phi(z)}\partial\xi(z)e^{-\phi(0)}\partial\xi(0) = e^{-\phi(z)}e^{-\phi(0)}\partial\xi(z)\partial\xi(0) \sim \mathcal{O}(1) \Rightarrow \partial\xi(z)\partial\xi(0) \sim \mathcal{O}(z)$$
$$\gamma(z)\gamma(0) \sim e^{\phi(z)}\eta(z)e^{\phi(0)}\eta(0) = e^{\phi(z)}e^{\phi(0)}\eta(z)\eta(0) \sim \mathcal{O}(1) \Rightarrow \eta(z)\eta(0) \sim \mathcal{O}(z)$$

It makes this theory look like a bc theory (although this will imply that $\partial\xi(z)\partial\xi(0) \sim \mathcal{O}(1)$ but the $\mathcal{O}(1)$ term should vanish due to the anti commutativity). Now, we determine the stress tensor of the ϕ theory. The stress tensor of $\beta\gamma$ theory (with β having a weight of λ) is as follows (I have excluded the normal order sign);

$$T(z) = (1-\lambda)\partial\beta(z)\gamma(z) - \lambda\beta(z)\partial\gamma(z)$$

The OPE of this $T(z)$ can be calculated with $\beta\gamma$ but we will calculate the fully contracted term only. The calculation is as follows:

$$T(z)\beta(0)\gamma(0) \sim (1-\lambda)\partial\beta(z)\gamma(z)\beta(0)\gamma(0) - \lambda\beta(z)\partial\gamma(z)\beta(0)\gamma(0) + \ldots = \frac{(1-\lambda)}{z^3} - \frac{\lambda}{z^3} + \ldots = \frac{1-2\lambda}{z^3} + \ldots$$

This can be compared with the following OPE;

$$-\frac{1}{2}\partial\phi(z)\partial\phi(z)\partial\phi(0) - \frac{1}{2}\partial\phi(z)\partial\phi(z)\partial\phi(0) + \frac{1}{2}(1-2\lambda)\partial^2\phi(z)\partial\phi(0) = \frac{1-2\lambda}{z^3} + \frac{\partial\phi(0)}{z^2} + \frac{\partial^2\phi(0)}{z}$$

It gives the conclusion that the stress tensor for the ϕ theory is as follows:

$$T_B^\phi(z) = -\frac{1}{2}\partial\phi(z)\partial\phi(z) + \frac{1-2\lambda}{2}\partial^2\phi(z) \tag{10.4.15}$$

The weights of the exponentials in 10.4.14 are calculated as follows:

$$\begin{aligned}
T_B^\phi(z)e^{\phi(0)} &\sim -\frac{1}{2}\partial\phi(z)\partial\phi(z)e^{\phi(0)} + \frac{1-2\lambda}{2}\partial^2\phi(z)e^{\phi(0)} \\
&\sim \frac{1/2-\lambda}{z^2}e^{\phi(0)} + \frac{\partial\phi(z)e^{\phi(0)}}{2z} \sim -\frac{\lambda}{z^2}e^{\phi(0)} + \mathcal{O}(z^{-1}), \\
T_B^\phi(z)e^{-\phi(0)} &\sim -\frac{1}{2}\partial\phi(z)\partial\phi(z)e^{-\phi(0)} + \frac{1-2\lambda}{2}\partial^2\phi(z)e^{-\phi(0)} \\
&\sim \frac{\lambda-1/2}{z^2}e^{-\phi(0)} - \frac{\partial\phi(z)e^{-\phi(0)}}{2z} \sim \frac{\lambda-1}{z^2}e^{-\phi(0)} + \mathcal{O}(z^{-1}).
\end{aligned}$$

which means that the weights of $\xi(z)$ and $\eta(z)$ have to be 0 and 1 respectively. Therefore, the $\eta\xi$ system is a $\lambda = 1bc$ system with η being like b and ξ being like c. Using (2.5.5), the stress tensor of $\eta\xi$ theory is as follows:

$$T_B^{\eta\xi} = - :\eta\partial\xi:(z) \tag{10.4.16}$$

We now calculate the central charges of the ϕ theory and the $\eta\xi$ theory. Observing (10.4.15), we see that it is a one-dimensional linear dilaton CFT with $\alpha' = 2$ and $V = (1-2\lambda)/2$. So, the central charge of ϕ theory is as follows:

$$c_\phi = 1 + 12V^2 = 1 + 12\left(\frac{1-2\lambda}{2}\right)^2 = 3(1-2\lambda)^2 + 1 \tag{10.4.17}$$

Moreover, since $\eta\xi$ is equivalent to a bc system with $\lambda = 1$ we can use 2.5.6 to get the following;

$$c_{\eta\xi} = 1 - 3(2(1)-1)^2 = -2 \tag{10.4.18}$$

So, the central charge of the $\beta\gamma$ system is $3(1-2\lambda)^2 + 1 - 2 = 3(1-2\lambda)^2 - 1$ which is the right central charge. The bosonized representation is

$$\beta = e^{-\phi}\partial\xi, \qquad \gamma = \eta e^{\phi},$$

where (η, ξ) is a fermionic pair of weights $(1, 0)$ and ϕ is a scalar with background charge fixed so that $e^{q\phi}$ has the conformal weight used below.

For string theory, the relevant value of λ is $3/2$. Using (10.4.9) and 10.4.8, we can identify

$\delta(\gamma(z))$ as the following;

$$\delta(\gamma(z)) \sim e^{-\phi(z)} \tag{10.4.19}$$

which has $h = 1/2$. A similar analysis applies to the vertex operator of the R ground state. Denoting the spin field by Σ, one has

$$|0;\mathbf{s}\rangle_R = \mathcal{V}_{\mathbf{s}}(0)|1\rangle = \Sigma(0)\Theta_{\mathbf{s}}(0)|1\rangle. \tag{10.4.20}$$

Using 10.4.6 and 10.4.5, we can derive the following;

$$\beta(z)\Sigma(0)|1\rangle = \sum_{r\in\mathbb{Z}} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{R}} = \sum_{r<0} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{R}} = \beta_{-1}z^{-1/2}|0\rangle_{\mathrm{R}} + \ldots \sim \mathcal{O}\left(z^{-1/2}\right) \sim \beta(z)\Sigma(0) \sim \mathcal{O}\left(z^{-1/2}\right)$$

$$\gamma(z)\Sigma(0)|1\rangle = \sum_{r\in\mathbb{Z}} \frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{R}} = \sum_{r<1} \frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{R}} = \gamma_0 z^{1/2}|0\rangle_{\mathrm{R}} + \ldots \sim \mathcal{O}\left(z^{1/2}\right) \sim \gamma(z)\Sigma(0) \sim \mathcal{O}\left(z^{1/2}\right)$$

We can compare these OPEs with the following OPEs;

$$e^{\phi(z)}e^{-\phi(0)/2} \sim \mathcal{O}\left(z^{1/2}\right), e^{-\phi(z)}e^{-\phi(0)/2} \sim \mathcal{O}\left(z^{-1/2}\right)$$

Using these OPEs and 10.4.14, we can get the following identification;

$$\Sigma(z) \sim e^{-\phi(z)/2} \tag{10.4.21}$$

The matter spin field $\Theta_{\mathbf{s}}$ in Eq. (10.4.20) is fixed just as explicitly. Pair the ten real worldsheet fermions into five complex fermions,

$$\psi_I^+ = \frac{1}{\sqrt{2}}\big(\psi^{2I} + i\psi^{2I+1}\big), \qquad \psi_I^- = \frac{1}{\sqrt{2}}\big(\psi^{2I} - i\psi^{2I+1}\big), \qquad I = 0, \ldots, 4.$$

Introduce five bosons H_I with

$$H_I(z)H_J(0) \sim -\delta_{IJ}\ln z, \qquad \psi_I^+(z) \simeq e^{+iH_I(z)}, \qquad \psi_I^-(z) \simeq e^{-iH_I(z)}.$$

For a Ramond ground state labelled by spin weights $s_I = \pm\frac{1}{2}$, define

$$\Theta_{\mathbf{s}}(z) =: \exp\left(i\sum_{I=0}^{4} s_I H_I(z)\right) : .$$

Then

$$\psi_I^\pm(z)\Theta_{\mathbf{s}}(0) \sim z^{\pm s_I} : \exp\left(i(\pm1 + s_I)H_I(0) i\sum_{J\neq I} s_J H_J(0)\right) : .$$

The half-integral powers are the branch cuts of the Ramond sector. They turn the fermion modes on the cylinder from half-integral NS modes into integral Ramond modes. The conformal weight follows from the exponential OPE:

$$h(\Theta_{\mathbf{s}}) = \frac{1}{2}\sum_{I=0}^{4} s_I^2 = \frac{5}{8}.$$

For the superghost boson at $\lambda = 3/2$,

$$T_\phi = -\frac{1}{2} : (\partial\phi)^2 : -\partial^2\phi, \qquad h(e^{q\phi}) = -\frac{1}{2}q(q+2),$$

so $h(e^{-\phi/2}) = 3/8$. Hence a holomorphic Ramond vertex in the $-1/2$ picture has

$$V_R^{(-1/2)}(z) = u_{\mathbf{s}}\, e^{-\phi/2}\Theta_{\mathbf{s}} e^{ik\cdot X}(z), \qquad h\big(e^{-\phi/2}\Theta_{\mathbf{s}}\big) = \frac{3}{8} + \frac{5}{8} = 1.$$

The plane-wave factor adds the usual momentum contribution. Thus the Ramond ground-state vertex is marginal exactly at $k^2 = 0$, with the open-string boundary normalization or the closed-string holomorphic normalization giving the same massless condition. The remaining zero-mode constraint is $G_0 V_R = 0$, equivalently

$$k_\mu \Gamma^\mu_{\mathbf{s}'\mathbf{s}} u_{\mathbf{s}} = 0,$$

which is the massless Dirac equation obtained again in Eq. (10.5.5) [3].

The worldsheet fermion number F is defined to be odd for β and γ because $\{F, T_F\} = 0$ and the BRST charge contains the product γT_F, which must be even. This justifies the negative fermion number of $|0\rangle_{\mathrm{NS}}$. In the bosonized description the charge is measured by the current associated with 10.4.11: the operator $e^{l\phi}$ carries ϕ-momentum l. The NS picture-changing vacuum $e^{-\phi}$ therefore contributes one unit of odd ghost fermion number. In the Ramond ground-state vertex $e^{-\phi/2}S_{\mathbf{s}}$, the half-integer ϕ-momentum is combined with the spin-field cocycle; the physical GSO parity is the total matter-plus-ghost parity, not the ϕ-momentum alone. Cocycle factors are understood throughout the bosonized formulas: they make exponentials associated with different fermions anticommute and keep OPEs mutually local.

8.5 Physical states

The constraints are applied on the physical states as follows:

$$L_n|\psi\rangle = 0, G_r|\psi\rangle = 0, \quad n \geq 0, r \geq 0 \tag{10.5.1}$$

where we only talk about the matter generators right now. The mass-shell condition is as follows:

$$L_0|\psi\rangle = H|\psi\rangle = 0 \tag{10.5.2}$$

where this generator is the matter+ghost generator. H follows from the zero mode $L_0 - a$, with normal-ordering constants $a_{\mathrm{NS}} = 1/2$ and $a_{\mathrm{R}} = 0$:

$$H = \begin{cases} \alpha' p^2 + N - \frac{1}{2}(\text{ Neveu Schwarz }) \\ \alpha' p^2 + N(\text{ Ramond }) \end{cases} \tag{10.5.3}$$

The constants in Eq. (10.5.3) are the regularized zero-point energies of the transverse light-cone fields. For one real boson,

$$E_0^B = \frac{1}{2}\sum_{n=1}^{\infty} n = \frac{1}{2}\zeta(-1) = -\frac{1}{24}.$$

For one real NS fermion the modes are $r \in \mathbb{Z} + 1/2$, and the fermionic sign gives

$$E_0^{F,\mathrm{NS}} = -\frac{1}{2}\sum_{r=1/2}^{\infty} r = -\frac{1}{2}\zeta\left(-1, \frac{1}{2}\right) = -\frac{1}{48}.$$

For one real Ramond fermion the nonzero modes are integral, hence

$$E_0^{F,\mathrm{R}} = -\frac{1}{2}\sum_{n=1}^{\infty} n = +\frac{1}{24}.$$

With $D-2$ transverse boson–fermion pairs,

$$E_0^{\mathrm{NS}} = (D-2)\left(-\frac{1}{24} - \frac{1}{48}\right) = -\frac{D-2}{16}, \qquad E_0^{\mathrm{R}} = (D-2)\left(-\frac{1}{24} + \frac{1}{24}\right) = 0.$$

Thus $L_0 = \alpha' p^2 + N + E_0$. In $D = 10$, $E_0^{\mathrm{NS}} = -1/2$ and $E_0^{\mathrm{R}} = 0$, which gives the two lines of Eq. (10.5.3). Conversely, demanding the first NS excitation $N = 1/2$ to be massless gives

$$\frac{1}{2} - \frac{D-2}{16} = 0, \qquad D = 10,$$

in agreement with the superghost central-charge calculation [3].

The NS ground state is $|k, 0\rangle_{\mathrm{NS}}$ and is labelled by a momentum k. The constraints (10.5.1) are trivially satisfied for this state. The mass shell condition gives the following;

$$L_0|k, 0\rangle_{\mathrm{NS}} = \left(\alpha' p^2 + N - \frac{1}{2}\right)|k, 0\rangle_{\mathrm{NS}} = 0 \Rightarrow k^2 = \frac{1}{2\alpha'}$$

a positive k^2 indicates a tachyon. We will call the theory built on this vacuum NS- (because $e^{\pi i F} = -1$). (Calculate the fermion number by adding ghost contribution). The first excited state is given as follows:

$$|e; k\rangle_{\mathrm{NS}} = e_\mu \psi^\mu_{-1/2}|0; k\rangle_{\mathrm{NS}} \tag{10.5.4}$$

where e_μ is the polarization vector. The constraints are calculated as follows:

$$L_m|e;k\rangle_{\rm NS} = \frac{1}{4}\sum_{r>0}(2r-m)\psi^\mu_{m-r}\psi_{\mu r}e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\rm NS} + \frac{1}{4}\sum_{r>0}(2r+m)\psi^\mu_r\psi_{\mu m+r}e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\rm NS}$$
$$= \frac{1}{4}(1-3m)\psi^\mu_{2m-\frac{1}{2}}e_\mu|0;k\rangle_{\rm NS} + \frac{1}{4}(m+1)\psi^\mu_{\frac{1}{2}}e_\mu|0;k\rangle_{\rm NS} = 0 \quad (m>0)$$

So, this constraint is trivially satisfied. The next constraint is imposed as follows:

$$G_r|e;k\rangle_{\rm NS} = \left(\alpha^\mu_0\psi_{\mu r} + \sum_{n>0}\alpha^\mu_n\psi_{\mu r-n} + \sum_{n>0}\alpha^\mu_{-n}\psi_{\mu r+n}\right)e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\rm NS}$$
$$= \alpha^\mu_0 e_\mu\delta_{r-\frac{1}{2}}|e;k\rangle_{\rm NS} + \sum_{n>0}\alpha^\mu_{-n}e_\mu\delta_{r+n-\frac{1}{2}}|e;k\rangle_{\rm NS} = \alpha^\mu_0 e_\mu\delta_{r-\frac{1}{2}}|e;k\rangle_{\rm NS}$$

So, the only non-trivial constraint comes from $G_{\frac{1}{2}}$. This constraint implies the following;

$$\alpha^\mu_0 e_\mu|e;k\rangle_{\rm NS} = 0 \Rightarrow k^\mu e_\mu = 0$$

Finally, the mass shell constraint gives the following;

$$\left(\alpha' p^2 + N - \frac{1}{2}\right)|e;k\rangle_{\rm NS} = \left(\alpha' k^2 + \frac{1}{2} - \frac{1}{2}\right)|e;k\rangle_{\rm NS} = 0 \Rightarrow k^2 = 0$$

The fermion number for this state has to be $+1$. We now show that $G_{-1/2}|e;k\rangle_{\rm NS}$ is a null state. It is done as follows:

$$\langle\psi|G_{-1/2}|e;k\rangle_{\rm NS} = \overline{\langle e;k|G_{-1/2}|\psi\rangle} = 0$$
$$L_m G_{-\frac{1}{2}}|e;k\rangle_{\rm NS} = G_{-\frac{1}{2}}L_m|e;k\rangle_{\rm NS} + \frac{m+1}{2}G_{m-\frac{1}{2}}|e;k\rangle_{\rm NS} = 0 \quad (m>0)$$
$$G_r G_{-\frac{1}{2}}|e;k\rangle_{\rm NS} = 2L_{r-\frac{1}{2}}|e;k\rangle_{\rm NS} = 0 \quad \left(r \geq \frac{1}{2}\right)$$

The form of this null state is as follows:

$$G_{-\frac{1}{2}}|e;k\rangle_{\rm NS} = \alpha^\mu_0\psi_{\mu-\frac{1}{2}}|e;k\rangle_{\rm NS} = \sqrt{2\alpha'}k^\mu\psi_{\mu-\frac{1}{2}}|e;k\rangle_{\rm NS}$$

Thus, the general first excited state is as follows:

$$(e_\mu + \lambda k_\mu)\,\psi^\mu_{-\frac{1}{2}}|e;k\rangle_{\rm NS} \Rightarrow e_\mu \sim e_\mu + \lambda k_\mu$$

We will refer to the theory built on this vacuum NS+ theory (because $e^{\pi i F} = 1$ on this vacuum). Now, a general Ramond vacuum can be expanded in terms of the basis states labelled by $\mathbf{s}$ as follows:

$$|u;k\rangle_R = |\mathbf{s};k\rangle_R u_{\mathbf{s}} \tag{10.5.5}$$

The mass shell condition calculation goes as before and we get that $k^2 = 0$ for the ground state. The L_m and G_r (for $r \neq 0$) constraints are again trivially satisfied. The G_0 constraint gives us the following;

$$G_0|\mathbf{s};k\rangle_R u_{\mathbf{s}} = \alpha^\mu_0\psi_{\mu 0}|\mathbf{s};k\rangle_R u_{\mathbf{s}} = \sqrt{\alpha'}\,|\mathbf{s}';k\rangle_R\, k^\mu\Gamma_{\mu\mathbf{s}'\mathbf{s}}u_{\mathbf{s}} = 0 \Rightarrow k\cdot\Gamma_{\mathbf{s}'\mathbf{s}}u_{\mathbf{s}} = 0$$

which is the massless Dirac equation. Now, since the state is massless, we can go to the frame where $k_0 = k_1$. In that frame, the Dirac equation becomes;

$$\left(k_0\Gamma^0 + k_1\Gamma^1\right)u = k_0\left(\Gamma^0 - \Gamma^0\Gamma^0\Gamma^1\right)u$$
$$= -k_0\Gamma^0\left(\Gamma^0\Gamma^1 - 1\right)u = -k_0\Gamma^0\left(2S_0 - 1\right)u$$
$$= -2k_0\Gamma^0\left(S_0 - \frac{1}{2}\right)u = 0.$$

and thus, only the states with $s_0 = 1/2$ survive. Now, using 21.1.40 with $k = 4$ and $l = 0$, we get the following decomposition for $SO(9,1) \rightarrow SO(1,1)\times SO(8)$;

$$\mathbf{16} = (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{1}', \mathbf{8}') = \left(\frac{1}{2}, \mathbf{8}\right) \oplus \left(-\frac{1}{2}, \mathbf{8}'\right)$$

$$\mathbf{16}' = (\mathbf{1}, \mathbf{8}') \oplus (\mathbf{1}', \mathbf{8}) = \left(\frac{1}{2}, \mathbf{8}'\right) \oplus \left(-\frac{1}{2}, \mathbf{8}\right)$$

So the massless Dirac equation allows a vacuum with $SO(8)$ representation of $\mathbf{8}$ with chirality $+1$ (which is the same as $e^{\pi i F}$ on the vacuum) and a vacuum with $SO(8)$ representation of $\mathbf{8}$ with $e^{\pi i F} = -1$. We will call these sectors R+ and R−. The open string sectors are then as follows:

Sector	$e^{\pi i F}$	m^2	$SO(8)$
NS+	1	0	$\mathbf{8}_v$
NS−	-1	$-1/2\alpha'$	$\mathbf{1}$
R+	1	0	$\mathbf{8}$
R−	-1	0	$\mathbf{8}'$

For the closed-string vacuum, level matching requires the left- and right-moving mass-shell conditions to give the same spacetime mass:

$$\frac{\alpha'}{4} m^2 = N - \nu = \tilde{N} - \tilde{\nu} \tag{10.5.6}$$

Since the mass of NS- vacuum is different from all the other sectors, it can't be coupled with any other sector. So, the only closed string sector that contains NS- is (NS-, NS-). We can make the following table to help us work out the possible closed string sectors;

Sector	N	ν	$N-\nu$	$\alpha' m^2/4$
NS+	1/2	1/2	0	0
NS-	0	1/2	$-1/2$	$-1/2$
R+	0	0	0	0
R-	0	0	0	0

So it is very clear that the allowed sectors are (we exclude sectors that are reflections of other sectors - i.e. that can be obtained by flipping left and right sectors -);

$$(\text{NS}-, \text{NS}-), (\text{NS}+, \text{NS}+), (\text{NS}+, \text{R}+), (\text{NS}+, \text{R}-), (\text{R}+, \text{R}+), (\text{R}+, \text{R}-), (\text{R}-, \text{R}-)$$

Among these sectors, (NS-, NS-) has the tachyon. Before writing down the spectrum of the other sectors, we need to work out the following decompositions. $\mathbf{8}_v \times \mathbf{8}_v$ is a tensor representation but can be broken

sector	$So(8)$ rep	decomposition	dimensions
(NS−, NS−)	$\mathbf{1}$	-	-
(NS+, NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	$[0] + (2) + [2]$	$\mathbf{1} + \mathbf{28} + \mathbf{35}$
(NS+, R+)	$\mathbf{8}_v \times \mathbf{8}'$	-	$\mathbf{8}' + \mathbf{56}$
(NS+, R−)	$\mathbf{8}_v \times \mathbf{8}'$	-	$\mathbf{8} + \mathbf{56}$
(R+, R+)	$\mathbf{8}' \times \mathbf{8}'$	$[0] + [2] + [4]_+$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_+$
(R−, R−)	$\mathbf{8}' \times \mathbf{8}'$	$[0] + [2] + [4]_-$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_-$
(R+, R−)	$\mathbf{8} \times \mathbf{8}'$	$[1] + [3]$	$\mathbf{8} + \mathbf{56}$

Table 1: Spectrum for closed string sectors

down to a traceless symmetric tensor with $8(8+1)/2 - 1 = 35$ components, an anti-symmetric tensor with $8(8-1)/2 = 28$ components and a pure trace with one component (just like the closed bosonic string massless state). So, we have;

$$\mathbf{8}_v \times \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35} \tag{10.5.7}$$

From 21.1.37, we get the following three decompositions for $k = 3$,

$$\begin{aligned} \mathbf{8} \times \mathbf{8} &= [0] + [2] + [4]_+ = \mathbf{1} + \mathbf{28} + \mathbf{35}_+ \\ \mathbf{8}' \times \mathbf{8}' &= [0] + [2] + [4]_- = \mathbf{1} + \mathbf{28} + \mathbf{35}_- \\ \mathbf{8} \times \mathbf{8}' &= [1] + [3] = \mathbf{8} + \mathbf{56} \end{aligned} \tag{10.5.8}$$

where we used the fact that the number of independent components in an antisymmetric tensor of rank r in d dimensions is $\binom{d}{r}$. We chose $d = 8$. Lastly, we talk about $\mathbf{8}_v \times \mathbf{8}$. We can take a general state in this representation and label it as $|i; \mathbf{s}\rangle$. We can now make eight different linear combinations, labeled by $\mathbf{s}'$ as follows:

$$\Gamma^i_{\mathbf{s}'\mathbf{s}}|i; \mathbf{s}\rangle$$

The chirality of this combination is different from $|i; \mathbf{s}\rangle$ which can be shown as follows:

$$\Gamma|i; \mathbf{s}\rangle = \alpha|i; \mathbf{s}\rangle \Rightarrow \Gamma\left(\Gamma^i_{\mathbf{s}'\mathbf{s}}|i; \mathbf{s}\rangle\right) = -\Gamma^i_{\mathbf{s}'\mathbf{s}}\Gamma|i; \mathbf{s}\rangle = -\alpha\Gamma^i_{\mathbf{s}'\mathbf{s}}|i; \mathbf{s}\rangle$$

Under the action of $SO(8)$, this combination maps to a similar combination because Clifford multiplication is an $SO(8)$-equivariant map $8_v \otimes 8_s \to 8_c$. Therefore, there is an $\mathbf{8}'$ representation there. The tensor product has dimension 64, so after the eight-dimensional image is removed the remaining irreducible component has dimension 56. So, we have the following;

$$\begin{aligned} \mathbf{8}_v \times \mathbf{8} &= \mathbf{8}' + \mathbf{56} \\ \mathbf{8}_v \times \mathbf{8}' &= \mathbf{8} + \mathbf{56}' \end{aligned} \tag{10.5.9}$$

The BRST treatment gives the same physical spectrum after the superghost sector is included; here we keep the light-cone representation labels because they make the spacetime multiplets explicit. The closed-string sectors are as follows.

8.6 The superstring theories in ten dimensions

Instead of naming the theories with ν and F, we adopt the more convenient numbers (α, F) where $\alpha = 1 - 2\nu$. This gets rid of the fractions in ν and makes α defined up to mod 2 only. In the open string case, we have the following correspondences;

(α, F)	ν	sector
$(0, 1)$	$1/2$	NS+
$(0, -1)$	$1/2$	NS-
$(0, 1)$	0	R+
$(0, -1)$	0	R-

For closed string, we label the sectors by four numbers ($\alpha, F, \bar{\alpha}, \bar{F}$). The correspondences are tabulated as follows:

$(\alpha, F, \tilde{\alpha}, \tilde{F})$	sector
$(0,0,0,0)$	(NS+, NS+)
$(0,0,0,1)$	-
$(0,0,1,0)$	(NS+, R+)
$(0,0,1,1)$	(NS+, R−)
$(0,1,0,0)$	-
$(0,1,0,1)$	(NS+, NS−)
$(0,1,1,0)$	-
$(0,1,1,1)$	-
$(1,0,0,0)$	(R+, NS+)
$(1,0,0,1)$	-
$(1,0,1,0)$	(R+, R+)
$(1,0,1,1)$	(R+, R−)
$(1,1,0,0)$	(R−, NS+)
$(1,1,0,1)$	-
$(1,1,1,0)$	(R−, R+)
$(1,1,1,1)$	(R−, R−)

Some of the possibilities do not belong to an allowed sector because they involve coupling NS- to another sector. Some sectors are reflections of other sectors. So, we still have only seven possible sectors. Now, we impose conditions to make allowed theories.

1- We want operators to be mutually local (i.e. they do not pick up a nontrivial phase when one operator circles around another). All of the operators cannot be kept at once because the vertex operator of the Ramond vacuum has branch cuts. Consider two operators in $\left(\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1\right)\left(\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2\right)$ sectors. Bosonizing the spin fields gives factors $e^{i\alpha H/2}$, and moving one operator around the other gives the monodromy $e^{2\pi i q_1 q_2}$, with the cocycle contribution recorded by the fermion numbers. The phase acquired by this circling is

$$\exp\left[\pi i\left(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1\right)\right] \tag{10.6.1}$$

Note that since $\alpha = 0$ for NS sector, if all the α 's vanish, then the phase is unity and it makes sense because there are no branch cuts in the NS sector. So, the first requirement is;

$$F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1 \in 2\mathbb{Z} \tag{10.6.2}$$

2- We recall that in the plane, Ramond operators have half-integer modes and NS operators have integer modes. So, for the operator products in the plane, we have the following;

$$\mathrm{R}\times\mathrm{R} = \mathrm{NS} \Rightarrow 1\times 1 = 0(= 1+1 \bmod 2)$$

$$\mathrm{R}\times\mathrm{NS} = \mathrm{R} \Rightarrow 1\times 0 = 1(= 1+0 \bmod 2)$$

$$\mathrm{NS}\times\mathrm{NS} = \mathrm{NS} \Rightarrow 0\times 0 = 0(= 0+0 \bmod 2)$$

where we wrote the same thing in terms of corresponding α values. So, we see that in the operator products, α 's just add up (up to mod2). It is easier to see that the fermion number also adds up and thus, F is also conserved mod 2 . Now, we can state our second condition. We want the OPEs to close and thus, if ($\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1$) and ($\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2$), then we should also have the following sector;

$$\left(\alpha_1+\alpha_2, F_1+F_2, \tilde{\alpha}_1+\tilde{\alpha}_2, \tilde{F}_1+\tilde{F}_2\right) \bmod 2$$

3- Modular transformations relate NS and R spin structures in the one-loop partition function. Hence a theory built only from NS sectors is not modular invariant; at least one left-moving and one right-moving Ramond sector must be present.

Now, if we assume that we have at least one NS-R sector, it has to be (R+,NS+) = $(1,0,0,0)$ or (R-,NS+)= ($1,1,0,0$) sector or both. The phase in 10.6.1 for these two sectors is as follows:

$$\exp[\pi i((0)(1)-(1)(1)-(0)(0)+(0)(0))] = -1$$

Therefore these two sectors cannot coexist in a mutually local theory. The third requirement also demands NS–R or R–R sectors, or both. Since an R–NS sector is already present by assumption, the presence of an R–R sector implies the operator product

$$\mathrm{R-NS} \times \mathrm{R-R} = \mathrm{NS-R}$$

So, we will have an NS sector in any case (i.e. whether or not we have an R sector). So, we have the following possibilities;

$$\begin{aligned} \mathrm{IIB}\ &: (\mathrm{R+,NS+}) \text{ with } (\mathrm{NS+,R+}) \Rightarrow (1,0,0,0) \text{ with } (0,0,1,0) \\ \mathrm{IIA} &: (\mathrm{R+,NS+}) \text{ with } (\mathrm{NS+,R-}) \Rightarrow (1,0,0,0) \text{ with } (0,0,1,1) \\ \mathrm{IIA}' &: (\mathrm{R-,NS+}) \text{ with } (\mathrm{NS+,R+}) \Rightarrow (1,1,0,0) \text{ with } (0,0,1,0) \\ \mathrm{IIB}' &: (\mathrm{R-,NS+}) \text{ with } (\mathrm{NS+,R-}) \Rightarrow (1,1,0,0) \text{ with } (0,0,1,1) \end{aligned}$$

where the possibilities have been labeled. Consider first the IIB possibility, with sectors $(1,0,0,0)$ and $(0,0,1,0)$. Condition 2 also requires the following sectors:

$$(1,0,0,0)+(1,0,0,0) = (0,0,0,0)(\mathrm{mod}2) = (\mathrm{NS+,NS+})$$
$$(1,0,0,0)+(0,0,1,0) = (1,0,1,0)(\mathrm{mod}2) = (\mathrm{R+,R+})$$

So, the full type IIB possibility is as follows:

$$\mathrm{IIB} : (\mathrm{R+,NS+}), (\mathrm{NS+,R+}), (\mathrm{NS+,NS+}), (\mathrm{R+,R+}) \tag{10.6.3}$$

Consider first the IIA choice. The sectors $(1,0,0,0)$ and $(0,0,1,1)$ are already present. Closure of OPEs then requires

$$(1,0,0,0)+(1,0,0,0) = (0,0,0,0)(\mathrm{mod}2) = (\mathrm{NS+,NS+})$$
$$(1,0,0,0)+(0,0,1,1) = (1,0,1,1)(\mathrm{mod}2) = (\mathrm{R+,R-})$$

After these two sectors are included, the remaining operator products are mutually local and close within the same set. The full IIA possibility is therefore

$$\mathrm{IIA} : (\mathrm{R+,NS+}), (\mathrm{NS+,R-}), (\mathrm{NS+,NS+}), (\mathrm{R+,R-}) \tag{10.6.4}$$

For the IIA$'$ choice, the sectors $(1,1,0,0)$ and $(0,0,1,0)$ are already present. Closure of OPEs requires

$$(1,1,0,0)+(1,1,0,0) = (0,0,0,0)(\mathrm{mod}2) = (\mathrm{NS+,NS+})$$
$$(1,1,0,0)+(0,0,1,1) = (1,1,1,0)(\mathrm{mod}2) = (\mathrm{R-,R+})$$

All other OPEs close after adding these two sectors, so the full IIA$'$ possibility is

$$\mathrm{IIA}' : (\mathrm{R-,NS+}), (\mathrm{NS+,R+}), (\mathrm{NS+,NS+}), (\mathrm{R-,R+}) \tag{10.6.5}$$

Finally, for the IIB$'$ choice, the sectors $(1,1,0,0)$ and $(0,0,1,1)$ are already present. Closure of OPEs requires

$$(1,1,0,0)+(1,1,0,0) = (0,0,0,0)(\mathrm{mod}2) = (\mathrm{NS+,NS+})$$
$$(1,1,0,0)+(0,0,1,1) = (1,1,1,1)(\mathrm{mod}2) = (\mathrm{R-,R-})$$

All other OPEs close after adding these two sectors, so the full IIB$'$ possibility is

$$\mathrm{IIB}' : (\mathrm{R-,NS+}), (\mathrm{NS+,R-}), (\mathrm{NS+,NS+}), (\mathrm{R-,R-}) \tag{10.6.6}$$

Under a spacetime reflection in one transverse axis, for example $X^2 \to -X^2$ together with $\psi^2 \to -\psi^2$ and $\tilde{\psi}^2 \to -\tilde{\psi}^2$, the worldsheet action and constraints are unchanged. The reflection reverses one spinor weight in the Ramond sector, flips the sign of S_1 in Eq. (10.2.14), and shifts the corresponding fermion number by one unit as in Eq. (10.2.15). Hence $e^{\pi i F}$ changes sign on Ramond ground states. The primed and unprimed choices are therefore related by a spacetime reflection: IIA to IIA$'$ and IIB to IIB$'$.

Now, if we assume that we have no $\mathrm{R-NS}$ sector, condition 3 requires us to have $(\mathrm{R+,R+}) = (1,0,1,0), (\mathrm{R+,R-}) = (1,0,1,1), (\mathrm{R-,R+}) = (1,1,1,0)$ or $(\mathrm{R-,R-}) = (1,1,1,1)$ sector. Condition 2 requires us to have (NS+,NS+) sector as well as shown below.

$$(1,0,1,0)+(1,0,1,0)=(0,0,0,0)=(\text{NS}+,\text{NS}+)$$
$$(1,0,1,1)+(1,0,1,1)=(0,0,0,0)=(\text{NS}+,\text{NS}+)$$
$$(1,1,1,0)+(1,1,1,0)=(0,0,0,0)=(\text{NS}+,\text{NS}+)$$
$$(1,1,1,1)+(1,1,1,1)=(0,0,0,0)=(\text{NS}+,\text{NS}+)$$

Since all α's and F's vanish for (NS+, NS+), mutual locality is automatic. However, the corresponding one-loop partition function is not closed under modular transformations unless the (NS−, NS−) sector is also included. Modular invariance then leaves the two type-0 possibilities

$$0\text{ A}:(\text{NS}+,\text{NS}+),(\text{R}+,\text{R}-),(\text{R}-,\text{R}+),(\text{NS}-,\text{NS}-) \tag{10.6.7}$$
$$0\text{ B}:(\text{NS}+,\text{NS}+),(\text{R}+,\text{R}+),(\text{R}-,\text{R}-),(\text{NS}-,\text{NS}-) \tag{10.6.8}$$

These theories contain tachyons, but they are modular invariant. For IIB theory, all chiralities are +1, and hence

$$\text{IIB }: e^{\pi iF}=e^{\pi i\tilde{F}}=1 \tag{10.6.9}$$

For IIA theory, the left-handed chiralities are positive but the right-handed chiralities are +1 for NS sector (for which $\tilde{\alpha}=0$) and -1 for R sector (for which $\tilde{\alpha}=1$). So, the right-handed chiralities are given by $(-1)^{\tilde{\alpha}}$. Thus, have the following;

$$\text{IIA }: e^{\pi iF}=1, e^{\pi i\tilde{F}}=(-1)^{\tilde{\alpha}} \tag{10.6.10}$$

These projections are known as GSO projections. The 0B theory has same chirality for left and right movers and it has the same kind of periodicity for left and right movers. So, we get;

$$0\text{ B}: e^{\pi iF}=e^{\pi i\tilde{F}},\quad \alpha=\tilde{\alpha} \tag{10.6.11}$$

This projection is called the diagonal GSO projection. Using Table 1, the spectra of the IIA and IIB theories are

$$\text{IIA }: [0]+[1]+[2]+[3]+(2)+\mathbf{8}'+\mathbf{56}+\mathbf{8}+\mathbf{56}' \tag{10.6.12}$$
$$\text{IIB }: [0]^2+[2]^2+(2)+[4]_++\mathbf{8}'^2+\mathbf{56}^2 \tag{10.6.13}$$

The closed unoriented type-I projection keeps the Ω-even part of the type-IIB spectrum. The resulting closed spectrum contains the $N=1$ supergravity multiplet, and open strings must be added to cancel the R–R tadpole. The closed unoriented type I has the spectrum;

$$\text{IA (closed)}:\ [0]+(2)+[2]+\mathbf{8}+\mathbf{56} \tag{10.6.14}$$

where [2] comes from the R–R spectrum instead of the (NS+, NS+) spectrum. The unoriented closed spectrum has $N=1$ supersymmetry, with a single gravitino, so the compatible open sector must supply the corresponding $N=1$ gauge multiplet instead of an independent second supersymmetry. The combined closed-plus-open spectrum is

$$\text{IA (closed, open)}: [0]+(2)+[2]+\mathbf{8}+\mathbf{56}+\mathbf{8}+\mathbf{56} \tag{10.6.15}$$

The opposite open-string chirality is absent because the oriented open-string GSO projection keeps one Ramond chirality on the boundary Hilbert space. Keeping both would spoil the mutual-locality and modular-invariance conditions used to define the type-I open sector.

For type-II strings, the requirement that the target space admit a spin structure follows from consistency of the worldsheet GSO projection. The obstruction is a mixed global anomaly involving $(-1)^{F_L}$ and the target-space background fields, detected by a spin-bordism invariant [245]. In the smooth-target case the anomaly cancellation condition reduces to $w_2(X)=0$, while for orbifolds it requires an equivariant spin structure. This is the same logic used here in a local oscillator language, but upgraded from a state-by-state chirality projection to a global condition on the allowed string background.

Heckman, McNamara, Parra-Martinez, and Torres study GSO-related defects predicted by the cobordism program, including an IIA/IIB domain wall and a stable R7-brane; long-string probes see these objects as defects in the worldsheet GSO structure, not ordinary smooth compactifications [322]. Wada and Yamaguchi analyze D-branes in the Dabholkar–Park orientifold background and use worldsheet anomalies of one-dimensional Majorana fermions on the boundary to recover the relative KR-theory classification of stable branes [310]. Both results extend the GSO projection beyond the oscillator table: the projection can define global defects and boundary anomaly quantities that classify allowed D-brane sectors.

8.7 Modular invariance

We can calculate the torus amplitude for superstrings. We have two species, bosons and fermions, and the one-loop vacuum trace includes the standard factor $(-1)^{\mathbf{F}}$ for spacetime fermions, the path-integral form of the minus sign of a fermion loop. So, we put $d = 10$ in 7.3.1 and use the following facts;

$$q\bar{q} = e^{-4\pi\tau_2} \Rightarrow e^{-\pi\tau_2\alpha' k^2} = (q\bar{q})^{\alpha' k^2/4}$$

Moreover,

$$m_i^2 = \frac{4}{\alpha'}(h_i - 1) = \frac{4}{\alpha'}\left(\bar{h}_i - 1\right)$$

So, using these two facts, 7.3.1 becomes (for $d = 10$);

$$Z_{T^2} = V_{10}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}\int\frac{d^{10}k}{(2\pi)^{10}}\sum_{i\in\mathcal{H}_\perp}(-1)^{\mathbf{F}_i} q^{\alpha'\left(k^2+m_i^2\right)/4}\bar{q}^{\alpha'\left(k^2+m_i^2\right)/4}$$

We now have to sum over all possible states. The states are labelled by occupancy numbers of the bosonic modes and fermionic modes. So, we have to add over all possible modes. We also have to do the k integration. We saw in Chapter 5 that the contribution from each X is as follows:

$$Z_X(\tau) = \left(4\pi^2\alpha'\tau_2\right)^{-1/2}|\eta(\tau)|^2$$

So, the total bosonic contribution will be Z_X^8 (because there are eight transverse directions). However, the k^0 and k^1 directions are still undone, and doing them will give an additional factor of $i\left(4\pi^2\alpha'\tau_2\right)^{-1}$ (i comes because we need to Wick rotate $k^0 \to ik^0$).
For the fermionic sector, we need to consider general periodicities. We can do the calculation for periodicities much more general than $\nu = 0$ (R sector) or $\nu = 1/2$ (NS sector). Such periodicities were considered in Section 8.3 when deriving the vertex operators for R vacuum. Recall that $\alpha = 1 - 2\nu$ and then, the following facts are relevant.
The ground state is;

$$|0\rangle_\nu, 0 \le \nu \le 1 \Rightarrow 0 \le \alpha \le 1$$

and the creation operators are as follows:

$$\psi_{n+\nu}, \tilde{\psi}_{n+1-\nu}, n = -1, -2, \ldots \text{ or } \psi_{-n+(1-\alpha)/2}, \tilde{\psi}_{-n+1-(1-\alpha)/2} = \tilde{\psi}_{-n+(1+\alpha)/2}, n = 1, 2, \ldots$$

The weight of the ground state is as follows:

$$h_\nu = \frac{1}{2}\left(\nu - \frac{1}{2}\right)^2 = \frac{\alpha^2}{8}$$

The contribution of these two fermions to the conformal weight (or the hamiltonian as hamiltonian is $L_0 + \bar{L}_0$) is as follows:

$$\frac{\alpha^2}{8} + \sum_{m=1}^{\infty}\left(m - \frac{1-\alpha}{2}\right)N_m + \sum_{m=1}^{\infty}\left(m - \frac{1+\alpha}{2}\right)\tilde{N}_m + \text{ zero point}$$

where N_m is the occupancy number of $\psi_{-m+(1-\alpha)/2}$ and $\tilde{N}_m$ is the occupancy number of $\tilde{\psi}_{-m+(1+\alpha)/2}$. Now, we have to determine the zero-point energy. The combination;

$$\frac{\alpha^2}{8} + \text{ zero point}$$

should be $2 \times (-1/48) = -1/24$ when $\alpha = 0$ because in this case, the fermions will be in the NS sector (i.e. they are anti-periodic fermions) and since they are two in number, we have a factor of 2 . So, zero point should be $-1/24$. So, the total contribution to the hamiltonian by these two fermions is;

$$\begin{aligned}
&\frac{\alpha^2}{8}+\sum_{m=1}^{\infty}\left(m-\frac{1-\alpha}{2}\right) N_m+\sum_{m=1}^{\infty}\left(m-\frac{1+\alpha}{2}\right) \tilde{N}_m-\frac{1}{24} \\
&=\sum_{m=1}^{\infty}\left(m-\frac{1-\alpha}{2}\right) N_m+\sum_{m=1}^{\infty}\left(m-\frac{1+\alpha}{2}\right) \tilde{N}_m+\frac{3\alpha^2-1}{24}
\end{aligned}$$

To take the trace, we proceed as follows:

$$\begin{aligned}
\operatorname{Tr}_\alpha\left(q^H\right) &= q^{(3\alpha^2-1)/24} \prod_{m=1}^{\infty} \sum_{N_m, \tilde{N}_m=0}^{1} q^{\left(m-\frac{1-\alpha}{2}\right)\tilde{N}_m} q^{\left(m-\frac{1+\alpha}{2}\right)\tilde{N}_m} \\
&= q^{(3\alpha^2-1)/24} \prod_{m=1}^{\infty}\left(1+q^{m-(1-\alpha)/2}\right)\left(1+q^{m-(1+\alpha)/2}\right).
\end{aligned}$$

The occupation numbers are only 0 and 1 because these are fermions. Now we want to define the fermion number for ψ and $\tilde{\psi}$. Since ψ 's and $\tilde{\psi}$'s are of the following form in the bosonization;

$$\psi^a=\frac{1}{\sqrt{2}}\left(\psi^{2a}+i\psi^{2a+1}\right), \tilde{\psi}^a=\frac{1}{\sqrt{2}}\left(\psi^{2a}-i\psi^{2a+1}\right), \quad a \in\{1,2,3,4\}$$

we see using (10.2.16), we see that F gives a fermion number +1 for ψ and -1 for $\tilde{\psi}$. In the bosonization picture, it is the momentum for H and the ground-state charge is $\alpha/2$. This follows from the OPE of $-i\partial H$ with the spin-field exponential, as shown below. So, Q mod 2 should be F. Define $Q=-i(\partial H(z))_{-1}$ and then, we have;

$$\begin{aligned}
-i\partial H(z)|0\rangle_\nu &= -i\partial H(z)\exp\left(i\left(-\nu+\frac{1}{2}\right)\mathcal{H}(0)\right)|1\rangle = -i\partial H(z)\exp\left(-i\frac{\alpha}{2}\mathcal{H}(0)\right)|1\rangle \\
&\sim \frac{\alpha}{2z}\exp\left(-i\frac{\alpha}{2}\mathcal{H}(0)\right)|1\rangle+\ldots=\frac{\alpha}{2z}|0\rangle_\nu+\ldots
\end{aligned}$$

where we used the OPE between $-i\partial H$ and the H exponential. This OPE can be converted to the following commutation relation;

$$\left[Q,\left[\exp\left(-i\frac{\alpha}{2}\mathcal{H}\right)\right]_n\right]=\frac{\alpha}{2}\left[\exp\left(-i\frac{\alpha}{2}\mathcal{H}\right)\right]_n, \qquad Q=-i\oint\frac{dz}{2\pi i}\partial H(z)$$

The subscript issue is removed by using the zero-mode charge Q, instead of a general oscillator mode. The OPE

$$-i\partial H(z)\, e^{-i\alpha H(0)/2} \sim \frac{\alpha}{2z} e^{-i\alpha H(0)/2} \tag{8.1}$$

implies the displayed commutator after the contour integral. A nonzero $(\partial H)_m$ mode would shift the Laurent mode label; the charge Q is the mode that measures the eigenvalue without shifting it.

Using Q, we can define a general partition function as follows:

$$Z_\beta^\alpha(\tau)=\operatorname{Tr}_\alpha\left(q^H e^{\pi i\beta Q}\right)$$

This partition function will have two differences. Firstly, we will have an additional factor because of the vacuum charge under Q. Secondly, it will have an additional factor in the terms that have a fermion occupancy number equal to 1 . So, we will have the following;

$$Z_\beta^\alpha(\tau)=q^{\left(3\alpha^2-1\right)/24} e^{\pi i\alpha\beta/2} \prod_{m=1}^{\infty}\left[1+e^{\pi i\beta} q^{m-(1-\alpha)/2}\right]\left[1+e^{-\pi i\beta} q^{m-(1+\alpha)/2}\right] \tag{10.7.1}$$

We can write this in a compact form. Using the generalized theta functions, we have;

$$\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(0,\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i\left(n+\frac{\alpha}{2}\right)^2\tau+\pi i\beta\left(n+\frac{\alpha}{2}\right)}=e^{\pi i\alpha^2\tau/4}e^{\pi i\alpha\beta/2}\sum_{n\in\mathbb{Z}}e^{\pi in^2\tau}e^{\pi in(\alpha\tau+\beta)}$$

$$=q^{\alpha^2/8}e^{\pi i\alpha\beta/2}\sum_{n\in\mathbb{Z}}q^{n^2/2}e^{\pi in(\alpha\tau+\beta)}=q^{\alpha^2/8}e^{\pi i\alpha\beta/2}\prod_{m=1}^{\infty}\left(1-q^m\right)\left(1+q^{m-1/2}e^{\pi i(\alpha\tau+\beta)}\right)\left(1+q^{m-1/2}e^{-\pi i(\alpha\tau+\beta)}\right)$$

$$=q^{\alpha^2/8-1/24}e^{\pi i\alpha\beta/2}\eta(\tau)\prod_{m=1}^{\infty}\left(1+q^{m-1/2}e^{\pi i(\alpha\tau+\beta)}\right)\left(1+q^{m-1/2}e^{-\pi i(\alpha\tau+\beta)}\right)$$

$$=q^{\left(3\alpha^2-1\right)/24}e^{\pi i\alpha\beta/2}\eta(\tau)\prod_{m=1}^{\infty}\left(1+q^{m-(1-\alpha)/2}e^{\pi i\beta}\right)\left(1+q^{m-(1+\alpha)/2}e^{-\pi i\beta}\right)=\eta(\tau)Z^{\alpha}_{\beta}(\tau)$$

$$\Rightarrow Z^{\alpha}_{\beta}(\tau)=\frac{1}{\eta(\tau)}\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(0,\tau)$$

Now, the four types of partition functions relevant to superstrings are as follows:

$$Z^0_0(\tau)=\mathrm{Tr}_{\mathrm{NS}}\left(q^H\right),Z^0_1(\tau)=\mathrm{Tr}_{\mathrm{NS}}\left(e^{\pi iF}q^H\right),Z^1_0(\tau)=\mathrm{Tr}_{\mathrm{R}}\left(q^H\right),Z^1_1(\tau)=\mathrm{Tr}_{\mathrm{R}}\left(e^{\pi iF}q^H\right)$$

The total partition function for all fermions is thus given as follows:

$$Z^{\pm}_{\psi}(\tau)=\frac{1}{2}\left[Z^0_0(\tau)^4-Z^0_1(\tau)^4-Z^1_0(\tau)^4\pm Z^1_1(\tau)^4\right]$$

Why the minus signs sit outside the fourth powers. The three signs have two different origins, and neither is inside the single-fermion traces. The GSO-projected trace over each sector is written with the spacetime-statistics sign $(-1)^{\mathbf{F}}$ made explicit:

$$Z_{\psi}=\mathrm{Tr}_{\mathrm{NS}}\frac{1+e^{\pi iF}}{2}q^H\;-\;\mathrm{Tr}_{\mathrm{R}}\frac{1\pm e^{\pi iF}}{2}q^H=\frac{1}{2}\left[Z^{04}_0-Z^{04}_1\right]-\frac{1}{2}\left[Z^{14}_0\mp Z^{14}_1\right]. \tag{8.2}$$

The sign in front of Z^{04}_1 is the GSO projector itself: $e^{\pi iF}$ acting on the eight transverse fermions gives one factor of $e^{\pi iF_{\mathrm{pair}}}$ per complex pair, and on the NS ground state the total eigenvalue is -1 (one may check this from the superconformal ghosts, which contribute $e^{-\pi i}$ through the $e^{-\phi}$ vacuum factor), so the projector $\frac{1}{2}(1+e^{\pi iF})$ produces $\frac{1}{2}[Z^{04}_0-Z^{04}_1]$ with the sign *outside* the product over the four pairs. The overall minus sign of the whole Ramond block is different in kind: it is the spacetime spin-statistics factor $(-1)^{\mathbf{F}}$ of Eq. (7.3.4), present because R-sector states are spacetime fermions and a one-loop vacuum amplitude weighs fermion loops negatively. Finally, the $\pm$ on Z^{14}_1 is the only genuine choice, the relative GSO phase in the Ramond sector, and it distinguishes type IIB (+, same chirality kept on both sides) from type IIA (−). Since Z^1_1 vanishes identically by Eq. (7.2.3), this choice is invisible in the vacuum amplitude and first matters in amplitudes with Ramond insertions, which is why both theories are separately modular invariant.

The $\pm$ refers to $e^{\pi iF}=\pm1$ projection. In type IIA theory, $e^{\pi iF}$ in the right handed sector is -1 and thus, we will have Z^{-*}_{ψ} and in type IIB theory, $e^{\pi iF}$ in the right handed sector is +1 and thus, we will have Z^{+*}_{ψ}. So, the total partition function becomes;

$$Z_{T^2}=V_{10}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}\frac{i}{4\pi i\alpha'\tau_2}Z^8_XZ^+_{\psi}Z^{\pm*}_{\psi}$$

The +sign is for IIB theory and the - sign is for IIA theory. We now check the modular invariance of this partition function. We see that under T transformation;

$$d^2\tau\to d^2\tau,\tau_2\to\tau_2\Rightarrow\frac{d^2\tau}{\tau_2^2}\to\frac{d^2\tau}{\tau_2^2}$$

Under S transformation, we have;

$$d\left(-\frac{1}{\tau}\right)=\frac{d\tau}{\tau^2},\quad d\left(-\frac{1}{\bar{\tau}}\right)=\frac{d\bar{\tau}}{\bar{\tau}^2},\quad \tau_2\to\frac{\tau_2}{|\tau|^2},\quad \bar{\tau}_2\to\frac{\bar{\tau}_2}{|\tau|^2}\Rightarrow\frac{d^2\tau}{\tau_2^2}\to\frac{d^2\tau}{\tau_2^2}$$

Thus, the measure is modular invariant. Moreover, Z_X is modular invariant as we already know. For Z_ψ, Z^α_β is the partition function for fermions with boundary conditions along the two cycles of the torus:

$$\psi(w+2\pi) = -e^{-\pi i\alpha}\psi(w), \psi(w+2\pi\tau) = -e^{-\pi i\beta}\psi(w) \tag{10.7.2}$$

This implies the following;

$$\psi(w+2\pi(\tau+1)) = \psi(w+2\pi\tau+2\pi) = -e^{-\pi i\alpha}\psi(w+2\pi\tau) = e^{-\pi i(\alpha+\beta)}\psi(w) = -e^{-\pi i(\alpha+\beta-1)}\psi(w)$$

This is just the second boundary condition in 10.7 .2 with the following changes;

$$\tau \to \tau+1, \quad \beta \to \alpha+\beta-1$$

So, it seems that the following is true;

$$Z^\alpha_\beta(\tau) = \xi Z^\alpha_{\alpha+\beta-1}(\tau+1)$$

where ξ is a nonzero phase. Therefore $\tau \to \tau+1$ maps one spin-structure partition function to another, and the above equation can be written as

$$Z^\alpha_\beta(\tau+1) = \xi^{-1} Z^\alpha_{\beta-\alpha+1}(\tau)$$

which explicitly gives the T transformation of Z^α_β. Now, we turn towards S transformation. Define $w' = w/\tau$ and $\psi'(w') = \psi(w)$. Then in terms of ψ' the boundary conditions become;

$$\psi'(w'+2\pi) = \psi'\left(\frac{w}{\tau}+2\pi\right) = \psi'\left(\frac{w+2\pi\tau}{\tau}\right) = \psi(w+2\pi\tau) = -e^{-\pi i\beta}\psi(w) = -e^{-\pi i\beta}\psi'(w')$$

$$\psi'\left(w'-\frac{2\pi}{\tau}\right) = \psi'\left(\frac{w}{\tau}-\frac{2\pi}{\tau}\right) = \psi'\left(\frac{w-2\pi}{\tau}\right) = \psi(w-2\pi) = -e^{i\pi\alpha}\psi(w) = -e^{i\pi\alpha}\psi'(w')$$

These conditions should give us $Z^\alpha_\beta(\tau)$ but with $\beta \to -\alpha, \alpha \to \beta, \tau \to -1/\tau$. So, we see that we should have;

$$Z^\alpha_\beta(\tau) = \zeta Z^\beta_{-\alpha}(-1/\tau)$$

where ζ is again a constant. Since ζ is independent of τ, we can set $\tau = i$ and then, $\tau = -1/\tau$. Thus, we can derive the following;

$$Z^\alpha_\beta(i) = \zeta Z^\beta_{-\alpha}(i) = \zeta^2 Z^{-\alpha}_{-\beta}(i) = \zeta^3 Z^{-\beta}_\alpha(i) = \zeta^4 Z^\alpha_\beta(i) \Rightarrow \zeta^4 = 1$$

So, we see that $\zeta^4 = 1$. We will see that $\zeta = 1$. It can be seen from the following calculation;

$$\begin{aligned}
Z^\beta_{-\alpha}(-1/\tau) &= \frac{1}{\eta(-1/\tau)}\vartheta\begin{bmatrix}\beta/2\\-\alpha/2\end{bmatrix} = \frac{1}{\sqrt{-i\tau}\eta(\tau)}\sum_{n\in\mathbb{Z}}\exp\left[-\frac{\pi i}{\tau}\left(n+\frac{\beta}{2}\right)^2 - \pi i\left(n+\frac{\beta}{2}\right)\alpha\right]\\
&= \frac{1}{\sqrt{-i\tau}\eta(\tau)}\exp\left(-\frac{\pi i}{4\tau}\left(\beta^2+2\alpha\beta\tau\right)\right)\sum_{n\in\mathbb{Z}}\exp\left[-\frac{\pi i n^2}{\tau} - \pi i\left(\frac{\beta}{\tau}+\alpha\right)n\right]\\
&= \frac{\sqrt{-i\tau}}{\sqrt{-i\tau}\eta(\tau)}\exp\left(-\frac{\pi i}{4\tau}\left(\beta^2+2\alpha\beta\tau\right)\right)\sum_{n\in\mathbb{Z}}\exp\left[i\pi\tau\left(n-\frac{1}{2}\left(\frac{\beta}{\tau}+\alpha\right)\right)^2\right]\\
&= \frac{1}{\eta(\tau)}\exp\left(-\frac{\pi i}{4\tau}\left(\beta^2+2\alpha\beta\tau\right)\right)\sum_{n\in\mathbb{Z}}\exp\left[i\pi\tau n^2 - i\pi\tau n\left(\frac{\beta}{\tau}+\alpha\right) + \frac{i\pi\tau}{4}\left(\frac{\beta}{\tau}+\alpha\right)^2\right]\\
&= \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}\exp\left[i\pi\tau\left(n^2-n\alpha+\frac{\alpha^2}{4}\right) - 2\pi i n\frac{\beta}{2}\right] = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}\exp\left[i\pi\tau\left(n-\frac{\alpha}{2}\right)^2 - 2\pi i n\frac{\beta}{2}\right]\\
&= \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}\exp\left[i\pi\tau\left(n+\frac{\alpha}{2}\right)^2 + 2\pi i n\frac{\beta}{2}\right] = Z^\alpha_\beta(\tau)
\end{aligned}$$

where in the last step, we changed the summation variable i.e. $n \to -n$. Also note from this calculation that $Z^{-\alpha}_{-\beta}(\tau) = Z^\alpha_\beta(\tau)$. Also, we have shown that $\zeta = 1$. To find ξ, we need 10.7.1 to calculate $Z^{\alpha}_{\alpha+\beta-1}(\tau+1)$. We establish the following identities first. We will use q_2 for $e^{2\pi i(\tau+1)}$. We start with the following results;

$$e^{\pi i(\beta+\alpha-1)}q_2^{m-(1-\alpha)/2} = e^{\pi i(\beta+\alpha-1)}q^{m-(1-\alpha)/2}e^{-\pi i(1-\alpha)} = e^{\pi i(\beta+2\alpha)}q^{m-(1-\alpha)/2}$$
$$e^{-\pi i(\beta+\alpha-1)}q_2^{m-(1+\alpha)/2} = e^{-\pi i(\beta+\alpha-1)}q^{m-(1+\alpha)/2}e^{-\pi i(1+\alpha)} = e^{-\pi i(\beta+2\alpha)}q^{m-(1+\alpha)/2}$$
$$q_2^{\left(3\alpha^2-1\right)/24} = q^{\left(3\alpha^2-1\right)/24}e^{\pi i\left(3\alpha^2-1\right)/12}$$
$$e^{\pi i\alpha(\beta+\alpha-1)/2} = e^{\pi i\alpha\beta/2}e^{\pi i\alpha(\alpha-1)/2} = e^{\pi i\alpha(\beta+2\alpha)/2}e^{-\pi i\alpha(\alpha+1)/2}$$

But this gives;

$$Z^{\alpha}_{\alpha+\beta-1}(\tau+1) = \exp\left[\pi i\left(\frac{3\alpha^2-1}{12}\right) - \frac{\pi i\alpha(\alpha+1)}{2}\right] Z^{\alpha}_{\beta+2\alpha}(\tau)$$

Resolution of the phase. This *is* the correct transformation; the mistake is the naive expectation $\xi = 1$. A direct computation from the theta representation settles it. Shifting $\tau \to \tau+1$ inside $\vartheta\left[\begin{smallmatrix}\alpha/2\\ \beta/2\end{smallmatrix}\right](0,\tau) = \sum_n \exp[\pi i(n+\frac{\alpha}{2})^2\tau + \pi i(n+\frac{\alpha}{2})\beta]$ and using $e^{\pi i n^2} = e^{\pi i n}$ gives

$$Z^{\alpha}_{\beta}(\tau+1) = \exp\left[-\frac{i\pi}{12} - \frac{i\pi\,\alpha(\alpha+2)}{4}\right] Z^{\alpha}_{\alpha+\beta+1}(\tau), \tag{8.3}$$

where the $-i\pi/12$ is the η-function phase and the rest comes from completing the square in the theta sum. Substituting $\beta \to \alpha+\beta-1$ and using $Z^{\alpha}_{\beta+2} = e^{\pi i\alpha}Z^{\alpha}_{\beta}$ reproduces exactly the boxed phase above, so the two computations agree. The phase is expected on physical grounds: the boundary-condition argument identifies the two path integrals only up to the ground-state energy phase $e^{2\pi i(h-c/24)}$ of the sector being propagated, and that is precisely what Eq. (8.3) records ($-1/12$ for the two transverse fermion pairs at $\alpha = 0$, shifted by the twisted ground-state weight $\alpha^2/8$ per pair otherwise). The phase-free statement is reserved for the full integrand: in $Z^{\pm}_{\psi}$ the fourth powers turn these phases into $e^{-i\pi/3}$ (NS) and $e^{2\pi i/3}$ (R), a relative $e^{\pi i}$ that is exactly compensated in the sum, and the leftover overall $e^{2\pi i/3}$ cancels against the conjugate right-movers, as verified below. Modular covariance with computable phases, not phase-free covariance, is the correct requirement sector by sector.

We can now show that we can always come to $\alpha = 1$ if we start from $\alpha = 0$ (for $\beta = 0, 1$ which are present in $Z^{\pm}_{\psi}$) and thus, the Ramond sector should be present due to modular invariance. We use the following two transformations of the partition functions;

$$Z^{\alpha}_{\beta} \xrightarrow{T} Z^{\alpha}_{\beta+\alpha-1}, \quad Z^{\alpha}_{\beta} \xrightarrow{S} Z^{\beta}_{-\alpha}$$

Moreover, we also use the previously noted result that $Z^{\alpha}_{\beta}(\tau) = Z^{-\alpha}_{-\beta}(\tau)$. The required transformations are as follows:

$$Z^0_1 \xrightarrow{S} Z^1_0$$

$$Z^0_0 \xrightarrow{T} Z^0_{-1} \xrightarrow{S} Z^{-1}_0 = Z^1_0$$

The second transformation can be done as following as well;

$$Z^0_0 \xrightarrow{T} Z^0_{-1} \xrightarrow{S} Z^{-1}_0 \xrightarrow{S} Z^0_1 \xrightarrow{S} Z^1_0$$

Thus Z^0_0 and Z^0_1 are both related by modular transformations to Z^1_0, and invertibility implies the converse relations. However, neither Z^0_0 nor Z^0_1 is related to Z^1_1 by these transformations. A modular-invariant closed-string theory therefore cannot be built from a single NS spin structure alone; the Ramond spin structures must be included in the left- and right-moving sectors. In addition,

$$Z^1_1 \xrightarrow{T} Z^1_1, \quad Z^1_1 \xrightarrow{S} Z^1_{-1}\begin{cases} \xrightarrow{T} Z^1_{-1} \\ \xrightarrow{S} Z^{-1}_{-1} = Z^1_1 \end{cases}$$

Thus Z^1_1 transforms into itself under modular transformations, up to phases. By itself it does not define a complete string partition function: it has fermion zero modes, gives a vanishing vacuum trace without insertions, and is not closed under the OPE requirements of the full theory. We now determine how $Z_\psi^\pm$ transforms under modular transformations, beginning with T:

$$Z_\psi^\pm(\tau+1) = \frac{1}{2}\left[Z_0^0(\tau+1)^4 - Z_1^0(\tau+1)^4 - Z_0^1(\tau+1)^4 \mp Z_1^1(\tau+1)^4\right]$$
$$= \frac{1}{2}\left[e^{-\pi i/3}Z_1^0(\tau)^4 - e^{-\pi i/3}Z_2^0(\tau)^4 - e^{2\pi i/3}Z_0^1(\tau)^4 \mp e^{2\pi i/3}Z_1^1(\tau)^4\right]$$
$$= \frac{1}{2}\left[-e^{2\pi i/3}Z_1^0(\tau)^4 + e^{2\pi i/3}Z_2^0(\tau)^4 - e^{2\pi i/3}Z_0^1(\tau)^4 \mp e^{2\pi i/3}Z_1^1(\tau)^4\right]$$
$$= \frac{1}{2}e^{2\pi i/3}\left[Z_2^0(\tau)^4 - Z_1^0(\tau)^4 - Z_0^1(\tau)^4 \mp Z_1^1(\tau)^4\right]$$

Now using 10.7.1, we see that;

$$Z_{\beta+2}^\alpha(\tau) = e^{\pi i\alpha}Z_\beta^\alpha(\tau) \Rightarrow Z_{\beta+2}^0(\tau) = Z_\beta^0(\tau) \Rightarrow Z_2^0(\tau) = Z_0^0(\tau)$$

So, we get the following;

$$Z_\psi^\pm(\tau+1) = \frac{1}{2}e^{2\pi i/3}\left[Z_0^0(\tau)^4 - Z_1^0(\tau)^4 - Z_0^1(\tau)^4 \mp Z_1^1(\tau)^4\right] = e^{2\pi i/3}Z_\psi^\pm(\tau)$$
$$\Rightarrow Z_\psi(\tau+1)Z_\psi^\star(\bar\tau+1) = Z_\psi(\tau)Z_\psi^\star(\bar\tau)$$

for all subscripts of Z_ψ. For S transformations, we proceed as follows:

$$Z_\psi^\pm(-1/\tau) = \frac{1}{2}\left[Z_0^0(-1/\tau)^4 - Z_1^0(-1/\tau)^4 - Z_0^1(-1/\tau)^4 \mp Z_1^1(-1/\tau)^4\right]$$
$$= \frac{1}{2}\left[Z_0^0(\tau)^4 - Z_0^{-1}(\tau)^4 - Z_1^0(\tau)^4 \mp Z_1^{-1}(\tau)^4\right]$$

Now, using the fact that $Z_0^{-1}(\tau) = Z_0^1(\tau)$ and $Z_1^{-1}(\tau) = e^{\pi i}Z_{-1}^{-1}(\tau) = -Z_{-1}^{-1}(\tau) = -Z_1^1(\tau)$, we get;

$$Z_\psi^\pm(-1/\tau) = \frac{1}{2}\left[Z_0^0(\tau)^4 - Z_0^1(\tau)^4 - Z_1^0(\tau)^4 \mp Z_1^1(\tau)^4\right] = Z_\psi^\pm(\tau)$$

So, $Z_\psi^\pm(\tau)$ is invariant under S transformation.
It is much easier to show that the $Z_\psi^\pm$is modular invariant by showing that it vanishes. Firstly, we see that;

$$Z_0^0(\tau)^4 - Z_0^1(\tau)^4 - Z_1^0(\tau)^4 = \frac{1}{\eta(\tau)^4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau)^4\right)$$
$$= \frac{1}{\eta(\tau)^4}\left(\vartheta_3(0\mid\tau)^4 - \vartheta_4(0\mid\tau)^4 - \vartheta_2(0\mid\tau)^4\right) = 0$$

where we used (7.2.2). Moreover, using (7.2.3), we can deduce that $Z_1^1(\tau) = 0$.

8.7.1 More on $c = 1$ CFT

The purpose of the $c = 1$ theory here is that the compact boson is the smallest laboratory where modular invariance, enhanced gauge symmetry, and orbifolding can all be seen exactly. A circle theory is labeled by a radius, with

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}, \qquad n, w \in \mathbb{Z}, \tag{8.4}$$

and its partition function is the Narain theta sum divided by $|\eta(\tau)|^2$. At the self-dual radius the fields with momentum-winding quantum numbers that were massive at generic radius become currents, schematically

$$J_L^3 \sim i\partial X_L, \qquad J_L^\pm \sim e^{\pm i\,2X_L/\sqrt{\alpha'}}, \tag{8.5}$$

with a similar right-moving copy. Thus the circle branch contains an $SU(2)_L \times SU(2)_R$ enhancement point, while the orbifold branch keeps only the states invariant under $X \mapsto -X$ and adds twisted-sector states at the fixed points. This leads naturally to the later heterotic construction: gauge enhancement is being read directly from allowed worldsheet vertex operators, not appended as a spacetime assumption. [10, 86].

8.8 Divergences of type I theory

The important divergence in type-I perturbation theory is not a random ultraviolet infinity. After open-closed duality, the long-cylinder limit is a tree-level exchange of massless closed-string fields between boundaries, crosscaps, and orientifold planes. The R–R part has the schematic form

$$\mathcal{A}_{\mathrm{RR}}^{\mathrm{massless}} \propto (N-32)^2 \int_0^\infty d\ell, \tag{8.6}$$

where N is the number of D9-branes and ℓ is the transverse-channel proper time. Thus the divergence is a tadpole: it says that the assumed flat background sources an uncancelled R–R field unless $N = 32$. The same condition gives the $SO(32)$ Chan–Paton group and is the perturbative string counterpart of ten-dimensional anomaly cancellation. [10, 167, 352].

8.8.1 The cylinder

We need to compute the cylinder, Möbius strip, and Klein bottle partition functions and their massless divergences. For the cylinder partition function, just write the bosonic part, set $d = 10$, and add an additional 1/2 because of the projection operator. Moreover, for the fermionic part, multiply Z_ψ^+. So, we get;

$$\begin{aligned}
Z_{C_2}^{II} &= iV_{10}n^2 \int_0^\infty \frac{dt}{4t} (8\pi\alpha' t)^{-5} \eta(it)^{-8} \cdot \frac{1}{2} \left[Z_0^0(it)^4 - Z_1^0(it)^4 - Z_0^1(it)^4 - Z_1^1(it)^4\right] \\
&= iV_{10}n^2 \int_0^\infty \frac{dt}{8t} (8\pi\alpha' t)^{-5} \eta(it)^{-8} \left[Z_0^0(it)^4 - Z_0^1(it)^4\right] \\
&\quad + iV_{10}n^2 \int_0^\infty \frac{dt}{8t} (8\pi\alpha' t)^{-5} \eta(it)^{-8} \left[-Z_1^0(it)^4 - Z_1^1(it)^4\right] \\
&= Z_{C_2,0} + Z_{C_2,1}.
\end{aligned}$$

where $Z_{C_2,0}$ and $Z_{C_2,1}$ equal the two terms shown above. Since there is no $e^{\pi i F}$ insertion in the definition of $Z_{C_2,0}$, the fermions in this trace are anti-periodic around the loop; this is the NS trace, just as in the torus discussion where the insertion of $e^{\pi i F}$ changes the spin structure around one cycle. So, $Z_{C_2,0}$ comes from the NS–NS sector. We now focus on $Z_{C_2,0}$ because supersymmetry pairs the NS–NS and R–R oscillator spectra with opposite signs, so $Z_{C_2,0} + Z_{C_2,1} = 0$ before the transverse-channel interpretation separates the tadpoles. The modular transformation of $Z_{C_2,0}$ is done as follows:

$$\begin{aligned}
&-iV_{10}n^2 \int_0^\infty \frac{d(\pi/s)}{8(\pi/s)} \left(\frac{8\pi^3\alpha'}{s}\right)^{-5} \eta\left(\frac{i\pi}{s}\right)^{-8} \left[Z_0^0\left(\frac{i\pi}{s}\right)^4 - Z_0^1\left(\frac{i\pi}{s}\right)^4\right] \\
&= -iV_{10}n^2 \int_0^\infty \frac{d(\pi/s)}{8(\pi/s)} \left(\frac{8\pi^3\alpha'}{s}\right)^{-5} \left(\frac{\pi}{s}\right)^4 \eta\left(\frac{is}{\pi}\right)^{-8} \left[Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_{-1}^0\left(\frac{is}{\pi}\right)^4\right] \\
&= \frac{iV_{10}n^2}{8\pi\left(8\pi^2\alpha'\right)^5} \int_0^\infty ds \eta\left(\frac{is}{\pi}\right)^{-8} \left[Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_1^0\left(\frac{is}{\pi}\right)^4\right]
\end{aligned}$$

where we used the modular transformations of η and Z_β^α and we also used the fact that $Z_{\beta+2}^0 = Z_\beta^0 \Rightarrow Z_{-1}^0 = Z_1^0$. Now, we extract the massless divergence from this partition function. We use the following fact;

$$\begin{aligned}
\eta(is/\pi)^{-8} &= \left(e^{-s/12}\right)^{-8} \left(1 - e^{-2s} + \ldots\right)^{-8} \\
&= e^{2s/3}\left(1 + 8e^{-2s} + \ldots\right) = e^{2s/3} + 8e^{-4s/3} + \ldots \\
&= e^{2s/3} + \mathcal{O}\left(e^{-4s/3}\right).
\end{aligned} \tag{10.8.1}$$

Moreover, we have;

$$Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_1^0\left(\frac{is}{\pi}\right)^4 = \frac{1}{\eta(is/\pi)^4}\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,is/\pi)^4 - \frac{1}{\eta(is/\pi)^4}\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(0,is/\pi)^4$$
$$= \frac{\vartheta_3(0,is/\pi)^4}{\eta(is/\pi)^4} - \frac{\vartheta_4(0,is/\pi)^4}{\eta(is/\pi)^4} = \frac{\vartheta_2(0,is/\pi)^4}{\eta(is/\pi)^4}$$

where we used 7.2.2. Now, we expand $\vartheta_2(0,\tau)$ as follows:

$$\vartheta_2(0,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} = q^{1/8} + \sum_{n=1}^{\infty} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} + \sum_{n=1}^{\infty} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} = 2q^{1/8} + \sum_{n=2}^{\infty} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} + \sum_{n=1}^{\infty} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}$$
$$= 2q^{1/8} + 2\sum_{n=1}^{\infty} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$$

where we merged the two sums by shifting the index $n \to n+1$ in the first sum. Continuing, we get;

$$\vartheta_2(0,\tau) = 2e^{\pi i\tau/4} + 2\sum_{n=1}^{\infty} e^{\pi i\tau\left(n+\frac{1}{2}\right)^2} = 2e^{\pi i\tau/4} + 2e^{9\pi i\tau/4} + \ldots$$
$$\Rightarrow \vartheta_2(0,is/\pi) = 2e^{-s/4} + 2e^{-9s/4} + \ldots = 2e^{-s/4}\left(1 + e^{-4s} + \ldots\right)$$
$$\Rightarrow \vartheta_2(0,is/\pi)^4 = 16e^{-s}\left(1 + e^{-4s} + \ldots\right)^4 = 16e^{-s}\left(1 + 4e^{-4s} + \ldots\right) = 16e^{-s} + 64e^{-4s} + \ldots$$

Moreover, we have;

$$\eta(is/\pi)^{-4} = e^{s/3}\left(1 - e^{-2s} + \ldots\right)^{-4} = e^{s/3} + 4e^{-5s/3} + \ldots$$

Therefore, we get;

$$\frac{\vartheta(0,is/\pi)^4}{\eta(is/\pi)^4} = \left(e^{s/3} + 4e^{-5s/3} + \ldots\right)\left(16e^{-s} + 64e^{-4s} + \ldots\right) = 16e^{-2s/3} + \mathcal{O}\left(e^{-8s/3}\right) \qquad (10.8.2)$$

Using 10.8.1 and 10.8.2, we get;

$$\eta(is/\pi)^{-8}\left[Z_0^0(is/\pi)^4 - Z_1^0(is/\pi)^4\right] = \eta(is/\pi)^{-8}\frac{\vartheta(0,is/\pi)^4}{\eta(is/\pi)^4}$$
$$= \left(e^{2s/3} + \mathcal{O}\left(e^{-4s/3}\right)\right)\left(16e^{-2s/3} + \mathcal{O}\left(e^{-8s/3}\right)\right) = 16 + \mathcal{O}\left(e^{-2s}\right) \qquad (10.8.3)$$

So, the massless divergence from the cylinder partition function becomes;

$$Z_{C_2} = \frac{16iV_{10}n^2}{8\pi\left(8\pi^2\alpha'\right)^5}\int_0^{\infty} ds$$

This is not a spacetime ultraviolet divergence. In the transverse channel it is the exchange of a massless closed-string state between two boundaries, and in the R–R sector it is the square of the D9-brane charge. The cylinder by itself is therefore inconsistent: flat space with only D9-branes sources an uncancelled R–R ten-form potential. The Klein bottle supplies the orientifold-plane charge, and the Möbius strip supplies the boundary–crosscap interference term; only the sum of the three amplitudes can cancel the tadpole. The final cancellation condition is the type-I result $n = 32$.

8.8.2 The Klein bottle

For the Klein bottle partition function, the bosonic part is already known. In $d = 10$ it is

$$Z_{K_2}^{\text{bosonic}} = iV_{10}\int_0^{\infty}\frac{dt}{4t}\left(4\pi\alpha' t\right)^{-5}\eta(2it)^{-8}$$

It remains to derive the fermionic part. In the torus case, the general partition function Z_β^α

includes $e^{i\beta Q} = e^{i\beta F}$ in the trace and corresponds to the boundary conditions

$$\psi(w+2\pi) = -e^{-\pi i\alpha}\psi(w), \quad \psi(w+2\pi\tau) = -e^{-\pi i\beta}\psi(w)$$

In the Klein bottle case, however, $\psi(w+2\pi it)$ cannot be proportional to $\psi(w)$, because $w + 2\pi it \sim -\bar{w}$. It must instead be proportional to $\tilde{\psi}(\bar{w})$. The trace must therefore include an operator implementing this boundary condition, together with Ω and the GSO projectors. The required insertion is

$$R = \Omega \exp(\pi i\beta F + \pi i\tilde{\beta}\tilde{F})$$

It gives the following boundary conditions;

$$\psi(w+2\pi it) = -e^{\pi i\beta}\tilde{\psi}(\bar{w}), \quad \psi(\bar{w}+2\pi it) = -e^{\pi i\tilde{\beta}}\psi(w)$$

This implies;

$$\psi(w+4\pi it) \sim -e^{\pi i\beta}\tilde{\psi}(\bar{w}+4\pi it) = e^{i\pi(\beta+\tilde{\beta})}\psi(w) \tag{10.8.4}$$

This agrees with the fact that for the Klein bottle,

$$w + 4\pi it \sim -\bar{w} + 2\pi it = -\overline{(w+2\pi it)} = -\overline{(-\bar{w})} = w$$

Since the fundamental region for Klein bottle can be taken to be;

$$0 \leq \sigma^1 < \pi, \quad 0 \leq \sigma^2 < 4\pi it$$

The Klein bottle trace pairs left- and right-moving fields through Ω, so only sectors with matching spacetime statistics contribute: NS–NS and R–R. The NS–NS sector corresponds to antiperiodic boundary conditions in Eq. (10.8.4), namely $(\beta, \tilde{\beta}) = (0,1)$ or $(1,0)$, so $R = \Omega e^{\pi iF}$ or $R = \Omega e^{\pi i\tilde{F}}$. The R–R sector corresponds to $(\beta, \tilde{\beta}) = (0,0)$ or $(1,1)$, giving $R = \Omega$ or $R = \Omega e^{i\pi F + i\pi\tilde{F}}$. The NS–NS and R–R partition functions are then

$$\mathrm{NS-NS}: Z_0^0(2it), \quad \mathrm{R-R}: -Z_0^1$$

and thus, the full Klein bottle partition function is;

$$Z_{K_2} = iV_{10}\int_0^\infty \frac{dt}{8t}\left(4\pi^2\alpha' t\right)^{-5}\eta(2it)^{-8}\left[Z_0^0(2it)^4 - Z_0^1(2it)^4\right]$$

where the minus sign is due to spacetime statistics, and the additional factor of $1/2$ is the GSO projection. Performing the modular transformation with $s = \pi/(2t)$ gives

$$\begin{aligned}
Z_{K_2} &= iV_{10}\int_0^\infty \frac{d(\pi/2s)}{8(\pi/2s)}\left(\frac{2\pi^3\alpha'}{s}\right)^{-5}\eta\left(\frac{i\pi}{s}\right)^{-8}\left[Z_0^0(i\pi/s)^4 - Z_0^1(i\pi/s)^4\right] \\
&= \frac{iV_{10}}{8\left(2\pi^3\alpha'\right)^5}\int_0^\infty \frac{ds}{s}s^5\left(\frac{s}{\pi}\right)^{-4}\eta\left(\frac{is}{\pi}\right)^{-8}\left[Z_0^0(is/\pi) - Z_{-1}^0(is/\pi)\right] \\
&= \frac{i2^{10}V_{10}}{8\pi\left(8\pi^2\alpha'\right)^5}\int_0^\infty ds\eta\left(\frac{is}{\pi}\right)^{-8}\left[Z_0^0(is/\pi) - Z_1^0(is/\pi)\right]
\end{aligned}$$

where we again used the fact that $Z_{-1}^0 = Z_1^0$. Now, we extract the massless divergence from this partition function. For that, we already have the expansion of the integrand in 10.8.3. So, we have;

$$Z_{K_2} = \frac{i2^{10}V_{10}}{8\pi\left(8\pi^2\alpha'\right)^5}\int_0^\infty ds\left(16 + \mathcal{O}\left(e^{-2s}\right)\right)$$

8.8.3 The Möbius strip

First determine the effect of Ω on fermionic fields. The doubling trick for open-string fermions starts from $\psi^\mu(\sigma^1 + i\sigma^2) = \psi^\mu(w)$ and $\tilde{\psi}^\mu(\sigma^1 + i\sigma^2) = \tilde{\psi}^\mu(\bar{w})$ on $0 \leq \sigma^1 \leq \pi$. Define the extended fermion $\Psi^\mu(\sigma^1, \sigma^2)$ by

$$\Psi^\mu\left(\sigma^1,\sigma^2\right)=\Psi^\mu(w)=\begin{cases}\psi^\mu\left(\sigma^1,\sigma^2\right)=\psi^\mu(w), & 0\le\sigma^1\le\pi\\ \tilde\psi^\mu\left(2\pi-\sigma^1,\sigma^2\right)=\tilde\psi^\mu(2\pi-\bar w), & \pi\le\sigma^1\le 2\pi\end{cases}$$

The two definitions agree at the boundaries by the open-string fermion boundary condition, so Ψ^μ is a single holomorphic fermion on the doubled strip. The operator Ω sends (σ^1,σ^2) to $(\pi-\sigma^1,\sigma^2)$, equivalently $w\mapsto\pi-\bar w$. Its action is therefore

$$\Omega\Psi^\mu(w)\Omega^{-1}=\Omega\psi^\mu(w)\Omega^{-1}=\tilde\psi^\mu(\pi-\bar w)=\tilde\psi\left(\pi-\sigma^1,\sigma^2\right)=\Psi^\mu\left(\sigma^1+\pi,\sigma^2\right)\quad\text{for }0\le\sigma^1\le\pi$$

$$\Omega\Psi^\mu(w)\Omega^{-1}=\Omega\tilde\psi^\mu(2\pi-\bar w)\Omega^{-1}=\tilde\psi^\mu(\pi-\bar w)=\tilde\psi\left(\pi-\sigma^1,\sigma^2\right)=\Psi^\mu\left(\sigma^1+\pi,\sigma^2\right)\quad\text{for }\pi\le\sigma^1\le 2\pi$$

The different domains can be organized directly in modes. In terms of modes, we have (with the mode proof fixed by the boundary conditions);

$$\Omega\psi^\mu_r\Omega^{-1}=e^{-\pi i r}\psi^\mu_r$$

In the NS sector, $r\in\mathbb{Z}+1/2$ and thus, $\Omega^2=-1$. For the NS vacuum, Ω^2 is the same as $e^{\pi iF}$ i.e. both are -1.[1] Now, the combined Ω and GSO projection operators get the following form in the NS sector;

$$\frac{1+\Omega}{2}\frac{1+e^{\pi iF}}{2}=\frac{1+\Omega}{2}\frac{1+\Omega^2}{2}=\frac{1+\Omega+\Omega^2+\Omega^3}{4}$$

The calculation reduces to the spin-structure sum that we now carry out.

Completing the Möbius amplitude and the $SO(32)$ selection. Only the terms of the projector with an odd power of Ω contribute to the Möbius strip (the even powers belong to the cylinder), so the trace to compute is $\operatorname{Tr}\left[\Omega\, e^{-2\pi t L_0}\right]$ over the open-string sectors, with the Chan–Paton factor contributing $\pm n$ exactly as in the bosonic case of Chapter 5: the upper sign for $SO(n)$, the lower for $Sp(n/2)$. Because Ω multiplies each fermionic mode ψ_r by $e^{-i\pi r}$ and each bosonic mode α_n by $(-1)^n$, its insertion is equivalent to evaluating the cylinder traces at the shifted modulus

$$q=e^{-2\pi t}\;\longrightarrow\;e^{i\pi}e^{-2\pi t},\qquad\text{i.e.}\qquad 2it\;\longrightarrow\;2it+\tfrac12,\tag{8.7}$$

so every η and ϑ function of the cylinder reappears with argument $2it+\frac12$. The Möbius vacuum amplitude is therefore

$$Z_{M_2}=\mp n\, iV_{10}\int_0^\infty\frac{dt}{8t}\left(8\pi^2\alpha' t\right)^{-5}\frac{1}{\eta(2it+\frac12)^8}\left[Z^0_0\left(2it+\tfrac12\right)^4-Z^0_1\left(2it+\tfrac12\right)^4\right],\tag{8.8}$$

the R–R piece entering with the relative minus sign required by spacetime statistics. To reach the transverse channel one uses the modular matrix $P=TST^2S$, which maps $2it+\frac12$ to the closed-string proper time ℓ with $t=1/8\ell$; P is built from the same S and T transformations already derived for Z^α_β, so no new transformation rules are needed. Expanding the transformed integrand for large ℓ, the massless R–R exchange between one boundary and one crosscap carries the coefficient $\mp2\times32\,n=\mp64\,n$: the $32=2^5$ is the crosscap-to-boundary normalization mismatch, and the factor 2 counts the two orientations of the exchange. Adding the three surfaces, the total massless R–R tadpole is proportional to

$$\underbrace{n^2}_{\text{cylinder}}\;\mp\;\underbrace{64\,n}_{\text{Möbius}}\;+\;\underbrace{1024}_{\text{Klein bottle}}\;=\;(n\mp32)^2,\tag{8.9}$$

which vanishes only for the upper sign with $n=32$: the unoriented open superstring is consistent only with gauge group $SO(32)$. In this counting the upper sign corresponds to the orthogonal projection of Ω on the Chan–Paton factors and the lower sign to the symplectic projection, so only $SO(32)$ survives. This is the superstring version of the $SO(8192)$ computation of Chapter 5, and unlike its bosonic ancestor the cancellation here is obligatory, the uncancelled tadpole sits in the R–R sector, where no Fischler–Susskind shift of the background can absorb it, because there is no source the vacuum could turn on without breaking ten-dimensional Lorentz invariance. The

[1] This holds on arbitrary NS states, not only the vacuum: acting on a state built from m fermionic raising modes, Ω^2 produces the phase $\prod e^{-2\pi i r_j}=(-1)^m$ from the half-integer moding, while $e^{\pi iF}$ produces $(-1)^F$ with $F=m+1$ counting the ghost contribution of the vacuum; both operators therefore agree state by state, which is what allows the projector to collapse to powers of Ω alone.

NS–NS channel carries the same square and cancels with it once supersymmetry is imposed, while the R–R channel is the essential condition because it is a charge-conservation law instead of a shift of the background [10, 352].

Chapter 9

The heterotic string

This chapter exploits a freedom that the previous one did not use: on a closed string the left- and right-moving sectors never mix locally, so they need not be the same theory. The heterotic string couples the right-moving half of the superstring to the left-moving half of the bosonic string, and the mismatch of central charges (-26 against -15) demands an internal left-moving system with $c = 16$. The question the chapter answers is why that system cannot be arbitrary. Realized on free fermions, consistency of the GSO projections allows exactly the $SO(32)$ and $E_8 \times E_8$ spectra; realized on bosons, modular invariance requires the internal momenta to lie on a sixteen-dimensional even self-dual Euclidean lattice, and such lattices are classified, there are precisely two, the root lattices just named. A spacetime gauge symmetry is thus fixed by a lattice condition on the worldsheet. To make the mechanism precise we develop the necessary Lie-algebra and current-algebra theory, roots, weights, level quantization, and the Sugawara construction, proving on the way that the fermionic and bosonic descriptions carry the same stress tensor, and we end with toroidal compactification and the BPS states that Chapter 13 will use as duality probes [10, 361].

The lattice logic extends to non-supersymmetric and cobordism-motivated settings. Altavista, Anastasi, Angius, Uranga, and Wang construct networks of trivalent ten-dimensional heterotic junctions by extending a $(0, 1)$ heterotic worldsheet description, organizing one-dimensional networks with graph theory and giving worldsheet realizations for more general graphs [250]. This is not needed for the perturbative classification below, but it extends the role of the heterotic worldsheet: the same left-moving current algebra and modular constraints that select $SO(32)$ and $E_8 \times E_8$ can also be used as local input at junctions between distinct ten-dimensional string sectors.

The quantization of chiral bosons on arbitrary Riemann surfaces refines the heterotic left-moving sector at the level of the path integral. For chiral scalars valued on a torus with lattice Γ, modular consistency requires the theta function

$$\Theta_\Gamma(\tau) = \sum_{p\in\Gamma} q^{p^2/2}$$

to transform without an anomalous multiplier under $SL(2, \mathbb{Z})$. This holds when Γ is even and self-dual:

$$p^2 \in 2\mathbb{Z}, \qquad \Gamma = \Gamma^*.$$

Hull and Lambert obtain this condition from the chiral-boson action and its higher-genus partition function [251]. The same arithmetic condition that classifies the heterotic gauge lattice also makes the chiral path integral globally well defined.

The gauge group stays tied to the chiral CFT. In the pure-spinor description a spacetime supersymmetry generator must be both BRST closed and conserved on the worldsheet,

$$Q_{\mathrm{BRST}} q_\alpha = 0, \qquad \bar\partial q_\alpha = 0.$$

In flat space these equations reproduce the usual ten-dimensional supersymmetry charge; in curved superspace they become differential constraints on a normalizable spinor superfield [287]. Narain compactifications can also be reconstructed from error-correcting codes through Construction A: a code $C \subset \mathbb{F}_p^n$ defines an integral lattice, and the heterotic condition is again evenness and self-duality of the resulting momentum lattice [269]. Simple-current extensions express the no-global-symmetry condition in worldsheet terms: gauging a center one-form symmetry corresponds to extending the holomorphic algebra by chiral simple currents [282]. Hence the gauge group is fixed by modularly consistent chiral CFT quantities, not by a separate spacetime input.

Cobordism-junction constructions also reinterpret ten-dimensional string sectors. Non-compact worldsheet CFTs can evade otherwise rigid perturbative obstructions and give RG flows connecting type 0A/0B and IIA/IIB worldsheet theories, though strongly coupled linear-dilaton regions appear in the domain-wall description [270]. The "branching" construction gives microscopic worldsheet junctions where several ten-dimensional string theories meet at a nine-dimensional locus, with extra degrees of freedom becoming light at the branch point and admitting a supercritical/tachyon-condensation interpretation [276]. Tachikawa's trivalent heterotic junction shows the same mechanism for the supersymmetric $E_8 \times E_8$, supersymmetric $SO(32)$, and non-supersymmetric $SO(16) \times SO(16)$ heterotic theories using a two-dimensional non-conformal $\mathcal{N} = (0,1)$ theory with three asymptotic ends [278]. For this chapter, the familiar heterotic current-algebra endpoints can be treated as asymptotic phases of more general worldsheet RG configurations.

9.1 World sheet supersymmetries

We want to classify all the worldsheet symmetry algebras with the following constraints;

- The spins of the holomorphic current are less than or equal to 2 . The higher spin currents (called W algebras) have been used as symmetry currents (as follows from the orbifold projection) but the constructions (e.g. of the BRST operator) are non-trivial because of the non-linear commutators. Some cases turn out to be special cases of bosonic strings.
- The spin is a multiple of 1/2. If they are not, then the OPE;

$$j(z)j(0) \sim \frac{d_{jj}}{z^{2h}}$$

(where d_{jj} is a constant) will be multi-valued. The string theories that relax this condition are fractional strings and it is not clear if these theories exist.

The only spins left then are as follows:

$$h = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

We can make algebras with different numbers of different types of currents. Before proceeding, we need the central charge of the ghost pair associated with a current of spin h. A constraint of spin h is accompanied by ghosts of weights h and $1-h$; applying the standard bc or $\beta\gamma$ central-charge formula gives

$$\begin{aligned} c_g &= (-1)^{2h+1}\left(3(2h-1)^2 - 1\right) \\ \Rightarrow c_0 &= -2, c_{1/2} = -1, c_1 = -2, c_{3/2} = 11, c_2 = -26 \end{aligned} \tag{11.1.1}$$

The Jacobi identity allows only the following algebras;

n_0	$n_{1/2}$	n_1	$n_{3/2}$	n_2	c_g
0	0	0	0	1	-26
0	0	0	1	1	$-26+11=-15$
0	0	1	2	1	$-2+2(11)-26=-6$
0	1	3	3	1	$-1+3(-2)+3(11)-26=0$
0	4	7	4	1	$4(-1)+7(-2)+4(11)-26=0$
1	4	6	4	1	$-2+4(-1)+6(-2)+4(11)-26=0$
0	0	3	4	1	$3(-2)+4(11)-26=12$

The first two possibilities are Virasoro and $N=1$ superconformal algebras. The third possibility is called $N=2$ superconformal algebra. After the first three, no algebra has a negative ghost central charge and thus, none of them will have a positive critical dimension. The needed $N=2$ superconformal quantities are the following..

9.2 The $SO(32)$ and $E_8\times E_8$ heterotic strings

The heterotic string appears when we choose the left- and right-moving algebras to be different. We will study the $(N,\tilde{N})=(0,1)$ heterotic case. The other possibilities are $(N,\tilde{N})=(0,2)$ or $(N,\tilde{N})=(1,2)$ heterotic theories; the order is kept ascending because the flipped possibility is realized by the worldsheet reflection $z\to\bar{z}$. To have $(N,\tilde{N})=(0,1)$, we need a bosonic string on the left-moving side and a fermionic string on the right-moving side. To keep the dimension the same, we need to have the following fields;

$$X_L^\mu(z)\quad(\mu=0,\ldots,9)\quad \text{for left moving side}$$
$$X_R^\mu(\bar{z}),\tilde{\psi}^\mu(\bar{z})\quad(\mu=0,\ldots,9)\quad \text{for right moving side} \tag{11.2.1}$$

However, if we do this, then the central charges for the ghosts are $(c_g,\tilde{c}_g)=(-26,-15)$. The central charge on the right is canceled out by the matter central charge but the central charge on the left will be canceled
out only if we put some internal CFT with $(c,\tilde{c})=(16,0)$ into the game. One way to do this is to introduce 32 fermion fields on the left moving side as follows:

$$X_L^\mu(z)\quad(\mu=0,\ldots,9),\lambda^A(z),\quad(A=1,\ldots 32)\ \text{for left moving side}$$
$$X_R^\mu(\bar{z}),\tilde{\psi}^\mu(\bar{z})\quad(\mu=0,\ldots,9)\ \text{for right moving side} \tag{11.2.2}$$

This will give us the following action;

$$S=\frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu+\lambda^A\bar{\partial}\lambda^A+\tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right) \tag{11.2.3}$$

Note that there won't be any $T_F(z)$ field now and thus, no fermion constraint on the right moving side. This forbids us to choose timelike directions for any $\lambda^A(z)$. The theory has $SO(1,9)\times SO(32)$ symmetry now where $SO(32)$ is the internal symmetry. The periodicity of $T_B(z)$ (which will contain λ^A 's instead of ψ^μ 's now) only requires to have the following boundary condition on $\lambda^A(w)$ (notice the w coordinate and hence notice that we are imposing this boundary condition on the cylinder field);

$$\lambda^A(w+2\pi)=\mathcal{O}^{AB}\lambda^B(w),\quad(\mathcal{O}\in SO(32))$$

The worldsheet current for spacetime supersymmetry is the dimension-one operator $\mathcal{V}_{\mathbf{s}}=e^{-\phi/2}S_{\mathbf{s}}$, where $S_{\mathbf{s}}=\exp(i\sum_a s_aH^a)$ is the spin field built by bosonizing the ten right-moving fermions. Now, every vertex operator in the R sector contains such a spin field, so the OPE of $\mathcal{V}_{\mathbf{s}}$ with itself should be single-valued. Thus,

$$\mathcal{V}_{\mathbf{s}}(z)\mathcal{V}_{\mathbf{s}}(0)\sim e^{-\phi(z)/2}\exp\left[\sum_a s_aH^a(z)\right]e^{-\phi(0)/2}\exp\left[\sum_a s_aH^a(0)\right]$$
$$\sim z^{-1/4}\prod_{j=0}^{4}e^{is_jH^j(z)}e^{is'_jH^j(z)}\sim z^{-\frac{1}{4}}z^{\mathbf{s}.\mathbf{s}'}e^{i\sum_a(s+s')_aH^a(0)}$$

The power of z should be an integer. So, we have the following condition;

$$\text{s.s'}\ -\frac{1}{4}\in\mathbb{Z}\ \text{(Ramond sector)}$$

In the NS sector, the corresponding locality condition is imposed between the spin field and the NS vertex operators; the s_a still label the spin field appearing in the spacetime-supercharge current, while the NS operator contributes vector weights instead of Ramond spin weights.

$$\text{s.s. }' - \frac{1}{2} \in \mathbb{Z} \text{ (NS sector)}$$

the appearance of $-1/2$ instead of $-1/4$ can be understood because of the appearance of $e^{-\phi(z)}$ in the NS vertex operators instead of $e^{-\phi(z)/2}$. The GSO projection is the simultaneous solution of this locality condition, closure of OPEs among the allowed sectors, and modular invariance of the one-loop partition function.

We now determine the states in the heterotic theory, beginning with the left-moving sector. This sector has $SO(8) \times SO(32)$ symmetry, with $SO(8)$ coming from the transverse X^i fields and $SO(32)$ from the λ^A fields. The normal-ordering constants in the left-moving sector are

$$\text{NS} : -\frac{8}{24} - \frac{32}{48} = -1, \quad \text{R} : -\frac{8}{24} + \frac{32}{24} = 1 \tag{11.2.4}$$

Therefore, the R sector has no massless states. So, the transverse Hamiltonian is as follows:

$$H = \frac{\alpha' p^2}{4} + N - 1(\text{NS}), \quad H = \frac{\alpha' p^2}{4} + N + 1(\text{R}) \tag{11.2.5}$$

Using the mass shell condition, we see that for $N = 0$, the NS ground state is a tachyon. The first excited states are as follows:

$$\lambda^A_{-1/2}|0\rangle_{\text{NS}}$$

but they are forbidden due to the GSO projection (because F is odd for this state). So, the actual first excited states are as follows:

$$\alpha^i_{-1}|0\rangle_{\text{NS}}, \quad \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{\text{NS}}$$

They satisfy the GSO projection and the mass shell condition implies that these are massless states. The α state lives in ($\mathbf{8}_v, \mathbf{1}$) representation of $SO(8) \times SO(32)$ and λ states lives in ($\mathbf{1}, [2]$) representation of $SO(8) \times SO(32)$. The λ 's live in the adjoint representation of $SO(32)$ with the dimension of 496 . So, the massless sector lives in $(\mathbf{8}_v, \mathbf{1}) + (\mathbf{1}, \mathbf{496})$.

Now, the right-handed sector (because of the GSO projection), has no tachyon. Moreover, the massless states (like type IIB theory) live in $\mathbf{8}_v + \mathbf{8}$ representation ($\mathbf{8}_v$ comes from the NS sector and $\mathbf{8}$ comes from the R sector). So, the massless states are as follows:

$$[(8_v, 1) + (1, 496)] \times (8_v + 8) \tag{11.2.6}$$

These states can be broken down using the results that we derived before. This breakdown can be divided into two parts. They are as follows:

$$\begin{aligned} (\mathbf{8}_v, \mathbf{1}) \times (\mathbf{8}_v + \mathbf{8}) &= (\mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}) && \text{(type I supergravity multiplet)} \\ (\mathbf{1}, \mathbf{496}) \times (\mathbf{8}_v + \mathbf{8}) &= (\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}, \mathbf{496}) && (\text{SO(32)N} = 1 \text{ gauge multiplet }) \end{aligned} \tag{11.2.8}$$

This massless spectrum matches the field content of type-I $N = 1$ supergravity coupled to an $SO(32)$ gauge multiplet. The equality is not yet a duality statement, because the perturbative strings and couplings are different; it becomes the type-I–heterotic strong-weak relation in Chapter 13.

Another heterotic theory is obtained by having the following boundary conditions;

$$\lambda^A(w + 2\pi) = \begin{cases} \eta\lambda^A(w) A = 1, \ldots, 16 \\ \eta'\lambda^A(w) A = 17, \ldots, 32 \end{cases} \tag{11.2.9}$$

where η and η' can be ± 1. So, we have four $\text{NS} - \text{NS}'$, $\text{NS} - \text{R}'$, $\text{R} - \text{NS}'$ and $\text{R} - \text{R}'$ sectors in the left-moving theory. In the left sector, the GSO projection is applied to both subsectors separately. The allowed sectors close under OPE because the two sets of sixteen fermions have separately conserved fermion parity, and the sum over the four spin structures is modular invariant by the same theta-function identities used in Chapter 8.

For the NS - NS' sector, the normal ordering constant is still - 1 and thus, we still have a tachyon in the left sector and the first allowed excited states are as follows:

$$\alpha^i_{-1}|0\rangle_{\text{NS\_NS}'}, \quad \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{\text{NS\_NS}'} \tag{11.2.10}$$

but because of the GSO projection, A and B should be from the same left subsector. So, the left sector has $SO(8) \times SO(16) \times SO(16)$ symmetry. The α state lives in ($\mathbf{8}_v, \mathbf{1}, \mathbf{1}$) representation of this symmetry and λ states live in either $(\mathbf{1}, \mathbf{120}, \mathbf{1})$ representation or $(\mathbf{1}, \mathbf{1}, \mathbf{120})$ representation. **120** comes from the fact that $SO(16)$ has an adjoint representation of dimension $16 \times 15/2 = 120$. For the $\mathrm{R}-\mathrm{NS}'$ sector, we have 16 periodic and 16 anti-periodic fermions and thus, the normal ordering constant is;

$$-\frac{8}{24} + \frac{16}{24} - \frac{16}{48} = 0$$

So, there is no tachyon in the left sector now. Since there is a R sector now, we have 16 zero modes i.e. λ_0^A for $A = 1, \ldots, 16$. So, the ground states are degenerate and they live in a $2^{16/2} = 2^8 = 256$ dimensional spinor representation of the first $SO(16)$. The GSO projection retains only one chirality and thus, only **128** is retained from $\mathbf{256} = \mathbf{128} + \mathbf{128}'$. These ground states thus live in $(\mathbf{1}, \mathbf{128}, \mathbf{1})$ representation. The same thing happens with NS-R sector but with $SO(16)'$ instead of $SO(16)$. Thus, the ground state in this sector lives in ($\mathbf{1}, \mathbf{1}, \mathbf{128}$) representation. Notice that **120** is the adjoint representation and **128** is a spinor representation of $SO(16)$. For the $\mathrm{R}-\mathrm{R}'$ sector, the normal ordering sector is again 1 and thus, there are no massless states in the $\mathrm{R}-\mathrm{R}'$ spectrum.
The left moving massless spectrum is thus written as follows:

$$(8_v, 1, 1) + (1, 120 + 128, 1) + (1, 1, 120 + 128)$$

Coupling this left spectrum with the right spectrum of $\mathbf{8}_v \times \mathbf{8}$, we get the following;

$$\begin{aligned}
(8_v \times 8_v, 1, 1) + (8_v, 120 + 128, 1) + (8_v, 1, 120 + 128) + (8 \times 8_v, 1, 1) + (8, 120 + 128, 1) + (8, 1, 120 + 128) \\
= (1, 1, 1) + (28, 1, 1) + (35, 1, 1) + (8', 1, 1) + (56, 1, 1) \\
+ (8_v, 120 + 128, 1) + (8_v, 1, 120 + 128) + (8, 120 + 128, 1) + (8, 1, 120 + 128)
\end{aligned}$$

Massless vectors must form the adjoint of the gauge algebra because their vertex operators are the currents generating the nonabelian symmetry; closure of their OPE is the worldsheet form of closure of the gauge algebra. Therefore we need a group with $120 + 128 = 248$ generators containing $SO(16)$ such that the adjoint decomposes as $\mathbf{120} \oplus \mathbf{128}$. The exceptional algebra E_8 has precisely this maximal $SO(16)$ embedding. So, the full symmetry in the left sector is $SO(8) \times E_8 \times E_8$. Thus, the full massless spectrum is as follows:

$$\begin{aligned}
(1, 1, 1) + (28, 1, 1) + (35, 1, 1) + (8', 1, 1) + (56, 1, 1) \\
+ (8_v, 248, 1) + (8_v, 1, 248) + (8, 248, 1) + (8, 1, 248)
\end{aligned}$$

In the bosonized description the same result is the statement that the allowed left-moving momenta fill the even self-dual lattice $\Gamma_{E_8} \oplus \Gamma_{E_8}$. Modular invariance of the left-moving partition function requires this lattice condition, and the roots of the lattice are exactly the dimension-one currents that generate $E_8 \times E_8$.

9.3 Other ten-dimensional heterotic strings

The heterotic string is allowed because its asymmetric left- and right-moving sectors are glued by a modular-invariant lattice. In toroidal compactification the momenta live in an even self-dual lattice,

$$\Gamma^{d+16,d} = \Gamma^{d+16,d\,*}, \qquad p_L^2 - p_R^2 \in 2\mathbb{Z}. \tag{9.1}$$

The condition is stronger than ordinary momentum quantization. It allows spacetime gauge symmetry to arise from worldsheet current algebras while preserving modular invariance of the torus partition function. [10, 86, 350, 361].

9.4 Lie-algebra background

Let the Lie algebra G generators be T^a. The Lie bracket of the corresponding Lie algebra $\mathfrak{g}$ is;

$$\left[T^a,T^b\right]=if_c^{ab}T^c \tag{11.4.1}$$

which satisfies the Jacobi identity;

$$\left[T^a,\left[T^b,T^c\right]\right]+\left[T^c,\left[T^a,T^b\right]\right]+\left[T^b,\left[T^c,T^a\right]\right]=0 \tag{11.4.2}$$

The finite Lie group element is as follows:

$$\mathbb{I}+i\theta_aT^a+\frac{1}{2!}\left(\theta_aT^a\right)^2+\ldots=\exp\left(i\theta_aT^a\right) \tag{11.4.3}$$

For a compact semisimple group, the Lie algebra has a nondegenerate invariant inner product, the Cartan–Killing form. In terms of generators,

$$d^{ab}=\left(T^a,T^b\right)=\mathrm{Tr}\left(\mathrm{ad}\left(T^a\right)\mathrm{ad}\left(T^b\right)\right) \tag{11.4.4}$$

where $\mathrm{ad}\left(T^a\right)$ means T^a in the adjoint representation. This product is symmetric due to the cyclic property of the trace;

$$\left(T^a,T^b\right)=\mathrm{Tr}\left(\mathrm{ad}\left(T^a\right)\mathrm{ad}\left(T^b\right)\right)=\mathrm{Tr}\left(\mathrm{ad}\left(T^b\right)\mathrm{ad}\left(T^a\right)\right)=\left(T^b,T^a\right) \tag{11.4.5}$$

This inner product has the following property as well;

$$\left(\left[T^a,T^b\right],T^c\right)+\left(T^b,\left[T^a,T^c\right]\right)=\mathrm{Tr}\left(T^aT^bT^c\right)-\mathrm{Tr}\left(T^bT^aT^c\right)+\mathrm{Tr}\left(T^bT^aT^c\right)-\mathrm{Tr}\left(T^bT^cT^a\right)=0 \tag{11.4.6}$$

where we will have to use the cyclic property on the last trace to cancel it with the first trace. The condition is equivalent to the following;

$$\left(\left[T^a,T^b\right],T^c\right)+\left(T^b,\left[T^a,T^c\right]\right)=if_d^{ab}\left(T^d,T^c\right)+if_d^{ac}\left(T^b,T^d\right)=0\Rightarrow f_d^{ab}d^{dc}+f_d^{ac}d^{bd}=0 \tag{11.4.7}$$

A subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a subset of $\mathfrak{g}$ such that if $X,Y\in\mathfrak{h}$, then $[X,Y]\in\mathfrak{h}$. An ideal I of $\mathfrak{g}$ is a subalgebra such if $X\in\mathfrak{g}$ and $Y\in I$, then $[X,Y]\in I$.
Now, if $\mathfrak{g}$ has no non-trivial ideals, then $\mathfrak{g}$ is called a simple Lie algebra. A Lie algebra $\mathfrak{g}$ is called semisimple if it has no abelian ideals. Cartan's criterion implies that a semisimple Lie algebra is a direct sum of simple Lie algebras, and on each simple factor the invariant bilinear form is unique up to an overall scale. We choose that scale so that $d^{ab}=\delta^{ab}$. We can generalize 11.4.4 to arbitrary representations as follows:

$$\mathrm{Tr}\left(t_r^at_r^b\right)=T_rd^{ab}$$

where t_r^a is the generator in r representation of $\mathfrak{g}$ and T_r is an r dependent constant. We see from (11.4.4) that adjoint representation, $T_r=1$.
We also see that the following is true;

$$\left[t_r^at_r^bd_{ab},t_r^c\right]=d_{ab}\left[t_r^a,t_r^c\right]t_r^b+d_{ab}t_r^a\left[t_r^b,t_r^c\right]=id_{ab}f_d^{ac}t_r^dt_r^b+id_{ab}f_d^{bc}t_r^at_r^d$$

Renaming the indices in the second term as $a\to d, d\to b, b\to a$, we get;

$$\left[t_r^at_r^bd_{ab},t_r^c\right]=-i\left(d_{da}f_b^{ca}+d_{ab}f_d^{ca}\right)t_r^dt_r^b=-i\left(f_{db}^c+f_{bd}^c\right)t_r^dt_r^b=0 \tag{11.4.8}$$

where we used the antisymmetry of $f^a{}_{bc}$ in the last step. So, $t_r^at_r^bd_{ab}$ commutes with all generators, and thus, (by Schur's lemma), it should be a constant but dependent on r. So, we have;

$$t_r^at_r^bd_{ab}=Q_r$$

where Q_r is a constant called the Casimir invariant of r. The well-known Lie groups series and their Lie algebras are as follows:

- $A_n=SO(n+1)$ for $n>1$. The matrices are unitary matrices with determinant 1. The infinitesimal $U\in SO(n)$ manipulation gives us the following;

$$U_{ab}=\mathbb{I}_{ab}+i\theta_\alpha T_{ab}^\alpha+\mathcal{O}\left(\theta^2\right)\Rightarrow U_{ab}^{-1}=U_{ab}^\dagger=\mathbb{I}_{ab}-i\theta_\alpha\left(T^{*\alpha}\right)_{ab}^T+\mathcal{O}\left(\theta^2\right)$$
$$\Rightarrow\mathbb{I}_{ab}=\left(UU^{-1}\right)_{ab}=\mathbb{I}+i\theta_\alpha\left(T_{ab}^\alpha-T_{ba}^{*\alpha}\right)+\mathcal{O}\left(\theta^2\right)\Rightarrow T_{ab}^\alpha=T_{ba}^{*\alpha}$$

Moreover, we have the following;

$$1 = \det U = \det\left(\mathbb{I} + i\theta_\alpha T^\alpha + \mathcal{O}\left(\theta^2\right)\right) = 1 + i\theta_\alpha \operatorname{Tr}\left(T^\alpha\right) + \mathcal{O}\left(\theta^2\right) \Rightarrow \operatorname{Tr}\left(T^\alpha\right) = 0$$

So, the generators are traceless Hermitian matrices.

- $B_n = SO(2n+1), D_n = SO(2n)$ for $n > 1$. The infinitesimal $M \in SO(2n)$ or $M \in SO(2n+1)$ is given as follows:

$$M_{ab} = \mathbb{I}_{ab} + \theta_\alpha T^\alpha_{ab} + \mathcal{O}\left(\theta^2\right) \Rightarrow M^{-1}_{ab} = M^T_{ab} = \mathbb{I}_{ab} + \theta_\alpha \left(T^\alpha\right)^T_{ab} + \mathcal{O}\left(\theta^2\right)$$

$$\Rightarrow \mathbb{I}_{ab} = \left(MM^T\right)_{ab} = \mathbb{I} + \theta_\alpha \left(T^\alpha_{ab} + \left(T^\alpha\right)^T_{ba}\right) + \mathcal{O}\left(\theta^2\right) \Rightarrow T^\alpha_{ab} + T^{T\alpha}_{ba} = 0$$

So, T is antisymmetric and thus, also traceless. Moreover, we have the following;

$$1 = \det M = \det\left(\mathbb{I} + \theta_\alpha T^\alpha + \mathcal{O}\left(\theta^2\right)\right) = 1 + \theta_\alpha \operatorname{Tr}\left(T^\alpha\right) + \mathcal{O}\left(\theta^2\right) \Rightarrow \operatorname{Tr}\left(T^\alpha\right) = 0$$

But this gives us no new information. So, the generators are antisymmetric matrices.

- $C_n = USp(k)$ which consists of $2n \times 2n$ unitary matrices such that if $U \in USp(n)$, then;

$$MUM^{-1} = \left(U^T\right)^{-1} \qquad \text{where } M = i\begin{bmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{bmatrix} \Rightarrow M^{-1} = M$$

The infinitesimal version of U is as follows:

$$U = \mathbb{I} + i\theta_\alpha T^\alpha + \mathcal{O}\left(\theta^2\right) \Rightarrow MUM^{-1} = \mathbb{I} + i\theta_\alpha M T^\alpha M^{-1} + \mathcal{O}\left(\theta^2\right) \tag{11.4.9}$$

Since U is unitary, T^α 's should be hermitian. Moreover, we have;

$$U^T = \mathbb{I} + i\theta_\alpha \left(T^\alpha\right)^T + \mathcal{O}\left(\theta^2\right) \Rightarrow \left(U^T\right)^{-1} = \mathbb{I} - i\theta_\alpha \left(T^\alpha\right)^T + \mathcal{O}\left(\theta^2\right) \tag{11.4.10}$$

Comparing 11.4.9 and 11.4.10, we get;

$$MT^\alpha M^{-1} = -\left(T^\alpha\right)^T \Rightarrow MTM^{-1} = -T^T \forall \alpha \tag{11.4.11}$$

So, the generators satisfy the condition in 11.4.11. We now constrain the form of T by this information. First of all, we write T in the block form (where all the matrices are $n \times n$ matrices);

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Now, using 11.4.11, we have;

$$-\begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix} = -\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \Rightarrow \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} -A^T & -C^T \\ -B^T & -D^T \end{pmatrix}$$

This implies;

$$D = -A^T, C = C^T, B = B^T \Rightarrow T = \begin{pmatrix} A & C \\ B & -A^T \end{pmatrix} \tag{11.4.12}$$

with B and C being symmetric matrices. Now, we use the fact that T is hermitian. Then, we have;

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow A = A^\dagger, B = C^\dagger, C = B^\dagger, D = D^\dagger \tag{11.4.13}$$

Therefore, using 11.4.13 and 11.4.12 we get;

$$D = D^\dagger = \left(-A^T\right)^\dagger = -A^*$$

$$B = C^\dagger = \left(C^T\right)^* = C^* \Rightarrow C = B^*$$

Therefore, we have;

$$T = \begin{pmatrix} A & B \\ B^\star & -A^\star \end{pmatrix} \text{ with } A^\dagger = A, B^T = B \tag{11.4.14}$$

The independent components of this compact symplectic algebra are counted by the Hermitian matrix A and the complex symmetric matrix B, giving $n^2+n(n+1)=n(2n+1)$ real generators. The non-compact real forms differ by the reality condition, but the root-system discussion below uses the compact form. We now divide the generators into a maximal commuting set $H=\{H^i\}$ and the remaining generators E^α. The number of elements in H is the rank of the algebra, and the subspace spanned by them is the Cartan subalgebra. The Lie algebra can then be written as

$$\left[H^i,H^j\right]=0,\quad \left[H^i,E^\alpha\right]=\alpha^i E^\alpha \tag{11.4.15}$$

First, H^j terms cannot appear in $\left[E^\alpha,E^\beta\right]$ when $\alpha+\beta\neq 0$. The most general form of this commutator is

$$\left[E^\alpha,E^\beta\right]=f_i^{\alpha\beta}H^i+\zeta_\mu^{\alpha\beta}E^\mu \tag{11.4.16}$$

with not all f 's vanishing. Taking commutator with H^j gives us the following;

$$\left[H^j,\left[E^\alpha,E^\beta\right]\right]=\zeta_\mu^{\alpha\beta}\left[H^j,E^\mu\right]=\zeta_\mu^{\alpha\beta}\mu^j E^\mu \tag{11.4.17}$$

We calculate this commutator using Jacobi identity and 11.4.15 as follows:

$$\left[H^j,\left[E^\alpha,E^\beta\right]\right]+\left[E^\beta,\left[H^j,E^\alpha\right]\right]+\left[E^\alpha,\left[E^\beta,H^j\right]\right]=0$$

$$\Rightarrow \left[H^j,\left[E^\alpha,E^\beta\right]\right]=\left[\left[H^j,E^\alpha\right],E^\beta\right]+\left[E^\alpha\left[H^j,E^\beta\right]\right]=\left(\alpha^j+\beta^j\right)\left[E^\alpha,E^\beta\right]=\left(\alpha^j+\beta^j\right)\left(f_i^{\alpha\beta}H^i+\zeta_\mu^{\alpha\beta}E^\mu\right) \tag{11.4.18}$$

Comparing 11.4.18 with 11.4.17, we see that if $\alpha+\beta\neq 0$, then all $f_i^{\alpha\beta}$ are zero. So, we have proved what we wanted to show. We also notice that;

$$\zeta_\mu^{\alpha\beta}\left(\alpha^j+\beta^j-\mu^j\right)=0\forall\alpha,\beta$$

Therefore, we see that $\zeta_\mu^{\alpha\beta}\neq 0$ only when $\alpha+\beta=\mu$ where μ is a root. If $\alpha+\beta$ is a root, then $\zeta_\mu^{\alpha\beta}=0$ for all μ except $\mu=\alpha+\beta$. We denote $\zeta_{\alpha+\beta}^{\alpha\beta}$ as $\epsilon(\alpha,\beta)$.
We might still have H in 11.4.16 if $\alpha+\beta=0$ and thus, we rewrite $f_i^{\alpha\beta}$ as f_i^α only because $f_i^{\alpha\beta}$ is non-zero only if $\beta=-\alpha$. We see by comparing 11.4.17) and 11.4.18 that if $\alpha+\beta=0$, then $\zeta_\mu^{\alpha\beta}\mu^j=0$ for all μ. But all μ can't be zero in general because that will make the whole lie algebra equal to Cartan subalgebra but we know that this isn't true in general. So, for $\beta=-\alpha$, all $\zeta_\mu^{\alpha\beta}$'s vanish. Therefore, we see that;

$$\begin{aligned}
\left[E^\alpha,E^\beta\right]&=\epsilon(\alpha,\beta)E^{\alpha+\beta}\qquad(\alpha+\beta\text{ is a root}),\\
\left[E^\alpha,E^\beta\right]&=f_i^\alpha H^i\qquad(\alpha+\beta=0),\\
\left[E^\alpha,E^\beta\right]&=0\qquad(\text{otherwise}).
\end{aligned}$$

It is easy to see that $\epsilon(\beta,\alpha)=-\epsilon(\alpha,\beta)$. The possible values of $\epsilon(\alpha,\beta)$ are ±1, because the normalization of the root generators can be chosen so that the nonzero structure constants in a root string are real and their magnitudes are fixed by the $SU(2)$ subalgebra generated by $\{E^\alpha,E^{-\alpha},\alpha\cdot H\}$. Using a particular (EEE) Jacobi identity, we have (with $\beta\neq\alpha$ and $\beta\neq-\alpha$);

$$\left[E^\alpha,\left[E^{-\alpha},E^\beta\right]\right]+\left[E^\beta,\left[E^\alpha,E^{-\alpha}\right]\right]+\left[E^{-\alpha},\left[E^{-\beta},E^\alpha\right]\right]=0\Rightarrow f_i^\alpha\beta^i=0$$

Similarly, for $\beta=\alpha$, the $SU(2)$ subalgebra associated with the root α fixes the Cartan element in the commutator. Normalizing H^i by the Killing form gives

$$f_i^\alpha=\frac{2\alpha_i}{\alpha^2}\Rightarrow\left[E^\alpha,E^{-\alpha}\right]=\frac{2\alpha_i}{\alpha^2}H^i=\frac{2\alpha\cdot H}{\alpha^2} \tag{11.4.19}$$

The matrices that represent H^i 's are mutually commuting and thus, they have simultaneous eigenvectors. Let the eigenvalues of the j-th eigenvector under the matrix representing H^i be w_j^i. We can get weight vectors where the k-th weight vector is;

$$\left(w_k^1,w_k^2,\ldots,w_k^{\operatorname{rank}(\mathfrak{g})}\right),\quad k=1,2,3,\ldots,\dim(r)$$

The number of weight vectors is equal to the dimension of the representation. For reference, we quote the α-th root vector as well;

$$\left(\alpha^1, \alpha^2, \ldots, \alpha^{\text{rank}}\left(\mathfrak{g}\right)\right), \quad \alpha = 1, \ldots, \dim \mathfrak{g}$$

$\dim \mathfrak{g}$ equals $\dim(r)$ when r is the adjoint representation. In the adjoint representation the Cartan generators act by commutator: $[H^i, E^\alpha] = \alpha^i E^\alpha$, so each root generator E^α is a weight vector of weight α. The remaining zero-weight states are the Cartan generators themselves. Examples of the weights and roots are as derived now.

- The vector representation of the Cartan subalgebra for $SO(2n)$ consists of the matrices that have all entries equal to zero but a 2×2 submatrix. H^j has the following non-zero 2×2 submatrix in $(2j-1, 2j)$ rows and columns;

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Therefore, we have;

$$H^j_{ab} = i\left(\delta_{a+1-2j}\delta_{b-2j} - \delta_{a-2j}\delta_{b+1-2j}\right), \quad a, b = 1, \ldots, 2n$$

These Cartan generators commute because each H^j acts on a distinct two-dimensional plane. The rank of $SO(2n)$ is therefore n. The eigenvectors of H^j have nonzero entries only in the $(2j-1, 2j)$ positions:

$$V^j_{\pm} = (0, \ldots, 0, 1, \mp i, \ldots 0)$$

We just manipulate the 2 dimensional spaces to find the eigenvalues;

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \mp i \end{bmatrix} = \begin{bmatrix} \pm 1 \\ -i \end{bmatrix} = \pm\begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

So, the eigenvalues of $V^j_{\pm}$ under H^k are as follows:

$$\omega^j_{\pm} = (0, \ldots, \pm 1, \ldots, 0)$$

where the nonzero entry is in the j-th position. So, $\omega^j_{\pm}$ are the weight vectors of the vector representation.

For $SO(2n)$, the adjoint representation is the antisymmetric tensor product $\wedge^2 V$ of the vector representation. Weights add in tensor products because H^i acts by the Leibniz rule on $v \otimes w$, giving eigenvalue $\omega^i_v + \omega^i_w$. So, there are four possibilities. We can add a ω^j_+ with ω^k_+ which gives us $+1$ in j-th and k-th positions. Moreover, we can add a ω^j_- with ω^k_- which gives us -1 in j-th and k-th positions. In all of these possibilities, $j \neq k$ because the antisymmetric tensor product kills $v_j \wedge v_j$. However, we add a ω^j_+ with ω^k_- which gives us a $+1$ in j-th position and -1 in k-th position for $j \neq k$. However, if $j = k$, then we get a zero root. There are k such roots. Since these are zero roots, they correspond to H^i for $(i = 1, \ldots, n)$. Therefore, the weight vectors for the adjoint representation which are also the root vectors are as follows:

$$(+1, +1, \underbrace{0, \ldots, 0}_{n-2}) + \text{ permutations} \tag{11.4.20}$$

$$(+1, -1, \underbrace{0, \ldots, 0}_{n-2}) + \text{ permutations} \tag{11.4.21}$$

$$(-1, -1, \underbrace{0, \ldots, 0}_{n-2}) + \text{ permutations} \tag{11.4.22}$$

$$(\underbrace{0, \ldots, 0}_{n})(\text{n of them }) \tag{11.4.23}$$

The spinor weights of $SO(2n)$ are the vectors with entries $\pm 1/2$. The two chiral spinor representations are distinguished by parity: one has an even number of plus signs and the other has an odd number of plus signs, each containing 2^{n-1} weights.

- $B_n = SO(2n+1)$ has the same Cartan generators with an additional row and column of zeros. The nonzero weights agree with the $SO(2n)$ vector weights, and the extra basis vector has zero eigenvalue under every H^i. Thus there is one additional weight,

$$\underbrace{(0,\ldots,0)}_{n}$$

The number of generators of $SO(2n)$ and $SO(2n+1)$ are as follows:
$SO(2n) : \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2+n, SO(2n+1) : \frac{(2n+1)(2n+2)}{2} = (n+1)(2n+1) = 2n^2+3n+1$
Thus $SO(2n+1)$ has $2n^2+3n+1-2n^2-n = 2n+1$ more generators than $SO(2n)$. The nonzero root vectors corresponding to the additional generators are

$$(\pm 1, \underbrace{0,\ldots,0}_{n-1}) + \text{ permutations}$$

The remaining classical and exceptional cases are summarized in the table below.
We also define the dual Coxeter number $h(g)$ as follows:

$$-\sum_{b,c} f_c^{ab} f_b^{dc} = h(g)\psi^2 d^{ad} \tag{11.4.24}$$

where $\psi^2 = \psi^i\psi^j d_{ij}$. It means that $\psi^2 d^{ad} = \psi^i\psi^j d_{ij} d^{ad}$ and thus, the normalization of ψ^2 doesn't change $\psi^2 d^{ad}$ and hence, the normalization also doesn't change $h(g)$. The properties of the Lie algebras are summarized as follows .

Group	Simply-laced	dimension	$h(g)$
$A_{n-1} = SU(n)(n > 1)$	Yes	$n^2 - 1$	n
$B_n = SO(2n+1)(n \geq 1)$	No	$2n(n+1)$	$2n-1$
$C_n = USp(n)(n \geq 1)$	No	$2n^2 + n$	$n+1$
$D_n = SO(2n)(n \geq 1)$	Yes	$n(2n+1)$	$2n-2$
E_6	Yes	78	12
E_7	Yes	133	18
E_8	Yes	248	30
F_4	?	52	9
G_2	?	14	4

9.4.1 Other useful facts for grand unification

The group-theory facts above become useful for grand unification because matter multiplets are organized by weights, while gauge bosons are organized by roots. The standard embeddings are

$$\mathbf{16}_{SO(10)} \longrightarrow \mathbf{10}_{SU(5)} \oplus \bar{\mathbf{5}}_{SU(5)} \oplus \mathbf{1}, \qquad \mathbf{27}_{E_6} \longrightarrow \mathbf{16}_{SO(10)} \oplus \mathbf{10}_{SO(10)} \oplus \mathbf{1}. \tag{9.2}$$

Therefore the spinor weights of $SO(2n)$ in the previous paragraph matter physically: for $SO(10)$ they place one Standard-Model family, plus a right-handed-neutrino singlet, into a single irreducible representation. In a heterotic compactification the same information is encoded in the allowed left-moving lattice momenta and current-algebra primaries.[1]

9.5 Current algebras

Let $j^a(z)$ be a field with conformal weights $(1,0)$ (also called a holomorphic current). Conformal invariance allows only a second-order pole proportional to the identity and a first-order pole proportional to another weight-one current, so the OPE has the form

[1]The branching rules in Eq. (9.2) are facts about representations, not yet a model. A grand-unified construction is fixed only once one chooses the chain — $SU(5) \subset SO(10)$, $SO(10) \subset E_6$, or a direct breaking of $SO(32)$ or $E_8 \times E_8$ — together with the Wilson line or orbifold shift that performs the breaking. With that choice the surviving gauge bosons are the roots invariant under the shift and the matter content follows from the branchings above; without it no phenomenological conclusion should be drawn. Chapter 15 carries this out for heterotic orbifolds.

$$j^a(z)j^b(0) \sim \frac{k^{ab}}{z^2} + \frac{ic_c^{ab}}{z} j^c(0)$$

where k^{ab} and c_c^{ab} are constants. Note that both terms have the same conformal weight as they should. The mode expansion of $j^a(z)$ is as follows:

$$j^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j_n^a \Rightarrow j_n^a = \oint_{C(0)} \frac{dz}{2\pi i} z^n j^a(z)$$

where $C(0)$ is the contour around 0 . The commutation relation can be derived via normal procedure as follows:

$$\begin{aligned}
\left[j_n^a, j_m^b\right] &= \oint_{C(0)} \frac{dz}{2\pi i} \oint_{C(0)} \frac{dw}{2\pi i} w^m z^n j^a(z) j^b(w) = \oint_{C(0)} \frac{dw}{2\pi i} w^m \oint_{C(w)} \frac{dz}{2\pi i} z^n \mathcal{R}\left(j^a(z)j^b(w)\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi i} w^m \oint_{C(w)} \frac{dz}{2\pi i} z^n \left(\frac{k^{ab}}{(z-w)^2} + \frac{ic_c^{ab}}{z-w} j^c(0)\right) = \oint_{C(0)} \frac{dw}{2\pi i} w^m \left(k^{ab} n w^{n-1} + ic_c^{ab} j^c(w) w^n\right) \\
&= nk^{ab} \oint_{C(0)} \frac{dw}{2\pi i} w^{m+n-1} + ic_c^{ab} \oint_{C(0)} \frac{dw}{2\pi i} j^a(w) w^{m+n} = ic_c^{ab} j_{m+n}^a + nk^{ab}\delta_{m+n} \qquad (11.5.1)
\end{aligned}$$

where $\mathcal{R}$ denotes radial ordering: the z-contour is deformed around w, and only the singular part of the OPE contributes to the commutator. 11.5.1 also implies;

$$\left[j_0^a, j_0^b\right] = ic_c^{ab} j_0^a \qquad (11.5.2)$$

Thus the j_0^a span a Lie algebra, denoted g, with $f^{ab}{}_c = c^{ab}{}_c$. Therefore the Jacobi identity gives

$$c_d^{bc} c_e^{ad} + c_d^{ab} c_e^{cd} + c_d^{ca} c_e^{bd} = 0 \qquad (11.5.3)$$

The $j_1^a j_0^b j_{-1}^c$ Jacobi identity gives us the following;

$$\left[j_1^a, \left[j_0^b, j_{-1}^c\right]\right] + \left[j_{-1}^c, \left[j_1^a, j_0^b\right]\right] + \left[j_0^b, \left[j_{-1}^c, j_1^a\right]\right] = 0$$

$$\Rightarrow \left(c_d^{bc} c_e^{ad} + c_d^{ab} c_e^{cd} + c_d^{ca} c_e^{bd}\right) j_0^e + k^{ad} c_d^{bc} + k^{cd} c_d^{ab} = 0 \Rightarrow k^{ad} c_d^{bc} + k^{cd} c_d^{ab} = 0$$

where we used 11.5.3 in the last step. Using 11.4.7) we see that if g is simple, then k^{ab} is proportional to d^{ab} since the inner product is unique for simple g. So, we have;

$$k^{ab} = \hat{k} d^{ab}$$

where $\hat{k}$ is a constant. We also see that $\hat{k}$ is positive if the theory is unitary. If $|1\rangle$ is the state corresponding to the unit operator, then we have;

$$\begin{aligned}
k^{ab} = \hat{k}\, d^{aa} &= \left[j_1^a, j_{-1}^a\right] = \left[j_1^a, j_{-1}^a\right] \langle 1|1\rangle \\
&= \langle 1| \left[j_1^a, j_{-1}^a\right] |1\rangle \\
&= \langle 1| j_1^a j_{-1}^a |1\rangle - \langle 1| j_{-1}^a j_1^a |1\rangle \\
&= \langle 1| j_1^a j_{-1}^a |1\rangle = \| j_{-1}^a |1\rangle \|^2 \geq 0.
\end{aligned}$$

where we used the fact that j_1 should annihilate $|1\rangle$. Now, for compact Lie algebra, the invariant form d^{ab} can be chosen positive definite, because the generators may be taken anti-hermitian and $-\mathrm{Tr}(T^a T^b)$ is positive on the compact real form. Thus, $\hat{k} \geq 0$. Since the weight of j^a is 1, the vertex operator of j_{-1}^a is $j^a(z)$ and thus, $\hat{k}$ vanishes only if $j^a(z)$ vanishes.

To show that $\hat{k}$ is quantized, take any root of g. Denote this root by α. Define three generators as follows:

$$J^3 = \frac{\alpha \cdot H}{\alpha^2}, J^{\pm} = E^{\pm\alpha}$$

Then, we can derive that;

$$\left[J^3, J^\pm\right] = \left[\frac{\alpha \cdot H}{\alpha^2}, E^{\pm\alpha}\right] = d_{ij}\frac{\alpha^i}{\alpha^2}\left[H^j, E^{\pm\alpha}\right] = \pm d_{ij}\frac{\alpha^i}{\alpha^2}\left[H^j E^{\pm\alpha}\right] = \pm E^{\pm\alpha} = \pm J^\pm$$

$$\left[J^+, J^-\right] = \frac{2\alpha \cdot H}{\alpha^2} = 2J^3$$

From the $SU(2)$ representation theory, we know that the representations of $SU(2)$ are labeled by the eigenvalue of J^3 and they are integers or half integers. So, $2J^3$ is always an integer. So, $2\alpha.H/\alpha^2$ has integer eigenvalues.

There is another $SU(2)$ subalgebra in the current algebra. For that, let's call the Cartan generators as H_0 and define the following;

$$J^3 = \frac{\alpha \cdot H_0 + \hat{k}}{\alpha^2}, \quad J^\pm = E_1^{\pm\alpha}$$

The 1 subscript on E denotes that it is made by the affine modes E_{-1}^{α} and $E_1^{-\alpha}$. Together with the shifted Cartan generator above, these modes obey the same $SU(2)$ commutation relations; the shift by $\hat{k}$ is the central term in the affine commutator. So, we deduce that $2\hat{k}/\alpha^2$ is also an integer for all roots. If this condition is satisfied for long roots, then it is automatically satisfied for short roots. So, we have;

$$k = \frac{2\hat{k}}{\psi^2} \in \mathbb{Z} \tag{11.5.4}$$

where ψ is a long root. k is called the level. Different references absorb factors of ψ^2 into the definition of the invariant form; in these conventions the long roots have the normalization used in Eq. (11.5.4). From now, we will take $d^{ab} = \delta^{ab}$. For $\mathfrak{g} = U(1) \times U(1) \times \ldots \times U(1)$, there are no roots (because the generators commute) and
thus, there is no concept of a level. Moreover, $c_c^{ab} = 0$ because the group is abelian (so, there is no z^{-1} term in the jj OPE). So, we can normalize the OPE to the following;

$$j^a(z)j^b(0) \sim \frac{\delta^{ab}}{z^2}$$

This OPE matches the OPE of the free boson current OPE which is as follows:

$$i\partial H^a(z) i\partial H^b(0) \sim \frac{\delta^{ab}}{z^2}$$

The equivalence $j^a \sim i\partial H^a$ is therefore consistent at the level of current OPEs; the corresponding Sugawara stress tensor agrees with the free-boson stress tensor after the normalization of the invariant form is fixed.

An example of currents is the fermion bilinear;

$$i\lambda^A\lambda^B$$

which lives in the vector $\times$ vector representation of $SO(n)$. The z^{-2} term of the OPE of this current is as follows:

$$i\lambda^A\lambda^B(z)i\lambda^A\lambda^B(w) = -\lambda^A\lambda^B(z)\lambda^A\lambda^B(w) \sim -\lambda^A\lambda^B(z)\lambda^A\lambda^B(w) = \frac{1}{(z-w)^2}$$

where we used the anti-commutativity of fermions and the OPE of λ^A which is as follows:

$$\lambda^A(z)\lambda^B(w) \sim \frac{\delta^{AB}}{z-w}$$

So, the coefficient of $1/(z-w)^2$ term is 1 . So, for this current, we compare this to the result of the general currents;

$$1 = \hat{k}d^{aa} = \hat{k}\delta^{aa} = \psi^2 k/2$$

d^{aa} appears because we took the OPE of a current with itself (i.e. $i\lambda^A\lambda^B$ with itself). Now, the long roots of $SO(n)$ have length squared equal to 2 (except $n = 3$). So, we have;

$$k = 1 \text{ for } i\lambda^A\lambda^B (n \neq 3)$$

For $n=3$, there is no separate long-root normalization in the same convention and $\psi^2=1$, so $k/2=1$ and $k=2$. The diagonal-current normalization accounts for this factor. Another example can be constructed from any real representation r. Take $\dim(r)$ real fermions, so that the bilinear current is antisymmetric and hermitian, and construct

$$\frac{1}{2}\lambda^A\lambda^B t_{r,AB}$$

The z^{-2} term in the OPE is as follows:

$$\begin{aligned}
&\frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w)\\
&\sim \frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w)\\
&\qquad+\frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w)\\
&=\frac{1}{4(z-w)^2}t_{r,AB}t_{r,CD}\left(-\delta^{AC}\delta^{BD}+\delta^{BC}\delta^{AD}\right)\\
&=\frac{1}{4(z-w)^2}\left(t_{r,AB}t_{r,BA}-t_{r,AB}t_{r,AB}\right)\\
&=\frac{1}{4(z-w)^2}\left(\operatorname{Tr}(t_rt_r)-\operatorname{Tr}(t_rt_r^T)\right).
\end{aligned}$$

Using $\operatorname{Tr}(t_rt_r)=-\operatorname{Tr}(t_rt_r^T)=T_r$, where the second equality follows from the antisymmetry $t_r^T=-t_r$, one obtains

$$\frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w)\sim\frac{T_r}{2(z-w)^2}\left(1+\mathcal{O}(z^{-1})\right).$$

Comparing it to the general case, we have;

$$\frac{T_r}{2}=k^{aa}=\hat{k}\,d^{aa}=\hat{k}\,\delta^{aa}=\hat{k}=\frac{\psi^2k}{2}\quad\Rightarrow\quad T_r=\psi^2k\quad\Rightarrow\quad k=\frac{T_r}{\psi^2}.$$

The remaining examples follow from the same index calculation with the corresponding representation matrices.

9.5.1 Sugawara construction

We now show that $:jj:(z)$ has the same OPE with $j^c(w)$ as the stress tensor $T_B(z)$ which is as follows:

$$T_B(z)j^c(w)\sim\frac{1}{(z-w)^2}j^c(w)+\frac{1}{z-w}\partial j^c(w)\tag{11.5.5}$$

since $j^c(w)$ is a Virasoro primary field and has conformal weight $(1,0)$. Now, $ij:(z)$ is defined as follows'

$$:jj:(z)=\lim_{z_1\to z}\left(j^a(z)j^a(z_1)-\frac{\hat{k}\delta^{aa}}{(z-z_1)^2}\right)=\lim_{z_1\to z}\left(j^a(z)j^a(z_1)-\frac{\hat{k}\dim(g)}{(z-z_1)^2}\right)$$

where the sum over a is implicit. We consider the following OPE such that z_1 and z_2 are far;

$$\begin{aligned}
&j^a(z_1)j^a(z_2)j^c(z_3)\sim j^a(z_1)j^a(z_2)j^c(z_3)+j^a(z_1)j^a(z_2)j^c(z_3)\\
&=\frac{\hat{k}j^c(z_2)}{(z_1-z_3)^2}+\frac{if^{acd}}{z_1-z_3}j^a(z_2)j^d(z_3)+\frac{\hat{k}j^c(z_1)}{(z_2-z_3)^2}+\frac{if^{acd}}{z_2-z_3}j^a(z_1)j^d(z_3)\\
&=\frac{\hat{k}j^c(z_2)}{(z_1-z_3)^2}+\frac{if^{acd}}{z_1-z_3}j^a(z_2)j^d(z_1)+\frac{\hat{k}j^c(z_1)}{(z_2-z_3)^2}+\frac{if^{acd}}{z_2-z_3}j^a(z_1)j^d(z_2)+\ (\text{holomorphic terms in } z_3)
\end{aligned}$$

Now, we expand this expression as a Laurent expansion in (z_2-z_1). We keep terms up to $(z_2-z_1)^0$ only. It is as follows:

$$\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}+\frac{if^{acd}}{z_1-z_3}\left[\frac{\hat{k}\delta^{ad}}{(z_2-z_1)^2}+\frac{if^{ade}j^e(z_1)}{z_2-z_1}\right]+\frac{\hat{k}j^c(z_1)}{(z_2-z_1+z_1-z_3)^2}+\frac{if^{acd}}{z_2-z_1+z_1-z_3}\left[\frac{\hat{k}\delta^{ad}}{(z_1-z_2)^2}+\frac{if^{ade}j^e(z_2)}{z_1-z_2}\right]$$
$$\sim\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}-\frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)}+\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}\left[1+\frac{(z_2-z_1)}{(z_1-z_3)}\right]^{-2}+\frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)}\left[1+\frac{(z_2-z_1)}{(z_1-z_3)}\right]^{-1}$$
$$\sim\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}-\frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)}+\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}+\frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)}-\frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)^2}$$
$$\sim\frac{2\hat{k}j^c(z_1)}{(z_1-z_3)^2}+\frac{f^{cad}f^{ead}j^e(z_1)}{(z_1-z_3)^2}$$

where we made heavy use of antisymmetry of f^{abc} and dropped all terms of the order of $\mathcal{O}(z_2-z_1)$ or higher. Using 11.4.24 (where we have kept all the indices upwards), we have;

$$f^{cad}f^{ead}=\psi^2h(g)\delta^{ce}$$

Therefore, after expanding $j^c(z_1)=j^c(z_3)+(z_1-z_3)\partial j^c(z_3)+\cdots$, we have

$$:jj:(z_1)\,j^c(z_3)\sim\frac{2\hat{k}+\psi^2h(g)}{(z_1-z_3)^2}j^c(z_1)\sim(k+h(g))\psi^2\left(\frac{1}{(z_1-z_3)^2}j^c(z_3)+\frac{1}{z_1-z_3}\partial j^c(z_3)\right)$$

where we used the fact $2\hat{k}=\psi^2k$. Therefore, we see that;

$$\frac{:jj:(z_1)}{(k+h(g))\psi^2}j^c(z_3)\sim\frac{1}{(z_1-z_3)^2}j^c(z_3)+\frac{1}{z_1-z_3}\partial j^c(z_3)$$

This OPE matches 11.5.5 and we can thus define;

$$T_B^s(z)=\frac{:jj:(z)}{(k+h(g))\psi^2}\tag{11.5.6}$$

where s stands for Sugawara. We can also derive the following result;

$$T_B^s(z_1)\,j^c(z_3)\sim\frac{1}{(z_1-z_3)^2}j^c(z_3)+\frac{1}{z_1-z_3}\partial j^c(z_3)$$
$$\Rightarrow j^c(z_3)\,T_B^s(z_1)\sim\frac{1}{(z_3-z_1)^2}j^c(z_1)+\frac{1}{z_3-z_1}\partial j^c(z_1)-\frac{1}{z_3-z_1}\partial j^c(z_1)=\frac{1}{(z_3-z_1)^2}j^c(z_1)$$

Now, we derive the following OPE where again z_2 and z_1 are far;

$$j^a(z_1)\,j^a(z_2)\,T_B^s(z_3)\sim j^a(z_1)\,j^a(z_2)\,T_B^s(z_3)+j^a(z_1)\,j^a(z_2)\,T_B^s(z_3)$$
$$=\frac{1}{(z_3-z_1)^2}j^a(z_1)\,j^a(z_2)+\frac{1}{z_3-z_1}\partial j^a(z_1)\,j^a(z_2)+\frac{1}{(z_3-z_2)^2}j^a(z_1)\,j^a(z_2)+\frac{1}{z_3-z_2}\partial j^a(z_2)\,j^a(z_1)\tag{11.5.7}$$

Before doing the expansion in z_2-z_1, we need the following OPEs;

$$j^a(z_1)\,j^a(z_2)\sim\frac{\hat{k}}{(z_1-z_2)^2}\Rightarrow\partial j^a(z_1)\,j^a(z_2)\sim\frac{-2\hat{k}}{(z_1-z_2)^3},\quad j^a(z_1)\,\partial j^a(z_2)\sim\frac{2\hat{k}}{(z_1-z_2)^3}$$

Now, we do the z_2-z_1 Laurent expansion in 11.5.7 and keep terms up to $(z_2-z_1)^0$ only. Double contractions give the fourth-order pole and single contractions reconstruct T_B^s and its derivative, so

$$T_B^s(z_1)\,T_B^s(z_2)\sim\frac{c^{g,k}}{2(z_1-z_3)^4}+\frac{2}{2(z_1-z_3)^2}T_B^s(z_3)+\frac{1}{z_1-z_3}\partial T_B^s(z_3)\,,c^{g,k}=\frac{k\dim(g)}{k+h(g)}$$

The normal-ordering constant vanishes here because the Sugawara tensor is defined by point splitting with the singular jj contraction subtracted; no zero-mode shift remains in the stress tensor itself. Since the concept of the nonabelian level is replaced by the normalization of the current for $U(1)$, we can use 11.5.4 and the identification $j=i\partial H$ to write the stress-energy tensor as follows (recall that we use $\alpha' = 2$ for H CFT);

$$T_B^s(z) = -\frac{1}{2} : \partial H \partial H : (z) = \frac{1}{2} : i\partial H i\partial H : (z) = \frac{1}{2} : jj : (z)$$

The total stress tensor T_B is the sum of the Sugawara tensor T_B^s and the tensor that is not made by currents, denoted T'_B. Since T'_B has nonsingular OPE with the currents, the singular part of $T_B T_B$ splits into the Sugawara contribution plus the remaining Virasoro tensor. Hence,

$$T'_B(z_1) T'_B(z_2) \sim \frac{c'}{2(z_1 - z_2)^4} + \frac{2T_B(z_2)}{(z_1 - z_2)^2} + \frac{1}{z_1 - z_2} \partial T'_B(z_2), \quad c' = c - c^{g,k}$$

If the non-current theory is unitary, then;

$$c - c^{g,k} \geq 0 \Rightarrow c \geq c^{g,k}$$

For any simply laced algebra, the dual Coxeter number satisfies

$$h(g) = \frac{\dim(g)}{\operatorname{rank}(g)} - 1$$

This follows from the Freudenthal–de Vries formula together with the fact that all long roots have the same length in a simply laced algebra; equivalently, it is checked on the ADE series and is preserved by the classification.

and therefore, we have;

$$c^{g,k} = k \dim(g) \left[k + \frac{\dim(g)}{\operatorname{rank}(g)} - 1 \right]^{-1} = \frac{k \dim(g)}{\dim(g) + (k-1)\operatorname{rank}(g)}$$

which means that;

$$c^{g,1} = \operatorname{rank}(g)$$

Now, if $g = E_8 \times E_8$ or $g = SO(32)$, then $\operatorname{rank}(g) = 16$ and thus $c = 16$ which is the same as the central charge of a theory of 32 fermions or 16 bosons. For this to happen in detail for the fermionic theory, the Sugawara tensor should be what we derived while deriving the λ part of T_B in heterotic theory i.e.;

$$T_B^s(z) = -\frac{1}{2} : \lambda^A \partial \lambda^A : (z) \tag{11.5.8}$$

To derive this, we consider the heterotic $SO(32)$ current as follows:

$$j^{(A,B)}(z) = i\lambda^A \lambda^B(z), \quad A, B = 1, \ldots, 32, A \neq B$$

We see that $A \leftrightarrow B$ gives the same current but with a negative sign. Now, $: j^{(A,B)} j^{(A,B)} : (z)$ is given as follows:

$$: j^{(A,B)} j^{(A,B)} : (z) = \lim_{z \to z_1} \left(\frac{1}{2} \sum_{A \neq B} j^{(A,B)}(z) j^{(A,B)}(z_1) - \frac{\dim(SO(32))}{(z - z_1)^2} \right)$$

$$= \lim_{z \to z_1} \left(\frac{1}{2} \sum_{A,B=1, A \neq B}^{32} j^{(A,B)}(z) j^{(A,B)}(z_1) - \frac{496}{(z - z_1)^2} \right)$$

where the factor of $1/2$ comes to avoid counting over equivalent currents. Now, we do the following calculation;

$$j^{(A,B)}(z) j^{(A,B)}(z_1) = - : \lambda^A \lambda^B : (z) : \lambda^A \lambda^B : (z_1)$$

$$= - : \lambda^A \lambda^B : (z) : \lambda^A \lambda^B : (z_1) - : \lambda^A \lambda^B : (z) : \lambda^A \lambda^B : (z_1) - : \lambda^A \lambda^B : (z) : \lambda^A \lambda^B : (z_1) + \mathcal{O}(z - z_1)$$

$$= \frac{1}{(z - z_1)^2} + \frac{\lambda^B(z) \lambda^B(z_1)}{z - z_1} + \frac{\lambda^A(z) \lambda^A(z_1)}{z - z_1} + \mathcal{O}(z - z_1)$$

$$= \frac{1}{(z - z_1)^2} - : \lambda^B \partial \lambda^B : (z) - : \lambda^A \partial \lambda^A : (z) + \mathcal{O}(z - z_1)$$

Now, dropping the $\mathcal{O}(z_1 - z_2)$ part, we use this result to get;

$$: j^{(A,B)} j^{(A,B)} : (z) = \lim_{z \to z_1} \left[\frac{1}{2} \sum_{A,B=1}^{32} {}_{A \neq B} \left(\frac{1}{(z - z_1)^2} - : \lambda^B \partial \lambda^B : (z) - : \lambda^A \partial \lambda^A : (z) \right) - \frac{496}{(z - z_1)^2} \right]$$

The first term in the sum isn't dependent on A or B and thus, we get $32 \times 31 = 992$ from the sum but the factor of $1/2$ makes it 496 and thus, it cancels the last term in the limit. Moreover, the second and third term in the sum gives the same sum but the factor of $1/2$ just reduces it to a single term. Moreover, these terms are independent of z_1 and thus, the limit is trivial. So, we get;

$$: j^{(A,B)} j^{(A,B)} : (z) = - \sum_{A,B=1}^{32} \sum_{A \neq B} : \lambda^A \partial \lambda^A : (z) = -31 : \lambda^A \partial \lambda^A : (z)$$

where in the last step, sum over A is implicit and the sum over B gives 31 . Now, using 11.5.6 with $\psi^2 = 2, k = 1, h(g) = 30$ (which is relevant for $SO(32)$), we get;

$$T_B^s(z) = -\frac{1}{(1 + 30)(2)} 31 : \lambda^A \partial \lambda^A : (z) = -\frac{1}{2} : \lambda^A \partial \lambda^A : (z)$$

So, we get 11.5 .8 at the end.
For any Lie algebra g, the Sugawara central charge obeys the bounds

$$\operatorname{rank}(g) \leq c^{g,h} \leq \dim(g)$$

The upper bound follows directly from $c^{g,k} = k \dim(g)/(k + h^\vee) \leq \dim(g)$. The lower bound is saturated by the large-level limit of the Cartan subalgebra contribution: each Cartan current contributes one free boson, so at least $\operatorname{rank}(g)$ units of central charge are present in any unitary affine theory.

9.5.2 Primary fields

For an affine current algebra, a primary field is a highest-weight field for the zero-mode algebra and transforms with only a simple pole under the current:

$$j^a(z) \Phi_\lambda(w) \sim \frac{(t^a_\lambda \Phi_\lambda)(w)}{z - w}. \tag{9.3}$$

The Sugawara stress tensor then gives its conformal weight

$$h_\lambda = \frac{C_2(\lambda)}{k + h^\vee}, \qquad (\lambda, \theta) \leq k \tag{9.4}$$

for an integrable highest weight λ, where θ is the highest root. At level $k = 1$ for simply laced algebras, the allowed primaries are tightly restricted; this is the worldsheet reason that the heterotic gauge algebra is not an arbitrary rank-16 Lie algebra. The later lattice discussion is the bosonic realization of the same statement: allowed momenta are weights, and the current algebra knows which weights are local with respect to the root lattice. [39, 138].

9.6 The bosonic construction and the toroidal compactification

If we have d non compact dimensions, then the l momenta are as follows:

$$(l_L^m, l_R^n) \; d \leq m \leq 25, d \leq n \leq 9$$

The right-moving GSO projection enforces the same modular-invariance conditions on the internal lattice Γ:

$$\Gamma = \Gamma^*, \quad l \circ l \in 2\mathbb{Z}$$

where $\circ$ is has signature of $(26 - d, 10 - d)$. For $d = 10$, all the signs in the $\circ$ inner product are positive, and thus, we have a Euclidean lattice. Even, self-dual Euclidean lattices exist only

for dimensions that are
multiples of 8. Since we want sixteen-dimensional lattices, the relevant even self-dual Euclidean lattices are:

$$\Gamma_{16} = (n_1, \dots n_{16}) \text{ or } \left(n_1 + \frac{1}{2}, \dots n_{16} + \frac{1}{2}\right), \sum_j n_j \in 2\mathbb{Z} \tag{11.6.1}$$

$$\Gamma_8 \times \Gamma_8 \text{ where } \Gamma_8 = (n_1, \dots n_8) \text{ or } \left(n_1 + \frac{1}{2}, \dots n_8 + \frac{1}{2}\right), \sum_j n_j \in 2\mathbb{Z} \tag{11.6.2}$$

Now, we investigate the massless spectrum of the heterotic strings. Since we have 26 bosons in the left sector, just like the bosonic theory and we have at least two non-compact dimensions, the normal ordering constant in the left sector is -1 . So, the massless states have $\partial X^\mu, \partial X^m$ and $e^{ik_L \cdot X_L}$ as the massless vertex operators which correspond to the following states;

$$\alpha^\mu_{-1} \left|k_L^2 = 0\right\rangle, \quad \alpha^m_{-1} \left|k_L^2 = 0\right\rangle, \quad \left|k_L^2 = 4/\alpha'\right\rangle \tag{11.6.3}$$

The first two of them have level 1 and the last one has level zero. Using (8.4.1), we see that if $k_R^2 = 0$ and $m^2 = 0$, then $N = 1$ and $k_L^2 = 0$ gives a massless state. Also, if $N = 0$ and $k_L^2 = 4/\alpha'$, then the state is again massless. If $k_L^2 = 4/\alpha'$ then $l_L^2 = \alpha'/2 \times 4/\alpha' = 2$. First state in 11.6.3) is in $\mathbf{8}_v$ representation of $SO(8)$. The second set of states has 16 different states and lives in the Cartan subalgebra of the gauge group because the currents ∂X^m commute with one another. The OPE of these currents is as follows:

$$\partial X^m(z) \partial X^n(0) = \frac{1}{z^2}\delta^{mn} + \dots \tag{11.6.4}$$

l_L^m are roots of the gauge group because the vertex operators $e^{il_L \cdot X_L}$ are dimension-one currents, and their OPEs with the Cartan currents ∂X^m have charges l_L^m.
The last state has $l_L^2 = 2$ where l_L is a 16 dimensional vector living in either Γ^{16} or $\Gamma^8 \times \Gamma^8$. From the $(n_1, \dots, n_{16})\, \Gamma^{16}$ lattice, we see that the only way that we can have $l_L^2 = 2$ is by having $n_i^2 = n_j^2 = 1$ for some i, j where $i \neq j$. So, the possible integer Γ^{16} lattices are as follows:

$$(1, 1, \underbrace{0, \dots, 0}_{14}) + \text{ permutations}, \ (1, -1, \underbrace{0, \dots, 0}_{14}) + \text{ permutations}, \ (-1, -1, \underbrace{0, \dots, 0}_{14}) + \text{ permutations}$$

These are nothing but the $SO(32)$ root lattices as shown in 11.4.20, 11.4.21 and 11.4.22 for $n = 16$. The half-integer Γ^{16} lattice does not show up in the massless spectrum of the $SO(32)$ theory because it belongs to the spinor conjugacy class, whose states are projected out of the massless current algebra by the chosen GSO projection.
However, this half-integer lattice is seen to have the following form;

$$(n_1, \dots, n_{16}) + \underbrace{\left(\frac{1}{2}, \dots, \frac{1}{2}\right)}_{16}, \sum_i n_i = 2\mathbb{Z}$$

So this lattice is just a sum of a spinor weight corresponding to 2^{15} spinor representation of $SO(32)$ and an arbitrary root of $SO(32)$. An arbitrary root means that it is an arbitrary integer-valued linear combination of the roots of $SO(32)$. Notice that we can replace an even number of $1/2$ in the weight above by $-1/2$ and replace n_i 's in the corresponding positions by $n_i + 1$. Since an even number of 1 's are added, the sum of all new n_i 's is still even, and thus, the root vector with new n_i 's is still an arbitrary root of $SO(32)$.
The integer lattices from $\Gamma^8 \times \Gamma^8$ that have $l_L^2 = 2$ are as follows:

$$(1,1,\underbrace{0,\ldots,0}_{6})+\text{ permutations }\times(\underbrace{0,\ldots,0}_{8}),(\underbrace{0,\ldots,0}_{8})\times(1,1,\underbrace{0,\ldots,0}_{6})+\text{ permutations}$$

$$(1,-1,\underbrace{0,\ldots,0}_{6})+\text{ permutations }\times(\underbrace{0,\ldots,0}_{8}),(\underbrace{0,\ldots,0}_{8})\times(1,-1,\underbrace{0,\ldots,0}_{6})+\text{ permutations}$$

$$(-1,-1,\underbrace{0,\ldots,0}_{6})+\text{ permutations }\times(\underbrace{0,\ldots,0}_{8}),(\underbrace{0,\ldots,0}_{8})\times(-1,-1,\underbrace{0,\ldots,0}_{6})+\text{ permutations}$$

The following lattices;

$$(\epsilon e_i)\times(\eta e_j)\,,\qquad \epsilon,\eta=\pm 1,\qquad i,j=1,\ldots,8,$$

are not allowed because each factor must be an allowed Γ^8 lattice, with $\sum_{i=1}^{8} n_i$ even as in Eq. (11.6.2). The allowed integer vectors in $\Gamma^8\times\Gamma^8$ form the $SO(16)\times SO(16)$ root lattice. It remains to include the half-integer $\Gamma^8\times\Gamma^8$ vectors. In the allowed vectors of this type, every n_i must be either 0 or -1. Indeed, if some $n_i\geq 1$, then

$$\left(n_i+\frac{1}{2}\right)^2\geq\frac{9}{4}$$

but this is already greater than 2, and the other squared components only increase l_L^2. A similar argument excludes $n_i<-1$. Thus the allowed Γ^8 half-integer vectors have all $n_i=0$ or an even number of $n_i=-1$, so that the parity condition on $\sum_i n_i$ is satisfied. Together with the integer roots, these vectors complete the $E_8\times E_8$ lattice. Since both $SO(32)$ and $E_8\times E_8$ are simply laced with roots of length squared 2, one has $\hat{k}=k$ for the corresponding current algebras. Comparing with Eq. (11.6.4) gives $k=1$ for both gauge groups. The following definitions are useful.

- The root lattice of a Lie algebra g is the set of all the integer-coefficient linear combinations of the roots of g. This lattice is denoted as Γ_g.
- Let r be a representation of g and λ be a highest weight of r. Then the weights of r lie in the coset $\lambda+\Gamma_g$; this coset, or the sublattice generated by all weights of the representation, is denoted by Γ_r.
- The union of Γ_r for all representations is called the weight lattice of g. It is denoted as Γ_w.

A useful result says that for a simply laced algebra (recall that they do not include B_n algebras i.e. $SO(2n+1)$ and C_n algebras i.e. $USp(n)$) we have

$$\Gamma_r\subset\Gamma_g^*\forall r$$

Indeed, a weight λ is defined by the condition that $2\lambda\cdot\alpha/\alpha^2\in\mathbb{Z}$ for every root α. For simply laced algebras all roots have $\alpha^2=2$, so $\lambda\cdot\alpha\in\mathbb{Z}$, which is exactly the statement that $\lambda\in\Gamma_g^*$.

Moreover, we have;

$$\Gamma_w=\Gamma_g^*$$

This implies that for E_8 group, $\Gamma_g=\Gamma_g^*=\Gamma_w$ and thus, the weight and root lattices are the same for E_8. This is true for all simply laced Lie algebras whose root lattice is self-dual. In the heterotic construction the left-moving internal bosons take their momenta in this even self-dual lattice, while the right-moving supersymmetric side supplies spacetime fermions and the mass-shell condition. This left-right asymmetry is precisely what allows ten-dimensional gauge groups with rank 16.

9.6.1 Toroidal compactification

Now, if we have $d<10$ non-compact dimensions, then the $l\circ l'$ inner product has $(26-d,10-d)$ signature. Just like the bosonic theory, we see that the inner product is invariant in $O(26-d,10-d,\mathbb{R})$ rotations but the mass formula and the constraint in 8.4.1 are invariant only under $O(26-d,\mathbb{R})\times O(10-d,\mathbb{R})$ rotations. Moreover, the compact momenta lattices are invariant under $O(26-d, 10-d, \mathbb{Z})$ transformations. In other words, we can start from a reference lattice Γ_0 such that all compact directions are orthogonal and all radii are $SU(2)\times SU(2)$ radii. Then, the following equality is true among the lattices;

$$\Lambda'\Lambda\Lambda''\Gamma_0 = \Lambda\Gamma_0$$

where we have;

$$\Lambda \in O(26-d, 10-d, \mathbb{R}), \quad \Lambda' \in O(26-d, \mathbb{R}) \times O(10-d, \mathbb{R}), \quad \Lambda'' \in O(26-d, 10-d, \mathbb{Z})$$

and therefore, the moduli space of the inequivalent theories is;

$$\frac{O(26-d, 10-d, \mathbb{R})}{O(26-d, \mathbb{R}) \times O(10-d, \mathbb{R}) \times O(26-d, 10-d, \mathbb{Z})}$$

where $O(26-d, 10-d, \mathbb{Z})$ elements act on elements of $O(26-d, 10-d, \mathbb{R})$ from the right. Now, we consider the massless spectrum of this theory. We can have the following three kinds of massless gauge bosons (the remaining massless states are organized by the same left-right lattice organization);

$$\alpha^m_{-1}\tilde{\psi}^\mu_{-1/2}|0\rangle, \quad m \in \{d, \dots, 25\}$$

$$\alpha^\mu_{-1}\tilde{\psi}^m_{-1/2}|0\rangle, \quad m \in \{d, \dots, 9\}$$

$$e^{ik_L\cdot X_L}\tilde{\psi}^\mu|0\rangle \quad m \in \{d, \dots, 9\}, l_L^2 = 2$$

The first two kinds of gauge bosons come from the dimensional reduction of the metric and antisymmetric tensor: $G_{\mu m}$ gives Kaluza–Klein gauge fields and $B_{\mu m}$ gives winding gauge fields. The third kind comes from purely left-moving dimension-one currents at enhanced-symmetry points of the Narain lattice. The vertex operators of first type of gauge bosons are $\partial X^m\tilde{\psi}^\mu$, the operators for second type of bosons are $\partial X^\mu\tilde{\psi}^m$ and the operator for the last one are $e^{ik_L\cdot X_L}\tilde{\psi}^\mu$. There are $26-d$ bosons of the first type, $10-d$ bosons of the second type, and $10-d$ bosons of the last type. Since all of these states have level $\tilde{N} = 1$, a nonzero l_R^2 would give a positive contribution $l_R^2/2$ to the right-moving mass formula and would make the state massive. For generic points, $l_L^2 \neq 2$ and thus, the gauge group is $U(1)^{26-d+10-d} = U(1)^{36-2d}$.

9.6.2 Supersymmetry and BPS states

After the Narain lattice has been fixed, supersymmetry adds a sharper invariant: a BPS state is one for which the mass is determined by a charge appearing in the supersymmetry algebra. Abstractly,

$$\{Q_\alpha, Q^\dagger_\beta\} = 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}P_\mu\mathcal{Z}_{\alpha\beta}, \qquad M \geq |Z|. \tag{9.5}$$

Saturation means that some linear combination of supercharges annihilates the state. In toroidal heterotic compactification this becomes a statement about the left-right lattice quantities: a perturbative half-BPS state has the right-moving sector in its supersymmetric ground configuration, so its mass is fixed by the charge vector instead of by arbitrary oscillator excitations. Schematically,

$$M^2_{\text{BPS}} \propto p_R^2, \qquad N_R = a_R, \qquad N_L - 1 + \frac{1}{2}p_L^2 = N_R - a_R + \frac{1}{2}p_R^2. \tag{9.6}$$

In both the current-algebra/lattice construction and the later D-brane BPS bounds, the protected mass follows from a charge algebra and the oscillator content is restricted by level matching.[2] [167, 174].

[2] Equation (9.6) is deliberately schematic, because the surrounding notes use several α' and normal-ordering conventions. A numerical application needs one fixed heterotic mass formula, with M^2, level matching, the lattice norm, and the right-moving NS/R intercept written in the same convention as Eq. (8.4.1) and Chapter 8; that same convention then carries through the BPS inequalities and enhanced-symmetry checks.

Chapter 10

Low-energy supergravities and anomalies

This chapter extracts the spacetime field theories determined by the massless superstring spectra. It begins with the common NS–NS action, then derives type IIA supergravity from eleven-dimensional supergravity, records the $SL(2,\mathbb{R})$ structure of type IIB, and states the type-I and heterotic low-energy actions in the conventions needed later. The second part studies ten-dimensional anomalies. The type-II theories pass the purely gravitational anomaly test through their chiral field content, while type-I and heterotic theories require Green–Schwarz factorization. Thus the spacetime low-energy description reproduces the same restrictions already found from modular invariance, lattice arithmetic, and tadpole cancellation [10, 121].

10.1 Low energy supergravity

Low-energy supergravity is obtained by keeping the massless string modes and expanding in $E\sqrt{\alpha'} \ll 1$. The NS–NS sector common to all closed superstrings begins with

$$S_{\text{NSNS}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + O(\alpha')\right]. \tag{10.1}$$

Type IIA and IIB differ by their R–R field content and by the chirality of the two ten-dimensional supercharges. The GSO projection chooses that massless spectrum in perturbation theory; the supergravity action is the unique two-derivative local theory compatible with the resulting supersymmetry and gauge invariances. [10, 162, 352].

10.1.1 Type IIA superstring

We will show the connection between 11-dimensional SUGRA and type IIA superstrings. The spectrum of 11-dimensional supergravity is counted in light-cone gauge, where the little group is $SO(9)$: the graviton is a traceless symmetric tensor, A_3 is a three-form, and the gravitino is a vector-spinor with the gamma-trace removed. Both the fermions and bosons have 128 states as follows:

$$\text{Bosons:}\quad \underbrace{\frac{9\times 8}{2} - 1}_{\text{graviton}} + \underbrace{\binom{9}{3}}_{A_3} = 44 + 84 = 128$$

$$\text{Fermions:}\quad \underbrace{9\times 16}_{\text{indices}} - \underbrace{16}_{\text{const.}} = 144 - 16 = 128$$

The gamma-trace constraint removes the unphysical spinor component of the gravitino, leaving the 128 fermionic states shown above. For the low-energy bosonic action, we have the following for eleven-dimensional supergravity:

$$S_{11} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa^2}\int A_3\wedge F_4\wedge F_4 \tag{12.1.1}$$

Now, for dimensional reduction, we choose the standard reduced metric form as follows:

$$G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}\left(dx^{10}+A_\mu dx^\mu\right), \quad \mu = 0,1,\dots 9$$
$$= \left(G_{\mu\nu}+e^{2\sigma}A_\mu A_\nu\right)dx^\mu dx^\nu + 2e^{2\sigma}A_\mu dx^\mu dx^{10} + e^{2\sigma}dx^{10}dx^{10}$$

where $G_{10,10} = e^{2\sigma}$. Substituting this metric ansatz into the eleven-dimensional action gives the reduced ten-dimensional action. The Ricci scalar decomposes as

$$R^{11} = R^{10} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}\left|F_2\right|^2$$

The block form of the metric is given as follows:

$$\begin{pmatrix} \tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu \\ e^{2\sigma}A_\mu & e^{2\sigma} \end{pmatrix} \text{ where } \tilde{G}_{\mu\nu} = G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu$$

The determinant of a block matrix is given as follows:

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)\det\left(D - CA^{-1}B\right) \Rightarrow \det\begin{bmatrix} \tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu \\ e^{2\sigma}A_\mu & e^{2\sigma} \end{bmatrix} = e^{2\sigma}\tilde{G} - e^{4\sigma}\tilde{G}\tilde{G}^{\mu\nu}A_\mu A_\nu$$

We can calculate $\tilde{G}^{\mu\nu}$ (i.e. the inverse of $\tilde{G}_{\mu\nu}$) by assuming a form;

$$\tilde{G}^{\mu\nu} = \alpha G^{\mu\nu} + \beta A^\mu A^\nu$$

and we readily get;

$$\alpha = 1, \quad \beta = -\frac{e^{2\sigma}}{1+e^{2\sigma}(A)^2}, \quad (A)^2 = A_\mu A^\mu \Rightarrow \tilde{G}^{\mu\nu} = G^{\mu\nu} - \frac{e^{2\sigma}}{1+e^{2\sigma}(A)^2}A^\mu A^\nu$$

Using this expression, the required determinant becomes;

$$\det\begin{bmatrix} \tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu \\ e^{2\sigma}A_\mu & e^{2\sigma} \end{bmatrix} = e^{2\sigma}\tilde{G}\left[1 - e^{2\sigma}\left(G^{\mu\nu} - \frac{e^{2\sigma}}{1+e^{2\sigma}(A)^2}A^\mu A^\nu\right)A_\mu A_\nu\right] = \frac{e^{2\sigma}\tilde{G}}{1+e^{2\sigma}(A)^2}$$

We now evaluate $\tilde{G}.\tilde{G}_{\mu\nu}$ is given as follows:

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu = G_{\mu\alpha}\left(\delta^\alpha_\nu + e^{2\sigma}A^\alpha A_\nu\right) \Rightarrow \tilde{G} = G\det(\mathbb{I}+M) \quad \text{ where } M^\alpha_\beta = e^{2\sigma}A^\alpha A_\beta$$

Now, we use the following identity;

$$\det(A) = e^{\mathrm{tr}(\ln A)} \Rightarrow \det(\mathbb{I}+M) = e^{\mathrm{tr}(\ln(\mathbb{I}+M))}$$

Now, we have;

$$(\ln(\mathbb{I}+M))^\alpha_\beta = \sum_{j=1}^{\infty}(-1)^j\frac{(M^j)^\alpha_\beta}{j} \Rightarrow \mathrm{tr}(\ln(\mathbb{I}+M)) = \sum_{j=1}^{\infty}(-1)^j\frac{\mathrm{tr}\,M^j}{j}$$

Now, we have;

$$\left(M^j\right)^\alpha_\beta = e^{2j\sigma}A^\alpha A_{\mu_1}A^{\mu_1}A_{\mu_2}\dots A^{\mu_{j-2}}A_{\mu_{j-1}}A^{\mu_{j-1}}A_\beta = A^{2(j-1)}A^\alpha A_\beta \Rightarrow \mathrm{tr}\left(M^j\right) = e^{2j\sigma}A^{2j}$$

which gives us the following;

$$\mathrm{tr}(\ln(\mathbb{I}+M)) = \sum_{j=1}^{\infty}(-1)^j\frac{e^{2j\sigma}A^{2j}}{j} = \ln\left(1+e^{2\sigma}A^2\right) \Rightarrow \det(\mathbb{I}+M) = 1+e^{2\sigma}A^2 \Rightarrow \tilde{G} = G\left(1+e^{2\sigma}A^2\right)$$

Therefore the determinant simplifies to

$$\det\begin{bmatrix} \tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu \\ e^{2\sigma}A_\mu & e^{2\sigma} \end{bmatrix} = \frac{e^{2\sigma}G\left(1+e^{2\sigma}(A)^2\right)}{1+e^{2\sigma}(A)^2} = e^{2\sigma}G$$

Since all the fields are independent of x^{10} direction, the x^{10} integral just gives $2\pi R$ and thus, the Ricci term in 12.1.1 becomes;

$$\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-G}R = \frac{2\pi R}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{\sigma}\left(R - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}\left|F_2\right|^2\right)$$
$$= \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}\left(e^{\sigma}R - 2\nabla^2 e^{\sigma} - \frac{1}{4}e^{3\sigma}\left|F_2\right|^2\right)$$

The total-derivative term may be dropped on a compact space or retained as a boundary term, depending on the boundary conditions; here $\kappa_0^2 = \kappa^2/(2\pi R)$. The reduction of the $\left|F_4\right|^2$ term in Eq. (12.1.1) is evaluated component by component as follows:

$$-\frac{1}{4.4!\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}G^{M_1N_1}\ldots G^{M_4N_4}F_{M_1\ldots M_4}F_{N_1\ldots N_4}$$
$$= -\frac{1}{96\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}G^{M_1N_1}\ldots G^{M_4N_4}F_{M_1\ldots M_4}F_{N_1\ldots N_4}$$

Now, the dot product is evaluated as follows:

$$G^{\mu_1\nu_1}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots\mu_4}F_{\nu_1\ldots\nu_4} + 8G^{\mu_1 d}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots\mu_4}F_{d\ldots\nu_4}$$
$$+4G^{dd}\ldots G^{\mu_4\nu_4}F_{d\ldots\mu_4}F_{d\ldots\nu_4} + 6G^{d\nu_1}G^{\mu_2 d}G^{\mu_2\nu_3}G^{\mu_4\nu_4}F_{d\mu_2\mu_3\mu_4}F_{\nu_1}d\nu_3\nu_4 \tag{12.1.2}$$

Consider the second term above. One kind of general term for this is;

$$G^{\mu_1\nu_1}\ldots G^{\mu_j d}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots\mu_j\ldots\mu_4}F_{\nu_1\ldots d\ldots\nu_4}$$

Interchange the indices μ_1 and μ_j, also rename the index ν_1 to ν_j. Now, we can do some permutations to interchange the first and j th index on both F 's, and thus, we get a factor of unity by doing this. So, we can transform such general terms for any j to the form written above. There are four terms like this (as $j = 1, 2, 3, 4$). Another kind of general term is as follows:

$$G^{\mu_1\nu_1}\ldots G^{d\nu_j}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots d\ldots\mu_4}F_{\nu_1\ldots\nu_j\ldots\nu_4}$$

In this term, we do interchange $\mu_k \leftrightarrow \nu_k$ for $k \neq j$ and set $\nu_j = \mu_j$. Moreover, by using the symmetry of G, we can get this kind of term to be like the general term of the first kind. Again, we have four terms like this. So, we have eight terms in total. Now, we can consider the case where one of μ_j 's is d and one of ν_k 's is d. One case arises when $j = k$. There are four terms like this. Again, we can show that all of these terms are
equal. That gives us the third term in 12.1.2). The last case arises when $j \neq k$. The general term, in this case, is as follows:

$$G^{\mu_1\nu_1}\ldots G^{d\nu_j}\ldots G^{\mu_k d}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots d\ldots\mu_k\ldots\mu_4}F_{\nu_1\ldots\nu_j\ldots d\ldots\nu_4}$$

We interchange d and μ_1 indices in the first F and similarly, we will interchange d and ν_2 in the second F. This may give us a power of -1 . This power is $2j - 3 + 2(k-1) - 3$ which is even and thus, no minus factors appear. Then, rename two indices as $\mu_1 \to \mu_j, \nu_2 \to \nu_k$ and interchange the indices $\nu_j \leftrightarrow \nu_1, \nu_k \leftrightarrow \nu_2$. After that, we interchange the indices in the F 's again. This time, interchange d with μ_2 and ν_j with ν_1. This may again give us a power of -1 . This power is $2(k-1) - 3 + 2j - 3$ which is again even. So, after this procedure, this general term becomes;

$$G^{d\nu_1}G^{\mu_2 d}G^{\mu_3\nu_3}G^{\mu_4\nu_4}F_{d\mu_2\mu_3\mu_4}F_{\nu_1 d\nu_3\nu_4}$$

Moreover, there are $\binom{4}{2} = 6$ terms like this and thus we have a factor of 6 in the last term of 12.1 .2 . The first term in 12.1 .2 gives us a term proportional to $\left|F_4\right|^2$ and thus, we get the term;

$$-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}\frac{1}{4!}G^{\mu_1\nu_1}\ldots G^{\mu_4\nu_4}F_{\mu_1\ldots\mu_4}F_{\nu_1\ldots\nu_4} = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}\left|F_4\right|^2$$

The third term in 12.1.2 gives us the following term;

$$-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}\frac{4}{4!}\left(e^{-2\sigma}+A^2\right)G^{\mu_2\nu_2}G^{\mu_3\nu_3}G^{\mu_4\nu_4}F_{d\mu_2\mu_3\mu_4}F_{d\nu_2\nu_3\nu_4}$$
$$=-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{-\sigma}\left|F_3\right|^2-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{\sigma}A^2\left|F_3\right|^2$$

The $A^2|F_3|^2$ term is not an independent kinetic term; it combines with the second term in Eq. (12.1.2) through the off-diagonal inverse metric components. The expression for G^{dd} follows from the same block inverse used to obtain $G^{d\mu}$. The special case needed here is

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix}^{-1}=\begin{bmatrix} A^{-1}+\frac{1}{k}A^{-1}bb^TA^{-1} & -\frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix}\qquad \text{where } k=c-b^TA^{-1}b$$

Only the component $G^{d\mu}$ is needed. It is

$$G^{d\mu}=\left(e^{2\sigma}\left[G^{\mu\nu}-\frac{e^{2\sigma}}{1+e^{2\sigma}A^2}A^{\mu}A^{\nu}\right]A_{\mu}A_{\nu}-1\right)^{-1}\left[\left(G^{\mu\nu}-\frac{e^{2\sigma}}{1+e^{2\sigma}A^2}A^{\mu}A^{\nu}\right)A_{\nu}\right]=-A^{\mu}$$

We can also calculate $1/k$ as follows:

$$\frac{1}{k}=e^{-2\sigma}+A^2$$

Now, the second term in 12.1.2 is as follows:

$$-8G^{\mu_1\nu_1}\dots G^{\mu_4\nu_4}\left(F_4\right)_{\mu_1\dots\mu_4}A_{\nu_1}\left(F_3\right)_{\nu_2\nu_3\nu_4}=-2G^{\mu_1\nu_1}\dots G^{\mu_4\nu_4}\left(F_4\right)_{\mu_1\dots\mu_4}\left(A\wedge F_3\right)_{\nu_1\nu_2\nu_2\nu_3}$$

where in the last line, we used the fact that only the antisymmetric part of AF_3 factor will survive this summation. The last term in 12.1.2) is as follows:

$$6G^{\mu_1\nu_1}G^{\mu_2\nu_2}G^{\mu_3\nu_3}G^{\mu_4\nu_4}A_{\mu_1}\left(F_3\right)_{\mu_2\mu_3\mu_4}A_{\nu_1}\left(F_3\right)_{\nu_2\nu_3\nu_4}$$

The A^2 term in form notation: the term comes from $(A_1\wedge F_3)^2$ inside the reduced $|F_4|^2$, so the two A's are not being wedged with each other as $A_1\wedge A_1$. They are contracted by the spacetime metric after the antisymmetric field strengths have been formed. In contrast, in the Chern–Simons term only the fully antisymmetrized wedge products survive; repeated one-forms in the same wedge vanish automatically. Thus the metric kinetic reduction may contain $A_{\mu}A^{\mu}$, while the Chern–Simons reduction keeps only the antisymmetric wedge combinations.

The Chern Simons terms are done as follows:

$$-\frac{1}{12\kappa^2}\frac{11!}{3!(4!)^2}\int dx^{\mu_1}\dots dx^{\mu_{11}}A_{[[\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6\mu_7]}F_{\mu_8\mu_9\mu_{10}\mu_{11}]}$$

Now, we concentrate on the index which is d. This can be in A (3 cases) or in F (8 cases). The dx^d integration gives $2\pi R$. Thus, we get the following;

$$-\frac{2\pi R}{12\kappa^2}\frac{11!}{3!(4!)^2}\left[3\frac{2!(4!)^2}{10!}\int A_2\wedge F_4\wedge F_4+8\cdot\frac{4!(3!)^2}{10!}\int A_3\wedge F_4\wedge F_3\right]$$
$$=-\frac{11}{12\kappa_0^2}\left[\int A_2\wedge F_4\wedge F_4+2\int A_3\wedge F_4\wedge F_3\right]$$

We prove that all of these three terms are equal which is shown as follows:

$$0=\int d\left(A_2\wedge F_4\wedge A_3\right)=\int\left(A_2\wedge F_4\wedge dA_3+dA_2\wedge F_4\wedge A_3\right)=\int\left(A_2\wedge F_4\wedge F_4+F_3\wedge F_4\wedge A_3\right)$$
$$\Rightarrow\int A_2\wedge F_4\wedge F_4=-\int F_3\wedge F_4\wedge A_3=\int A_3\wedge F_4\wedge F_3$$

using the identities

$$d^2=0\Rightarrow d\left(F_p\right)=d^2A_{p-1}=0$$
$$\omega_p\wedge\omega_q=(-1)^{pq}\omega_q\wedge\omega_p$$

where ω_p is any p-form. With the standard ten-dimensional normalization of the reduced forms, the Chern–Simons term becomes

$$= -\frac{11}{4\kappa_0^2}\int A_2\wedge F_4\wedge F_4 = -\frac{11}{4\kappa_0^2}\int A_3\wedge F_4\wedge F_3$$

Thus the action in ten dimensions is

$$S = S_1 + S_2 + S_3$$

where

$$S_1 = \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}\left(e^{\sigma}R - \frac{1}{4}e^{3\sigma}\left|F_2\right|^2\right)$$

$$S_2 = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\left(e^{-\sigma}\left|F_3\right|^2 + e^{\sigma}\left|\tilde{F}_4\right|^2\right)$$

$$S_3 = -\frac{1}{4\kappa_0^2}\int A_2\wedge F_4\wedge F_4 = -\frac{1}{4\kappa_0^2}\int A_3\wedge F_3\wedge F_4$$

where $\tilde{F}_4 = dF_3 - A_1\wedge F_3$. We now show that the Chern-Simons term is gauge invariant;

$$\begin{aligned} A_3\wedge F_3\wedge F_4 &\to (A_3 + d\lambda_2)\wedge F_3\wedge F_4 \\ &= A_3\wedge F_3\wedge F_4 + d(\lambda_2\wedge F_3\wedge F_4) \\ &= A_3\wedge F_3\wedge F_4 + \text{total derivative.} \end{aligned}$$

The gauge invariance of the term depending on $\tilde{F}_4$ is as follows. Under the A_3 gauge transformation,

$$\delta\tilde{F}_4 = d\left(d\lambda_2\right) = 0$$

However, in the transformation of A_1, we have;

$$\delta\tilde{F}_4 = -d\lambda_0\wedge F_3 = -d\left(\lambda_0\wedge F_3\right)$$

which is not zero in general. This is canceled by assigning a simultaneous transformation to A_3:

$$\delta A_3 = \lambda_0\wedge F_3$$

This additional transformation is the ten-dimensional remnant of the higher-dimensional reparametrization of x^{10}. We see that $\tilde{F}_4$ satisfies a non-trivial Bianchi identity as follows:

$$d\tilde{F}_4 = -d\left(A_1\wedge F_3\right) = -dA_1\wedge F_3 = -F_2\wedge F_3$$

We want to find the relation between σ and the dilaton ϕ now. We redefine the metric as follows:

$$G_{\mu\nu}^{\text{old}} = e^{-\sigma}G_{\mu\nu}^{\text{new}}$$

After the rescaling, the old and new labels are suppressed. The determinant and scalar curvature transform as

$$G \to e^{-10\sigma}G \Rightarrow \sqrt{-G}\to e^{-5\sigma}\sqrt{-G}$$

$$R \to e^{\sigma}R - \frac{9}{2}e^{3\sigma}\nabla_\mu\nabla^\mu e^{-2\sigma}$$

Now, the R becomes as follows in this transformation;

$$\sqrt{-G}e^{\sigma}R \to \sqrt{-G}e^{-3\sigma}R$$

However, we want this to be as follows:

$$\sqrt{-G}e^{-2\phi}R \Rightarrow \sigma = \frac{2\phi}{3}$$

Now, can expand the R transformation as follows:

$$e^{\sigma}R-\frac{9}{2}e^{\sigma}\left[4\partial_{\mu}\sigma\partial^{\mu}\sigma-2\nabla_{\mu}\partial^{\mu}\sigma\right]=e^{\sigma}R-\frac{9}{2}\left(6e^{\sigma}\partial_{\mu}\sigma\partial^{\mu}\sigma-2\nabla_{\mu}\left(e^{\sigma}\partial^{\mu}\sigma\right)\right)=e^{\sigma}R-27\partial_{\mu}\sigma\partial^{\mu}\sigma+\text{ derivative}$$
$$=e^{-2\phi/3}R-12e^{\sigma}\partial_{\mu}\phi\partial^{\mu}\phi+\text{ derivative}$$

After the Weyl transformation and the integration by parts, the dilaton kinetic term combines with the original scalar kinetic term to give the standard string-frame coefficient $+4(\partial\phi)^2$. In addition, we have the following;

$$|F_p|^2=G^{\mu_1\nu_1}\dots G^{\mu_p\nu_p}F_{\mu_1\dots\mu_p}F_{\nu_1\dots\nu_p}\to e^{p\sigma}G^{\mu_1\nu_1}\dots G^{\mu_p\nu_p}F_{\mu_1\dots\mu_p}F_{\nu_1\dots\nu_p}=e^{p\sigma}|F_p|^2$$

So, the actions in new metric become as follows:

$$S_1=\frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}e^{-2\phi}\left(R+4\partial_{\mu}\phi\partial^{\mu}\phi-\frac{1}{4}e^{2\phi}|F_2|^2\right)$$
$$S_2=-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}e^{-2\phi}\left(|F_3|^2+e^{2\phi}\left|\tilde{F}_4\right|^2\right)$$

The Chern Simons term doesn't change because it is independent of metric. Now, in type II A theory, the bosonic spectrum is as follows:

$$(\mathrm{NS}+,\mathrm{NS}+):[0]+[2]+(2)$$
$$(\mathrm{R}+,\mathrm{R}-):[1]+[3]$$

So, in the (NS+,NS+) sector, we have a dilaton, a graviton and a two form field (which we will now call B_2 and its field strength is called H_3). In (R+, R−) sector, we have a one-form field and a three-form field which we will call C_1, C_3 and their corresponding field strengths are F_2, F_4. So, we can arrange the three actions above as NS sector action and R sector action as follows:

$$S_{\mathrm{NS}}=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}\left(R+4G^{\mu\nu}\partial_{\mu}\phi\,\partial_{\nu}\phi-\frac{1}{2}|H_3|^2\right)\tag{12.1.3}$$
$$S_{\mathrm{R}}=-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\left(|F_2|^2+\left|\tilde{F}_4\right|^2\right)\tag{12.1.4}$$
$$S_{\mathrm{CS}}=-\frac{1}{4\kappa_0^2}\int B_2\wedge F_4\wedge F_4.\tag{12.1.5}$$

This is the massless type-IIA bosonic action in the conventions used in this chapter.[1] The remaining fermionic terms are fixed by $N=2$ supersymmetry in ten dimensions and are not needed for the bosonic duality and brane calculations below.

10.1.2 Massive type IIA supergravity

The bosonic field content of the massless IIA theory is as follows:

$$\underbrace{G_{\mu\nu},H_3=dB_2,\phi}_{\text{(NS+,NS+) sector}}\;\underbrace{F_2=dC_1,F_4=dC_3}_{\text{(R+,R-) sector}}\quad\underbrace{F_6=dC_5,F_8=dC_7}_{\text{Poincare duals}}$$

Following the democratic formulation of Ramond–Ramond fields, one may include a ten-form field strength F_{10}. In ten dimensions its Hodge dual is a zero-form, so it carries no propagating local degree of freedom; it is instead a constant flux, the Romans mass. If it is present, then the EOM for this field strength is;

$$d*F_{10}=0\Rightarrow *F_{10}=\text{ constant}$$

[1] In string frame the tree-level NS–NS action carries $e^{-2\phi}$, not $e^{+2\phi}$, the same Euler-characteristic factor that gives g_s^{-2} at sphere level. The H_3 term is a kinetic term $-\frac{1}{2}|H_3|^2$ and should not be multiplied by an extra ϕ. The R–R kinetic terms do not carry $e^{-2\phi}$, which is why D-brane charges and tensions compare differently in string and Einstein frames.

where the last step follows because $*F_{10}$ is a scalar. The first-order action that implements this condition is as follows:

$$S_{IIA}^{\text{massive}} = \tilde{S}_{IIA} - \frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,M^2 + \frac{1}{4\kappa_{10}^2}\int MF_{10}. \tag{12.1.6}$$

where $\tilde{S}_{IIA}$ is the sum of (12.1.3), 12.1.4 and 12.1.5) but with the following changes;

$$F_2 \to F_2 + MB_2, \quad F_4 \to F_4 + \frac{1}{2}MB_2\wedge B_2 \Rightarrow \tilde{F}_4 \to \tilde{F}_4 + \frac{1}{2}MB_2\wedge B_2$$

We see that M is just an auxiliary field as it has no derivatives in 12.1.6.

10.1.3 Type IIB supergravity

The field content of type IIB is as follows:

$$\underbrace{G_{\mu\nu}, H_3\,(=dB_2)\,,\phi}_{\text{(NS+,NS+) sector}}\,\underbrace{F_1 = dC_0, F_3 = dC_2, F_5 = dC_4}_{\text{(R}\pm\text{,R}\pm\text{) sectors}}\,\underbrace{F_5 = dC_4, F_7 = dC_6, F_9 = dC_8}_{\text{Poincare duals}}$$

Since we are working in ten dimensions which is 2 mod 4, we can impose one of the following conditions;

$$*F_5 = \pm F_5$$

as we saw in Appendix A. Moreover, since $|F_5|^2 = 0$ after imposing self-duality, one cannot write an ordinary covariant kinetic term for it. The standard treatment is therefore a pseudo-action: vary the action first and impose $*F_5 = F_5$ on the equations of motion afterward.[2] The following actions give the bosonic low-energy theory for IIB in that sense:

$$S_{\text{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}\left(R + 4G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\left|H_3\right|^2\right) \tag{12.1.7}$$

$$S_{\text{R}} = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\left(\left|F_1\right|^2 + \left|\tilde{F}_3\right|^2 + \frac{1}{2}\left|\tilde{F}_5\right|^2\right) \tag{12.1.8}$$

$$S_{\text{CS}} = -\frac{1}{4\kappa_0^2}\int C_4\wedge H_3\wedge F_3. \tag{12.1.9}$$

where we have;

$$\tilde{F}_3 = F_3 - C_0\wedge H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$$

From these definitions, we can deduce the EOM and Bianchi identity for $\tilde{F}_5$ as follows:

$$d*\tilde{F}_5 = d*F_5 - \frac{1}{2}d*(C_2\wedge H_3) + \frac{1}{2}d*(B_2\wedge F_3) = -\frac{1}{2}d*(C_2\wedge H_3) + \frac{1}{2}d*(B_2\wedge F_3) = H_3\wedge F_3$$

The last equality follows from varying the pseudo-action with respect to the four-form potential and using the Bianchi identities for H_3 and F_3. Therefore $d\tilde{F}_5 = d*\tilde{F}_5$, so $\tilde{F}_5 = *\tilde{F}_5$ is consistent with both the equation of motion and the Bianchi identity, although it is not implied by them alone; $\tilde{F}_5$ and $*\tilde{F}_5$ could differ by an exact form. If the self-duality condition is imposed directly in the action, the equations for the other fields are altered. It is therefore imposed only after deriving the equations of motion.

We now investigate the symmetry of the low-energy action. Define the following;

$$G_{E\,\mu\nu} = e^{-\phi/2}G_{\mu\nu}, \quad \tau = C_0 + ie^{-\phi}, \quad \mathcal{M}_{ij} = \frac{1}{\operatorname{Im}(\tau)}\begin{bmatrix} |\tau|^2 & -\operatorname{Re}(\tau) \\ -\operatorname{Re}(\tau) & 1 \end{bmatrix}, \quad F_3^i = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}$$

This implies that;

[2] The factor 1/2 in the $\tilde{F}_5$ kinetic term of Eq. (12.1.8) prevents double-counting the self-dual degrees of freedom, which is needed because $|\tilde{F}_5|^2$ vanishes once $\tilde{F}_5 = *\tilde{F}_5$ is imposed on shell.

$$\mathrm{Re}(\tau) = C_0, \quad \mathrm{Im}(\tau) = e^{-\phi}$$

We now derive a couple of results. The first on is as follows:

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3$$
$$= -\frac{1}{4\kappa_{10}^2} \int \left(C_4 \wedge F_3^1 \wedge F_3^2 - C_4 \wedge F_3^2 \wedge F_3^1\right) = -\frac{\epsilon_{ij}}{4\kappa_{10}^2} \int C_4 \wedge F_3^i \wedge F_3^j. \tag{12.1.10}$$

The second one is as follows:

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_E} \left(-\frac{1}{4} \left|\tilde{F}_5\right|^2\right)$$
$$= -\frac{1}{8\kappa_{10}^2} \int d^{10}x e^{-5\phi/2} \sqrt{-G} \left(\tilde{F}_5\right)_{\alpha_1 \ldots \alpha_5} \left(\tilde{F}_5\right)_{\beta_1 \ldots \beta_5} G_E^{\alpha_1\beta_1} \ldots G_E^{\alpha_5\beta_5}$$
$$= -\frac{1}{8\kappa_{10}^2} \int d^{10}x e^{-5\phi/2} \sqrt{-G} \left(\tilde{F}_5\right)_{\alpha_1 \ldots \alpha_5} \left(\tilde{F}_5\right)_{\beta_1 \ldots \beta_5} e^{5\phi/2} G^{\alpha_1\beta_1} \ldots G^{\alpha_5\beta_5}$$
$$= -\frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(\tilde{F}_5\right)_{\alpha_1 \ldots \alpha_5} \left(\tilde{F}_5\right)_{\beta_1 \ldots \beta_5} G^{\alpha_1\beta_1} \ldots G^{\alpha_5\beta_5} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(\frac{1}{2} \left|\tilde{F}_5\right|^2\right).$$

$$\tag{12.1.11}$$

where the last $\left|\tilde{F}_5\right|^2$ uses $G_{\mu\nu}$ instead of $G_{E\mu\nu}$ to contract the indices. Now, we derive another result.

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_E} \left(-\frac{\mathcal{M}_{ij}}{2} F_3^i \cdot F_3^j\right)$$
$$= -\frac{1}{4\kappa_{10}^2} \int d^{10}x e^{-5\phi/2} \sqrt{-G}\, \mathcal{M}_{ij} \left(F_3^i\right)_{\alpha_1 \ldots \alpha_3} \left(F_3^j\right)_{\beta_1 \ldots \beta_3} e^{3\phi/2} G^{\alpha_1\beta_1} \ldots G^{\alpha_3\beta_3}$$
$$= -\frac{1}{4\kappa_{10}^2} \int d^{10}x e^{-\phi} \sqrt{-G}\, \mathcal{M}_{ij} \left(F_3^i\right)_{\alpha_1 \ldots \alpha_3} \left(F_3^j\right)_{\beta_1 \ldots \beta_3} G^{\alpha_1\beta_1} \ldots G^{\alpha_3\beta_3} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x e^{-\phi} \sqrt{-G}\, \mathcal{M}_{ij} F_3^i \cdot F_3^j$$

where in the last step, G metric is used instead of G_E to contract the indices. We manipulate this final expression as follows:

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}F_3^i.F_3^j = -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\left(\frac{|\tau|^2}{\mathrm{Im}(\tau)}|H_3|^2+\frac{1}{\mathrm{Im}(\tau)}|F_3|^2-\frac{2\,\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}H_3.F_3\right)$$
$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}F_3^i\cdot F_3^j = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(\left(C_0^2+e^{-2\phi}\right)|H_3|^2+|F_3|^2-2C_0H_3.F_3\right)$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right)+-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(C_0^2|H_3|^2+|F_3|^2-2C_0H_3.F_3\right)$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right)+-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,|F_3-C_0\wedge H_3|^2$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right)+-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left|\tilde{F}_3\right|^2$$

Therefore, we have;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(-\frac{\mathcal{M}_{ij}}{2}F_3^i\cdot F_3^j\right) = \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right)+-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left|\tilde{F}_3\right|^2 \tag{12.1.12}$$

Another result is derived as follows:

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\,\frac{\partial_\mu\bar{\tau}\,\partial^\mu\tau}{(\mathrm{Im}\,\tau)^2}$$
$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-5\phi/2}\sqrt{-G}\,\frac{e^{\phi/2}G^{\mu\nu}\left(\partial_\mu C_0+ie^{-\phi}\partial_\mu\phi\right)\left(\partial_\nu C_0-ie^{-\phi}\partial_\nu\phi\right)}{e^{-2\phi}}$$
$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-5\phi/2}\sqrt{-G}\,\frac{e^{\phi/2}|F_1|^2+e^{-3\phi/2}G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi}{e^{-2\phi}}$$
$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,|F_1|^2+\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}\left(-\frac{1}{2}G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right). \tag{12.1.13}$$

Before deriving the last result, we use the fact that in $n\neq 2$ dimensions, if two metrics are related as

$$\tilde{g} = e^{2\phi}g$$

then the Ricci tensors are related as follows:

$$\tilde{R} = e^{-2\phi}\left(R-\frac{4(n-1)}{n-2}e^{-(n-2)\phi/2}g^{\mu\nu}\nabla_\mu\nabla_\nu e^{(n-2)\phi/2}\right)$$

also, recall that for a scalar ϕ, $\nabla_\mu\phi=\partial_\mu\phi$. Plugging in $n=10$ and rescaling ϕ as $\phi\to-\phi/4$ (to match the relation between G and G_E that we have);

$$R_E = e^{\phi/2}\left(R-\frac{9}{2}e^{\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi}\right) = e^{\phi/2}R-\frac{9}{2}e^{3\phi/2}G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi}$$

Now, we do the following manipulation;

$$G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi} = G^{\mu\nu}\nabla_\mu\left(-\partial_\nu\phi e^{-\phi}\right) = -G^{\mu\nu}\left(\partial_\mu\partial_\nu\phi-\Gamma^\alpha_{\mu\nu}\partial_\alpha\phi\right)e^{-\phi}+G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi e^{-\phi}$$
$$= -G^{\mu\nu}\left(\nabla_\mu\nabla_\nu\phi\right)e^{-\phi}+\partial_\mu\phi\partial^\mu\phi e^{-\phi}$$

Thus, the expression of R_E becomes;

$$R_E = e^{\phi/2}\left[R+\frac{9}{2}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi-\frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]$$

Now, we derive the last result that we need;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}R_E = \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-5\phi/2}\sqrt{-G}e^{\phi/2}\left[R+\frac{9}{2}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi-\frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left[R-\frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]+\frac{9}{4\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$$

The integrand of the last term can be written as follows:

$$G^{\mu\nu}\nabla_\mu\left(e^{-2\phi}\nabla_\nu\phi\right) = G^{\mu\nu}\left(-2\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right)+e^{-2\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$$
$$\Rightarrow e^{-2\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi = G^{\mu\nu}\nabla_\mu\left(e^{-2\phi}\nabla_\nu\phi\right)+2G^{\mu\nu}\left(\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right) = \nabla^\mu\left(e^{-2\phi}\nabla_\nu\phi\right)+2G^{\mu\nu}\left(\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right)$$

The first term in this expression will give a surface term and thus, we omit this term. Therefore, we get;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}R_E = \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left[R-\frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]+\frac{9}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left[R+\frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right] \qquad (12.1.14)$$

Note that the dilaton kinetic term from the sum of 12.1.13 and 12.1.14 is;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(4\partial_\mu\phi\partial^\mu\phi\right)$$

which is exactly the term in 12.1.7). Now, using 12.1.10 to 12.1.14, it is easily seen that the following action gives the same action as the sum of (12.1.7), 12.1.8) and (12.1.9);

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(R_E-\frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\operatorname{Im}\tau)^2}-\frac{\mathcal{M}_{ij}}{2}F_3^i\cdot F_3^j-\frac{1}{4}\left|\tilde{F}_5\right|^2\right)-\frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4\wedge F_3^i\wedge F_3^j \qquad (12.1.15)$$

We will now show that this action is invariant under the following $SL(2,\mathbb{R})$ transformations;

$$\tau\to\tau' = \frac{a\tau+b}{c\tau+d}(ad-bc=1), \quad F_3^i\to F_3^{i'} = \Lambda^i_jF_3^j \quad \Lambda = \begin{bmatrix} d & c \\ b & a \end{bmatrix}\in SL(2,\mathbb{R}) \qquad (12.1.16)$$

with the other fields not changing. We show the invariance by showing the invariance of individual terms. We start with the following;

$$\partial_\mu\bar{\tau}'\partial^\mu\tau' = \partial_\mu\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)\partial^\mu\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)\partial_\mu\bar{\tau}(ad-bc)\partial^\mu\tau}{|c\tau+d|^4} = \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{|c\tau+d|^4}$$

Moreover, we have;

$$\operatorname{Im}(\tau') = \frac{1}{2i}(\tau'-\bar{\tau}') = \frac{1}{2i}\left(\frac{a\tau+b}{c\tau+d}-\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{1}{2i}\frac{(ad-bc)\tau-(ad-bc)\bar{\tau}}{|c\tau+d|^2} = \frac{1}{2i}\frac{\tau-\bar{\tau}}{|c\tau+d|^2} = \frac{\operatorname{Im}(\tau)}{|c\tau+d|^2}$$
$$\Rightarrow (\operatorname{Im}(\tau'))^2 = \frac{(\operatorname{Im}(\tau))^2}{|c\tau+d|^4}$$

Therefore, we have;

$$\frac{\partial_\mu\bar{\tau}'\partial^\mu\tau'}{(\operatorname{Im}(\tau'))^2} = \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{(\operatorname{Im}(\tau))^2}$$

We now see how $\mathcal{M}_{ij}$ transforms. For this purpose, we first calculate the following;

$$\Lambda = \begin{bmatrix} d & c \\ b & a \end{bmatrix}\Rightarrow\Lambda^{-1} = \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}\Rightarrow\left(\Lambda^{-1}\right)^T = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

$$\Rightarrow \left(\Lambda^T\right)^{-1}\mathcal{M}\Lambda^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} a & -c \\ -b & d \end{bmatrix}$$
$$= \begin{bmatrix} a^2M_{11} - 2abM_{12} + b^2M_{22} & -acM_{11} + (ad+bc)M_{12} - bdM_{22} \\ -acM_{11} + (ad+bc)M_{12} - bdM_{22} & c^2M_{11} - 2cdM_{12} + d^2M_{22} \end{bmatrix}$$

where we used the fact that $M_{12} = M_{21}$. We see that $\left(\Lambda^{-1}\right)^T \mathcal{M}\Lambda^{-1}$ is symmetric too. We now claim that $\mathcal{M}$ transforms as follows:

$$\mathcal{M}' = \left(\Lambda^{-1}\right)^T \mathcal{M}\Lambda^{-1}$$

We now prove this claim. For this purpose, we see the following;

$$\operatorname{Re}(\tau') = \frac{1}{2}\left(\frac{a\tau+b}{c\tau+d} + \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{2ac|\tau|^2 + (ad+bc)(\tau+\bar{\tau}) + 2bd}{2|c\tau+d|^2} = \frac{(ad+bc)\operatorname{Re}(\tau) + ac|\tau|^2 + bd}{|c\tau+d|^2}$$
$$\Rightarrow -\frac{\operatorname{Re}\tau'}{\operatorname{Im}\tau'} = -\frac{(ad+bc)\operatorname{Re}(\tau) + ac|\tau|^2 + bd}{\operatorname{Im}(\tau)} = -(ad+bc)\frac{\operatorname{Re}(\tau)}{\operatorname{Im}(\tau)} - ac\frac{|\tau|^2}{\operatorname{Im}(\tau)} - bd\frac{1}{\operatorname{Im}(\tau)}$$
$$\Rightarrow M'_{12} = (ad+bc)M_{12} - acM_{11} - bdM_{22}$$

The result is consistent with our claim but the claim is not proven completely. We continue by calculating the transformations of the other components of $\mathcal{M}$. We see that;

$$|\tau'|^2 = \left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{a^2|\tau|^2 + 2ab\operatorname{Re}\tau + b^2}{|c\tau+d|^2} \Rightarrow \frac{|\tau'|^2}{\operatorname{Im}\tau'} = \frac{a^2|\tau|^2 + 2ab\operatorname{Re}\tau + b^2}{\operatorname{Im}\tau}$$
$$\Rightarrow M'_{11} = a^2M_{11} - 2abM_{12} + b^2M_{22}$$

Finally, we also see that;

$$M'_{22} = \frac{1}{\operatorname{Im}\tau'} = \frac{|c\tau+d|^2}{\operatorname{Im}\tau} = \frac{c^2|\tau|^2 + 2cd\operatorname{Re}\tau + d^2}{\operatorname{Im}\tau} = c^2M_{11} - 2cdM_{12} + d^2M_{22}$$

Therefore, our claim was correct. Now, we can see that the following term is invariant;
$\mathcal{M}_{ij}F_3^iF_3^j = F_3^i\mathcal{M}_{ij}F_3^j \to \Lambda^i{}_kF_3^k\left(\left(\Lambda^{-1}\right)^T\mathcal{M}\Lambda^{-1}\right)_{ij}\Lambda^j_lF_3^l = F_3^k\left(\Lambda^T\left(\Lambda^{-1}\right)^T\mathcal{M}\Lambda^{-1}\Lambda\right)_{kl}F_3^l = F_3^k\mathcal{M}_{kl}F_3^l = \mathcal{M}_{kl}F_3^kF_3^l$
Moreover, we see that;

$$\epsilon_{ij}\int C_4\wedge F_3^i\wedge F_3^j \to \epsilon_{ij}\Lambda^i_l\Lambda^j_k\int C_4\wedge F_3^l\wedge F_3^k = \epsilon_{lk}\det(\Lambda)\int C_4\wedge F_3^l\wedge F_3^k = \epsilon_{lk}\int C_4\wedge F_3^l\wedge F_3^k$$

So, we see that 12.1.15 is invariant under 12.1.16).

10.1.4 Type I superstring

Type I is best read as the unoriented open-and-closed descendant of type IIB. The projection by worldsheet parity Ω keeps the $N=1$ closed-string supergravity multiplet, and validity then requires open strings with Chan-Paton factors. The low-energy field content is therefore ten-dimensional $N=1$ supergravity coupled to Yang-Mills, with gauge group fixed by tadpole and anomaly cancellation:

$$\mathcal{S}_{\mathrm{I}} \sim \int d^{10}x\,\sqrt{-G}\left[e^{-2\Phi}\left(R + 4(\nabla\Phi)^2\right) - \frac{1}{4g_{\mathrm{YM}}^2}\operatorname{Tr}F_{\mu\nu}F^{\mu\nu} + \cdots\right], \qquad G_{\mathrm{CP}} = SO(32). \quad (10.2)$$

The cylinder, Möbius, and Klein-bottle amplitudes give the same restriction: the open-string Chan–Paton multiplicity is not arbitrary because massless closed-string tadpoles and ten-dimensional gauge-gravitational anomalies must cancel. [10, 149].

10.1.5 Heterotic strings

The heterotic low-energy theory is also ten-dimensional $N=1$ supergravity coupled to Yang-Mills, but the gauge algebra comes from the left-moving current algebra instead of from open-string endpoints. The two anomaly-free choices are

$$G_{\text{het}} = SO(32) \quad \text{or} \quad E_8 \times E_8. \tag{10.3}$$

The spacetime remnant of the worldsheet construction is the modified three-form field strength

$$H = dB - \frac{\alpha'}{4}\left(\omega_{3Y} - \omega_{3L}\right), \qquad dH = \frac{\alpha'}{4}\left(\operatorname{tr} R^2 - \operatorname{Tr} F^2\right). \tag{10.4}$$

Thus the same current algebra that produced the rank-16 gauge group also supplies the Green–Schwarz structure needed for anomaly cancellation. The gauge group is forced by worldsheet modular invariance, and the same restriction reappears as a spacetime anomaly condition.[3] [149, 150].

10.2 Anomalies

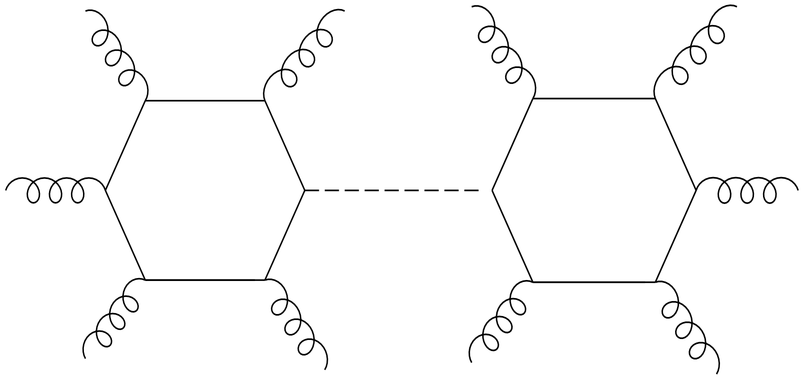


Figure 10.1: The ten-dimensional hexagon diagram responsible for gauge and gravitational anomalies, shown with its factorized form. When the anomaly polynomial factorizes as $I_{12} = X_4 X_8$, a Green–Schwarz counterterm built from the two-form cancels it. From Ref. [374].

Anomaly cancellation is the spacetime analogue of modular invariance, and the one-loop hexagon diagram responsible is drawn in Figure 10.1. In ten-dimensional type-I and heterotic theories the anomaly polynomial must factorize schematically as

$$I_{12} = X_4 X_8, \qquad X_4 = \operatorname{tr} F^2 - \operatorname{tr} R^2. \tag{10.5}$$

Only after this factorization can the Green–Schwarz mechanism cancel the gauge and gravitational anomaly. This shows that not every modular-looking spectrum defines an allowed spacetime theory. [10].

10.2.1 Type II anomalies

Type IIB is the sharper test case, because it is chiral with no gauge group to hide behind. Its potentially anomalous fields are exactly three: two left-handed gravitini, two right-handed dilatini, and the self-dual four-form, and each contributes a twelve-form to the purely gravitational anomaly polynomial,

$$I_{12}^{\text{IIB}} = 2\, I_{3/2}(R) + 2\left(-I_{1/2}(R)\right) + I_{\text{SD}}(R) = 0, \tag{10.6}$$

where $I_{3/2}$, $I_{1/2}$, I_{SD} are the standard gravitino, spinor, and self-dual-tensor anomaly polynomials of Alvarez-Gaumé and Witten. The cancellation is exact and term by term nontrivial, the three contributions are very different polynomials in the Pontryagin classes, and their sum vanishes only for precisely this field content in precisely ten dimensions. No Green–Schwarz mechanism is needed or available: the spectrum GSO-selected by modular invariance in Chapter 8 is anomaly-free on its own. Type IIA, being non-chiral, has nothing to cancel. This is a direct instance of

[3] Equation (10.4) depends on whether Tr is the adjoint trace or a normalized fundamental trace. Type-I and heterotic actions can be compared only after the Green–Schwarz term, the F^2 and F^4 terms, and the anomaly polynomial have all been translated to the same trace convention.

the general claim: a worldsheet consistency condition (the modular-invariant GSO sum) lands, without adjustment, exactly on the one chiral multiplet content that spacetime consistency permits [10, 149].

10.2.2 Type I and heterotic anomalies

The $N = 1$ theories are chiral *and* gauged, so their anomaly polynomial contains gauge, gravitational, and mixed pieces, and it does not vanish identically, it must be cancelled. The cancellation happens in two steps. First, for a gauge group with exactly 496 generators and no sixth-order Casimir independent of the lower ones, the twelve-form factorizes,

$$I_{12} = X_4 \wedge X_8, \qquad X_4 = \operatorname{tr} R^2 - \frac{1}{30} \operatorname{Tr} F^2, \tag{10.7}$$

and the only compact possibilities are $SO(32)$ and $E_8 \times E_8$ (with the trivial $U(1)^{496}$ and $E_8 \times U(1)^{248}$ ruled out by other consistency requirements). Second, a factorized anomaly can be cancelled by the Green–Schwarz counterterm $\int B_2 \wedge X_8$ together with the modified transformation of B_2, which is possible in string theory because the modified field strength $H = dB - \frac{\alpha'}{4}(\omega_{3Y} - \omega_{3L})$ is exactly what the worldsheet computation of the previous subsection delivers. The point worth stressing is that the string never needed the anomaly analysis: tadpole cancellation (type I) and the even self-dual lattice classification (heterotic) had already produced this precise list of groups from the worldsheet. Spacetime and worldsheet consistency select the same theories by what look like unrelated arguments [149, 150].

10.2.3 Relation to string theory

The anomaly discussion enters string theory through the massless fields produced by the one-loop spectrum. In ten-dimensional type-I and heterotic theories the relevant twelve-form anomaly polynomial must factorize,

$$I_{12} = X_4 X_8, \tag{10.8}$$

so that a Green–Schwarz term involving the two-form can cancel the anomaly. Therefore the ten-dimensional gauge groups are highly restricted. The same calculation is related back to the annulus, Möbius, and Klein-bottle tadpoles: the worldsheet amplitudes and the spacetime anomaly polynomial are two ways of detecting whether the massless background has been chosen consistently. [10, 38].

Hosseini, Tachikawa, and Zhang revisit type-I anomaly cancellation by treating the R–R fields through boundary modes of an eleven-dimensional bulk theory and a quadratic refinement of the differential KO-theory pairing, then extend the analysis to the $usp(32)$ Sugimoto and $u(32)$ Sagnotti strings [315]. Lescano's nonrelativistic heterotic work shows that four-derivative gravitational corrections still carry a finite $SO(8)$ Green–Schwarz mechanism, but that the gravitational Green–Schwarz transformation can be trivialized by field redefinitions in the nonrelativistic limit [321, 323]. Larotonda and Lin revisit anomaly inflow in the ten-dimensional Sugimoto model and find evidence that the global gauge group is $Sp(16)/\mathbb{Z}_2$, constrained by charged brane spectra and inflow [311]. The factorized polynomial $I_{12} = X_4 X_8$ is then the local shadow of a global problem involving R–R differential cohomology, gauge-group topology, brane inflow, and possible nonrelativistic limits.

Chapter 11

Superstring amplitudes and supermoduli

This chapter develops the interaction technology for superstrings. Worldsheet superspace packages the matter fields and superconformal transformations; bosonization of the superghosts introduces picture number; physical vertex operators are represented in pictures related by the picture-changing operator; and amplitudes are written as integrals over supermoduli space. Tree-level three- and four-point amplitudes show how the GSO projection removes the bosonic tachyon poles and reproduces Yang–Mills theory at low energy, while the one-loop spin-structure sum gives the modular test for the interacting superstring. The final subsection records the non-renormalization statements only for protected quantities, where supersymmetry and duality make the claim precise [10, 121].

Recent amplitude work clarifies how the supermoduli technology below interfaces with modern field-theory and pure-spinor methods. Berkovits gives a manifestly spacetime-supersymmetric pure-spinor prescription and relates it by a $U(5)$-covariant field redefinition to the RNS prescription, with the pure spinor parametrizing the choice of $SO(10)/U(5)$ and avoiding the usual pure-spinor ghost pole subtleties [294]. At one loop, chiral splitting now gives a direct map from field-theory BCJ numerators to superstring moduli-space integrands through at least seven points, with monodromy constraints fixing coefficients not visible in the naive field-theory limit [292]. A later n-point construction identifies a complete one-loop worldsheet basis whose non-cusp kinematic coefficients are pieces of one-loop BCJ numerators; possible cusp-form contributions first enter only at sufficiently high modular weight [303]. These results support Eq. (11.2): the superstring integrand is a worldsheet object, but its kinematic numerators can often be organized by field-theory color-kinematics structures.

11.1 Superspace and superfields

A general coordinate transformation $z \to z'$ will transform the derivative ∂ as follows:

$$\partial = \frac{\partial z'}{\partial z}\partial' + \frac{\partial \bar{z}'}{\partial z}\bar{\partial}'$$

A conformal transformation is one for which ∂ is proportional to ∂'. Define superderivatives as follows:

$$D_\theta = \partial_\theta + \theta\partial_z, \quad \bar{D}_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$$

Then the general transformation for D_θ is as follows:

$$\begin{aligned} D_\theta &= \left(\partial_\theta\theta'\partial_{\theta'} + \partial_\theta z'\partial_z + \partial_\theta\bar{\theta}'\partial_{\bar{\theta}'} + \partial_\theta\bar{z}'\partial_{\bar{z}'}\right) + \theta'\left(\partial\theta'\partial_{\theta'} + \partial z'\partial_z + \partial\bar{\theta}'\partial_{\bar{\theta}'} + \partial\bar{z}'\partial_{\bar{z}'}\right) \\ &= D_\theta\theta'\partial_{\theta'} + D_\theta z'\partial_{z'} + D_\theta\bar{\theta}'\partial_{\bar{\theta}'} + D_\theta\bar{z}'\partial_{\bar{z}'} \end{aligned}$$

Now, we can say that the superconformal transformation is a transformation that sends D_θ to a multiple of itself. This implies that;

$$D_\theta\bar{\theta}' = D_\theta\bar{z}' = 0, \quad D_\theta z' = \theta' D_\theta\theta'$$

We now write z' and θ' in terms of z and θ. Now, we can expand both of them as a Taylor expansion in θ and it will give us two terms. Both of these terms will depend on z. These functions will be of different Grassmannian characters to preserve the Grassmannian character of the field. So, we have;

$$z'(z,\theta) = f(z) + \theta m(z), \theta'(z,\theta) = g(z) + \theta h(z)$$

where $f(z), h(z)$ are grassman even functions and $m(z), g(z)$ are grassman odd functions. We can constrain these fields by using the condition $D_\theta z' = \theta' D_\theta \theta'$. We proceed as follows:

$$\theta' D_\theta \theta' = (g(z) + \theta h(z))(h(z) + \theta \partial_z g(z)) = g(z)h(z) + \theta\left(h^2(z) - g(z)\partial_z g(z)\right), \quad D_\theta z' = m(z) + \theta \partial_z f(z)$$

$$\Rightarrow m(z) = g(z)h(z), h(z) = \pm\sqrt{g(z)\partial_z g(z) + \partial_z f(z)}$$

Thus, we have the following;

$$z'(z,\theta) = f(z) + \theta g(z)h(z), \theta'(z,\theta) = g(z) + \theta h(z), \quad h(z) = \pm\sqrt{g(z)\partial_z g(z) + \partial_z f(z)} \tag{12.3.1}$$

For the infinitesimal transformations, we consider $f(z) = 1 + \epsilon v(z), g(z) = -i\epsilon\eta(z)$. It gives us;

$$h(z) = 1 + \frac{\epsilon}{2}\partial v(z), g(z)h(z) = -i\epsilon\eta(z), \Rightarrow \delta z = \epsilon(v(z) - i\theta\eta(z)), \delta\theta = \epsilon\left(-i\eta(z) + \frac{1}{2}\theta\partial v(z)\right)$$

(Check that it satisfies superconformal algebra) A tensor superfield of weight $(h,\tilde{h})$ transforms as follows:

$$(D_\theta\theta')^{2h}\left(D_{\bar{\theta}}\bar{\theta}'\right)^{2\tilde{h}}\phi'(\mathbf{z},\mathbf{z}') = \phi(\mathbf{z},\mathbf{z})$$

where $\mathbf{z} = (z,\theta)$. For the transformation

$$\delta z = \epsilon\theta\eta(z), \quad \delta\theta = \epsilon\eta(z)$$

We see that;

$$D_\theta\theta' = 1 + \theta\epsilon\partial_z\eta, D_{\bar{\theta}}\bar{\theta}' = 1 + \bar{\theta}\epsilon\partial_{\bar{z}}\tilde{\eta}$$

and thus, we have;

$$(1 + \theta\epsilon\partial_z\eta)^{2h}\left(1 + \bar{\theta}\epsilon\partial_{\bar{z}}\tilde{\eta}\right)^{2\tilde{h}}\left(\phi'(\mathbf{z},\bar{\mathbf{z}}) + \theta\epsilon\eta\partial_z\phi' + \bar{\theta}\epsilon\tilde{\eta}\partial_{\bar{z}}\phi' + \epsilon\eta\partial_\theta\phi' + \epsilon\tilde{\eta}\partial_{\bar{\theta}}\phi'\right) = \phi(\mathbf{z},\bar{\mathbf{z}})$$

$$\Rightarrow \delta\phi(\mathbf{z},\bar{\mathbf{z}}) = -\epsilon\left(2h\theta\partial_z\eta + \eta Q_\theta + 2\tilde{h}\bar{\theta}\partial_{\bar{z}}\tilde{\eta} + \tilde{\eta}Q_{\bar{\theta}}\right)\phi'(\mathbf{z},\bar{\mathbf{z}}) = -\epsilon\left(2h\theta\partial_z\eta + \eta Q_\theta + 2\tilde{h}\bar{\theta}\partial_{\bar{z}}\tilde{\eta} + \tilde{\eta}Q_{\bar{\theta}}\right)\phi(\mathbf{z},\bar{\mathbf{z}}) + \mathcal{O}\left(\epsilon^2\right) \tag{12.3.2}$$

where the binomial theorem has been used and where $Q_\theta = \partial_\theta - \theta\partial_z, Q_{\bar{\theta}} = \partial_{\bar{\theta}} - \theta\partial_{\bar{z}}$. The holomorphic part of ϕ, denoted $\phi(\mathbf{z})$, is expanded as follows:

$$\phi(\mathbf{z}) = \mathcal{O}(z) + \theta\Psi(z) \Rightarrow \delta\phi(\mathbf{z}) = \delta\mathcal{O}(z) + \theta\delta\Psi(z)$$

But from the result in 12.3.2, we see that;

$$\begin{aligned}\delta\phi(\mathbf{z}) = -\epsilon\left(2h\theta\partial_z\eta\phi(\mathbf{z}) + \eta\partial_\theta\phi(\mathbf{z}) - \eta\theta\partial_z\phi(\mathbf{z})\right) &= -\epsilon\left(2h\theta\partial_z\eta\mathcal{O}(z) + \eta\Psi(z) - \eta\theta\partial_z\mathcal{O}(z)\right)\\ &= -\epsilon\eta\Psi(z) - \epsilon\theta\left(2h\partial_z\eta\mathcal{O}(z) + \eta\partial_z\mathcal{O}(z)\right)\end{aligned}$$

Comparing these two results for the variation of the field, we get;

$$\delta\mathcal{O} = -\epsilon\eta\Psi, \quad \delta\Psi = -\epsilon\left(2h\partial_z\eta\mathcal{O}(z) + \eta\partial_z\mathcal{O}(z)\right)$$

A general superconfromal variation of a field $\mathcal{A}(z,\bar{z})$ is given as follows:

$$\delta\mathcal{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial^n\eta(z)G_{n-1/2} + \left(\partial^n\eta(z)\right)^*\tilde{G}_{n-1/2}\right)\mathcal{A}(z,\bar{z}) \tag{12.3.3}$$

Comparing 12.3.3 with the variations above, we get;

$$G_{-1/2}\mathcal{O}=\Psi, G_r\mathcal{O}=0\left(r\geq\frac{1}{2}\right)$$

$$G_{-1/2}\Psi=\partial_z\mathcal{O}, G_{1/2}\Psi=2h\mathcal{O}, G_r\Psi=0\left(r\geq\frac{3}{2}\right)$$

Now, we consider purely conformal transformations i.e.

$$z'=z+\epsilon v(z), \bar{z}'=\bar{z}+\epsilon\bar{v}(\bar{z}), \theta'=\theta+\frac{1}{2}\epsilon\theta\partial_z v(z), \bar{\theta}'=\bar{\theta}+\frac{1}{2}\epsilon\bar{\theta}\partial_{\bar{z}}\bar{v}(\bar{z})$$

Then, the variation in the tensor field is given as follows:

$$\delta\phi(\mathbf{z},\overline{\mathbf{z}})=-\epsilon\left(h\partial_z v+\tilde{h}\partial_{\bar{z}}\bar{v}+v\partial_z+\bar{v}\partial_{\bar{z}}+\frac{1}{2}\theta\partial_z v\partial_\theta+\frac{1}{2}\bar{\theta}\partial_{\bar{z}}\bar{v}\partial_{\bar{\theta}}\right)\phi(\mathbf{z},\overline{\mathbf{z}})+\mathcal{O}\left(\epsilon^2\right)$$

Now, again we consider the holomorphic part of $\phi(\mathbf{z},\overline{\mathbf{z}})$ and then, the variation becomes;

$$\delta\phi(\mathbf{z})=-\epsilon\left(h\partial_z v\mathcal{O}+v\partial_z\mathcal{O}\right)-\epsilon\theta\left[\left(h+\frac{1}{2}\right)\partial_z v\Psi+v\partial_z\Psi\right]$$

$$\Rightarrow\delta\mathcal{O}=-\epsilon\left(h\partial_z v\mathcal{O}+v\partial_z\mathcal{O}\right), \delta\Psi=-\epsilon\left[\left(h+\frac{1}{2}\right)\partial_z v\Psi+v\partial_z\Psi\right]$$

A general conformal transformation of an operator is given as follows:

$$\delta\mathcal{A}(z,\bar{z})=-\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial^n v(z)L_{n-1}+\left(\partial^n v(z)\right)^*\tilde{L}_{n-1}\right)\mathcal{A}(z,\bar{z}) \tag{12.3.4}$$

Comparing 12.3.4 with the variations above, we see that;

$$L_{-1}\mathcal{O}=\partial_z\mathcal{O},\quad L_0\mathcal{O}=h\mathcal{O},\quad L_n\mathcal{O}=0\quad(n>0)$$

$$L_{-1}\Psi=\partial_z\Psi,\quad L_0\Psi=\left(h+\frac{1}{2}\right)\Psi,\quad L_n\Psi=0\quad(n>0)$$

Thus $\mathcal{O}$ and Ψ have conformal dimensions h and $h+1/2$, respectively. The rigid worldsheet supersymmetry transformation is generated by $G_{-1/2}$.

11.1.1 Actions and backgrounds

The super-Jacobian of the superconformal transformation in Eq. (12.3.1) is

$$\left|\begin{array}{cc}\partial z'/\partial z & \partial z'/\partial\theta\\ \partial\theta'/\partial z & \partial\theta'/\partial\theta\end{array}\right|=\left|\begin{array}{cc}\partial_z f+\theta\partial_z(gh) & gh\\ \partial g+\theta\partial h & h\end{array}\right|=\left(\partial_z f+g\partial_z g\right)\left(h+\theta\partial_z g\right)=h^2(z)D_\theta\theta'$$

The extra factor is the Berezinian contribution from the odd coordinate; superconformal transformations preserve the distribution generated by D_θ, not the ordinary bosonic Jacobian alone. The measure transforms as follows:

$$dz'd\theta'=D_\theta\theta' dzd\theta$$

The weight of this measure is $(-1/2,0)$ because the weight of θ is $(1/2,0)$. Including the measure $d\bar{z}d\bar{\theta}$, we see that the total measure has the weight $(-1/2,-1/2)$. Thus, the weight of the lagrangian density is $(1/2,1/2)$. An example of such a lagrangian density is;

$$D_{\bar{\theta}}\mathbf{X}^\mu(z,\bar{z})D_\theta\mathbf{X}_\mu(z,\bar{z})$$

because the weight of $D_\theta\mathbf{X}^\mu(z,\bar{z})$ is $(1/2,0)$. Here, $\mathbf{X}^\mu(z,\bar{z})$ is a superfield. Its Taylor expansion can be done as follows:

$$\mathbf{X}^\mu(z,\bar{z})=X^\mu(z,\bar{z})+i\theta\psi^\mu(z)+i\bar{\theta}\bar{\psi}^\mu(z)+\theta\bar{\theta}F(z,\bar{z})$$

Here, the first three terms are easily understood. The fourth term is a possible term that can appear in Taylor's expansion but we will see that the F field is an auxiliary field. To write the action in terms of the components, we need the following small calculations;

$$\begin{aligned} D_\theta \mathbf{X}^\mu(\mathbf{z},\overline{\mathbf{z}}) &= (\partial_\theta + \theta\partial_z)\left(X^\mu(z,\bar z) + i\theta\psi^\mu(z) + i\bar\theta\bar\psi^\mu(z) + \theta\bar\theta F(z,\bar z)\right) = \left(i\psi^\mu + \bar\theta F^\mu\right) + \theta\left(\partial_z X^\mu + i\bar\theta\partial_z\bar\psi^\mu\right) \\ &= i\psi^\mu + \theta\partial_z X^\mu + \bar\theta F^\mu + i\theta\bar\theta\partial_z\bar\psi^\mu \end{aligned} \tag{12.3.5}$$

Similarly,

$$\begin{aligned} D_{\bar\theta} \mathbf{X}^\mu(\mathbf{z},\overline{\mathbf{z}}) &= \left(\partial_{\bar\theta} + \bar\theta\partial_{\bar z}\right)\left(X^\mu(z,\bar z) + i\theta\psi^\mu(z) + i\bar\theta\bar\psi^\mu(z) + \theta\bar\theta F(z,\bar z)\right) = \left(i\psi^\mu - \theta F^\mu\right) + \bar\theta\left(\partial_{\bar z} X^\mu + i\theta\partial_z\psi^\mu\right) \\ &= i\bar\psi^\mu + \bar\theta\partial_{\bar z} X^\mu - \theta F^\mu - i\theta\bar\theta\partial_{\bar z}\psi^\mu \end{aligned} \tag{12.3.6}$$

Now, we calculate $D_{\bar\theta}\mathbf{X}^\mu D_\theta \mathbf{X}_\mu$ with retain only the terms that contain $\theta\bar\theta$ because the other terms vanish under $d^2\theta$ integration. It gives us;

$$D_{\bar\theta}\mathbf{X}^\mu(\mathbf{z},\overline{\mathbf{z}}) D_\theta \mathbf{X}_\mu(\mathbf{z},\overline{\mathbf{z}}) = \bar\theta\theta\left(\bar\psi^\mu\partial_z\bar\psi_\mu + \partial_z X^\mu \partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar z}\psi_\mu\right) + (\text{ non } \theta\bar\theta \text{ terms }) \tag{12.3.7}$$

and thus, we get (after doing the $d^2\theta$ integration);

$$\begin{aligned} S =& \frac{1}{4\pi}\int d^2z d^2\theta \bar\theta\theta\left(\bar\psi^\mu\partial_z\bar\psi_\mu + \partial_z X^\mu\partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar z}\psi_\mu\right) \\ =& \frac{1}{4\pi}\int d^2z\left(\bar\psi^\mu\partial_z\bar\psi_\mu + \partial_z X^\mu\partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar z}\psi_\mu\right) \end{aligned} \tag{12.3.8}$$

The vanishing of the variation of this action w.r.t F^μ gives;

$$\frac{1}{4\pi}\int d^2z F_\mu \delta F^\mu = 0 \Rightarrow F_\mu = 0 \tag{12.3.9}$$

Varying X^μ in the superfield action and integrating the two superderivatives by parts gives the following equation of motion:

$$D_\theta D_{\bar\theta} X^\mu = 0 \tag{12.3.10}$$

To derive the OPE of the superfield that is invariant under rigid supersymmetry (i.e. when η is not dependent on z), we first derive the combinations that are invariant under rigid supersymmetry. The rigid supersymmetry is given as follows:

$$\delta z = -i\epsilon\theta\eta, \quad \delta\theta = -i\epsilon\eta \tag{12.3.11}$$

Now, we calculate the transformation of the following quantities under supersymmetry;

$$\begin{aligned} z_1 - z_2 &\to (z_1 - i\epsilon\theta_1\eta) - (z_2 - i\epsilon\theta_2\eta) = z_1 - z_2 - i\epsilon(\theta_1 - \theta_2)\eta \\ \theta_1 - \theta_2 &\to (\theta_1 - i\epsilon\eta) - (\theta_2 - i\epsilon\eta) = \theta_1 - \theta_2 \end{aligned} \tag{12.3.12}$$

$$\begin{aligned} \theta_1\theta_2 &\to (\theta_1 - i\epsilon\eta)(\theta_2 - i\epsilon\eta) = \theta_1\theta_2 - i\epsilon(\theta_1 - \theta_2)\eta \\ &\Rightarrow z_1 - z_2 - \theta_1\theta_2 \to z_1 - z_2 - \theta_1\theta_2 \end{aligned} \tag{12.3.13}$$

So, we see that the quantities that are invariant under rigid supersymmetry are $\theta_1 - \theta_2$ and $z_1 - z_2 - \theta_1\theta_2$ (and their conjugates). So, the OPE can depend on these quantities only. The OPE can be calculated as follows (we set $\alpha' = 2$ in this section);

$$\begin{aligned} \mathbf{X}^\mu(\mathbf{z}_1,\overline{\mathbf{z}}_1)\mathbf{X}^\nu(\mathbf{z}_2,\overline{\mathbf{z}}_2) &= \left(X^\mu(z_1,\bar z_1) + i\theta_1\psi^\mu(z_1) + i\bar\theta_1\bar\psi^\mu(z_1)\right)\left(X^\mu(z_2,\bar z_2) + i\theta_2\psi^\mu(z_2) + i\bar\theta_2\bar\psi^\mu(z_2)\right) \\ &\sim -\eta^{\mu\nu}\ln|z_1 - z_2|^2 - \theta_1\theta_2\frac{\eta^{\mu\nu}}{z_1 - z_2} - \bar\theta_1\bar\theta_2\frac{\eta^{\mu\nu}}{\bar z_1 - \bar z_2} \end{aligned} \tag{12.3.14}$$

where the auxiliary field has been omitted. Including it would only add terms proportional to its algebraic equation of motion, and after eliminating it the singular OPE is unchanged. Now, we expand the following function as Taylor's expansion in θ_1 and $\bar\theta_1$;

$$\ln|(z_1 - z_2 - \theta_1\theta_2)|^2 = \ln\left[(z_1 - z_2 - \theta_1\theta_2)\left(\bar z_1 - \bar z_2 - \bar\theta_1\bar\theta_2\right)\right] = \ln|z_1 - z_2|^2 + \frac{\theta_2}{z_1 - z_2}\theta_1 + \frac{\bar\theta_2}{\bar z_1 - \bar z_2}\bar\theta_1$$

$$= \ln|z_1 - z_2|^2 + \frac{\theta_1\theta_2}{z_1 - z_2} + \frac{\bar{\theta}_1\bar{\theta}_2}{z_1 - z_2}0 \tag{12.3.15}$$

Using the above two equations, we have;

$$\mathbf{X}^\mu(\mathbf{z}_1, \bar{\mathbf{z}}_1)\mathbf{X}^\nu(\mathbf{z}_2, \bar{\mathbf{z}}_2) \sim -\eta^{\mu\nu}\ln|z_1 - z_2 - \theta_1\theta_2|^2 \tag{12.3.16}$$

We now write the ghost action in terms of two superfields i.e. B and C with weights $(\lambda - 1/2, 0)$ and $(1-\lambda, 0)$. It means that the weight of $BD_{\bar{\theta}}C$ is

$$\left(\lambda - \frac{1}{2}, 0\right) + (1-\lambda, 0) + \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) \tag{12.3.17}$$

Therefore, the following action has superconformal invariance;

$$S_{BC} = \frac{1}{2\pi}\int d^2z d^2\theta BD_{\bar{\theta}}C \tag{12.3.18}$$

The equations of motion are easily seen to be;

$$D_{\bar{\theta}}B = D_{\bar{\theta}}C = 0 \tag{12.3.19}$$

Now, we see that;

$$D_{\bar{\theta}}B = 0 \Rightarrow D_{\bar{\theta}}^2 B = 0 \Rightarrow \left(\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}\right)\left(\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}\right)B = \partial_{\bar{z}}B - \bar{\theta}\partial_{\bar{z}}\partial_{\bar{\theta}}B + \bar{\theta}\partial_{\bar{z}}\partial_{\bar{\theta}}B = \partial_{\bar{z}}B = 0 \Rightarrow \partial_{\bar{\theta}}B = 0 \tag{12.3.20}$$

The last equality follows because $D_{\bar{\theta}}B = \partial_{\bar{z}}B = 0$. Similarly, we can show that;

$$\partial_{\bar{\theta}}C = \partial_{\bar{z}}C = 0 \tag{12.3.21}$$

Therefore, the Taylor expansion of $B(\mathbf{z})$ and $C(\mathbf{z})$ are as follows:

$$B(\mathbf{z}) = \beta(z) + \theta b(z), \quad C(\mathbf{z}) = c(z) + \theta\gamma(z) \tag{12.3.22}$$

We see that the weights of all these components are exactly the weights that we find in bc ghost system and $\beta\gamma$ ghost system. The OPE is as follows:

$$B(\mathbf{z}_1)C(\mathbf{z}_2) = (\beta(z_1) + \theta_1 b(z_1))(c(z_2) + \theta_2\gamma(z_2)) = \theta_2\beta(z_1)\gamma(z_2) + \theta_1 b(z_1)c(z_2)$$

$$\sim -\theta_2\frac{1}{z_1 - z_2} + \theta_1\frac{1}{z_1 - z_2} = \frac{\theta_1 - \theta_2}{z_1 - z_2} \tag{12.3.23}$$

We can now do the following manipulation to show that this OPE is invariant under rigid supersymmetry;

$$\frac{\theta_1 - \theta_2}{z_1 - z_2 - \theta_1\theta_2} = \frac{\theta_1 - \theta_2}{z_1 - z_2}\left(1 - \frac{\theta_1 - \theta_2}{z_1 - z_2}\right)^{-1} = \frac{\theta_1 - \theta_2}{z_1 - z_2}\left(1 - \frac{\theta_1 + \theta_2}{z_1 - z_2}\right) = \frac{\theta_1 - \theta_2}{z_1 - z_2} \tag{12.3.24}$$

Therefore, we have;

$$B(\mathbf{z}_1)C(\mathbf{z}_2) \sim \frac{\theta_1 - \theta_2}{z_1 - z_2 - \theta_1\theta_2}$$

So we see that the OPE is written in terms of the quantities which are invariant under rigid supersymmetry. Now, we write down the sigma model action using superfields. Performing the Berezin integral over $\theta, \bar{\theta}$ reproduces the component action:

$$S = \frac{1}{4\pi}\int d^2z d^2\theta\left[G_{\mu\nu}(\mathbf{X}) + B_{\mu\nu}(\mathbf{X})\right]D_{\bar{\theta}}\mathbf{X}^\nu D_\theta\mathbf{X}^\mu \tag{12.3.25}$$

$$= \frac{1}{4\pi}\int d^2z\left[(G_{\mu\nu}(X) + B_{\mu\nu}(X))\partial_z X^\mu\partial_{\bar{z}}X_\mu + G_{\mu\nu}\left(\psi^\mu\mathcal{D}_{\bar{z}}\psi^\nu + \bar{\psi}^\mu\mathcal{D}_z\bar{\psi}^\nu\right) + \frac{1}{2}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\right] \tag{12.3.26}$$

where the covariant derivatives are defined as follows:

$$\mathcal{D}_{\bar{z}}\psi^\nu = \partial_{\bar{z}}\psi^\nu + \left[\Gamma^\nu_{\rho\sigma}(X) + \frac{1}{2}H^\nu_{\rho\sigma}(X)\right]\partial_{\bar{z}}X^\rho\psi^\sigma \tag{12.3.27}$$

$$\mathcal{D}_z\bar{\psi}^\nu = \partial_z\psi^\nu + \left[\Gamma^\nu_{\rho\sigma}(X) - \frac{1}{2}H^\nu_{\rho\sigma}(X)\right]\partial_z X^\rho\bar{\psi}^\sigma \tag{12.3.28}$$

The R–R sector is treated through the corresponding spacetime potentials, while the dilaton dependence enters through the choice of string or Einstein frame.
In the heterotic string case, we have $\bar{\psi}^\mu$ in the left sector but no ψ^μ in the right sector. But there are λ^A fields in the left sector. So, the superfields required in this case are as follows:

$$\mathbf{X}^\mu(\mathbf{z},\bar{\mathbf{z}}) = X^\mu(z,\bar{z}) + i\bar{\theta}\bar{\psi}^\mu, \quad \boldsymbol{\lambda}^A(\mathbf{z}) = \lambda^A + \theta G^A$$

(Write more about the heterotic case and anomalies)

11.1.2 Vertex operators

A naive idea of pictures is as follows. Positions of vertex operators in bosonic strings can be fixed but they come with an additional $c(z)\tilde{c}(\bar{z})$ factor. The positions of the vertex operators can be unfixed and they will have an d^2z integral with them. In the superstring case, we call these different categories of vertex operators to be in different pictures. The analogue of $c(z)\tilde{c}(\bar{z})$ in the NS sector case is (as follows from the orbifold projection);

$$e^{-\phi(z)-\tilde{\phi}(\bar{z})}$$

which is the NS vacuum vertex operator. In the superstring case, we integrate over $(\theta,\bar{\theta})$ in addition to $(z,\bar{z})$. Integrating over $d^2\theta$ extracts the $\theta\bar{\theta}$ component; denote this component by Ψ. The same component is obtained by applying $G_{-1/2}\tilde{G}_{-1/2}$ to the lowest-weight component, denoted $\mathcal{O}$. Thus

$$G_{-1/2}\tilde{G}_{-1/2}\mathcal{O} = \Psi$$

So, the $d^2\theta$ integration can be replaced with $G_{-1/2}\tilde{G}_{-1/2}$ acting on lowest weight components of the superfield. However, in the unfixed operators, we won't have $e^{-\phi(z)-\tilde{\phi}(\bar{z})}$ factor, just like the bosonic case where $c(z)\tilde{c}(\bar{z})$ is absent from the vertex operators whose position aren't fixed. So, we see that the $(\phi,\tilde{\phi})$ charges of the unfixed operators is $(0,0)$ and it is $(-1,-1)$ for the fixed operators. This fact is referred to by saying that unfixed operators are in the $(0,0)$ picture and the fixed operators are in the $(-1,-1)$ picture. In other words, having ($\phi,\tilde{\phi}$) charge equal to ($q,\tilde{q}$) is defined as being in the ($q,\tilde{q}$) picture.
As an example, we derive the $(0,0)$ and $(-1,-1)$ picture vertex operators for the following massless states in the NS - NS sector;

$$\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|0;k\rangle_{\text{NS}} \tag{12.3.29}$$

The unfixed vertex operators for these states are as follows:

$$\mathcal{V}^{-1,-1} = g_c e^{ik\cdot X}e^{-\phi-\tilde{\phi}}\psi^\mu\tilde{\psi}^\nu$$

where g_c is the closed string coupling constant. To find the vertex operators in the $(0,0)$ picture, we apply the $G_{-1/2}\tilde{G}_{-1/2}$ operator on this vertex operators and get rid of the $e^{-\phi-\tilde{\phi}}$ factor. For that, we will need the following;

$$G_r = \sum_{n\in\mathbb{Z}}\alpha^\mu_n\psi_{\mu r-n} \Rightarrow G_{-1/2} = \sum_{n\in\mathbb{Z}}\alpha^\mu_n\psi_{\mu,-1/2-n} = \alpha^\mu_0\psi_{\mu,-1/2} + \sum_{n=1}^{\infty}\left(\alpha^\mu_n\psi_{\mu,-1/2-n} + \alpha^\mu_{-n}\psi_{\mu,-1/2+n}\right) \tag{12.3.30}$$

where the commas have been placed to differentiate between Lorentz indices and Fourier indices. For the state in 12.3.29, no term in the first term in the sum in 12.3.30) contributes because all the α 's in that term are annihilation operators. From the second term in the sum, only $n=1$ term contributes because all the other ψ modes annihilate the state in 12.3 .29 . Similar arguments also apply for $\tilde{G}_{-1/2}$ Lastly, the zero mode term in $G_{-1/2}$ also contributes. So, effectively, we have;

$$G_{-1/2}\tilde{G}_{-1/2}\psi^\alpha_{-1/2}\tilde{\psi}^\beta_{-1/2}|0;k\rangle_{\text{NS}} = \left(\alpha^\mu_0\psi_{\mu,-1/2} + \alpha^\mu_{-1}\psi_{\mu,1/2}\right)\left(\tilde{\alpha}^\nu_0\tilde{\psi}_{\nu,-1/2} + \tilde{\alpha}^\nu_{-1}\tilde{\psi}_{\nu,1/2}\right)\psi^\alpha_{-1/2}\tilde{\psi}^\beta_{-1/2}|0;k\rangle_{\text{NS}}$$

Now, using the anticommutation relations for the fermionic modes, we have;

$$
\begin{aligned}
&-\left(\alpha_0^{\mu}\psi_{\mu,-1/2}\psi^{\alpha}_{-1/2}+\alpha^{\mu}_{-1}\delta^{\alpha}_{\mu}\right)\left(\tilde{\alpha}_0^{\nu}\tilde{\psi}_{\nu,-1/2}\tilde{\psi}^{\beta}_{-1/2}+\tilde{\alpha}^{\nu}_{-1}\delta^{\beta}_{\nu}\right)|0;k\rangle_{\text{NS}}\\
&=-\left(\alpha_0\cdot\psi_{-1/2}\psi^{\alpha}_{-1/2}+\alpha^{\alpha}_{-1}\right)\left(\tilde{\alpha}_0\cdot\tilde{\psi}_{-1/2}\tilde{\psi}^{\beta}_{-1/2}+\tilde{\alpha}^{\beta}_{-1}\right)|0;k\rangle_{\text{NS}} \qquad (12.3.31)
\end{aligned}
$$

where the overall minus sign comes because ψ^{α} travels through the $\tilde{\psi}$ bracket and both of the terms in the bracket contain a single spinor mode. Now, the vertex operators corresponding to these states are as follows:

$$\mathcal{V}^{0,0}=-g_c\left(k\cdot\psi\psi^{\alpha}+i\partial_z X^{\alpha}\right)\left(k\cdot\tilde{\psi}\tilde{\psi}^{\beta}+i\partial_{\bar{z}}X^{\beta}\right)e^{ik\cdot X}$$

This vertex operator is in the $(0,0)$ picture because there is no $(\phi,\tilde{\phi})$ charge. This expression is in $\alpha'=2$ convention. Similarly for open strings, we have the following massless state;

$$\psi^{\mu}_{-1/2}|0;k;a\rangle_{\text{NS}}$$

where a is the Chan Paton index. Now, the vertex operator in the -1 picture corresponding to this state is;

$$\mathcal{V}^{-1}=g_o e^{-\phi}\psi^{\mu}t^{a}e^{ik\cdot X} \qquad (12.3.32)$$

with t^a being the gauge group generator and with $d^2zd^2\theta$ integrations suppressed. Now, the picture 0 vertex operator is derived just like the calculation above. We get;

$$
\begin{aligned}
G_{-1/2}\psi^{\mu}|0;k;a\rangle&=\left(\alpha_0\cdot\psi_{-1/2}\psi^{\mu}_{-1/2}+\alpha^{\mu}_{-1}\right)|0;k;a\rangle_{\text{NS}}\Rightarrow\mathcal{V}^{0}=g_o t^{a}\left(\partial_z X^{\mu}+2k\cdot\psi\psi^{\mu}\right)\\
&=g_0\left(2\alpha'\right)^{-1/2}t^{a}\left(i\dot{X}^{\mu}+2\alpha' k\cdot\psi\psi^{\mu}\right)e^{ik.X} \qquad (12.3.33)
\end{aligned}
$$

The α' factors and the boundary derivative convention are fixed by matching the normalization of the boundary two-point function. Heterotic gauge vertices use the same spacetime NS operator together with the left-moving current algebra.

11.2 Tree level amplitudes

On the bosonic sphere, the conformal Killing group fixes three complex vertex positions. Gauge fixing therefore supplies three insertions of $c(z)\widetilde{c}(\bar{z})$. In the RNS superstring one must also account for the fermionic superconformal Killing directions. Equivalently, the zero modes of the c and γ ghosts determine the required insertions. On the sphere, two closed-string NS vertices carry $e^{-\phi-\widetilde{\phi}}$, so two NS–NS vertices are taken in the $(-1,-1)$ picture and the rest in the $(0,0)$ picture. This is the background-charge condition of the bosonized superghost ϕ. For open strings on the disc, two vertex operators are taken in the -1 picture and the rest in the 0 picture.
The vertex operator of the R ground state is

$$\mathcal{V}_{\mathbf{s}}=e^{-\phi/2}\Theta_{\mathbf{s}} \qquad (12.4.1)$$

and thus, it has a ϕ charge of $-1/2$. The background charge of the bosonized superghost fixes the total ϕ charge on the sphere to be -2. Thus an amplitude with two Ramond vertices uses $-1/2$ picture for each fermion and -1 picture for one NS vertex, with the remaining NS vertices in the zero picture.

The charge counting is direct. For the RNS superghost at $\lambda=3/2$,

$$T_{\phi}=-\frac{1}{2}:(\partial\phi)^2:-\partial^2\phi,\qquad \phi(z)\phi(0)\sim-\ln z.$$

The improvement term is a background charge $Q=2$. Splitting $\phi=\phi_0+\phi'$ on a genus-g surface, a correlator of exponentials contains the zero-mode factor

$$\left\langle\prod_i e^{q_i\phi(z_i)}\right\rangle_g\propto\int d\phi_0\,\exp\left[\left(\sum_i q_i+2(1-g)\right)\phi_0\right]\left\langle\prod_i e^{q_i\phi'(z_i)}\right\rangle.$$

The zero-mode integral imposes the neutrality condition

$$\sum_i q_i = -2(1-g).$$

On the sphere this gives $\sum_i q_i = -2$ in each holomorphic sector. An NS vertex in the -1 picture contributes $q = -1$, while a Ramond vertex in the $-1/2$ picture contributes $q = -1/2$. Therefore

$$q_{\text{NS}}^{(-1)} + q_{\text{NS}}^{(-1)} = -2, \qquad q_R^{(-1/2)} + q_R^{(-1/2)} + q_{\text{NS}}^{(-1)} = -2.$$

These equations give the standard tree-level picture assignments. Picture changing replaces $e^{-\phi}$ or $e^{-\phi/2}$ factors by BRST-equivalent operators with one unit higher ϕ-charge, through $X(z) = \{Q_B, \xi(z)\}$, but it cannot change the total background-charge constraint [3].

11.2.1 Three point amplitudes

Type I disc amplitude: Type I disc amplitude for three bosons is as follows:

$$\frac{1}{\alpha' g_o^2}\left\langle c\mathcal{V}^{-1}(x_1)\, c\mathcal{V}^{-1}(x_2)\, c\mathcal{V}^{0}(x_3)\right\rangle + (1 \leftrightarrow 2) \tag{12.4.2}$$

where we fixed the $(\theta, \bar{t})$ coordinates of two operators and the $(z, \bar{z})$ coordinates of all three of them. The overall normalization is taken to be the same as C_{D^2} in 6.5.4. This normalization was derived using unitarity. The $1 \leftrightarrow 2$ term comes due to summing over the other ordering. For calculating this amplitude, we need the following expectation values;

$$\langle c(x_1)\, c(x_2)\, c(x_3)\rangle = x_{12}x_{23}x_{31}, \left\langle e^{-\phi/2}(x_1)\, e^{-\phi/2}(x_2)\right\rangle = x_{12}^{-1} \tag{12.4.3}$$

The c expectation value was calculated in Chapter 4 while the ϕ expectation value was calculated in (10.4.13). We have chosen a normalization for this OPE. Using (12.3.32) and 12.3.33, we see that we also need the following expectation value;

$$\left\langle \psi^\mu e^{ik_1\cdot X}(x_1)\, \psi^\nu e^{ik_2\cdot X}(x_2)\left(i\dot{X}^\rho + 2\alpha' k\cdot\psi\psi^\rho e^{ik\cdot X}(x_3)\right)\right\rangle$$

This expectation value is obtained by Wick contracting the fermions and using the standard exponential correlator for the X^μ fields:

$$\begin{aligned}&\left\langle \psi^\mu e^{ik_1.X}(x_1)\, \psi^\nu e^{ik_2.X}(x_2)\left(i\dot{X}^\rho + 2\alpha' k.\psi\psi^\rho e^{ik.X}(x_3)\right)\right\rangle \\ &= -2i\alpha'(2\pi)^{10}\delta\left(\sum_i k_i\right)\left(\frac{\eta^{\mu\nu}k_1^\rho}{x_{12}x_{13}} + \frac{\eta^{\mu\nu}k_2^\rho}{x_{12}x_{23}} + \frac{\eta^{\nu\rho}k_3^\mu - \eta^{\mu\rho}k_3^\nu}{x_{13}x_{23}}\right)\end{aligned}$$

11.2.2 Four point amplitudes

The four-point superstring amplitude is the first place where the picture assignment, Chan-Paton trace, and low-energy Yang-Mills limit meet. For open strings on the disc the schematic ordered answer has the form

$$\mathcal{A}_4^{\text{open}} \sim g_o^2\, \text{Tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\, K(1,2,3,4)\, \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} \;+\; \text{orderings}, \tag{11.1}$$

where K is the supersymmetric kinematic factor built from polarizations and momenta. The absence of the bosonic tachyon pole is not accidental; it is the GSO projection expressed in the analytic structure of the amplitude. In the $\alpha' \to 0$ limit, Eq. (11.1) reduces to the Yang-Mills four-gluon amplitude plus controlled higher-derivative corrections. [17].

11.3 General amplitudes

A general superstring amplitude is a supermoduli-space integral with ghosts and superghosts inserted to divide out superdiffeomorphisms and super-Weyl transformations. Schematically,

$$\mathcal{A}_{g,n} = \int_{\mathcal{SM}_{g,n}} \left\langle \prod_r (b, \mu_r) \prod_a \delta((\beta, \chi_a)) \prod_{i=1}^{n} V_i \right\rangle . \tag{11.2}$$

The b insertions pair with even Beltrami differentials μ_r, while the β insertions pair with odd gravitino deformations χ_a. The superstring analogue of Eq. (6.19); the extra structure is exactly what picture number is keeping track of in practical calculations. [72, 121].

11.3.1 Pictures

Picture number labels the superghost sector for the bosonized superghost system. With $\beta = e^{-\phi}\partial\xi$ and $\gamma = \eta e^{\phi}$, a change of picture is generated by the picture-changing operator

$$X(z) = \{Q_B, \xi(z)\} = c\partial\xi + e^{\phi}T_F + \cdots . \tag{11.3}$$

On the sphere, the background charge of ϕ requires total left-moving picture number -2 and, for a closed-string amplitude, total right-moving picture number -2 as well:

$$\sum_i q_i = -2, \qquad \sum_i \tilde{q}_i = -2. \tag{11.4}$$

Thus two closed NS–NS vertices may be put in the $(-1,-1)$ picture and the rest in $(0,0)$. Picture-changing operators can move picture charge among the vertices without changing the physical amplitude, provided the total picture number is kept fixed.[1] [10].

11.3.2 Super-Riemann surfaces

A super-Riemann surface is a complex supermanifold of dimension $(1|1)$ equipped with a rank $(0|1)$ distribution whose square generates the even tangent direction. Locally, with coordinates $(z|\theta)$, the distinguished derivative is

$$D_\theta = \partial_\theta + \theta\,\partial_z, \qquad D_\theta^2 = \partial_z. \tag{11.5}$$

For genus $g \geq 2$ and no Ramond punctures, the supermoduli space has dimension

$$\dim \mathcal{SM}_g = (3g-3\,|\,2g-2). \tag{11.6}$$

Therefore the measure in Eq. (11.2) contains both ordinary b-ghost insertions and superghost insertions: the former integrate even complex-structure moduli, while the latter integrate odd supermoduli. [121, 127].

11.3.3 The measure on supermoduli space

In computations one often chooses a slice through supermoduli space and represents the odd directions by gravitino wavefunctions χ_a. Integrating over the odd moduli produces the $\delta(\beta)$ insertions already displayed in Eq. (11.2). A useful local form of the measure is

$$d\mu_{\text{super}} \sim \prod_r dm^r\, d\bar{m}^r \left\langle \prod_r (b, \mu_r)(\bar{b}, \bar{\mu}_r) \prod_a \delta((\beta, \chi_a)) \prod_{\bar{a}} \delta\big((\bar{\beta}, \bar{\chi}_{\bar{a}})\big) \right\rangle . \tag{11.7}$$

The key point is slice independence: changing the representatives μ_r, χ_a must change the integrand by a BRST-exact term plus possible boundary contributions. Here superstring perturbation theory is more delicate than the bosonic case. [73, 121].

[1] Picture-changing operators are BRST exact only away from singular collisions. When they collide with each other or with vertex operators, extra contact terms can appear, and this matters at higher genus because the amplitude is integrated over supermoduli, where a local prescription can miss boundary or contact contributions. Picture independence is then a patchwise statement unless a vertical-integration or Sen-style supermoduli prescription is specified.

11.4 One-loop amplitudes

At one loop the superstring amplitude is organized as a modular integral over the torus modulus and a spin-structure sum. Schematically,

$$\mathcal{A}_{1,n} = \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \sum_{\delta} \eta_\delta \left\langle \prod_i V_i \right\rangle_{\tau,\delta} . \tag{11.8}$$

The phases η_δ implement the GSO projection. Modular invariance is the central check: after $\tau \mapsto (a\tau+b)/(c\tau+d)$, the spin structures mix, and only the projected sum defines an allowed vacuum amplitude. This is the superstring version of the torus modular test already used for the bosonic string, now with spacetime supersymmetry visible through cancellations among spin structures. [10, 121].

The arithmetic content of Eq. (11.8) appears in the low-energy expansion of the genus-one integral

$$\mathcal{A}_1 = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \mathcal{I}(\tau, \bar{\tau}; k_i, \epsilon_i).$$

After expanding $\mathcal{I}$ in powers of $\alpha' k_i \cdot k_j$, the coefficients are modular graph forms. These forms can be reduced to iterated integrals of holomorphic Eisenstein series, giving explicit terms such as $\alpha'^8 \zeta_3 \zeta_5$ in the analytic four-graviton amplitude [297]. Pinching rules in the chiral-splitting representation show how the same integrand approaches the worldline limit as $\alpha' \to 0$, including reducible supergravity diagrams at fixed loop momentum [291]. Related one-loop four- and five-point computations produce single-valued multiple zeta values, theta lifts, L-values of cusp forms, and S-duality-compatible charge sectors in the IIB effective action [295, 296, 305]. In the forward limit, a Mellin transform of the superstring amplitude reduces to the Riemann zeta function; finite momentum transfer deforms this relation while preserving the dispersion interpretation of the EFT expansion [308]. The modular integral therefore controls both finiteness and the arithmetic coefficients of higher-derivative interactions.

11.4.1 Non-renormalization theorems

Non-renormalization statements are stated as protection statements, not as a claim that quantum corrections vanish everywhere. A typical protected term has the schematic form

$$S_{\text{eff}} \supset \int d^D x \sqrt{-g}\, f(\phi)\, \mathcal{O}_{\text{BPS}}, \tag{11.9}$$

where supersymmetry, duality, and charge quantization restrict the function $f(\phi)$. In the D-brane chapters this matters because BPS tensions and R-R charges can be compared across weak and strong coupling. Generic higher-derivative terms are not protected in the same way, so the text must separate protected quantities from ordinary corrections. [39, 352].

Chapter 12

D-branes and Ramond–Ramond charge

The bosonic-string D-branes of Chapter 7 entered only as hypersurfaces defined by boundary conditions; this chapter shows that their superstring counterparts are charged dynamical objects, and they are the objects on which the strong-coupling analysis of the next chapter is built. We first track T-duality through the Ramond sector and find that it flips one spacetime chirality, so type IIA and IIB are exchanged and their D-branes are even- and odd-dimensional respectively, exactly matching the Ramond–Ramond potentials available to couple to them. The cylinder amplitude between two branes, separated into its NS–NS and R–R channels, then has three consequences: its overall vanishing verifies that parallel branes are supersymmetric, its NS–NS part fixes the tension, and its R–R part fixes the charge, which we show saturates the Dirac quantization condition with the minimal unit, D-branes carry precisely the charges that Ramond–Ramond gauge invariance requires to exist. The remainder of the chapter is the dynamics: which brane pairs preserve supersymmetry (the $\#_{ND} \in \{0, 4, 8\}$ rule, derived twice), branes at angles, slow relative motion and the substringy distances it probes, and the BPS bound states, the (p, q) strings and D0–Dp systems, whose exact masses are used throughout Chapter 13 [351, 352].

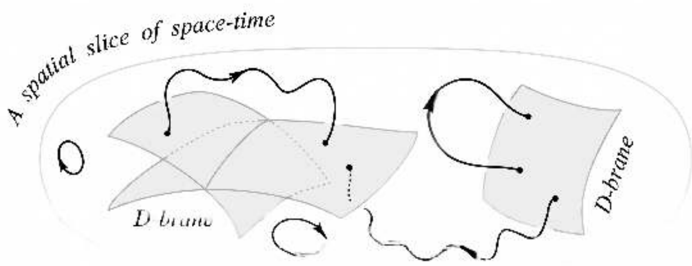


Figure 12.1: A Dp-brane is a dynamical hypersurface in spacetime on which open strings can end. Its worldvolume supports open-string fields, while its tension and Ramond–Ramond charge source the surrounding closed-string fields.

The two ways of speaking about a D-brane, as a boundary condition and as a dynamical object, are reconciled by its massless open-string modes (Figure 12.1). A Dp-brane imposes Dirichlet conditions on the $9 - p$ transverse coordinates and Neumann conditions along its worldvolume, which by itself looks like a rigid hypersurface. The lightest open strings on it are a worldvolume gauge field A_a from the Neumann directions together with $9 - p$ scalars X^i from the Dirichlet directions. Those scalars are the collective coordinates of the brane, the Goldstone modes of the translations it breaks, so their expectation values fix its transverse position and their fluctuations let it bend and move. Since string theory contains gravity, no object can stay perfectly rigid, and the rigid hypersurface is only the vacuum; the brane is a dynamical object whose shape is a set of fields governed by the Dirac–Born–Infeld action. For N coincident branes the position becomes an $N \times N$ matrix X^i in the adjoint of $U(N)$; its eigenvalues are the locations of the individual branes, and separating them breaks $U(N)$ to $U(1)^N$ with the off-diagonal gauge bosons realized as stretched strings. Viewed instead through the closed string, the same object is a BPS soliton that sources the metric, dilaton, and Ramond–Ramond fields, and the equivalence of the two pictures is the open/closed duality of the cylinder.

We do know that under T duality (in the 9 -th direction say), we have to do the following change;

$$X'^9_{\mathrm{R}}(\bar{z}) = -X^9_{\mathrm{R}}(\bar{z})$$

This implies sending all the Fourier components of X_{R} to negative of themselves. This will send $\tilde{T}_{\mathrm{F}}$ to negative of itself and it won't change any other stress tensor. This will disrupt the superconformal algebra. Since $\tilde{T}_{\mathrm{F}}$ contains $\tilde{\psi}^\mu \partial X^\mu_{\mathrm{R}}$, we can also do the following change;

$$\tilde{\psi}'^9(\bar{z}) = -\psi^9(\bar{z})$$

to save the superconformal algebra. Using the definition of $\tilde{F}$, we see that $\tilde{F}$ contains $\tilde{S}_4$ and $\tilde{S}_4$ contains $\tilde{\psi}^9_r$. Thus, the value of $\tilde{S}_4$ changes on its eigenstates by its negative. Since the possible values of $\tilde{S}_4$ are $\pm 1/2$, $\tilde{S}_4$ going negative to itself means that there is a difference of ± 1 and thus, there is a difference of ± 1 in $\tilde{F}$. Hence, $e^{\pi i \tilde{F}}$ (which is just chirality on the ground state) gets reversed on the Ramond right-handed ground state. So, we see that type IIA and type IIB are T duals of each other (where T duality is done on an odd number of dimensions). Now, we relate $\mathrm{R}-\mathrm{R}$ fields in T dual theories. Consider the vertex operators of $\mathrm{R}-\mathrm{R}$ ground states;

$$\psi_0^{[\mu_1} \ldots \psi_0^{\mu_p]}|\alpha; k\rangle \propto \Gamma^{[\mu_1} \ldots \Gamma^{\mu_p]} \mathcal{V}_\alpha |1\rangle$$

So, we need to ask how $\mathcal{V}_\alpha$ transforms under T-duality in the 9 direction. Since $\mathcal{V}_\alpha$ has a spinor index, it should transform like a spinor under parity in the 9 direction. The corresponding operator is $\beta^9_{\alpha\beta}$. This operator anticommutes with Γ^9 but commutes with all the other Γ's because it is the spin representation of a reflection in the 9-direction. So, we have an obvious solution of the following;

$$\beta^9 = \Gamma^9 \Gamma \Rightarrow \beta^9 \Gamma^9 = \Gamma^9 \Gamma \Gamma^9 = -\Gamma^9 \beta^9, \beta^9 \Gamma^\mu = \Gamma^\mu \beta^9 \quad \mu \neq 9$$

So, we now see that under T duality, if $\mu = 9$ is present in the R vertex operator, then it gets absorbed by the extra Γ^9 coming from the transformation of $\mathcal{V}$ and if there is no $\mu = 9$ present in the vertex operator, then it get a $\mu = 9$ index. This transforms between type IIA and type IIB $\mathrm{R}-\mathrm{R}$ fields. For more than one T dualized direction, the parity operators get multiplied;

$$\beta = \prod_m \beta^m$$

We derive a couple of identities now. Firstly, we see that;

$$\beta^m \beta^n = \Gamma^m \Gamma \Gamma^n \Gamma = \Gamma^n \Gamma^m \Gamma \Gamma = -\Gamma^n \Gamma \Gamma^m \Gamma = -\beta^n \beta^m, \quad (m \neq n)$$

Thus T-dualities in different directions anticommute on Ramond ground states. The physical duality group is recovered after including the corresponding fermion-number phase, so the apparent sign is a spinorial action and not a contradiction in the spacetime duality.

12.1 T-duality of type II strings

The calculation above shows the essential type-II fact: T-duality along an odd number of directions flips the chirality of one Ramond ground state, so type IIA and type IIB are exchanged. At the level of Ramond-Ramond potentials, a T-duality along y shifts whether y is an index of the potential:

$$C^{(p+1)}_{\mu_1 \cdots \mu_p y} \longleftrightarrow C'^{(p)}_{\mu_1 \cdots \mu_p}, \qquad C^{(p)}_{\mu_1 \cdots \mu_p} \longleftrightarrow C'^{(p+1)}_{\mu_1 \cdots \mu_p y}. \tag{12.1}$$

The form-field version of the boundary-condition statement: a Dp-brane wrapped along the dualized direction becomes a D$(p-1)$-brane, while a Dp-brane transverse to that direction becomes a D$(p+1)$-brane. Consequently type IIA carries even-p D-branes and type IIB carries odd-p D-branes, matching the RR potential parities. [10, 167].

12.2 T-duality of the type I string

(Include the supersymmetry charges thing and the worldsheet supercurrent thing). D-p branes naturally couple to $p+1$ form fields. They can couple to $R-R$ fields;

$$\int C_{p+1}$$

D-branes are charged under R–R potentials; the NS–NS two-form B_2 couples electrically to the fundamental string and magnetically to the NS5-brane, not to D-brane charge. Type IIA has $1,3,5,7$ and 9 form potentials and thus has Dp-branes with p even. Type IIB has $0,2,4,6,8$ and 10 form fields and thus has Dp-branes with p odd.

12.2.1 New connections between string theories

The supersymmetry algebra for extended objects contains central charges carried by strings and branes. The allowed charges are the p-form charges compatible with the ten-dimensional spinor bilinears, so the algebra takes the following schematic form:

$$\begin{aligned}
\{Q_\alpha,\bar{Q}_\beta\} &= -2\left[P_M+\frac{Q^{\rm NS}_M}{2\pi\alpha'}\right]\Gamma^M_{\alpha\beta}\Rightarrow\left\{Q_\alpha,Q^\dagger_\beta\right\}=2\left[P_M+\frac{Q^{\rm NS}_M}{2\pi\alpha'}\right]\left(\Gamma^0\Gamma^M\right)_{\alpha\beta}\\
\left\{\tilde{Q}_\alpha,\bar{\tilde{Q}}_\beta\right\} &= -2\left[P_M-\frac{Q^{\rm NS}_M}{2\pi\alpha'}\right]\Gamma^M_{\alpha\beta}\Rightarrow\left\{\tilde{Q}_\alpha,\tilde{Q}^\dagger_\beta\right\}=2\left[P_M-\frac{Q^{\rm NS}_M}{2\pi\alpha'}\right]\left(\Gamma^0\Gamma^M\right)_{\alpha\beta}\\
\left\{Q_\alpha,\bar{\tilde{Q}}_\beta\right\} &= -2\sum_p\tfrac{\tau_p}{p!}Q^{\rm R}_{M_1\dots M_p}\left(\beta^{M_1}\dots\beta^{M_p}\right)_{\alpha\beta}\Rightarrow\left\{Q_\alpha,\tilde{Q}^\dagger_\beta\right\}=2\sum_p\tfrac{\tau_p}{p!}Q^{\rm R}_{M_1\dots M_p}\left(\beta^{M_1}\dots\beta^{M_p}\Gamma^0\right)_{\alpha\beta}.
\end{aligned}\tag{13.2.3}$$

where $Q^{\rm NS}$ and $Q^{\rm R}$ are NS and R charges. Recall that charges carried by extended objects have multiple indices.

Connection to type-I divergences and BPS algebra: the same massless closed-string exchange that produced tadpoles in the annulus/Möbius/Klein-bottle channel is the spacetime exchange sourced by D-brane tension and R–R charge. In a supersymmetric D-brane, the NS–NS tension contribution and R–R charge contribution are related by the central terms in Eq. (13.2.3). A D-instanton is the $p=-1$ member of the same charge family: its Euclidean worldvolume is zero-dimensional, so its action scales like $1/g_s$, matching the expected nonperturbative factor $e^{-\text{const}/g_s}$. [167, 352].

12.3 D-brane action and charges

The superstring D-brane tension is obtained from the cylinder amplitude for an open string stretched between two branes, extracting the massless divergence in the closed-string channel. The amplitude has NS–NS and R–R contributions. The NS–NS part follows the type-I divergence calculation of Section 8.8, with the momentum integral restricted to the $p+1$ worldvolume directions and with the extra stretching contribution from the separation of the branes. Following the same normalization as in Subsection 7.7.2, one obtains

$$\begin{aligned}
\mathcal{A}_{\rm NS-NS} &= \frac{8iV_{p+1}}{\left(8\pi^2\alpha'\right)^{(p+1)/2}}\int_0^\infty\frac{dt}{t^{p+3/2}}e^{-ty^2/2\pi\alpha'}t^4=\frac{8iV_{p+1}}{\left(8\pi^2\alpha'\right)^{(p+1)/2}}\int_0^\infty dt t^{(5-p)/2}e^{-ty^2/2\pi\alpha'}\\
&=\frac{8^2iV_{p+1}}{8\pi\left(8\pi^2\alpha'\right)^5}\int_0^\infty\pi\frac{dt}{t^2}\left(8\pi^2\alpha' t\right)^{(9-p)/2}e^{-ty^2/2\pi\alpha'}=\frac{4\times 16iV_{p+1}}{8\pi\left(8\pi^2\alpha'\right)^5}\int_0^\infty\pi\frac{dt}{t^2}\left(8\pi^2\alpha' t\right)^{(9-p)/2}e^{-ty^2/2\pi\alpha'}
\end{aligned}$$

The last line rewrites the expression in the standard transverse-channel normalization used in the earlier tadpole calculations. Evaluating the same massless divergence in its simpler form gives

$$\mathcal{A}_{\mathrm{NS-NS}} = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt t^{(5-p)/2} e^{-ty^2/2\pi\alpha'} = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \left(\frac{2\pi\alpha'}{y^2}\right)^{(7-p)/2} \int_0^\infty d\mu \mu^{(5-p)/2} e^{-\mu}$$
$$= \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}} \left(\frac{2\pi\alpha'}{y^2}\right)^{(7-p)/2} \Gamma\left(\frac{7-p}{2}\right)$$

Now, we use 8.7.6 with $d = 9 - p$ to get;

$$G_{9-p}(y) = \frac{1}{4\pi^{(9-p)/2}|y|^{7-p}}\Gamma\left(\frac{7-p}{2}\right) \Rightarrow \Gamma\left(\frac{7-p}{2}\right) = 4\pi^{(9-p)/2}|y|^{7-p}G_{9-p}(y)$$

Putting this equation into the expression of $\mathcal{A}_{\text{NS-NS}}$, we get;

$$\mathcal{A}_{\mathrm{NS-NS}} = 2\pi i V_{p+1} \left(4\pi^2\alpha'\right)^{3-p} G_{9-p}(y) \tag{13.3.1}$$

Now, for the field theory calculation, the expectation values become the following by putting $D = 10$ in 8.7.10 and 8.7.11;

$$\langle\tilde{\phi}\tilde{\phi}\rangle = -\frac{2i\kappa^2}{k^2}$$

(Do the graviton expectation and the field theory calculation). The field theory result is given as follows:

$$\frac{2i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} = 2i\kappa^2\tau_p^2 V_{p+1} G_{9-p}(y) \tag{13.3.2}$$

Equality of 13.3.1 and 13.3.2, we get;

$\tau_p^2 = \frac{\pi\left(4\pi^2\alpha'\right)^{3-p}}{\kappa^2} \Rightarrow \tau_{p-1}^2 = \frac{\pi\left(4\pi^2\alpha'\right)^{3-(p-1)}}{\kappa^2} = \frac{\pi\left(4\pi^2\alpha'\right)^{3-p}}{\kappa^2}4\pi\alpha' = 4\pi\alpha'\tau_p^2 \Rightarrow \tau_{p-1} = 2\pi\sqrt{\alpha'}\tau_p \Rightarrow$ $T_{p-1} = 2\pi\sqrt{\alpha'}T_p$

So, we see that the recursion relation 8.7.5 is still satisfied (recall that τ_p and T_p are related linearly by the exponential of dilaton expectation value). To guess the low energy R$-$R exchange action is easy. We have two $D-p$ branes (with couple to C_{p+1} field and the action for this field is);

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,|F_{p+2}|^2$$

Moreover, the branes are sources for C_{p+1} fields and thus, the action of the coupling of branes to C_{p+1} field is (with coupling constant μ_p);

$$\mu_p\int C_{p+1}$$

So, the total action is;

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,|F_{p+2}|^2 + \mu_p\int C_{p+1}$$

The propagator for the C_{p+1} field is easy to read as the normalization is canonical. One can redefine a new C_{p+1} field as $\left(\sqrt{2}\kappa_{10}\right)^{-1}C_{p+1}$ and then, the normalization and propagator are the standard photon normalization and propagator i.e. i/k^2. Be careful about the different signature of the metric used usually in QFT. The reason there is no minus sign in the propagator here. The propagator is as follows:

$$2\kappa_{10}^2(\text{ standard propagator }) = \frac{2i\kappa_{10}^2}{k^2}$$

The field-theory amplitude is obtained by multiplying the two brane sources by the C_{p+1} propagator and integrating over the common worldvolume, which supplies the factor V_{p+1}:

$$-2i\kappa_{10}^2\mu_p^2 G_{9-p}(y)V_{p+1}$$

Since this should be negative of the NS - NS field theory result, using 13.3.1 we have;

$$-2i\kappa_{10}^2\mu_p^2 G_{9-p}(y)V_{p+1} = -2\pi i V_{p+1}\left(4\pi^2\alpha'\right)^{3-p} 2G_{9-p}(y) \Rightarrow \mu_p^2 = \frac{\pi}{\kappa_{10}^2}\left(4\pi\alpha'\right)^{3-p} = \frac{\kappa^2}{\kappa_{10}^2}\tau_p^2 = e^{2\phi_0}\tau_p^2 = T_p^2 \tag{13.3.3}$$

In the second-last step we used the normalization relation between the ten-dimensional gravitational coupling and the string-frame coupling, $\kappa = e^{\phi_0}\kappa_{10}$, together with the equality of BPS tension and R–R charge. Orientifold planes obey the same calculation with boundary states replaced by crosscap states; their signs and factors of 2 are fixed by the Möbius and Klein-bottle amplitudes of Chapter 8.

The D-brane tension derived from the cylinder can also be read as a central charge. For a BPS object the supersymmetry algebra has the schematic form

$$\{Q, Q^\dagger\} = 2(H - Z), \qquad H \geq |Z|.$$

Saturation gives $M = |Z|$. The open-bosonic-SFT computation of the D0-brane mass obtains this central term from the spontaneously broken Poincare algebra, matching the normalization extracted from the annulus/cylinder channel [248]. For charged BPS black holes, the index is coupling independent only when the ultraviolet cutoff is placed at the string scale, so the full tower above the EFT cutoff is part of the protected quantity [244]. D-brane tension and R–R charge are therefore fixed by the same algebraic and worldsheet quantities that enter the cylinder computation.

The same protected quantities appear in nonperturbative settings. In topological string theory the nonperturbative sectors are labeled by D-brane charge vectors γ, and the transseries has the schematic form

$$F_{\text{top}} \sim \sum_\gamma \Omega(\gamma)\,\exp\left[-\frac{Z_\gamma}{g_s}\right]\left(1 + \sum_{n\geq 1} a_{\gamma,n}g_s^n\right),$$

where $\Omega(\gamma)$ is interpreted as a BPS/DT multiplicity; modular resurgence organizes these multiplicities into monodromy orbits [264]. Fractional instantons on a twisted four-torus arise from wrapped intersecting D-branes, and their local moduli space is a hyper-Kahler Higgs branch [257]. Pure D-brane black-hole counting evaluates helicity traces in four-charge systems and separates BPS branches from non-BPS branches with no zero-energy extremum [252]. The recurring inputs are integer charge, the central charge Z_γ, and the geometry of the brane moduli space.

12.3.1 Dirac quantization condition

We first review the Dirac condition for normal electrodynamics. Since the electrical charge is carried by point particles (or 0 branes), the field strength should be F_2 and since normal electrodynamics is in four dimensions, the dual field strength is also a two-form which in turn implies that magnetic sources should also be 0 branes or point particles.
Consider a magnetic monopole with magnetic charge μ_m and a sphere S^2 around it. The flux through S^2 is calculated as follows:

$$\mu_m = \int_{S^2} B_i\, dA_i = \int_{S^2} \frac{1}{2}\epsilon_{ijk}F_{jk}\, dA_i. \tag{13.3.4}$$

Now, we do the following manipulation;

$$\begin{aligned} dA_i &= \left(dx^1 \times dx^2\right)_i = \epsilon_{i\ell m}\, dx^1_\ell dx^2_m \\ \Rightarrow \quad \epsilon_{ijk}dA_i &= \epsilon_{ijk}\epsilon_{i\ell m}\, dx^1_\ell dx^2_m = (\delta_{\ell j}\delta_{mk} - \delta_{\ell k}\delta_{mj})\, dx^1_\ell dx^2_m \\ &= dx^1_j dx^2_k - dx^1_k dx^2_j = 2\, dx^1_j \wedge dx^2_k. \end{aligned} \tag{13.3.5}$$

Using (13.3.5) in 13.3.4, we get;

$$\int_{S^2} \epsilon_{ijk}F_{jk}\, dx^1_j \wedge dx^2_k = \int_{S^2} F_2 = \mu_m. \tag{13.3.6}$$

Let P be an oriented closed loop followed by the electric probe, and let D be any oriented two-surface in the patch of A_1 such that $\partial D = P$. With this convention Stokes' theorem gives

$$\oint_P A_1 = \int_D F_2. \tag{12.2}$$

Including an electrically charged particle in this field with charge μ_e, we get the following phase;

$$\exp\left(i\mu_e \oint_P A_1\right) = \exp\left(i\mu_e \oint_D F_2\right) = \exp\left(i\mu_e\mu_m\right)$$

To ensure the single-valuedness of this quantity, we need to ensure that;

$$\mu_e\mu_m = 2\pi n$$

Now, we generalize this discussion to p-branes. If the source is a p-brane, then the field strength is F_{p+2}, the dual strength is $F_{10-p-2} = F_{8-p}$ and thus, the source of such a field is a $8-p-2 = (6-p)$ brane. We call this brane a magnetic brane and p brane an electric brane. Now, in 3 space dimensions, a point is surrounded by S^2 (two dimensions), and a line is surrounded by S^1 (one dimension). The dimension of the sphere S^k surrounding an extended entity of p dimensions in a d dimensional space is given by the formula;

$$k = d - p - 1 \tag{13.3.7}$$

For 9 space dimensions, a $6-p$ magnetic brane is surrounded by a sphere of dimension

$$9 - (6-p) - 1 = p + 2$$

which makes sense as it is the number of indices on F_{p+2} and we expected the generalization of (13.3.6) to be something like F_{p+2} integrated over S^{p+2}. The normalization follows from the kinetic term $-\frac{1}{4\kappa_{10}^2}\int |F_{p+2}|^2$: varying C_{p+1} in the presence of a magnetic $(6-p)$-brane gives $dF_{p+2} = 2\kappa_{10}^2\mu_{6-p}\,\delta_{p+3}$. Integrating this Bianchi identity over the linking sphere gives

$$\int_{S^{p+2}} F_{p+2} = 2\kappa_{10}^2\mu_{6-p}.$$

An argument like before gives us the following condition;

$$\exp\left(i\mu_p \oint_{S^{p+2}} F_{p+2}\right) = \exp\left(2i\kappa_{10}^2\mu_p\mu_{6-p}\right) \quad \text{single valued} \Rightarrow \mu_p\mu_{6-p} = \frac{\pi n}{\kappa_{10}^2}. \tag{13.3.8}$$

Using (13.3.3) we get the following;

$$\mu_p = \frac{\sqrt{\pi}}{\kappa_{10}}\left(4\pi\alpha'\right)^{(3-p)/2}, \quad \mu_{6-p} = \frac{\sqrt{\pi}}{\kappa_{10}}\left(4\pi\alpha'\right)^{(p-3)/2} \Rightarrow \mu_p\mu_{6-p} = \frac{\pi}{\kappa_{10}^2}$$

So, the charges in 13.3.3 satisfy 13.3.8) for $n = 1$.

12.3.2 D-brane actions

The massless (NS+, NS+) spectrum is the same as the bosonic massless spectrum, so the arguments of Section 7.7 apply. The coupling of the D-brane to NS–NS fields is therefore the DBI action. Since the tension equals the R–R charge in the BPS normalization above, T_p may be written as μ_p. For multiple branes the low-energy fields become matrix-valued, and the action includes a Chan–Paton trace. The NS–NS action is

$$S_{\text{NS-NS}} = -\mu_P \int d^{p+1}\xi \,\mathrm{Tr}\left[-e^{-\phi}\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)\right] \tag{13.3.9}$$

The flat directions of this action are the commuting transverse scalar matrices. When $[X^i, X^j] = 0$, the matrices can be diagonalized, and their eigenvalues are the positions of separated branes; noncommuting vevs describe genuinely coincident-brane configurations. The coupling to $\mathrm{R}-\mathrm{R}$ fields is the Chern–Simons coupling obtained by summing all wedge products of the worldvolume field strength and the pullback of the background potentials:

$$S_{\mathrm{R-R}} = i\mu_p \int_{p+1} \mathrm{Tr}\left[\exp\left[2\pi\alpha' F_2 + B_2\right] \wedge \sum_q C_q\right] \tag{13.3.10}$$

The displayed action is the part needed in this chapter. The full supersymmetric D-brane action also contains fermions, pullbacks written with covariant embedding fields, and curvature corrections such as the A-roof genus factors in the Chern–Simons term; these begin at higher order in α' and are not used in the present charge and tension calculation. To find the YM coupling on the brane,

12.3.3 Coupling constants

We now calculate the ratio τ_{F1} to τ_{D1} i.e. the ration of tension of fundamental string to the tension in $D1$ brane. It is as follows:

$$\frac{\tau_{F1}}{\tau_{D1}} = \frac{e^{-\phi_0}/2\pi\alpha'}{e^{-\phi_0}\sqrt{\pi}\left(4\pi^2\alpha'\right)/\kappa} = \frac{\kappa}{8\pi^{7/2}\alpha'^2}$$

Now, as shown by tree-level factorization, g_c is proportional to κ and thus;

$$g \propto \frac{\tau_{F1}}{\tau_{D1}}$$

Although we fixed the normalization between g_o and g_c in Chapter 7, the remaining convention is the normalization of $g_c \propto e^{\phi}$. So, we define g_c to be the ratio above;

$$g_c = \frac{\tau_{F1}}{\tau_{D1}}$$

Then, using the expression for τ_{F1}/τ_{D1}, we get;

$$\kappa = 8 g_c \pi^{7/2}\alpha'^2 \Rightarrow \kappa^2 = \frac{1}{2}2^7 g_c^2 \pi^7 \alpha'^4 = \frac{1}{2} g_c^2 (2\pi)^7 \alpha'^4$$

The string tension can then be written in terms of g_c as follows:

$$\tau_p = \frac{\sqrt{\pi}}{\kappa}\left(4\pi\alpha'\right)^{(3-p)/2} = \frac{\sqrt{\pi}\left(4\pi\alpha'\right)^{(3-p)/2}\sqrt{2}}{g_c (2\pi)^{7/2}\alpha'^2} = (2\pi)^{-p}\alpha'^{-(1+p)/2} g_c^{-1} \tag{13.3.11}$$

Notice that the D-brane tension has dimension (length)$^{-(p+1)}$, as required for an object with a $(p+1)$-dimensional worldvolume action. Only $p=-1$, the D-instanton, gives a dimensionless action without integrating over a spatial worldvolume. To derive the Yang-Mills coupling on the D-brane, we expand 13.3.9 for $G_{ab} = \eta_{ab}$, $B_{ab} = 0$ and $\phi = \phi_0$. We also use the following identity;
$\det(-\mathbb{I} + M) = -1 + \mathrm{Tr}(M) + \mathcal{O}\left(M^2\right) \Rightarrow \det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right) = -1 + 2\pi\alpha' F_{ab}F^{ba} + \mathcal{O}\left(F^2\right) \sim -1 - 2\pi\alpha' F_{ab}F^{ab} = -1 - 2\pi\alpha' F^2$
So, the expansion of 13.3.9 is as follows:

$$-\mu_p \int d^{p+1}\xi\, \mathrm{Tr}\left[e^{-\phi_0}\left(1 + 2\pi\alpha' F^2\right)\right] = -\mu_p \int d^{p+1}\xi\, \mathrm{Tr}\left[e^{-\phi_0}\right] - 2\pi\alpha'\mu_p \int d^{p+1}\xi\, \mathrm{Tr}\left[e^{-\phi_0}F^2\right]$$

Two normalization points deserve care here. The dilaton is a closed-string background and is proportional to the identity in Chan–Paton space, so it should be pulled outside the trace,

$$\mathrm{Tr}\left(e^{-\phi_0}\mathbf{1}\, F_{ab}F^{ab}\right) = e^{-\phi_0}\,\mathrm{Tr}(F_{ab}F^{ab}). \tag{12.3}$$

The other issue is the determinant expansion. For an antisymmetric F,

$$\sqrt{-\det(\eta + 2\pi\alpha' F)} = 1 + \frac{(2\pi\alpha')^2}{4}F_{ab}F^{ab} + O(F^4), \tag{12.4}$$

with no term linear in F. Hence

$$\frac{1}{g_{\mathrm{YM}}^2} = T_p e^{-\phi_0}(2\pi\alpha')^2, \qquad g_{\mathrm{YM}}^2 = (2\pi)^{p-2} g_s (\alpha')^{(p-3)/2}, \tag{12.5}$$

when $T_p = 1/[(2\pi)^p(\alpha')^{(p+1)/2}]$ is defined without the explicit $1/g_s$. This is the normalization used in the D3 check in Eq. (13.11). The action then becomes;

$$-2\pi\alpha'\tau_p \int d^{p+1}\xi \operatorname{Tr}\left[F^2\right] = \frac{8\pi\alpha'\tau_p}{-4}\int d^{p+1}\xi \operatorname{Tr}\left[F^2\right] = -\frac{1}{4}\frac{(2\pi\alpha')^2}{\pi\alpha'/2}\tau_p \int d^{p+1}\xi \operatorname{Tr}\left[F^2\right]$$

The denominator $\pi\alpha'/2$ is only a trace-normalization convention: the preceding blue formula uses the standard Yang–Mills normalization $-\frac{1}{4g_{\mathrm{YM}}^2}\operatorname{Tr} F_{ab}F^{ab}$. Now, if we read g_{YM}^2, we get;

$$\frac{1}{g_{YM}^2} = (2\pi\alpha')^2\,\tau_p \Rightarrow g_{YM}^2 = \frac{1}{(2\pi\alpha')^2\,\tau_p} = (2\pi)^{-2}\alpha'^{-2}\tau_p^{-1} = (2\pi)^{-2}\alpha'^{-2}(2\pi)^p\alpha'^{(1+p)/2}g_c = (2\pi)^{p-2}\alpha'^{(p-3)/2}g_c$$

We see that this coupling is dimensionless only for $p = 3$ i.e. three-dimensional space case which we already know is true.
(The relation between g_{YM}, κ and α' requires some orientifold based reasoning. Do that.), The result is;

$$\frac{g_{YM}^2}{\kappa} = 2(2\pi)^{7/2}\alpha' \quad \text{(Type I)}$$

The type-I Born–Infeld action is obtained by applying the orientifold projection to the open-string gauge sector and keeping the invariant Chan–Paton fields.

12.4 D-brane interactions: statics

Consider two branes of possibly different dimensions, a Dp-brane and a Dp'-brane. Let the sets of coordinates with Dirichlet boundary conditions on the two branes be S_D and S'_D, respectively, and let the corresponding Neumann sets be S_N and S'_N. Then the DD directions are $S_D \cap S'_D$, and similarly for NN, DN, and ND directions. The unbroken supersymmetries preserved by the first and second D-brane are

$$Q_\alpha + \left(\beta^\perp Q\right)_\alpha, \quad Q_\alpha + \left(\beta'^\perp Q\right)_\alpha = Q_\alpha + \left(\beta^\perp\beta^{\perp-1}\beta'^\perp Q\right)_\alpha \tag{13.4.1}$$

and thus, the supersymmetries unbroken by both D-branes are the ones which are generated by Q_α such that;

$$\beta^{\perp-1}\beta'^\perp Q_\alpha = Q_\alpha \tag{13.4.2}$$

because if this happens, then the spinors unbroken by the second brane and the first brane are the same as can be seen from 13.4.1. So, we need to find the number of solutions of 13.4.2. Now, $\beta^{\perp-1}\beta'^\perp$ is a reflection in precisely the DN and ND directions: directions that are Neumann for both branes or Dirichlet for both branes appear in both reflection operators and cancel, while mixed directions appear in only one operator. The total number of DN and ND directions is denoted as $\#_{ND}$. Now, we prove that $\#_{ND}$ is even. The number of elements of different relevant sets is as follows:

$$n\left(S_D\right) = D - p, \quad n\left(S_N\right) = p, \quad n\left(S'_D\right) = D - p', \quad n\left(S'_N\right) = p'$$

Now, instead of using a Venn diagram, write the four intersections as

$$\begin{aligned} a &= n(S_D \cap S'_D), \quad b = n(S_D \cap S'_N), \\ c &= n(S_N \cap S'_D), \quad d = n(S_N \cap S'_N), \end{aligned} \qquad \#_{ND} = b + c. \tag{12.6}$$

The set constraints are

$$a + b = D - p, \qquad a + c = D - p', \qquad b + d = p', \qquad c + d = p. \tag{12.7}$$

Subtracting the first two equations gives $b - c = p' - p$, so $\#_{ND} = b + c$ has the same parity as $p' - p$.[1] where we used the fact that the number of elements in $S_D \cap S'_N$ and $S'_D \cap S_N$ is $\#_{ND}$

[1] Two equivalent countings appear here. In Eq. (12.6) the four disjoint sets NN, DD, ND, DN have cardinalities d, a, b, c obeying the constraints of Eq. (12.7), with $\#_{ND} = b + c$ the number of mixed directions. The μ, ξ variables of the paragraph below are combinations of the same four cardinalities, $\mu = n(S_N \cap S'_N)$ and $\xi = n(S_N \cap S'_D)$, so the two computations are one argument written in different variables; both give $\#_{ND} \equiv p - p' \pmod 2$, hence $\#_{ND}$ even within a single type II theory. The supersymmetry projection depends only on the total $\#_{ND}$ and never on b and c separately, because the reflection $\beta^{\perp-1}\beta'^\perp$ acts in the ND and DN planes together.

and the total number of elements in all sets is $D.\xi$ and μ are undetermined coefficients but they should be non-negative integers because $n\left(S_N \cap S'_N\right) = \mu$ and $n\left(S_N \cap S'_D\right) = \xi$. Now, we apply the constraints that $n\left(S_D\right) = D - p$ and $n\left(S_N\right) = p$, Both of these constraints give the same condition i.e.;

$$\mu + \xi = D - p$$

Now, applying the constraints that $n\left(S'_D\right) = D - p'$ and $n\left(S'_N\right) = p'$ give the same result i.e.;

$$\#_{ND} - \mu + \xi = D - p' \Rightarrow \xi - \mu = D - p' - \#_{ND}$$

So, we get two simultaneous equations for ξ and μ. They are easily solved to give the following;

$$\xi = D - \frac{(p + p') + \#_{ND}}{2}, \quad \mu = \frac{(p - p') + \#_{ND}}{2}$$

Now, p and p' are both even or odd because they are in type II A theory or type II B theory which have even and odd branes respectively. So, $p + p'$ and $p - p'$ are both even. Therefore, from the expression of μ we deduce that $p - p' + \#_{ND}$ is a non-negative even integer (because μ is a non-negative integer). Since $p - p'$ is even, $\#_{ND}$ is even as well. $\#_{ND}$ is non-negative because it is the number of DN and ND directions. The expression of ξ tells the same story. So, we can write $\#_{ND} = 2j$ where j is a non-negative integer.
Since $\beta^{\perp -1}\beta^{\perp'}$ is the reflection in ND and DN directions, we can pair these directions to give j pairs (since they are even in number) and since a reflection of the coordinate axis of a two-dimensional plane is a rotation by π degrees, we can write $\beta^{\perp -1}\beta^{\perp'}$ as;

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[i\pi\left(J_1 + \ldots + J_j\right)\right]$$

For the spinor representation, J_i has eigenvalues $\pm 1/2$, and thus $e^{\pi i J_i}$ has eigenvalues $\pm i$. If j is odd, the product has eigenvalues $\pm i$, so no spinor can have eigenvalue $+1$. Therefore j must be even. This condition is necessary, and the detailed counting below determines how many eigenvectors with eigenvalue $+1$ remain.
So, $\#_{ND}$ is a multiple of 4 i.e. $\#_{ND} \in \{0, 4, 8\}$. We now count the unbroken supersymmetries corresponding to each case. First take the $\#_{ND} = 8$ case. So, we have;

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[\pi i\left(J_1 + J_2 + J_3 + J_4\right)\right]$$

where J_i 's are $\pm 1/2$. Different supersymmetries are counted by counting different $\tilde{Q}_\alpha$'s and different $\tilde{Q}_\alpha$'s are counted by counting different (J_1, J_2, J_3, J_4) tuples such that $J_1 + J_2 + J_3 + J_4$ is even. This sum is even when all J_i 's are $1/2$ or $-1/2$ which gives us two tuples. This sum is also even when any two J_i 's are $1/2$ and the other two J_i 's are $-1/2$. This second possibility can happen in $\binom{4}{2} = 6$ ways and thus, we have 6 more tuples. The first brane already relates Q and $\tilde{Q}$ and reduces the thirty-two supercharges to sixteen; the counting above is the additional projection imposed by the second brane, leaving eight unbroken supersymmetries.
For the $\#_{ND} = 4$ case, one has

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[\pi i\left(J_1 + J_2\right)\right]$$

So, we need (J_1, J_2) tuples such that $J_1 + J_2$ is even. This can only happen when (J_1, J_2) = ($1/2, -1/2$) or $(J_1, J_2) = (-1/2, 1/2)$. The two unaffected transverse spin labels remain free, giving an additional factor 2^2; hence $2 \times 2^2 = 8$ supercharges survive.
The $\#_{ND} = 0$ case is trivial. In this case, $\beta^{\perp -1}\beta^{\perp'} = 1$, there is no condition that comes from the second brane. Therefore, the supersymmetries allowed by the first brane are unbroken. This is 16 in number, as expected for parallel BPS branes related by translations or T-dualities.
The same multiple-of-four condition can also be read from the open-string NS zero-point energy: only for $\#_{ND} = 0, 4, 8$ does the GSO-projected light spectrum fit into a supersymmetric multiplet without a residual tachyon.

12.4.1 Branes at general angles

We start with the example of two $D - 4$ branes. They are taken to be in $(2, 4, 6, 8)$ directions with separation in 1 direction. To visualize the configuration, one may group $(2, 4, 6, 8)$ as the brane directions and use direction 1 as the separation direction. Now, rotate one brane by ϕ_1 in the (2,3) plane, ϕ_2 in the (4,5) plane, ϕ_3 in the (6,7) plane, and ϕ_4 in the (8,9) plane. There is no rotation in any plane involving direction 1 because that would make the separation position dependent. All of these rotations are collectively called ρ. The unbroken supersymmetry is the intersection of the supersymmetry preserved by the unrotated brane and the supersymmetry preserved by the rotated brane:

$$Q_\alpha + \left(\rho^{-1}\beta^\perp\rho Q\right)_\alpha = Q_\alpha + \left(\beta^\perp\beta^{\perp-1}\rho^{-1}\beta^\perp\rho Q\right)_\alpha$$

So, the unbroken supersymmetry is generated by spinors that is left invariant by the following operator;

$$\beta^{\perp-1}\rho^{-1}\beta^\perp\rho = \beta^{\perp-1}\beta^\perp\rho^2 = \rho^2$$

The first equality follows because conjugating a reflection by a rotation gives the reflection in the rotated plane; in the spinor representation this is the same group relation lifted from $SO(8)$ to Spin(8). In the spinor representation, this rotation ρ is represented as follows:

$$\rho = \exp\left(i\sum_{a=1}^{4} s_a\phi_a\right) \Rightarrow \rho^2 = \exp\left(2i\sum_{a=1}^{4} s_a\phi_a\right) \qquad \text{where } s_a \in \{-1/2, 1/2\} \tag{13.4.3}$$

So, we want this phase to be 1 . In other words, we want;

$$2s_1\phi_1 + 2s_2\phi_2 + 2s_3\phi_3 + 2s_4\phi_4 = 0(\text{mod}2\pi)$$

where $2s_i$'s can take the values equal to ± 1. We see that if this condition is satisfied for a $(2s_1, \ldots, 2s_4)$ tuple, then it is also satisfied for $(-2s_1, \ldots, -2s_4)$ tuple. So, we can focus our attention to the cases where $2s_1 = 1$ so that we don't consider equivalent tuples more than once. We can tabulate all the inequivalent conditions as follows:

s_1	s_2	s_3	s_4	Condition
1	-1	-1	-1	$\phi_1 - \phi_2 - \phi_3 - \phi_4 = 0(\text{mod}2\pi)$
1	-1	-1	1	$\phi_1 - \phi_2 - \phi_3 + \phi_4 = 0(\text{mod}2\pi)$
1	-1	1	-1	$\phi_1 - \phi_2 + \phi_3 - \phi_4 = 0(\text{mod}2\pi)$
1	-1	1	1	$\phi_1 - \phi_2 + \phi_3 + \phi_4 = 0(\text{mod}2\pi)$
1	1	-1	-1	$\phi_1 + \phi_2 - \phi_3 - \phi_4 = 0(\text{mod}2\pi)$
1	1	-1	1	$\phi_1 + \phi_2 - \phi_3 + \phi_4 = 0(\text{mod}2\pi)$
1	1	1	-1	$\phi_1 + \phi_2 + \phi_3 - \phi_4 = 0(\text{mod}2\pi)$
1	1	1	1	$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0(\text{mod}2\pi)$

Some possibilities are considered as follows:

- The last condition allows the $(2s_1, 2s_2, 2s_3, 2s_4) = (1,1,1,1)$ and $(2s_1, 2s_2, 2s_3, 2s_4) = (-1,-1,-1,-1)$ tuples and thus, two supersymmetries are allowed out of the original 16. So, this case breaks seveneights of supersymmetry.

- If we want $\phi_1+\phi_2+\phi_3 = \phi_4 = 0(\bmod 2\pi)$ then this is satisfied by last two rows in the table and thus, we have $2\times 2 = 4$ unbroken supersymmetries (a factor of two arises because we also count the equivalent tuples that have $s_1 = -1$).

- If we want $\phi_1 + \phi_2 = \phi_3 + \phi_4 = 0(\text{mod}2\pi)$ then this is satisfied by the last row and the fifth row in the table and thus, we have $2\times 2 = 4$ unbroken supersymmetries.

- If we want $\phi_1+\phi_2 = \phi_3 = \phi_4 = 0(\bmod 2\pi)$, then this condition is satisfied by the last four rows in the table. Thus, we have $2\times 4 = 8$ unbroken supersymmetries.

- We now consider the case when k of ϕ_i 's are $\pi/2$ and the rest of them are zeros. If $k = 0$ then all the cases in the table are satisfied and thus, $2\times 8 = 16$ supersymmetries are unbroken. The same as $\#_{ND} = 0$ case.

- If $k = 1$ or $k = 3$, none of the conditions in the table can be satisfied, and all supersymmetries are broken. These are the angle-space counterparts of the $\#_{ND} = 2$ and $\#_{ND} = 6$ cases: no common projector eigenspace remains.

- If $k=2$, then there are four rows in the table above which are satisfied. (the preserved supersymmetries are counted by the common projector eigenspaces). So, we have $2\times 4=8$ supersymmetries unbroken. The same as the case $\#_{ND}=4$.
- The $k=4$ case is satisfied for rows $2,3,5,8$. This case also has $2\times 4=8$ unbroken supersymmetries. It can be taken to be the same case $\#_{ND}=8$ case.

Now, we define the following complex coordinates to make sense of the rotations;

$$Z^1=X^2+iX^3,\quad Z^2=X^4+iX^5,\quad Z^3=X^6+iX^7,\quad Z^4=X^8+iX^9$$

We will denote the conjugate of these complex numbers as $\overline{Z^a}=Z^{\bar{a}}$. So, out of the full $SO(8)$ rotation group in eight dimensions, only the $U(4)$ subgroup retains the complex structure. The rotation ρ in particular is;

$$\rho=\operatorname{diag}\left(e^{i\phi_1},e^{i\phi_2},e^{i\phi_3},e^{i\phi_4}\right)\Rightarrow\det\rho=e^{i(\phi_1+\phi_2+\phi_3+\phi_4)}$$

Now, we consider the various cases that we discussed before.

- If $\phi_1+\phi_2+\phi_3+\phi_4=0(\text{mod}2\pi)$ then $\det\rho=1$ an thus, $\rho\in SU(4)$. This case has 2 unbroken supersymmetries.
- If $\phi_1+\phi_2+\phi_3=\phi_4=0(\text{mod}2\pi)$ then $\det\rho=1$ an thus, $\rho\in SU(3)$ ($\phi_4=0(\text{mod}2\pi)$ and thus, it doesn't give any factor in the group). This case has 8 unbroken supersymmetries.
- If $\phi_1+\phi_2=\phi_3+\phi_4=0(\text{mod}2\pi)$ then $\det\rho=1$ an thus, $\rho\in SU(2)\times SU(2)$. This case has 8 unbroken supersymmetries.
- If $\phi_1+\phi_2=\phi_3=\phi_4=0(\bmod 2\pi)$ then $\det\rho=1$ an thus, $\rho\in SU(2)$ ($\phi_3=\phi_4=0(\text{mod}2\pi)$ and thus, they don't give any factor in the group). This case has 8 unbroken supersymmetries.

We now write the boundary conditions in terms of the Z^a 's. In the unrotated brane version, the brane extends in the following directions;

$$(2,4,6,8)=\left(\operatorname{Re}\left(Z^1\right),\operatorname{Re}\left(Z^2\right),\operatorname{Re}\left(Z^3\right),\operatorname{Re}\left(Z^4\right)\right)$$

and thus the following is true;

$$X^3=\operatorname{Im}\left(Z^1\right)=0,X^5=\operatorname{Im}\left(Z^2\right)=0,X^7=\operatorname{Im}\left(Z^3\right)=0,X^9=\operatorname{Im}\left(Z^4\right)=0$$

The Neumann boundary conditions for directions parallel to the D-brane are thus, as follows:

$$\partial_1\operatorname{Re}\left(Z^1\right)=0,\quad\partial_1\operatorname{Re}\left(Z^2\right)=0,\quad\partial_1\operatorname{Re}\left(Z^3\right)=0,\quad\partial_1\operatorname{Re}\left(Z^4\right)=0$$

So, the boundary conditions are as follows:

$$\partial_1\operatorname{Re}\left(Z^a\right)=\operatorname{Im}\left(Z^a\right)=0,\quad a\in\{1,\ldots,4\}$$

In the rotated brane version, Z^a changes to $e^{\phi_a}Z^a$ and thus, if we undo these rotations i.e. if we consider $e^{-\phi_a}Z^a$, the boundary conditions then become the boundary conditions for the unrotated brane. So, the boundary conditions for the rotated brane are as follows:

$$\partial_1\operatorname{Re}\left(e^{-\phi_a}Z^a\right)=\operatorname{Im}\left(e^{-\phi_a}Z^a\right)=0,\quad a\in\{1,\ldots,4\}$$

If we let $\sigma^1=0$ endpoint on the unrotated brane and $\sigma^1=\pi$ endpoint on the rotated brane, then we have the following boundary conditions;

$$\sigma^1=0:\quad\partial_1\operatorname{Re}\left(Z^a\right)=\operatorname{Im}\left(Z^a\right)=0,\quad a\in\{1,\ldots,4\}$$

$$\sigma^1=\pi:\quad\partial_1\operatorname{Re}\left(e^{-\phi_a}Z^a\right)=\operatorname{Im}\left(e^{-\phi_a}Z^a\right)=0,\quad a\in\{1,\ldots,4\}$$

These boundary conditions give the following mode expansion. The doubling trick replaces the open string on the strip by a holomorphic field on the doubled surface, with monodromy $e^{2i\phi_a}$ after one circuit; hence the oscillator modes are shifted by $\nu_a=\phi_a/\pi$:

$$Z^a(w,\bar{w}) = \mathcal{Z}^a(w) + \overline{\mathcal{Z}^a}(\bar{w}) \text{ where } \mathcal{Z}^a(w) = i\sqrt{\frac{\alpha'}{2}}\sum_{\mathbb{Z}+\nu_a}\frac{\alpha^a_r}{r}e^{irw}$$

where $\nu_a = \phi_a/\pi$. It also gives the following partition function for a single Z scalar. The ground-state exponent is the regularized zero-point energy of two real bosons with fractional modes $m+\nu$ and $m+1-\nu$:

$$\mathcal{Z}_{Z,\text{ rotated}} = q^{E_0}\prod_{m=0}^{\infty}\left(1-q^{m+\phi/\pi}\right)^{-1}\left(1-q^{m+1-\phi/\pi}\right)^{-1} \qquad \text{where } E_0 = \frac{1}{24} - \frac{1}{2}\left(\frac{\phi}{\pi}-\frac{1}{2}\right)^2 \tag{13.4.4}$$

Note that if $\phi = 0$, then $E_0 = -1/12$ which one would expect for a Z scalar (because one Z contains two X 's and the partition function for one X has the power $q^{-1/24}$). We can write this partition function in terms of ϑ functions as well. For this purpose, consider the definition of $\vartheta_{11}(\nu,\tau)$ with $\tau = it$. We get the following;

$$\vartheta_{11}(\nu,it) = -2e^{-\pi t/4}\sin(\pi\nu)\prod_{m=1}^{\infty}\left(1-q^m\right)\left(1-zq^m\right)\left(1-z^{-1}q^m\right)$$

$$= -2e^{\pi it/4}q^{-1/24}\sin(\pi\nu)\eta(it)\prod_{m=1}^{\infty}\left(1-zq^m\right)\left(1-z^{-1}q^m\right) \qquad \text{where } q = e^{-2\pi t} \quad z = e^{2\pi i\nu}$$

To match this expression with the expression in (13.4.4), we should choose $z = q^{\phi/\pi}$ and thus, $\nu = i\phi t/\pi$. So, we consider the following;

$$\vartheta_{11}(i\phi t/\pi,it) = -2e^{-\pi t/4}q^{-1/24}\sin(i\phi t)\eta(it)\prod_{m=1}^{\infty}\left(1-q^{m+\phi/\pi}\right)\left(1-q^{m-\phi/\pi}\right)$$

$$= -2q^{1/8-1/24}\frac{1}{2i}\left(e^{-\phi t}-e^{\phi t}\right)\eta(it)\frac{1}{1-q^{\phi/\pi}}\prod_{m=0}^{\infty}\left(1-q^{m+\phi/\pi}\right)\left(1-q^{m+1-\phi/\pi}\right)$$

$$= -iq^{1/8-1/24-\phi/2\pi+E_0}\eta(it)\frac{1}{\mathcal{Z}_{Z\text{ rotated}}}$$

where in the last step, we used 13.4 .4 . Now, let's calculate the power of q that is appearing;

$$\frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+E_0 = \frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+\frac{1}{24}-\frac{1}{2}\left(\frac{\phi}{\pi}-\frac{1}{2}\right)^2 = \frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+\frac{1}{24}-\frac{\phi^2}{2\pi^2}-\frac{1}{8}+\frac{\phi}{2\pi} = -\frac{\phi^2}{2\pi^2}$$

and thus, the q factor can be written as follows:

$$q^{1/8-1/24-\phi/2\pi+E_0} = q^{-\phi^2/2\pi^2} = e^{\phi^2 t/\pi}$$

Therefore, we get;

$$\vartheta_{11}(i\phi t/\pi,it) = -ie^{\phi^2 t/\pi}\eta(it)\frac{1}{\mathcal{Z}_{Z\text{ rotated}}} \Rightarrow \mathcal{Z}_{Z\text{ rotated}} = \frac{-ie^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(i\phi t/\pi,it)} \tag{13.4.5}$$

Similarly, the fermionic partition function is obtained by the same shifted-mode product, with the spin-structure labels (α,β) determining the periodicity and the insertion of $(-1)^F$:

$$Z^{\alpha}_{\beta}(\phi,it) = \frac{\vartheta_{\alpha\beta}(i\phi t/\pi,it)}{\exp\left(\phi^2 t/\pi\right)\eta(it)} \tag{13.4.6}$$

Since we are talking about the open string only, we have only one fermionic sector and thus, right now, we only need the generalized $Z^+_\psi(it)$ right now. It is as follows:

$$Z^+(\phi,it) = \frac{1}{2}\left(\prod_{a=1}^{4}Z^0_0\left(\phi_a,it\right) - \prod_{a=1}^{4}Z^0_1\left(\phi_a,it\right) - \prod_{a=1}^{4}Z^1_0\left(\phi_a,it\right) - \prod_{a=1}^{4}Z^1_1\left(\phi_a,it\right)\right)$$

The following identity is the angled-brane generalization of Eq. (7.2.2):

$$\frac{1}{2}\left(\prod_{a=1}^{4} Z_0^0\left(\phi_a, it\right)-\prod_{a=1}^{4} Z_1^0\left(\phi_a, it\right)-\prod_{a=1}^{4} Z_0^1\left(\phi_a, it\right)-\prod_{a=1}^{4} Z_1^1\left(\phi_a, it\right)\right)=\prod_{a=1}^{4} Z_1^1\left(\phi_a^{\prime}, it\right)$$

where;

$$\begin{aligned}
& \phi_1^{\prime}=\frac{1}{2}\left(\phi_1+\phi_2+\phi_3+\phi_4\right), \phi_2^{\prime}=\frac{1}{2}\left(\phi_1+\phi_2-\phi_3-\phi_4\right) \\
& \phi_3^{\prime}=\frac{1}{2}\left(\phi_1-\phi_2+\phi_3-\phi_4\right), \phi_2^{\prime}=\frac{1}{2}\left(\phi_1-\phi_2-\phi_3+\phi_4\right)
\end{aligned}$$

Notice using previous analysis that if $\phi_1^{\prime}=0$, then there are only two unbroken supersymmetries but if $\phi_2^{\prime}, \phi_3^{\prime}$ or $\phi_4^{\prime}$ is zero, then there might be four or eight unbroken supersymmetries. So, we see that the total partition function coming from all bosonic and fermionic excitations is as follows:

$$\begin{aligned}
\prod_{a=1}^{4} \frac{-i e^{\phi_a^2 t / \pi} \eta(it)}{\vartheta_{11}\left(i \phi_a t / \pi, it\right)} \prod_{a=1}^{4} Z_1^1\left(\phi_a^{\prime}, it\right) & =\prod_{a=1}^{4} \frac{-i e^{\phi_a^2 t / \pi} \eta(it)}{\vartheta_{11}\left(i \phi_a t / \pi, it\right)} \prod_{a=1}^{4} \frac{\vartheta_{11}\left(i \phi_{\prime} t / \pi, it\right)}{\exp \left(\phi_a^{\prime 2} t / \pi\right) \eta(it)} \\
& =\prod_{a=1}^{4} \exp \left(\frac{t}{\pi}\left(\phi_a^2-\phi_a^{\prime 2}\right)\right) \prod_{a=1}^{4} \frac{\vartheta_{11}\left(i \phi_{\prime} t / \pi, it\right)}{\vartheta_{11}\left(i \phi_a t / \pi, it\right)}
\end{aligned}$$

The exponential prefactor is not discarded. It combines with the exponential factors in the product representation and modular properties of ϑ_{11}. In the standard branes-at-angles expression one uses the Riemann/Mumford theta identity to rewrite the GSO-summed fermion product in terms of the primed angles $\phi_a^{\prime}$. After this rewrite, the remaining Gaussian dependence in the open-string channel is the physical stretched-string factor

$$\exp \left[-\frac{t\, y^2}{2 \pi \alpha^{\prime}}\right], \tag{12.8}$$

while angle-dependent zero-point factors are encoded in the theta-function ratio. This is also why the parallel-brane limit must be taken with $\vartheta_{11}(\nu \mid \tau) \sim 2 \pi \nu\, \eta(\tau)^3$: the zero of ϑ_{11} is replaced by the translational zero mode along the common brane direction.

The potential is as follows:

$$V=-\int_0^{\infty} \frac{dt}{t}\left(8 \pi \alpha^{\prime} t\right)^{-1 / 2} \exp \left(-\frac{t y_1^2}{2 \pi \alpha^{\prime}}\right) \prod_{a=1}^{4} \frac{\vartheta_{11}\left(i \phi_a^{\prime} t / \pi, it\right)}{\vartheta_{11}\left(i \phi_a t / \pi, it\right)} \tag{13.4.7}$$

The dominant contribution comes when y_1 is minimum i.e. when strings are near the point of closest approach of the branes. The sign of the potential is not definite as the following quantity doesn't have a definite sign;

$$-\prod_{a=1}^{4} \frac{\sin \left(\phi_a^{\prime} t\right)}{\sin \left(\phi_a t\right)}$$

sin functions comes the definition of ϑ_{11}. Everything else in the integrand is positive definite. In the definition of $\vartheta_{11}(\nu, it)$, sending $\nu \rightarrow-\nu$ keeps the product part invariant but sends $\sin \pi \nu$ to $-\sin \pi \nu$. Thus, $\vartheta_{11}(\nu, it)$ is an odd function of ν and thus, it is zero at $\nu=0$. Now, we investigate the cases when ϕ_a 's or ϕ_a 's are zero because the ϑ_{11} functions vanish in these cases. If $\phi_a=0$ for some a, then the branes become parallel in some direction. For example, if $\phi_4=0$, then the branes become parallel in the 8th direction. The strings can now move in this direction. Since ϕ_4 is the rotation angle in the $8-9$ plane, the 8 and 9 directions become like directions when the branes are parallel with the branes extending in the 8th direction and the 9th direction being perpendicular to the brane. If we look at the partition function of the parallel branes for $p=4$, the contribution of the $8-9$ directions should be;

$$L \eta(it)^{-2}\left(8 \pi^2 \alpha^{\prime} t\right)^{-1 / 2}$$

where L is the length of the noncompact 8 th direction However, for $\phi_4 \neq 0$, the contribution of $8-9$ directions to the partition function is in 13.4.5. So, we need to do the following replacement;

$$
\begin{aligned}
\frac{-\left.ie^{\phi_4^2 t/\pi}\right|_{\phi_4=0}\eta(it)}{\vartheta_{11}\left(i\phi_4 t/\pi, it\right)} &= \frac{-i\eta(it)}{\vartheta_{11}\left(i\phi_4 t/\pi, it\right)} \\
&\to L\eta(it)^{-2}\left(8\pi^2\alpha' t\right)^{-1/2}, \\
\Rightarrow \quad \vartheta_{11}\left(i\phi_a t/\pi, it\right)^{-1} &\to iL\eta(it)^{-3}\left(8\pi^2\alpha' t\right)^{-1/2}.
\end{aligned}
$$

In order to generalize this picture to branes with some other number of dimensions, we use T-duality. If we T-dualize in the 8th direction, then the branes lose a dimension and we have 3-branes that are rotated in $2-3, 4-5$ and $6-7$ directions. These branes might have separations in the 8th and 9th directions. The fermion oscillator partition function is unaffected because T-duality changes a boundary condition but not the local fermion moding in the rotated planes. The 8th and 9th directions are now Dirichlet separation directions, so their contribution to the partition function is

$$
\eta(it)^{-2}\exp\left[-\frac{t\left(y_8^2+y_9^2\right)}{2\pi\alpha'}\right]
$$

and thus, we have to do the following replacement;

$$
\begin{aligned}
\frac{-\left.ie^{\phi_4^2 t/\pi}\right|_{\phi_4=0}\eta(it)}{\vartheta_{11}\left(i\phi_4 t/\pi, it\right)} = \frac{-i\eta(it)}{\vartheta_{11}\left(i\phi_4 t/\pi, it\right)} &\to \eta(it)^{-2}\exp\left[-\frac{t\left(y_8^2+y_9^2\right)}{2\pi\alpha'}\right] \\
\Rightarrow \vartheta_{11}\left(i\phi_a t/\pi, it\right)^{-1} \to i\eta(it)^{-3}\exp\left[-\frac{t\left(y_8^2+y_9^2\right)}{2\pi\alpha'}\right]&
\end{aligned}
$$

T-dualizing in the 9th direction causes the brane to extend in the 9th direction as well. This gives a factor of L_9, the length of the 9th direction, and a factor of $\left(8\pi^2\alpha' t\right)^{-1/2}$ from the zero-mode momentum integral along a Neumann direction. There is no additional η-factor because the oscillator contribution of that coordinate was already present before T-duality; only its zero-mode measure changes. Thus T-duality gives the potential for branes of different dimensions. A "rotated" pair of different-dimensional branes means that, after T-dualizing to a common reference frame, their Neumann subspaces meet at the angles appearing in the oscillator shifts.

The potential in Eq. (13.4.7) vanishes if one of the angles ϕ_a vanishes. The phases in Eq. (13.4.3), labelled by $(2s_1, 2s_2, 2s_3, 2s_4)$, then determine the remaining supersymmetry. If $\phi_1' = 0$, the phases $(\pm1, \pm1, \pm1, \pm1)$ are unity and two supersymmetries remain unbroken. If $\phi_2' = 0$, the phases $(\pm1, \pm1, \mp1, \mp1)$ give the same result. For $\phi_3' = 0$ and $\phi_4' = 0$, the relevant phases are $(\pm1, \mp1, \pm1, \mp1)$ and $(\pm1, \mp1, \mp1, \pm1)$, respectively, again leaving two unbroken supersymmetries. These cases account for eight of the sixteen possible phase assignments. The remaining assignments have an odd number of minus signs; when one of them is unity, the sign-reversed assignment is also unity, so two supersymmetries survive. In these cases none of the ϕ_a' vanish, and the potential in Eq. (13.4.7) is nonzero. The large-separation and tachyonic limits are then extracted by expanding the annulus integrand in the appropriate closed- and open-string channels.

12.5 D-brane interactions: dynamics

The previous section determined which brane configurations preserve supersymmetry; this section studies their relative motion by analytic continuation. A brane moving with constant velocity $v = \tanh u$ is equivalent, in the worldsheet calculation, to a brane rotated by an imaginary angle, $\phi \to -iu$, so the branes-at-angles partition functions and supersymmetry projections of the previous section apply directly to scattering. The annulus amplitude gives two results. First, the leading velocity dependence of the interaction between identical branes starts at v^4/r^{7-p}: the static (v^0) force cancels by supersymmetry, and the absence of a v^2 term states that the moduli-space metric of a BPS system is flat at this order. Second, the range over which the amplitude varies shows that slowly moving D-branes probe distances $r \sim \ell_s\sqrt{v}$, below the string length. Combining this with the quantum spreading of a D0-brane wavepacket gives the lower bound

$\delta x \gtrsim g^{1/3}\ell_s$, the eleven-dimensional Planck length that appears again in the strong-coupling analysis of Chapter 13 [351, 352].

12.5.1 D-brane scattering

The relative motion of D Branes is described by analytically continuing the angle variable to the rapidity variable. If only ϕ_1 is nonzero so that the rotation is in the $X^2 - X^3$ plane, the following equation is satisfied;

$$X^3 = \tan\phi_1 X^2$$

we define;

$$X^3 \to iX^0, \quad \phi_1 = -iu$$

where u is the rapidity. So, the following is satisfied;

$$X^3 = \tan(-iu)\left(iX^0\right) = -i\tanh(u)\left(iX^0\right) = \tanh(u)X^0$$

The interaction amplitude for general p is obtained from the branes-at-angles partition function by the analytic continuation $\phi_1 = -iu$ and by including the zero-mode volume of the common p-dimensional spatial directions:

$$\begin{aligned}\mathcal{A} = -iV_p \int_0^\infty \frac{dt}{t}\left(8\pi^2\alpha' t\right)^{-p/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \\ \times \frac{\vartheta_{11}(ut/2\pi, it)^4}{\eta(it)^9\vartheta_{11}(ut/\pi, it)}\end{aligned} \tag{13.5.1}$$

Using modular transformation, we have;

$$\begin{aligned}\vartheta_{11}\left(\frac{ut}{2\pi}, it\right) &= it^{-1/2}e^{-u^2t^2/4\pi}\vartheta_{11}\left(-\frac{iu}{2\pi}, \frac{i}{t}\right) \\ &= -it^{-1/2}e^{-u^2t^2/4\pi}\vartheta_{11}\left(\frac{iu}{2\pi}, \frac{i}{t}\right).\end{aligned}$$

Moreover,

$$\vartheta_{11}\left(\frac{ut}{\pi}, it\right) = -it^{-1/2}e^{-u^2t^2/\pi}\vartheta_{11}\left(\frac{iu}{\pi}, \frac{i}{t}\right).$$

and,

$$\eta(it) = t^{-1/2}\eta\left(\frac{i}{t}\right)$$

Using these transformed quantities, we can write the amplitude as follows:

$$\begin{aligned}\mathcal{A} = -iV_p \int_0^\infty \frac{dt}{t}\left(8\pi^2\alpha' t\right)^{-p/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) t^{-2} \\ \times \frac{e^{-u^2t^2/\pi}\vartheta_{11}(iu/2\pi, i/t)^4}{t^{-9/2}\eta(i/t)^9(-i)t^{-1/2}e^{-u^2t^2/\pi}\vartheta_{11}(iu/\pi, i/t)} \\ = \frac{V_p}{(8\pi^2\alpha')^{p/2}}\int_0^\infty \frac{dt}{t}t^{(6-p)/2}\exp\left(-\frac{ty^2}{2\pi\alpha'}\right)\frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9\vartheta_{11}(iu/\pi, i/t)}.\end{aligned}$$

We can write this amplitude as an integral over a worldline. For that purpose, we manipulate this amplitude as follows:

$$
\begin{aligned}
\mathcal{A} &= \frac{V_p}{(8\pi^2\alpha')^{p/2}} \int_0^\infty \frac{dt}{t}\, t^{(6-p)/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \\
&= \frac{2\pi\sqrt{2\alpha'}\, V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{1/2}}\, t^{(5-p)/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \\
&= \frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \\
&\quad \times \left[\exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\pi}{\tanh u} \sqrt{\frac{2\alpha'}{t}}\right].
\end{aligned}
$$

Now, the factor in the last parenthesis can be written as follows:

$$
\begin{aligned}
&\exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\pi}{\tanh u} \sqrt{\frac{2\alpha'}{t}} \\
&= \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \int_{-\infty}^\infty d\tau\, \exp\left(-\frac{t\tau^2 \tanh^2 u}{2\pi\alpha'}\right) \\
&= \int_{-\infty}^\infty d\tau\, \exp\left(-\frac{t(y^2 + \tau^2 \tanh^2 u)}{2\pi\alpha'}\right) \\
&= \int_{-\infty}^\infty d\tau\, \exp\left(-\frac{t(y^2 + \tau^2 v^2)}{2\pi\alpha'}\right) = \int_{-\infty}^\infty d\tau\, \exp\left(-\frac{tr^2}{2\pi\alpha'}\right).
\end{aligned}
$$

where

$$
v = \tanh u, \quad r^2(\tau) = y^2 + v^2\tau^2
$$

So, we get;

$$
\begin{aligned}
\mathcal{A} &= \frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \\
&\quad \times \int_{-\infty}^\infty d\tau\, \exp\left(-\frac{tr^2}{2\pi\alpha'}\right) \\
&= -i \int_{-\infty}^\infty d\tau\, \frac{2iV_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \\
&\quad \times \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \exp\left(-\frac{tr^2}{2\pi\alpha'}\right). \\
&= -i \int_{-\infty}^\infty d\tau V(r,\tau)
\end{aligned}
$$

where

$$
V(r,\tau) = i\frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \exp\left(-\frac{tr^2}{2\pi\alpha'}\right) \tag{13.5.2}
$$

Now, the expansion of $\vartheta_{11}(\nu, \tau)$ is done as follows:

$$
\begin{aligned}
\vartheta_{11}(\nu, \tau) &= -2e^{\pi i\tau/4} \sin(\pi\nu) \prod_{m=1}^\infty (1-q^m)(1-zq^m)(1-z^{-1}q^m) \\
&= -2e^{\pi i\tau/4} \left(\pi\nu + \frac{\pi^3\nu^3}{6} + \ldots\right) \prod_{m=1}^\infty (1-q^m) \\
&\quad \times (1 - q^m - 2\pi i\nu q^m + \ldots)(1 - q^m + 2\pi i\nu q^m + \ldots)
\end{aligned}
$$

$$= -2e^{\pi i\tau/4}\nu\pi\left(1+\frac{\pi^2\nu^2}{6}+\mathcal{O}(\nu^2)\right)\prod_{m=1}^{\infty}(1-q^m)$$

$$\times\prod_{m=1}^{\infty}\left[(1-q^m)^2+(1-q^m)2\pi i\nu q^m-(1-q^m)2\pi i\nu q^m+\mathcal{O}(\nu^2)\right]$$

$$= -2\nu\pi e^{\pi i\tau/4}q^{-1/8}\eta(\tau)^3+\mathcal{O}(\nu^2) = -2\pi\nu\eta(\tau)^3+\mathcal{O}(\nu^2).$$

The result implies;

$$\vartheta_{11}\left(\frac{iu}{2\pi},\frac{i}{t}\right) = -2\pi\eta(i/t)^3\frac{iu}{2\pi}+\mathcal{O}\left(u^2\right) = -iu\eta(i/t)^3+\mathcal{O}\left(u^2\right)$$

Similarly, we have;

$$\vartheta_{11}\left(\frac{iu}{\pi},\frac{i}{t}\right) = -2\pi\eta(i/t)^3\frac{iu}{\pi}+\mathcal{O}\left(u^2\right) = -2iu\eta(i/t)^3+\mathcal{O}\left(u^2\right)$$

Since $v=\tanh u = u+\mathcal{O}\left(u^3\right)$, we have $v\sim u$ for small u. So, we have;

$$\vartheta_{11}\left(\frac{iu}{2\pi},\frac{i}{t}\right) = -iv\eta(i/t)^3+\mathcal{O}\left(v^2\right)$$

Similarly, we have;

$$\vartheta_{11}\left(\frac{iu}{\pi},\frac{i}{t}\right) = -2iv\eta(i/t)^3+\mathcal{O}\left(v^2\right)$$

Using these expansions, 13.5.2 becomes;

$$V(r,v) = -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt t^{(5-p)/2}e^{-tr^2/2\pi\alpha'}+\mathcal{O}\left(v^6\right)$$

$$= -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{r^2}\right)^{(7-p)/2}\int_0^\infty d\mu\mu^{(5-p)/2}e^{-\mu}+\mathcal{O}\left(v^6\right)$$

$$= -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{r^2}\right)^{\frac{7-p}{2}}\Gamma\left(\frac{7-p}{2}\right)+\mathcal{O}\left(v^6\right) = -\frac{v^4}{r^{7-p}}\frac{V_p}{\alpha'^{p-3}}2^{2-2p}\pi^{(5-3p)/2}\Gamma\left(\frac{7-p}{2}\right)+\mathcal{O}\left(v^6\right) \tag{13.5.3}$$

Using these results, we see that the leading v term in 13.5.2 is proportional to v^4. The v^0 term cancels because the static branes are BPS, and the v^2 term cancels by the same theta-function identity that cancels the force between parallel supersymmetric branes; the first nonzero term is therefore v^4. Now, we investigate the small r limit. Since we have $-tr^2/2\pi\alpha'$ in the exponential in (13.5.2), the dominant contribution comes when $t\sim 2\pi\alpha'/r^2$. So, small r corresponds to large t, i.e. to light open strings stretched between the branes. To see the small r behavior, we expand (13.5.2) at large t, keeping the full dependence on ut. We get;

$$V(\nu,\tau)\sim -2V_p\int_0^\infty\frac{dt}{(8\pi\alpha' t)^{(p+1)/2}}\exp\left(-\frac{tr^2}{2\pi\alpha'}\right)\frac{\tanh u\sin^4 ut/2}{\sin ut}$$

As we saw in 13.5.3, the structure of the v^4 term is v^4/r^{7-p}. The higher terms have higher powers of v accompanied by stronger inverse powers of r, because each additional power of ut in the large-t integrand produces an additional power of $1/r^2$ after the Schwinger integral. So, no matter how small v is, at small enough r, the dominant term will not be the v^4 term. The apparent singularity is smoothed when $ut\approx 1$, because the trigonometric factors in the exact expression regulate the expansion parameter. Since $u\approx v$ for small v, and since the dominant contribution comes when $t\approx 2\pi\alpha'/r^2$, we have;

$$ut\sim 1\Rightarrow\frac{2\pi\alpha' v}{r^2}\sim\frac{\alpha' v}{r^2}\sim 1\Rightarrow r\sim\sqrt{\alpha' v}=l_s\sqrt{v} \tag{13.5.4}$$

Thus a slow-moving D-brane can probe distances smaller than the string length l_s. The scattering time is

$$\delta t \sim \frac{r}{v} \sim \sqrt{\frac{\alpha'}{v}}$$

Denote the uncertainty in distance by δx. For a D-brane with velocity v, it is of order $\sqrt{\alpha' v}$ as found above. Therefore

$$\delta x \delta t \sim \sqrt{\frac{\alpha'}{v}}\sqrt{\alpha' v} = \alpha'$$

In general, the uncertainty cannot be smaller than δx, so the value of $\delta x\, \delta t$ obtained above is a lower bound. Hence

$$\delta x \delta t \geq \alpha' \tag{13.5.5}$$

This uncertainty relation between coordinates hints at non-commutative geometry.
We can find a minimum distance that a $D0$ brane can probe. Since it is a particle, the wavepacket that minimizes the uncertainty in the Heisenberg uncertainty relation is the Gaussian wavepacket. For the gaussian wavepacket, we have the following;

$$\delta x \delta p \sim \hbar = 1 \Rightarrow \delta x \delta p \sim 1$$

But, for small v, we have $\delta p = m\delta v$ and thus, we have;

$$m\delta x \delta v \sim 1 \Rightarrow \delta x \sim \frac{1}{m\delta v}$$

Now, we use 13.3.11 with $p = 0$ to get the mass of the D0 brane. We get;

$$m = \tau_0 = \alpha'^{-1/2} g^{-1} \Rightarrow \delta x \sim \frac{g\sqrt{\alpha'}}{\delta v}$$

So, the uncertainty relation for the $D0$ brane should be;

$$\delta x \delta v \geq g\sqrt{\alpha'} \tag{13.5.6}$$

For an order-of-magnitude probe estimate at small velocity, take $\delta v \sim v$. Then

$$\delta x \geq \frac{g\sqrt{\alpha'}}{v}$$

Moreover, we have an inequality from 13.5.4 which reads;

$$\delta x \geq \sqrt{\alpha' v}$$

Imposing these two inequalities gives us a region and the lowest possible value of δx in this region is found when we have;

$$\frac{g\sqrt{\alpha'}}{v} = \sqrt{\alpha' v} \Rightarrow v^{3/2} = g \Rightarrow v = g^{2/3}$$

Using this value of v in the inequality corresponding to 13.5.4, we get;

$$\delta x \geq \alpha'^{1/2} g^{1/3} \tag{13.5.7}$$

12.5.2 D0-brane quantum mechanics

The intended subject here is the quantum mechanics of slowly moving D0-branes. For N coincident D0-branes the low-energy open-string degrees of freedom are the dimensional reduction of ten-dimensional super Yang–Mills theory to one time dimension. The nine transverse coordinates become Hermitian matrices $X^i(t)$, and the center-of-mass motion is the trace part while the relative positions and strings stretched between the branes live in the non-abelian part. Schematically,

$$S_{D0} = \frac{1}{2g_s\ell_s}\int dt\, \mathrm{Tr}\left[(D_t X^i)^2 + \frac{1}{2(2\pi\alpha')^2}[X^i, X^j]^2 + \text{fermions}\right], \qquad i = 1, \ldots, 9. \tag{12.9}$$

The commuting branch $[X^i, X^j] = 0$ describes separated D0-branes, while non-commuting matrices describe genuinely stringy bound configurations. At strong coupling, the D0-branes are described by matrix degrees of freedom that carry eleven-dimensional momentum. [173, 177].

12.5.3 The $\#_{ND} = 4$ system

The $\#_{ND} = 4$ case is special because the open-string zero-point energy in the NS sector is shifted to zero. The zero-point energy shift can be written as

$$E_0^{\mathrm{NS}} = -\frac{1}{2} + \frac{\#_{ND}}{8}, \tag{12.10}$$

so $\#_{ND} = 4$ gives massless bosons instead of a tachyon. The local CFT computation therefore explains the earlier spinor projection result: the brane intersection supports a supersymmetric hypermultiplet instead of an instability. Boundary conditions, zero-point energies, and preserved supercharges give the same conclusion. [38, 107].

12.6 D-brane interactions: bound states

This section asks when a collection of branes and strings forms a bound state, and the supersymmetry algebra makes the question quantitative. For each pair of charges the anticommutator of supercharges, Eq. (13.2.3), bounds the mass by a function of the charges; a configuration is a genuine bound state when its mass lies strictly below the sum of the constituents' masses, and it is a *BPS* bound state when it saturates the algebra's bound, in which case its mass is exact and can be followed to strong coupling. The cases worked out below span the possibilities: the F1–D1 system binds with the square-root tension law $\sqrt{p^2 + q^2/g^2}$, strictly below $p + q/g$, and these (p, q) strings become the $SL(2, \mathbb{Z})$ multiplet of Chapter 13; D0–D0 binds only at threshold, where the existence of the bound state is a hard question about matrix quantum mechanics that M-theory answers by requiring one graviton state for every N; D0–D4 preserves supersymmetry ($\#_{ND} = 4$) with the D0 realized as a gauge instanton on the D4; and D0–D2, D0–D6, D0–D8 illustrate that charge can be carried by worldvolume flux even when the naive open-string system is non-supersymmetric. The DBI–Chern–Simons couplings quoted in Eq. (12.11) below tracks which lower brane charges a flux configuration induces.

$$S_{Dp} = -T_p \int d^{p+1}\xi\, e^{-\Phi} \sqrt{-\det\!\big(P[G+B]_{ab} + 2\pi\alpha' F_{ab}\big)} + \mu_p \int P\left[\sum_q C_q\right] e^{B+2\pi\alpha' F}. \tag{12.11}$$

[38, 351, 352].

12.6.1 F-string–D-string bound states

We start with p F strings and qD strings. We assume that all of them are extended in the 1 direction and are in rest. In this case, we have $P_M = -M\delta_{M,0}$, $Q_M^{NS} = pL_1\delta_{M,1}$ and $Q_M^R = qL_1\delta_{M,1}$ where L_1 is the length of the system. The susy algebra in 13.2.1 to 13.2.2 becomes;

$$\begin{gathered}
\left\{Q_\alpha, Q_\beta^\dagger\right\} = -2M\left(\Gamma^0\right)^2_{\alpha\beta} + \frac{2pL_1}{2\pi\alpha'}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta} = 2M\delta_{\alpha\beta} + \frac{2pL_1}{2\pi\alpha'}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta} \\
\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta^\dagger\right\} = -2M\left(\Gamma^0\right)^2_{\alpha\beta} - \frac{2pL_1}{2\pi\alpha'}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta} = 2M\delta_{\alpha\beta} - \frac{2pL_1}{2\pi\alpha'}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta} \\
\left\{Q_\alpha, \tilde{Q}_\beta^\dagger\right\} = 2\frac{\tau_1}{1!}L_1 q\left(\beta^1\Gamma^0\right)_{\alpha\beta} = 2\frac{L_1 q}{2\pi\alpha' g}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta}.
\end{gathered}$$

The last equality uses the spatial convention $\beta^1 = -\Gamma^1$ together with $\{\Gamma^0, \Gamma^1\} = 0$, so $\beta^1\Gamma^0 = -\Gamma^1\Gamma^0 = \Gamma^0\Gamma^1$. The color marks the two spinor slots; the 1-direction is the common F/D-string direction.

These results can be summarized as follows:

$$\frac{1}{2}\left\{\begin{pmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{pmatrix}, \begin{pmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{pmatrix}\right\} = M\delta_{\alpha\beta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{L_1}{2\pi\alpha'}\left(\Gamma^0\Gamma^1\right)_{\alpha\beta}\begin{pmatrix} p & q/g \\ q/g & -p \end{pmatrix}. \tag{13.6.1}$$

The left side of the equation is non-negative because for any pair of test spinors v_α and $\tilde{v}_\alpha$, its expectation value is a sum of norms of linear combinations of Q and $\tilde{Q}$. Therefore, the eigenvalues of the right side must also be non-negative. We now notice that;

$$S_0 = \Gamma^{0+}\Gamma^{0-} - \frac{1}{2} = \frac{1}{4}\left(\Gamma^0 + \Gamma^1\right)\left(-\Gamma^0 + \Gamma^1\right) - \frac{1}{2} = \frac{1}{2}\Gamma^0\Gamma^1.$$

Since we know that the eigenvalues of S_0 are $\pm 1/2$ and thus, the eigenvalues of $\Gamma^0\Gamma^1$ are ± 1. Therefore, the eigenvalues of the right-hand side of 13.6.1 are as follows:

$$M \pm \frac{L_1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g^2}}.$$

The reason this diagonalizes so simply is that $\Gamma^0\Gamma^1$ acts on the spinor indices while the 2×2 matrix acts only on the $(Q, \tilde{Q})$ doublet. These two factors commute, $(\Gamma^0\Gamma^1)^2 = 1$, and the doublet matrix has eigenvalues $\pm\sqrt{p^2 + q^2/g^2}$. Since the anticommutator is positive, the negative-sign eigenvalue gives the BPS bound

$$\frac{M}{L_1} \geq \frac{1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g^2}}. \tag{13.6.2}$$

For a separated collection of p F-strings and q D-strings the mass per unit length is additive,

$$T_{\text{separated}} = \frac{|p|}{2\pi\alpha'} + \frac{|q|}{2\pi\alpha' g}.$$

The supersymmetric (p, q) string is instead a bound state that saturates Eq. (13.6.2):

$$T_{(p,q)} = \frac{1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g^2}}.$$

This is the same square-root formula derived from the type-IIB $SL(2,\mathbb{Z})$ tension formula in Chapter 13.

We have the following cases;

$$M(1,0) = \frac{1}{2\pi\alpha'} \geq \frac{1}{2\pi\alpha'}$$

$$M(0,1) = \frac{1}{2\pi\alpha' g} \geq \frac{1}{2\pi\alpha' g}$$

$$T_{\text{separated}}(1,1) = \frac{1 + g^{-1}}{2\pi\alpha'} \geq T_{(1,1)}$$

The separated $(1,1)$ configuration exceeds the BPS bound, while the true $(1,1)$ bound state saturates it at any value of g where the BPS description is valid.

The protected strong-coupling quantities can now be tied to the duality checks above.

12.6.2 The D0–Dp BPS bound

We again derive the BPS bound for this system. Let Dp brane be extending in $(1, \ldots, p)$ directions. In the $D0$ and Dp system, $Q^{NS} = 0$ and there are two Ramond charges. We thus have the following susy algebra;

$$\left\{Q_\alpha, Q_\beta^\dagger\right\} = -2M\left(\Gamma^0\right)^2_{\alpha\beta} = 2M\delta_{\alpha\beta}$$
$$\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta^\dagger\right\} = -2M\left(\Gamma^0\right)^2_{\alpha\beta} = 2M\delta_{\alpha\beta}$$
$$\left\{Q_\alpha, \tilde{Q}_\beta^\dagger\right\} = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \frac{\tau_p}{p!}Q^R_{M_1\ldots M_p}\left(\beta^{M_1}\ldots\beta^{M_p}\Gamma^0\right)_{\alpha\beta}\right) = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \tau_p Q^R_{1\ldots p}\left(\beta\Gamma^0\right)_{\alpha\beta}\right)$$

where the last step uses the fact that the wrapped brane has support only along the ordered spatial directions $1,\ldots,p$, so the antisymmetric charge tensor contributes through its $Q^R_{1\ldots p}$ component. Moreover, $\beta = \beta^1\cdots\beta^p$. Wrapping the D$p$-brane on a p-torus of volume V_p gives $Q^R_{1,\ldots,p} = V_p$ in these conventions. Therefore

$$\left\{Q_\alpha, \tilde{Q}_\beta^\dagger\right\} = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\beta\Gamma^0\right)_{\alpha\beta}\right) = 2Z_{\alpha\gamma}\Gamma^0_{\gamma\beta}$$

where $Z_{\alpha\gamma} = \tau_0\delta_{\alpha\gamma} + \tau_p V_p\beta_{\alpha\gamma}$. Hermitian conjugation of the preceding mixed anticommutator gives

$$\left\{\tilde{Q}_\alpha, Q_\beta^\dagger\right\} = -2Z^\dagger_{\alpha\gamma}\Gamma^0_{\gamma\beta}$$

The susy algebra can be written as follows:

$$\frac{1}{2}\left\{\begin{pmatrix} Q_\alpha \\ \tilde{Q}_\alpha\end{pmatrix}, \begin{pmatrix} Q_\beta^\dagger & \tilde{Q}_\beta^\dagger\end{pmatrix}\right\} = M\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta_{\alpha\beta} + \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0\end{pmatrix}\Gamma^0_{\gamma\beta} \tag{13.6.3}$$

The eigenvalues of the right-hand side should again be non-negative and thus, we get;

$$M\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta_{\alpha\beta} + \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0\end{pmatrix}\Gamma^0_{\gamma\beta} \geq 0 \Rightarrow M\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta_{\alpha\beta} \geq -\begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0\end{pmatrix}\Gamma^0_{\gamma\beta}$$
$$\Rightarrow M^2\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta_{\alpha\beta} \geq \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0\end{pmatrix}\Gamma^0_{\gamma\sigma}\begin{pmatrix} 0 & Z_{\sigma\lambda} \\ -Z^\dagger_{\sigma\lambda} & 0\end{pmatrix}\Gamma^0_{\lambda\beta}$$
$$= \begin{pmatrix} -Z_{\alpha\gamma}\Gamma^0_{\gamma\sigma}Z^\dagger_{\sigma\lambda}\Gamma^0_{\lambda\beta} & 0 \\ 0 & -Z^\dagger_{\alpha\gamma}\Gamma^0_{\gamma\sigma}Z_{\sigma\lambda}\Gamma^0_{\lambda\beta}\end{pmatrix} \tag{13.6.4}$$

Now, we do the following manipulation;

$$\begin{aligned}\left(Z^\dagger\Gamma^0\right)_{\alpha\beta} =& \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\beta^\dagger\Gamma^0\right)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\beta^{\dagger p}\ldots\beta^{\dagger 1}\Gamma^0\right)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\beta^p\ldots\beta^1\Gamma^0\right)_{\alpha\beta} \\ &= \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\Gamma^0\beta^p\ldots\beta^1\right)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p\left(\Gamma^0\beta^{\dagger p}\ldots\beta^{\dagger 1}\right)_{\alpha\beta} = \left(\Gamma^0 Z^\dagger\right)_{\alpha\beta}\end{aligned} \tag{13.6.5}$$

using the following identities, valid with the present gamma-matrix conventions:

$$\beta^{\dagger m} = (\Gamma\Gamma^m)^\dagger = \Gamma^{m\dagger}\Gamma^\dagger = \Gamma^0\Gamma^m\Gamma^0\Gamma^0 = \Gamma\Gamma^m = \beta^m \tag{13.6.6}$$
$$\beta^m\Gamma^0 = \Gamma\Gamma^m\Gamma^0 = \Gamma^0\Gamma\Gamma^m = \Gamma^0\beta^m \tag{13.6.7}$$

Similarly, we can also derive the following;

$$\left(Z\Gamma^0\right)_{\alpha\beta} = \left(\Gamma^0 Z\right)_{\alpha\beta} \tag{13.6.8}$$

Using 13.6.5 and 13.6.8 in 13.6.4, we get;

$$M^2\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta_{\alpha\beta} \geq \begin{pmatrix} \left(ZZ^\dagger\right)_{\alpha\beta} & 0 \\ 0 & \left(Z^\dagger Z\right)_{\alpha\beta}\end{pmatrix}$$

where we can easily see that;

$$Z = \tau_0 + \tau_p V_p\beta, \quad Z^\dagger = \tau_0 + \tau_p V_p\beta^\dagger \Rightarrow ZZ^\dagger = \tau_0^2 + \tau_0\tau_p V_p\left(\beta + \beta^\dagger\right) + \tau_p^2 V_p^2\beta\beta^\dagger \tag{13.6.9}$$

Now, we notice the following result;

$$\beta^\dagger = \left(\beta^1 \dots \beta^p\right)^\dagger = \beta^{\dagger p} \dots \beta^{\dagger 1} = \beta^p \dots \beta^1 = (-1)^{p(p-1)/2} \beta^1 \dots \beta^p \tag{13.6.10}$$

where the last step uses

$$\beta^m \beta^n = \Gamma\Gamma^m \Gamma\Gamma^n = -\Gamma\Gamma^m \Gamma\Gamma^n = -\beta^n \beta^m \text{ for } m \neq n$$

and I also used the following fact;

$$1 + 2 + \dots + (p-1) = \frac{p(p-1)}{2}$$

From 13.6.10, we can see that β is hermitian is p is a multiple of 4 . Moreover, we also note that;

$$(\beta^m)^2 = \Gamma\Gamma^m\Gamma\Gamma^m = -\Gamma\Gamma^m\Gamma^m\Gamma = -1 \Rightarrow \beta^2 = \beta^1 \dots \beta^p \beta^1 \dots \beta^p = (-1)^{p(p-1)/2} \beta^p \dots \beta^1 \beta^1 \dots \beta^p$$
$$= (-1)^{p(p-1)/2}(-1)^p = (-1)^{p(p+1)/2}$$

We see that if p is a multiple of 4, then $\beta^2 = 1$. So, we see that for p a multiple of $4, 13.6.9$ becomes;

$$ZZ^\dagger = Z^\dagger Z = \tau_0^2 + 2\tau_0\tau_p V_p \beta + \tau_p^2 V_P^2 = (\tau_0 + \tau_p V_p \beta)^2$$

The eigenvalues of β are ± 1. For the supersymmetry preserved by the bound state, the relevant projector takes the $+1$ eigenspace, and the BPS bound is

$$M \geq \tau_0 + \tau_p V_p$$

We note that a $D0$ and Dp state saturates this BPS bound and thus, some supersymmetry is unbroken. To make sense of this result, we note that for a $D0 - Dp$ system, the only Dirichlet directions can be the p directions on the Dp brane and all of these directions have to be Neumann directions on the $D0$ brane. So, in this case, $\#_{ND} = p$. We saw that if $\#_{ND}$ is a multiple of 4, then some supersymmetry is unbroken. This is exactly what we are seeing here i.e. for p a multiple of 4, the BPS bound is saturated.
Similarly, we can easily see that if p is even but not a multiple of 4 (i.e. if $p = 4k + 2$), then we have;

$$\beta^\dagger = -\beta, \quad \beta^2 = -1$$

In this case, 13.6.9 becomes;

$$ZZ^\dagger = Z^\dagger Z = \tau_0^2 + \tau_p^2 V_p^2$$

and thus, the BPS bound is easily seen to be;

$$M^2 \geq \tau_0^2 + \tau_p^2 V_p^2 \Rightarrow M \geq \sqrt{\tau_0^2 + \tau_p^2 V_p^2}$$

This bound is not saturated by a $D0 - Dp$ system and it is consistent with the fact that for $\#_{ND} = 4k + 2$, the system is not supersymmetric.

12.6.3 D0–D0 bound states

For N D0-branes the BPS mass is additive,

$$M_{ND0} = N\tau_0 = \frac{N}{g_s\sqrt{\alpha'}}. \tag{12.12}$$

The formula follows directly from the D-brane tension, but the real question is whether the interacting quantum mechanics has a normalizable threshold bound state after the free center of mass is removed. The low-energy theory is the dimensional reduction of ten-dimensional $U(N)$ super-Yang–Mills to one time dimension. Writing the nine adjoint scalars as matrices $X^i(t)$, the bosonic Hamiltonian has the schematic form

$$H_{\text{bos}} = \frac{1}{2g_{\text{QM}}^2} \text{Tr}(D_t X^i)^2 + \frac{1}{4g_{\text{QM}}^2 (2\pi\alpha')^2} \text{Tr}\,[X^i, X^j]^2. \tag{12.13}$$

The trace part is the free center of mass. The $SU(N)$ part controls relative separations and off-diagonal strings. If X^i are simultaneously diagonal, the eigenvalues are the positions of separated D0-branes. When two eigenvalues approach each other, off-diagonal strings become light and the interaction cannot be ignored. Hence the bound-state statement is a statement about the supersymmetric ground state of the $SU(N)$ matrix quantum mechanics, not about adding classical point masses. This is the same protected object that appears as a Kaluza–Klein momentum state after the type-IIA/M-theory lift [177, 353, 364].

Derivation. The BPS interpretation is fixed by the eleven-dimensional lift. Type-IIA D0-branes carry one unit of momentum around the M-theory circle, so

$$P_{11} = \frac{N}{R_{11}}, \qquad R_{11} = g_s \ell_s. \tag{12.14}$$

This gives $M = N/R_{11} = N/(g_s\ell_s) = N/(g_s\sqrt{\alpha'})$, exactly Eq. (12.12). A normalizable threshold bound state is required because eleven-dimensional supergravity contains one particle state for each integer unit of Kaluza–Klein momentum, instead of a continuum of unrelated multi-particle fragments.

Closed-string emission from a D0–open-string bound configuration.

There is a useful way to say what it means for an open string stretched between D0-branes to radiate a closed string. Classically, an open string can split when its profile self-intersects; after the splitting, the outgoing open-string pieces carry kinks where the joining point used to be. In the D0-brane problem the endpoints are not free in space: they obey Dirichlet conditions in the spatial directions and remain attached to the D0-branes. The D0-branes therefore specify the boundary state of the open string, while the emitted closed string is an ordinary bulk excitation of the same worldsheet theory.

Quantum mechanically, the inclusive decay probability is most cleanly read from unitarity. One computes the forward two-point amplitude of the initial open-string state on the first surface that can contain closed-string emission and reabsorption. Topologically this surface has one boundary and one handle,

$$g = 1, \qquad b = 1, \qquad \chi = 2 - 2g - b = -1. \tag{12.15}$$

It may be pictured as a disk boundary carrying two open-string vertex operators, V and $\bar{V}$, with a bulk handle through which the closed-string intermediate states propagate. If the resulting Lorentz-invariant two-point amplitude is denoted by $\mathcal{M}_2$, the optical theorem gives the total decay rate in the rest frame of an initial open-string state of mass M:

$$\Gamma_{\text{inclusive}} = \frac{1}{M}\,\mathrm{Im}\,\mathcal{M}_2. \tag{12.16}$$

The imaginary part is important. It isolates the physical cuts of the amplitude, namely the channels in which real closed strings and the final open-string state can go on shell. Thus the decay calculation is not a separate rule added to string theory; it is the same factorization and unitarity logic used earlier for ordinary string amplitudes.

One can also ask a less inclusive question: what is the probability to emit one specified closed-string state, for example a graviton? Then it is enough to study a disk amplitude with open-string vertices on the boundary and closed-string vertices inserted in the bulk. The relevant channel is the one in which the closed-string state goes on shell, and the residue of that pole gives the transition amplitude for

$$\text{open string on D0-branes} \longrightarrow \text{open string on D0-branes} + \text{closed string}. \tag{12.17}$$

If only massless closed strings are wanted, one keeps only the massless closed-string multiplet in the intermediate-state sum. The D0-branes mainly change the boundary conditions and the open-string spectrum; the logic of emission remains open–closed factorization on a worldsheet with boundary. This is the same mechanism behind the statement that an interacting open-string theory cannot be consistently separated from the closed-string sector.

12.6.4 D0–D2 bound states

A pointlike D0 placed near a flat D2 has $\#_{ND} = 2$, so the bare two-brane open-string system is not the BPS object of interest. The BPS description is a D2-brane carrying magnetic worldvolume flux. The Chern–Simons coupling is

$$S_{\rm CS}^{D2} \supset \mu_2 \int_{D2} C_1 \wedge (2\pi\alpha' F), \qquad k = \frac{1}{2\pi}\int_{D2} F \in \mathbb{Z}. \tag{12.18}$$

Thus a flux sector with integer k carries k units of D0 charge. The energy follows from the DBI action. For a flat D2 of area V_2 and constant magnetic field F_{12},

$$E_{D2/F} = \tau_2 V_2 \sqrt{1 + (2\pi\alpha' F_{12})^2}, \qquad F_{12} = \frac{2\pi k}{V_2}. \tag{12.19}$$

Using $\tau_0 = 4\pi^2\alpha'\tau_2$, this becomes

$$E_{D2/F} = \sqrt{(\tau_2 V_2)^2 + (k\tau_0)^2}. \tag{12.20}$$

The square-root form is the central-charge result for a single D2 carrying dissolved D0 charge. The tachyonic separated D0–D2 system and the smooth flux configuration are two descriptions of different points along the decay process, so their supersymmetry properties should not be conflated [118].

Derivation. The equality $\tau_0 = 4\pi^2\alpha'\tau_2$ follows from $\tau_p = [(2\pi)^p g_s (\alpha')^{(p+1)/2}]^{-1}$. Substituting $F_{12} = 2\pi k/V_2$ into the DBI energy gives

$$\tau_2^2 V_2^2 \left[1 + \left(\frac{4\pi^2\alpha' k}{V_2}\right)^2\right] = (\tau_2 V_2)^2 + (k\tau_0)^2. \tag{12.21}$$

The fluxed D2 is therefore lighter than a separated D2 plus k D0-branes for finite V_2, because $\sqrt{a^2+b^2} < a+b$. The missing binding energy is carried by the tachyon condensation/dissolution process.

12.6.5 D0–D4 bound states

The D0–D4 system is the canonical $\#_{ND} = 4$ example. On the D4-brane, D0 charge is the instanton number of the worldvolume gauge field:

$$k = \frac{1}{8\pi^2}\int_{D4} \mathrm{Tr}\, F \wedge F, \qquad S_{\rm CS}^{D4} \supset \mu_4 \int_{D4} C_1 \wedge \frac{(2\pi\alpha')^2}{2}\mathrm{Tr}\, F \wedge F. \tag{12.22}$$

The Yang–Mills action on the D4 satisfies the instanton bound

$$\frac{1}{4g_{YM}^2}\int d^4x\, \mathrm{Tr}\, F_{ij}F_{ij} = \frac{1}{8g_{YM}^2}\int \mathrm{Tr}(F \mp *F)^2 \pm \frac{1}{4g_{YM}^2}\int \mathrm{Tr}\, F \wedge F. \tag{12.23}$$

It is saturated by self-dual or anti-self-dual field strength. Thus a D0 dissolved in a D4 is a Yang–Mills instanton, and the instanton moduli space is the Higgs branch of the D0–D4 open-string quantum mechanics. The $\#_{ND} = 4$ condition gives massless modes that can resolve the small-instanton singularity [101, 118].

Derivation. The DBI and Chern–Simons normalizations agree with the D0 charge. Expanding the D4 Wess–Zumino term gives

$$\mu_4 \frac{(2\pi\alpha')^2}{2}\int C_1 \wedge \mathrm{Tr} F \wedge F = \mu_0\, k \int C_1, \tag{12.24}$$

using $\mu_0 = (2\pi)^4(\alpha')^2\mu_4$ and $k = (8\pi^2)^{-1}\int \mathrm{Tr} F \wedge F$. This shows algebraically that instanton number is D0 charge.

12.6.6 D-branes as instantons

D-branes appear as instantons in two related senses. First, a Euclidean D-brane wrapping an internal cycle contributes to the path integral with weight

$$\exp(-S_E + i\int C), \qquad S_E = \frac{\tau_p}{g_s^0}\mathrm{Vol}_{p+1}(\Sigma) + \cdots, \tag{12.25}$$

where the displayed volume term is written in the convention in which τ_p already includes the appropriate power of g_s^{-1}. Second, a lower-dimensional D-brane charge may appear as a gauge instanton on a higher-dimensional D-brane. Both statements are encoded in the Wess–Zumino coupling

$$S_{\mathrm{WZ}} = \mu_p \int_{Dp} C \wedge \mathrm{Tr}\, \exp(2\pi\alpha' F + B) \sqrt{\frac{\widehat{A}(R_T)}{\widehat{A}(R_N)}}. \tag{12.26}$$

The exponential collects the Chern character of the gauge bundle. Its expansion gives D$(p-2)$, D$(p-4)$, and lower charges on the same brane. The curvature factor supplies the gravitational correction to the charge. These topological invariants explain why D-brane charge is naturally refined by K-theory [101, 105].

12.6.7 D0–D6 bound states

A bare D0–D6 pair has $\#_{ND} = 6$, so it does not preserve supersymmetry by the elementary projection rule. Supersymmetric configurations with the same total charge can still be built using flux on the D6-brane. The relevant term in the Wess–Zumino expansion is

$$\mu_6 \int_{D6} C \wedge e^{2\pi\alpha' F} \supset \mu_6 \int_{D6} C_1 \wedge \frac{(2\pi\alpha' F)^3}{3!}. \tag{12.27}$$

For a diagonal flux on three orthogonal two-planes, $F = F_1 dx^1 \wedge dx^2 + F_2 dx^3 \wedge dx^4 + F_3 dx^5 \wedge dx^6$, the induced D0 charge is proportional to

$$Q_{D0} \propto \int_{D6} F \wedge F \wedge F = 6 \int F_1 F_2 F_3 \, d^6x. \tag{12.28}$$

The same flux also induces D2 and D4 charges through the lower powers of F. After T-duality this system is described by branes at angles, and supersymmetry requires the angle condition $\theta_1 + \theta_2 + \theta_3 = 0$ modulo 2π, with $\tan\theta_i = 2\pi\alpha' F_i$. Hence a fluxed D6 can be BPS, while the elementary separated D0–D6 pair is a different, non-BPS system [118].

12.6.8 D0–D8 bound states

The D0–D8 system must be treated with the background included. A D8-brane is a domain wall of massive type-IIA supergravity. Across it, the Romans mass jumps by one unit,

$$dF_0 = \mu_8 \, \delta_{D8}, \qquad \Delta F_0 = 1 \tag{12.29}$$

up to the normalization of the R–R flux quantum. Thus the two sides of the D8-brane are different massive-IIA regions. When a D0 crosses a D8, one fundamental string is created between them. This string creation is required by charge conservation and anomaly inflow. Therefore a statement about a D0–D8 bound state is meaningful only after the Romans mass, orientifold planes, and global tadpole conditions are specified. In type-I′ constructions these ingredients can produce supersymmetric configurations, but the isolated flat-space D0–D8 pair should not be analyzed as if it were the D0–D4 system with more ND directions.

Chapter 13

Strong coupling: dualities, M-theory, and holography

Everything so far has been an expansion in g_s; this chapter asks what happens outside that expansion: what are the five superstring theories when the coupling is large? The method is the one prepared in Chapter 12. BPS masses and tensions are fixed by charges appearing in the supersymmetry algebra, so they can be computed at weak coupling and trusted at strong coupling; matching the protected spectra of two theories is then evidence, and matching their charge lattices, low-energy actions, and anomaly structures supports the identification of the two theories as the same theory in different variables. Carried out case by case this yields the duality web: type IIB maps to itself under $SL(2,\mathbb{Z})$, with the fundamental string and D-string rotated into a family of (p,q) strings; strongly coupled type IIA grows an eleventh dimension of radius $R_{11} = g_s \ell_s$ whose Kaluza–Klein modes are D0-branes, pointing to eleven-dimensional M-theory with its membrane and five-brane; and type I and heterotic $SO(32)$ exchange strong and weak coupling. The chapter closes with two applications: matrix theory as a proposed definition of M-theory, and the D-brane counting of black-hole entropy, where Cardy's formula reproduces the Bekenstein–Hawking area law exactly [167, 353].

Why the D-string has only transverse propagating fields. The D-string worldvolume is $1+1$-dimensional. A gauge field A_a in two dimensions has field strength F_{01}, but no transverse photon polarization. In Hamiltonian language A_0 imposes Gauss' law, and A_1 has no local oscillator after gauge fixing; only electric flux sectors remain. Thus the local propagating bosons are the eight transverse scalars X^i, $i = 2, \ldots, 9$. For a worldsheet fermion u, the massless Dirac equation

$$(\Gamma^0 \partial_0 + \Gamma^1 \partial_1) u = 0 \tag{13.1}$$

implies, after multiplying by Γ^1,

$$\Gamma^1 \Gamma^0 \partial_0 u + \partial_1 u = 0 \qquad \Longleftrightarrow \qquad \Gamma^0 \Gamma^1 \partial_0 u = \partial_1 u, \tag{13.2}$$

where $(\Gamma^1)^2 = +1$ and $\Gamma^1\Gamma^0 = -\Gamma^0\Gamma^1$. A right/left mover obeys $(\partial_0 \mp \partial_1)u = 0$, hence

$$\boxed{\Gamma^0 \Gamma^1 u_\pm = \pm u_\pm, \qquad 2S_0 = \Gamma^0\Gamma^1.} \tag{13.3}$$

This removes the earlier "constant of integration" worry: for a definite nonzero-frequency chiral mode the relation is algebraic. Zero modes are handled by the same chirality projection on the constant spinor. The ten-dimensional Majorana–Weyl spinor therefore decomposes under $SO(1,1) \times SO(8)$ as

$$\mathbf{16} \longrightarrow \left(+\frac{1}{2}, \mathbf{8}\right) \oplus \left(-\frac{1}{2}, \mathbf{8}'\right), \tag{13.4}$$

with the assignment of $\mathbf{8}$ versus $\mathbf{8}'$ depending on the ten-dimensional chirality convention.

(p,q)-string tension and frame convention. Type IIB combines the axion and dilaton into

$$\boxed{\tau = C_0 + i e^{-\phi} = C_0 + \frac{i}{g_s}, \qquad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).} \tag{13.5}$$

The NS–NS and R–R two-forms transform as a doublet, so an object carrying p units of F1 charge and q units of D1 charge has Einstein-frame tension

$$T^E_{(p,q)} = \frac{1}{2\pi\alpha'}\frac{|p+q\tau|}{\sqrt{\operatorname{Im}\tau}}. \tag{13.6}$$

Derivation. The Einstein-frame expression is fixed by the $SL(2,\mathbb{Z})$-invariant charge norm. The charge vector (p,q) pairs with the axio-dilaton through $|p+q\tau|$, while the factor $(\operatorname{Im}\tau)^{-1/2}$ converts the string-frame tension to Einstein frame. Setting $C_0 = 0$ gives $T^E_{(1,0)} = (2\pi\alpha')^{-1}g_s^{1/2}$ for the F-string and $T^E_{(0,1)} = (2\pi\alpha')^{-1}g_s^{-1/2}$ for the D-string, which are the string-frame tensions multiplied by the common worldsheet Weyl factor.

This formula resolves the red normalization warning around Eq. (13.25). In string frame, the F1 tension is $T^S_{F1} = 1/(2\pi\alpha')$, while the D1 tension is $T^S_{D1} = 1/(2\pi\alpha' g_s)$. Passing to Einstein frame multiplies a string worldsheet tension by $e^{\phi/2} = g_s^{1/2}$. Thus

$$T^E_{(1,0)} = \frac{g_s^{1/2}}{2\pi\alpha'}, \qquad T^E_{(0,1)} = \frac{1}{2\pi\alpha' g_s^{1/2}}, \tag{13.7}$$

which is exactly Eq. (13.6) at $C_0 = 0$. The frame must be named whenever exact powers of g_s are compared [167, 171].

D3 gauge coupling and Montonen–Olive normalization. Expand the D3 DBI action in string frame,

$$S_{\text{DBI}} = -T_3\int d^4x\, e^{-\phi}\sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})}. \tag{13.8}$$

Using

$$\sqrt{\det(1+M)} = 1 - \frac{1}{4}(2\pi\alpha')^2 F_{ab}F^{ab} + \cdots \tag{13.9}$$

for antisymmetric $M_{ab} = 2\pi\alpha' F_{ab}$, the gauge kinetic term is

$$S_{\text{kin}} = -\frac{T_3 e^{-\phi}(2\pi\alpha')^2}{4}\int d^4x\, F_{ab}F^{ab}. \tag{13.10}$$

With $T_3 = 1/[(2\pi)^3\alpha'^2]$ in the convention where the explicit $e^{-\phi}$ sits in DBI,

$$\frac{1}{g^2_{\text{YM}}} = \frac{1}{2\pi g_s}, \qquad g^2_{\text{YM}} = 2\pi g_s. \tag{13.11}$$

Some Yang–Mills conventions absorb a factor of 2 into the trace normalization and quote $g^2_{\text{YM}} = 4\pi g_s$. That is why Eq. (13.27) was marked as convention-dependent. The Chern–Simons coupling $\mu_3\int C_0(2\pi\alpha' F)\wedge(2\pi\alpha' F)/2$ supplies the theta angle, so the physical statement is

$$\tau_{\text{YM}} \text{ is proportional to } C_0 + ie^{-\phi}, \tag{13.12}$$

with the proportionality fixed by the chosen trace convention. Therefore IIB $SL(2,\mathbb{Z})$ becomes electric-magnetic duality of the D3 worldvolume theory [167, 174].

IIA/M-theory radius and eleven-dimensional Planck length. The D0-brane mass in type IIA string frame is

$$M_{D0} = \frac{1}{g_s\ell_s}. \tag{13.13}$$

If this is the first Kaluza–Klein momentum mode around the eleventh circle, then $M_{KK} = 1/R_{11}$. Matching the two gives

$$R_{11} = g_s\ell_s. \tag{13.14}$$

The eleven-dimensional Planck scale follows by reducing the Einstein–Hilbert term. Since

$$2\kappa^2_{11} = (2\pi)^8\ell^9_{11}, \qquad 2\kappa^2_{10} = (2\pi)^7 g_s^2\ell_s^8, \qquad \kappa^2_{11} = (2\pi R_{11})\kappa^2_{10}, \tag{13.15}$$

one obtains

$$\ell^3_{11} = g_s\ell_s^3, \qquad \ell_{11} = g_s^{1/3}\ell_s. \tag{13.16}$$

These two equations are the quantitative core of the statement that strong-coupling type IIA decompactifies into an eleven-dimensional theory [352, 353].

Type-I/heterotic tension matching. The $SO(32)$ type-I/heterotic map is fixed by matching

Einstein-frame actions and BPS string tensions. If

$$g_I = \frac{1}{g_H}, \tag{13.17}$$

then the heterotic fundamental string maps to the type-I D1-brane. String-frame tensions give

$$T_{\text{het F1}} = \frac{1}{2\pi\alpha'_H}, \qquad T_{\text{I D1}} = \frac{1}{2\pi\alpha'_I g_I}. \tag{13.18}$$

Equating the dual objects gives

$$\alpha'_I = g_H\,\alpha'_H. \tag{13.19}$$

This is why Eq. (13.30) must include both the inverse coupling and the rescaling of α'. Matching only $g_I = 1/g_H$ would leave the tension normalization wrong [167, 170].

Matrix-theory mass scale and black-hole entropy check. For N D0-branes, the low-energy matrices $X^i(t)$ have potential schematically

$$V \sim \frac{1}{g_s \ell_s} \operatorname{Tr} \frac{[X^i, X^j]^2}{(2\pi\alpha')^2}. \tag{13.20}$$

On a diagonal background $X^i = \operatorname{diag}(x^i_1, \ldots, x^i_N)$, an off-diagonal fluctuation connecting a and b has mass of order

$$m_{ab} \sim \frac{|x_a - x_b|}{2\pi\alpha'}, \tag{13.21}$$

the energy of a string stretched between the two D0-branes. This is the microscopic reason that noncommuting matrices encode light stretched strings when branes approach each other [177, 364].

For the D1–D5–momentum black hole, the weak-coupling CFT has central charge $c = 6n_1n_5$. Cardy's formula gives

$$S_{\text{micro}} = 2\pi\sqrt{\frac{c\,n_p}{6}} = 2\pi\sqrt{n_1 n_5 n_p}, \tag{13.22}$$

Derivation. For the D1–D5 CFT the effective central charge is $c = 6Q_1Q_5$. If N units of momentum are placed in one chiral sector, Cardy's formula gives

$$S = 2\pi\sqrt{\frac{cN}{6}} = 2\pi\sqrt{Q_1Q_5N}. \tag{13.23}$$

The supergravity area of the corresponding five-dimensional extremal black hole gives the same entropy, $S_{BH} = A/(4G_5)$. The equality is protected because the index/counting can be moved to weak coupling, where the D-brane CFT is calculable, while the area is computed in the strong-coupling geometric regime.

which matches the leading Bekenstein–Hawking entropy of the corresponding extremal black hole.

13.1 The type IIB string and $SL(2,\mathbb{Z})$ duality

To see the $SL(2,\mathbb{Z})$ duality of type IIB theory, consider a D-string. Its worldsheet is two-dimensional. As for D-branes generally, the longitudinal excitations correspond to the world-volume gauge field. In two dimensions, however, the gauge field has no propagating photon; A_0 imposes Gauss' law and A_1 carries only global electric-flux sectors after gauge fixing. The local propagating excitations are therefore transverse. The Dirac equation for these excitations is

$$\left(\Gamma^0\partial_0 + \Gamma^1\partial_1\right) u = 0 \tag{14.1.1}$$

Multiplying from the left by Γ^1, we have;

$$\left(\Gamma^1\Gamma^0\partial_0 + \left(\Gamma^1\right)^2\partial_1\right) u = 0 \Rightarrow \Gamma^0\Gamma^1\partial_0 u = \partial_1 u \tag{14.1.2}$$

For left and right-handed spinors, we have the following;

$$(\partial_0 \mp \partial_1)\, u = 0 \Rightarrow \partial_1 u = \pm\partial_0 u \tag{14.1.3}$$

Using (14.1.3) in 14.1.2, we have;

$$\Gamma^0\Gamma^1\partial_0 u = \pm\partial_0 u \Rightarrow \Gamma^0\Gamma^1 u = \pm u$$

For a definite chiral mode the relation above is algebraic; zero modes obey the same chirality projection on the constant spinor. From appendix B, we know that $2S_0 = \Gamma^0\Gamma^1$. Therefore, we see that the s_0 eigenvalue of u has to be $\pm 1/2$. Therefore, the Majorana Weyl fermion living in **16** decomposes as follows:

$$16 \to \left(\frac{1}{2}, 8\right) \oplus \left(-\frac{1}{2}, 8'\right)$$

Infinite string has the same decomposition (complete).

13.1.1 $SL(2,\mathbb{Z})$ duality

Type IIB duality is organized by the axio-dilaton

$$\tau = C_0 + ie^{-\phi} = C_0 + \frac{i}{g_s}, \qquad \tau \longmapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{13.24}$$

The NS–NS and R–R two-forms transform as a doublet, so the fundamental string and D-string combine into (p,q) strings. In Einstein-frame conventions their protected tension is proportional to

$$T_{(p,q)} = \frac{1}{2\pi\alpha'}\frac{|p + q\tau|}{\sqrt{\operatorname{Im}\tau}}. \tag{13.25}$$

It makes precise the paragraph above: the D-string has the same transverse worldsheet field content as the fundamental string because it is one member of the same duality orbit. [167, 171].

Einstein-frame normalization: Eq. (13.25) is the Einstein-frame (p,q)-string tension. The string-frame F1 and D1 tensions are $1/(2\pi\alpha')$ and $1/(2\pi\alpha' g_s)$; converting frames gives the $g_s^{\pm 1/2}$ factors derived in Eq. (13.6).

13.1.2 The IIB NS5-brane

The IIB NS5-brane is the magnetic dual of the fundamental string and is mapped by $SL(2,\mathbb{Z})$ to the D5-brane. The tension scalings in string frame already show the strong-coupling content:

$$T_{\mathrm{D5}} \sim \frac{1}{g_s(\alpha')^3}, \qquad T_{\mathrm{NS5}} \sim \frac{1}{g_s^2(\alpha')^3}. \tag{13.26}$$

Thus the NS5-brane is nonperturbative from the fundamental-string viewpoint, while the D5-brane is the object naturally coupled to the R–R six-form. S-duality therefore exchanges the electric and magnetic completions of the full two-form charge lattice, not just two isolated objects. [167, 168].

The single-NS5 construction gives a worldsheet realization of a strongly coupled magnetic object. The decoupled $n_5 = 1$ background is written as a gauged WZW coset

$$\frac{\mathcal{G}}{\mathcal{H}} = \frac{PSU(1,1|2) \times \mathbb{R}^{5,1}}{U(1) \times U(1)},$$

with an additional dynamical constraint that removes the obstruction usually encountered for a controlled $n_5 = 1$ description [229]. A related embedding interpolates between the fivebrane throat and an AdS_3 region dual to a single-trace $T\bar{T}$-deformed symmetric product. The NS5-brane remains the magnetic dual required by charge quantization, and the coset description shows how exact worldsheet methods can still access the corresponding little-string regime.

13.1.3 D3-branes and Montonen–Olive duality

The D3-brane is self-dual under type IIB $SL(2,\mathbb{Z})$. Its low-energy worldvolume theory is four-dimensional $\mathcal{N} = 4$ super Yang–Mills, and the IIB axio-dilaton becomes the complexified gauge coupling up to convention-dependent factors:

$$\tau_{\rm YM} = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2} \sim C_0 + ie^{-\phi}. \tag{13.27}$$

Therefore IIB S-duality acts on the D3 worldvolume as Montonen–Olive electric-magnetic duality. It identifies the worldvolume electric-magnetic duality with the spacetime S-duality action on the D3-brane system. [167, 174].

D3 gauge-coupling normalization: the proportionality in Eq. (13.27) is fixed only after choosing the Yang–Mills trace normalization. With the DBI normalization used in Eq. (13.11), $g_{\rm YM}^2 = 2\pi g_s$; the common $4\pi g_s$ version uses a different trace convention.

13.2 U-duality

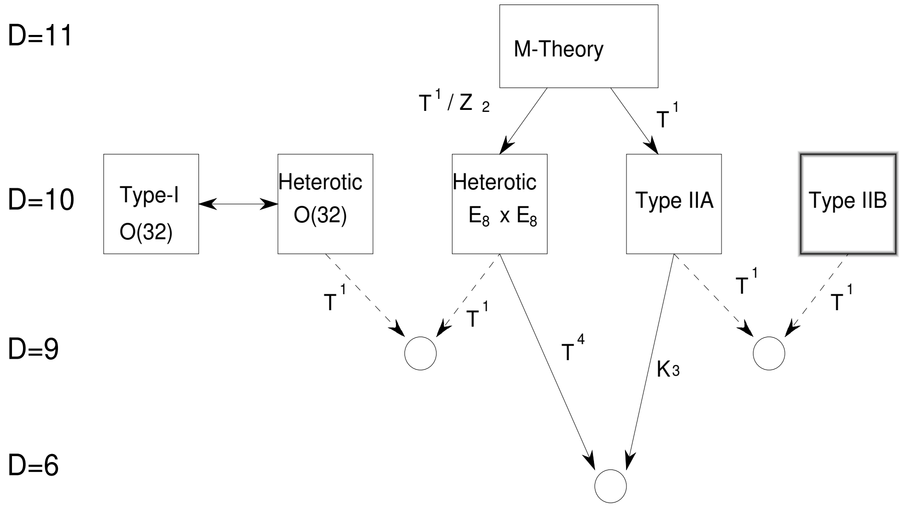


Figure 13.1: The web of string dualities. The five ten-dimensional superstring theories and eleven-dimensional M-theory are limits of a single structure, connected by T- and S-dualities and by compactification on circles, on T^4, and on $K3$. From Ref. [374].

U-duality is the nonperturbative duality group obtained after compactification, combining T-duality with strong-coupling S-duality, the two edges of the duality web in Figure 13.1. For type II theory on a torus T^d, the expected arithmetic group is the integer form of the split exceptional group,

$$\begin{array}{c|c} d & G(\mathbb{Z}) \\ \hline 1 & SL(2,\mathbb{Z}) \\ 2 & SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z}) \\ 3 & SL(5,\mathbb{Z}) \\ 4 & SO(5,5;\mathbb{Z}) \\ 5 & E_{6(6)}(\mathbb{Z}) \\ 6 & E_{7(7)}(\mathbb{Z}) \end{array} \tag{13.28}$$

The physical content is that momentum, winding, brane wrapping numbers, and flux charges must assemble into multiplets of the same arithmetic group. Therefore BPS spectra are the right test objects: their masses are protected enough to compare across descriptions. [173, 174].

13.2.1 U-duality and bound states

A U-duality-covariant bound-state statement has two ingredients: the charge vector must transform in the appropriate lattice representation, and the BPS mass must be an invariant constructed from the moduli and charges. Schematically,

$$Q \in \Gamma_{\rm charge}, \qquad M_{\rm BPS}^2(Q,\phi) = M_{\rm BPS}^2(gQ, g\cdot\phi), \qquad g \in G(\mathbb{Z}). \tag{13.29}$$

The strong-coupling continuation of the D0-Dp discussion: what looked like different bound states in perturbative language become different components of one charge orbit. [119, 174].

13.3 $SO(32)$ type I–heterotic duality

The $SO(32)$ type I string and the $SO(32)$ heterotic string are related by strong-weak duality. At the schematic level,

$$g_I = \frac{1}{g_H}, \qquad \alpha'_I = g_H\,\alpha'_H \tag{13.30}$$

after matching the Einstein-frame metric. The heterotic fundamental string maps to the type I D1-brane, and the shared $SO(32)$ gauge group is protected by anomaly cancellation. At this stage, a perturbative current algebra on one side becomes D-brane/R–R physics on the other. [149, 167].

13.3.1 Quantitative tests

Useful quantitative tests of type I–heterotic duality include matching the ten-dimensional low-energy actions in Einstein frame, comparing BPS string and five-brane tensions, matching anomaly-canceling Green–Schwarz terms, and comparing protected threshold corrections after compactification. The guiding structure is

$$S^E_{\text{het}}[g_H, \Phi_H, A] \quad\longleftrightarrow\quad S^E_{\text{I}}[g_I = 1/g_H, \Phi_I, A], \tag{13.31}$$

with the dilaton dependence redistributed between tree-level, disk, crosscap, and loop effects. This duality statement is tested by normalization checks on tensions, charges, and anomaly terms. [149, 167, 170].

13.3.2 Type I D5-branes

Type I D5-branes are the strong-dual objects related to heterotic five-brane physics. In compactifications they are also tied to small-instanton transitions: gauge instanton number can shrink to a brane-like object, changing the nonperturbative spectrum while preserving the anomaly constraints. The relevant charge relation is again a Wess–Zumino coupling of the form

$$\mu_5 \int_{D5} C \wedge \operatorname{Tr} e^{2\pi\alpha' F}, \tag{13.32}$$

with orientifold projection and $SO(32)$ Chan–Paton structure added. The construction is tied to the D-brane-as-instanton discussion, because the same charge lattice and tension normalizations appear in both descriptions [85, 167].

13.4 The type IIA string and M-theory

In type IIA string theory the string coupling determines the radius of the eleventh dimension (Figure 13.2):

$$R_{11} = g_s \ell_s, \qquad \ell_{11} = g_s^{1/3} \ell_s. \tag{13.33}$$

As $g_s \to \infty$, the circle radius grows without bound. The strong-coupling limit is therefore described by an eleven-dimensional theory instead of by a divergent ten-dimensional perturbation series. [352, 353].

13.4.1 U-duality and F-theory

U-duality combines the perturbative T-duality group with nonperturbative strong–weak dualities. For type-II strings compactified on a torus, the charges include momentum, winding, brane wrapping numbers, and fluxes. In four-dimensional type-II compactification on T^6, the discrete group acting on the full charge lattice is conventionally written as

$$E_{7(7)}(\mathbb{Z}), \qquad \Gamma_{\text{charge}} \subset \mathbf{56}. \tag{13.34}$$

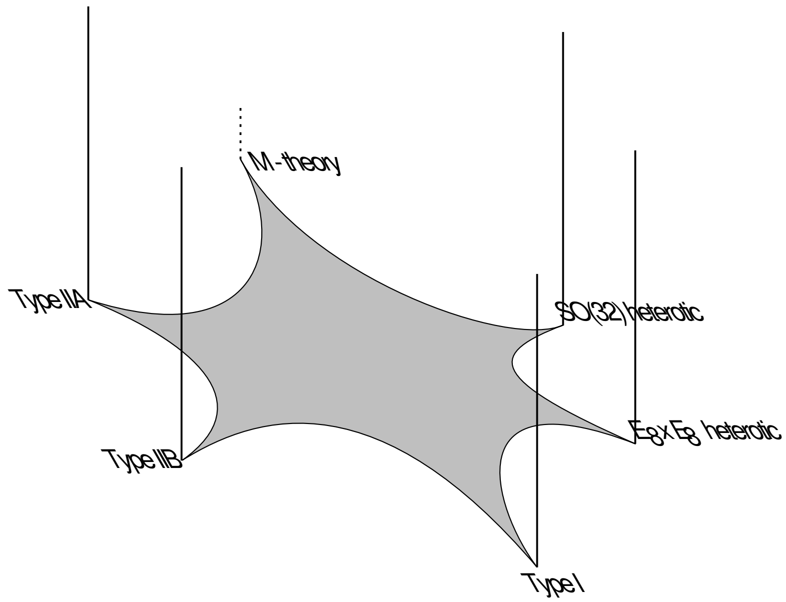


Figure 13.2: The moduli space of string/M-theory. Each cusp is a weak-coupling limit, one of the five ten-dimensional superstring theories or eleven-dimensional M-theory, and the interior interpolates smoothly between them. What look like six distinct theories are boundary regions of one connected moduli space. From Ref. [374].

The continuous group $E_{7(7)}$ appears in the classical supergravity limit. The integer subgroup preserves charge quantization in the quantum theory. The arithmetic group is therefore a symmetry of the allowed charge lattice, not a continuous field redefinition. [371].

F-theory gives a geometric notation for the varying axio-dilaton of type-IIB string theory. The complex coupling

$$\tau = C_0 + ie^{-\Phi} \tag{13.35}$$

transforms under $SL(2,\mathbb{Z})$. F-theory represents this $SL(2,\mathbb{Z})$ action as the modular transformation of an auxiliary elliptic curve. The elliptic fiber is not an ordinary pair of large spacetime dimensions; its complex structure records the IIB coupling. Degeneration loci of the elliptic fibration correspond to seven-brane sources, so the geometry encodes both the local value of τ and its monodromy around branes. [372].

13.4.2 IIA branes from 11 dimensions

A D-brane appears in type-IIA after reducing M-theory objects along or transverse to the eleventh circle. Its low-energy worldvolume dynamics is governed by the Dirac–Born–Infeld and Chern–Simons action already written in Eq. (12.11), whose DBI term carries the open-string gauge field and transverse scalars and whose Chern–Simons term supplies the coupling to the Ramond–Ramond potentials. The content of the present reduction is that the eleven-dimensional origin must be matched at the level of particle states, brane tensions, and charges [38, 351, 352].

13.5 $E_8 \times E_8$ heterotic string at strong coupling

One theory remains unaccounted for in the duality web, and its strong-coupling limit is of a different kind. The $E_8 \times E_8$ heterotic string cannot be S-dual to any of the ten-dimensional theories already placed, its gauge group matches none of them perturbatively, and the resolution (Hořava–Witten) is that its strong-coupling limit grows an eleventh dimension that is an *interval* and not a circle:

$$E_8 \times E_8 \text{ heterotic at } g_s \gg 1 \quad \simeq \quad \text{M-theory on } \mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2, \qquad L_{11} = g_s \ell_s, \tag{13.36}$$

with one E_8 gauge multiplet living on each ten-dimensional boundary wall. The consistency structure mirrors the type-I construction in eleven-dimensional language: the walls are gravitational anomaly sources, and anomaly inflow from the bulk Chern–Simons term $C_3 \wedge G_4 \wedge G_4$

cancels the boundary anomaly precisely when each wall carries a rank-248 group with the E_8 trace identities, the Green–Schwarz factorization of Chapter 10 rederived as bulk-boundary matching. The parallel with type IIA is exact but orbifolded: circle compactification of M-theory gives IIA, interval compactification gives $E_8 \times E_8$, and the two heterotic theories, T-dual to each other after further circle compactification, thereby enter the same duality network [167, 353].

13.6 Matrix theory

The letter M has not had a single fixed meaning in the literature. It is often associated with membrane, mystery, magic, or matrix. The word "matrix" becomes precise in the BFSS proposal, where M-theory in the infinite-momentum frame is described by the large-N limit of the quantum mechanics of N D0-branes. The bosonic part of the D0-brane matrix model may be written schematically as

$$S_{\mathrm{BFSS}} \sim \frac{1}{g_s \ell_s} \int dt \,\mathrm{Tr}\left[(D_t X^i)^2 + \frac{1}{(2\pi\alpha')^2}[X^i, X^j]^2\right] + \text{fermions}, \qquad i = 1, \ldots, 9. \tag{13.37}$$

The nine Hermitian matrices $X^i(t)$ are the transverse coordinates of the D0-branes. When all X^i commute, they can be diagonalized simultaneously and their eigenvalues give ordinary D0-brane positions. The off-diagonal entries describe strings stretched between different D0-branes; for two well-separated diagonal eigenvalues x^i_a and x^i_b, the corresponding stretched string has a mass scale of order

$$m_{ab} \sim \frac{\sqrt{(x^i_a - x^i_b)(x^i_a - x^i_b)}}{2\pi\alpha'}. \tag{13.38}$$

Non-commuting matrices therefore describe configurations that cannot be reduced to a set of classical points.

The BFSS proposal is stronger than the strong-coupling relation $R_{11} = g_s \ell_s$, $\ell_{11} = g_s^{1/3} \ell_s$ of Eq. (13.33). That relation only identifies the type-IIA coupling with the radius of the eleventh dimension, whereas Eq. (13.37) asserts that in a particular light-cone frame the entire dynamics is captured by a supersymmetric matrix quantum mechanics. The letter M is therefore not fixed to the word "matrix"; Matrix theory is one formulation of M-theory in a special light-cone, large-N limit [177, 353, 364, 368].

The decoupling-limit worldsheet formalism derives the BFSS light sector by cancelling the D0-brane rest energy against a near-critical R–R one-form. Schematically,

$$E^{\mathrm{finite}}_{\mathrm{D0}} = \lim_{\epsilon \to 0} \left(\frac{1}{g_s(\epsilon)\ell_s(\epsilon)} - C_0(\epsilon)\right),$$

so D0-branes and their off-diagonal open strings remain light. Their low-energy action is the matrix quantum mechanics

$$S_{\mathrm{BFSS}} = \frac{1}{g^2_{\mathrm{YM}}} \int dt \,\mathrm{Tr}\left[\frac{1}{2}(D_t X^i)^2 + \frac{1}{4}[X^i, X^j]^2 + \cdots\right], \qquad g^2_{\mathrm{YM}} \sim \frac{g_s}{\ell_s^3}.$$

The fundamental string in the same scaling develops nodal worldsheet degenerations [224]. Related work on twisted BFSS and IKKT identifies BV–BRST cohomological twists of matrix models with planar twists of type-II strings [234]. The matrix Hamiltonian is therefore selected by the critical scaling of the string background.

In the two-dimensional maximally supersymmetric $U(N)$ Yang–Mills description of matrix string theory, the infrared fixed point is the symmetric-product orbifold $(\mathbb{R}^8)^N/S_N$. Bajaj fixes the Wilson coefficient of the leading irrelevant Dijkgraaf–Verlinde–Verlinde operator by a non-renormalization theorem in the ultraviolet gauge theory, obtaining a first-principles relation between g_{YM} and the type-IIA string coupling [285]. This matters for Eq. (13.37): the matrix model is not only expected to reproduce the qualitative string spectrum, but also specific coefficients of the deformation away from the free symmetric-orbifold CFT.

13.6.1 The M-theory membrane

The membrane in M-theory is the M2-brane. Its bosonic action is the volume of a three-dimensional worldvolume plus the electric coupling to the eleven-dimensional three-form field,

$$S_{M2} = -T_{M2} \int d^3\xi \sqrt{-\det g_{ab}} + T_{M2} \int C_3, \qquad T_{M2} = \frac{1}{(2\pi)^2 \ell_{11}^3}. \tag{13.39}$$

The first term is the membrane analogue of the Nambu–Goto action. The second term is the electric coupling to the eleven-dimensional gauge potential C_3. The magnetic dual object is the M5-brane.

The reduction to type IIA gives a direct normalization check. If the M2 wraps the eleventh circle, its effective string tension is

$$T_{\text{wrapped}} = (2\pi R_{11}) T_{M2} = \frac{2\pi g_s \ell_s}{(2\pi)^2 g_s \ell_s^3} = \frac{1}{2\pi \ell_s^2} = T_{F1}, \tag{13.40}$$

where $\ell_{11}^3 = g_s \ell_s^3$ has been used. If the M2 is not wrapped on the circle, it reduces to the type-IIA D2-brane. For this reason the membrane discussion is centered on the M2-brane and its reductions, instead of on the D-brane DBI action alone. [352, 353].

13.6.2 Finite n and compactification

In Matrix theory, finite N labels a sector with fixed light-cone momentum instead of the whole uncompactified theory. One writes schematically

$$p^+ = \frac{N}{R_-}. \tag{13.41}$$

Recovering the uncompactified light-cone theory requires a suitable large-N limit. Compactification changes the matrix model because winding sectors and dual brane descriptions become dynamical. For example, D0-branes on a spatial circle are T-dual to D1-branes on the dual circle, so the matrices become fields on the dual circle.

Matrix compactification is not the elementary momentum-winding exchange of the ordinary closed-string circle spectrum, Eq. (1.45); the matrix description must instead reproduce compactified M-theory after the appropriate duality chain. [10, 177, 364].

Maskalaniec, Yan, and Zorba show that the D-particle decoupling limits leading to BFSS-like matrix quantum mechanics naturally produce non-Lorentzian supergravity backgrounds, and they relate the dynamics to current-algebra anomalies of the associated fundamental-string worldsheet [331]. At large N, D-particle backreaction deforms the non-Lorentzian description back toward Lorentzian type IIA, while at moderately large N the D-particles decouple at leading order. Matrix theory is then not only a quantum-mechanical guess for M-theory, but the low-energy target geometry selected by a precise D-brane decoupling limit.

13.7 Black-hole quantum mechanics

Black-hole quantum mechanics asks for the microscopic Hilbert space whose number of states gives the Bekenstein–Hawking entropy

$$S_{\text{BH}} = \frac{A_{\text{hor}}}{4G_N}. \tag{13.42}$$

The simplest string-theory calculation is the supersymmetric five-dimensional D1–D5–momentum system. If the integer charges are n_1, n_5, and n_p, the weak-coupling D-brane conformal field theory gives the degeneracy

$$S_{\text{micro}} = 2\pi \sqrt{n_1 n_5 n_p}. \tag{13.43}$$

The corresponding extremal black-hole solution gives the same leading entropy from Eq. (13.42). The equality is meaningful because the charges and supersymmetry protect the index while the coupling is varied.

For Matrix theory the microscopic system is a supersymmetric quantum mechanics instead of an ordinary field theory in fixed spacetime. The black-hole question is then phrased as a question about the spectrum and thermal states of the matrix Hamiltonian. In both descriptions the quantity being compared with geometry is a count of quantum states with fixed conserved charges [177, 352, 365].

Nanda, Shanmugapriya, and Virmani construct an index saddle for the D1–D5–P black string and its decoupling limit, obtaining a complex finite-temperature BTZ saddle that computes the dual CFT index after a chain of string dualities [346]. Larsen and Sharma similarly argue that the gravitational representative of the supersymmetric index in $AdS_3 \times S^3$ is a smooth complex BPS saddle, not a naive Euclidean continuation of the Lorentzian black hole [339]. Ruiperez derives first-order α' corrections to the heterotic BMPV black hole and finds entropy agreement with supersymmetric-index methods once Lorentz Chern–Simons terms are handled correctly [347]. Finally, Saha, Sen, and Shanmugapriya compute logarithms of charge ratios in BPS black-hole entropy and match microscopic results in theories where those quantities are known [340]. Together these refine Eq. (13.43): the tests now compare not only the leading Cardy entropy but also complex index saddles, α' corrections, and logarithmic charge-ratio terms.

13.7.1 A correspondence principle

The correspondence principle compares a highly excited string with a black hole at the coupling where the Schwarzschild radius becomes of order the string length,

$$R_S \sim \ell_s. \tag{13.44}$$

In D non-compact spacetime dimensions the Schwarzschild scaling is

$$R_S^{D-3} \sim G_D M, \qquad S_{\rm BH} \sim \frac{R_S^{D-2}}{G_D} \sim M R_S. \tag{13.45}$$

A highly excited free string has entropy of order

$$S_{\rm string} \sim M\ell_s. \tag{13.46}$$

At the crossover point $R_S \sim \ell_s$, Eq. (13.45) gives $S_{\rm BH} \sim M\ell_s$, which has the same scaling as Eq. (13.46). The comparison is a scaling argument, not an exact BPS equality. It explains why a weakly coupled long-string description and a strongly gravitating black-hole description can meet at a smooth parametric boundary [352, 366].

13.7.2 The information paradox

The information paradox is the tension between semiclassical Hawking evaporation and unitary quantum mechanics. Semiclassically, a pure collapsing state seems to produce nearly thermal radiation,

$$|\psi\rangle\langle\psi| \quad \longrightarrow \quad \rho_{\rm thermal}, \qquad S(\rho_{\rm thermal}) > 0. \tag{13.47}$$

Unitary evolution instead preserves the von Neumann entropy of the full closed system,

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \qquad S(\rho(t)) = S(\rho(0)). \tag{13.48}$$

The problem is not the appearance of thermal radiation after coarse graining. The problem is whether the complete radiation state contains the correlations required to purify it.

D-brane state counting shows that black-hole charge sectors can have explicit microscopic Hilbert spaces, as in Eq. (13.43). Matrix theory and holographic descriptions similarly replace a purely geometric description by a quantum system with a Hamiltonian and a Hilbert space. These tools do not by themselves complete the evaporation calculation, but they specify the microscopic degrees of freedom whose unitary evolution must be understood [177, 365].

The entanglement-worldsheet proposal relates the open/closed charge discussion to entropy functionals. The starting point is the AdS_3/CFT_2 interval entropy,

$$S_{\rm EE} = \frac{c}{3}\log\frac{\ell}{\epsilon}.$$

Expanding the proposed bilocal entropy action near coincident points gives a local two-dimensional action of Polyakov type,

$$S_{\rm ws} \sim \frac{1}{4\pi\alpha'}\int d^2\sigma \left(\sqrt{h}\, h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu \frac{1}{4\pi}\int \sqrt{h}\,\Phi R^{(2)},$$

with G, B, and Φ read from the entropy kernel [231]. In this formulation open-string charge is associated with entanglement entropy, while closed-string charge is compared with Bekenstein–Hawking entropy. The construction is separate from the D1–D5 count, but it uses the same open/closed channel relation as the cylinder amplitude.

13.8 Holography and AdS/CFT

AdS/CFT gives a second nonperturbative formulation, complementary to Matrix theory. For N coincident D3-branes, the open-string low-energy limit gives four-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills theory, while the near-horizon closed-string geometry is type-IIB string theory on $AdS_5 \times S^5$. Maldacena's conjecture identifies these two limits:

$$\mathcal{N} = 4 \ SU(N) \ \text{SYM} \quad \longleftrightarrow \quad \text{type IIB string theory on } AdS_5 \times S^5. \tag{13.49}$$

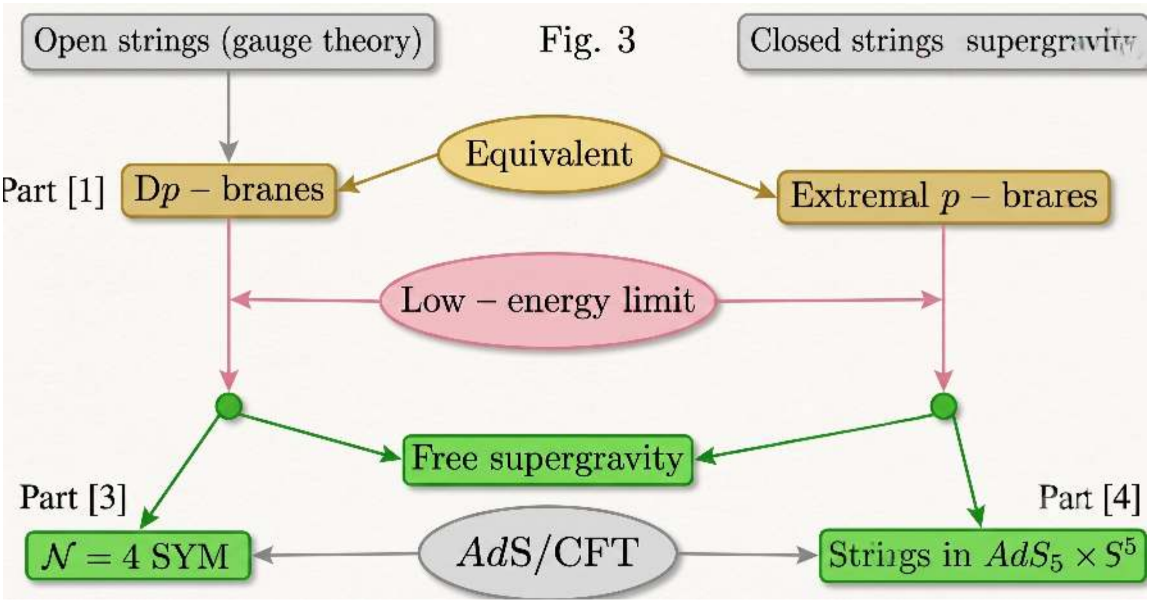


Figure 13.3: The AdS/CFT limit can be read from both sides of a stack of D3-branes. The open-string low-energy limit gives $\mathcal{N} = 4$ super-Yang–Mills theory, while the closed-string near-horizon limit gives type-IIB strings, or supergravity at strong 't Hooft coupling, on $AdS_5 \times S^5$.

The parameters of this low-energy gauge/gravity map (Figure 13.3) obey

$$\frac{L^4}{\alpha'^2} \sim g_{\mathrm{YM}}^2 N, \qquad g_s \sim g_{\mathrm{YM}}^2, \tag{13.50}$$

up to convention-dependent numerical factors. Large N and large 't Hooft coupling give the classical supergravity regime; finite N and finite coupling correspond to quantum and stringy corrections. This is again the recurring theme, that a gravitational background is defined by the Hilbert space and operator algebra of a lower-dimensional quantum theory, so spacetime dynamics is governed by a non-gravitational quantum system [369, 370].

Worldsheet derivations of holography keep the radial coordinate tied to field-theory parameters. In the worldline-to-worldsheet construction, Schwinger parameters t_i of dense multi-loop graphs combine into an emergent radial variable z, so the continuum limit produces a sigma-model action of the form

$$S_{\mathrm{ws}} \sim \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{ab} G_{MN}(X, z) \partial_a X^M \partial_b X^N.$$

The asymptotically AdS metric arises after reorganizing the graph moduli [223]. In exact AdS_3 backgrounds with NS–NS flux, the worldsheet theory contains an $SL(2, \mathbb{R})_k$ current algebra,

$$J^a(z) J^b(0) \sim \frac{k\eta^{ab}}{2z^2} + \frac{i f^{ab}{}_c J^c(0)}{z},$$

and spectra are built from representations and spectral flow [225, 243, 247]. A near-boundary scaling of Euclidean AdS_3 strings with B-field gives a Lorentz-invariant celestial CFT_2 sector, but not a full ISO(3, 1)-invariant celestial theory because the target-space conformal group is absent [228]. Holographic statements in these examples follow from explicit worldsheet CFT quantities.

Three recent AdS_3 constructions test this statement precisely. A tensionless hybrid-string construction on $AdS_3 \times S^3 \times S^3 \times S^1$ gives a Wakimoto-like free-field realization of $\mathfrak{d}(2, 1; \alpha)_1$, and its ghost-projected partition function matches the single-particle partition function of a symmetric-orbifold CFT, with DDF operators and BRST physical-state conditions included [255]. Deformed BTZ radiance in single-trace $T\bar{T}$ holography uses an exact gauged-WZW worldsheet background interpolating between AdS_3 and a linear-dilaton region; long-string emission probabilities are governed by ΔS_{BH}, and thermodynamic consistency fixes the background B-field value needed to match twisted-sector spectra [262]. A charged two-dimensional black hole near extremality, described by a dimensionally reduced $(SL(2, \mathbb{R}) \times U(1))/U(1)$ WZW model, gives one-loop entropy corrections from the torus partition function and a worldsheet realization of the black-hole/string transition [256]. All three examples turn a target-space statement–spectrum matching, radiance, or entropy correction–into a direct worldsheet calculation.

The AdS_3 worldsheet description is controlled by representation theory, not by a supergravity expansion alone. For NS–NS flux, physical states are organized by $SL(2, \mathbb{R})_k$ modules and spectral flow,

$$J_n^3 \to J_n^3 - \frac{kw}{2}\delta_{n,0}, \qquad L_0 \to L_0 - wJ_0^3 - \frac{kw^2}{4},$$

with integer w labeling long-string sectors [265]. A superspace Polyakov path integral on super Riemann surfaces gives a free-field realization of the supersymmetric $SL(2, \mathbb{R})$ model and computes near-boundary long-string correlators without relying on the usual picture-changing-operator construction [275]. Exact and strongly coupled worldsheet CFTs can also produce target spaces with Euclidean wormholes, double cones, or Einstein–Rosen bridges, including cases where the throat scale is stringy [274]. These examples show how AdS_3/CFT_2 quantities can be computed directly in the worldsheet theory.

Open strings and topological strings also serve as controlled probes of holographic dynamics. Takeda derives a completely positive Lindblad equation for holographic Brownian motion by starting from the influence functional of a trailing string endpoint, with explicit coefficients for BTZ and AdS_5 black-brane backgrounds [344]. Troost's equivariant-interpolation analysis relates Gromov–Witten theory on $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{C}^2$ to several known holographic points, including the symmetric-orbifold point dual to $AdS_3 \times S^3 \times \mathbb{C}^2$ at string-scale radius [345]. Mazenc and Sarkar's open-closed-open triality in twisted holography shows that two transverse brane stacks in the B-model on the resolved conifold give equivalent open-string field-theory descriptions, with integrating out one open sector reproducing its backreaction on the geometry seen by the other [336]. In each case worldsheet or open-string degrees of freedom encode dissipative probes, topological interpolations, or geometric backreaction without first passing through a classical supergravity limit.

Chapter 14

Advanced conformal field theory

In the first thirteen chapters conformal field theory was a means of calculation; the last five treat it as the object of study, because the compactifications we now want to control are defined by CFTs that need not have any geometric description at all. This chapter assembles the exact structure theory. Starting from the Ward identity and the Virasoro algebra as the residue algebra of the stress-tensor OPE, we classify highest-weight representations and meet the theories where symmetry determines everything: the minimal models, whose null vectors close the operator algebra on finitely many primaries, and the WZW and coset models, where the current algebras of Chapter 9 reappear as complete theories in their own right. Rational CFT organizes these examples: characters transform in finite modular representations, fusion rules are given by the Verlinde formula, and this is precisely the technology that orbifold and Gepner constructions will consume in the following chapters. The chapter ends away from criticality: the renormalization group on the space of two-dimensional theories, Zamolodchikov's c-theorem ordering its flows, and the Landau–Ginzburg description that will later bridge minimal models to Calabi–Yau geometry [359, 362].

Detailed CFT-first derivations of the conformal Killing equation, the $SO(d+1,1)$ embedding-space construction, radial quantization, the OPE-to-commutator dictionary, and the free-field central charges are collected in Ref. [4]. The discussion below uses those results in their world-sheet role: as constraints on string spectra, modular sums, current algebras, and compactification input.

Until now, conformal field theory has been a means of quantizing the string and computing its amplitudes. In the compactifications that follow it becomes the definition of the background itself: an orbifold, a Gepner model, or a Calabi–Yau sigma model is specified by its worldsheet CFT, sometimes with no large-radius geometry at all. Deciding which of these theories are consistent string vacua needs the exact representation theory, modular properties, and operator-algebra structure assembled in this chapter, and each result is used directly in the four chapters that follow: the $N=1$ algebra and minimal models in the superstring compactifications, rational CFT and the Verlinde formula in orbifolds, and the Landau–Ginzburg description as the bridge from minimal models to Calabi–Yau geometry.

Derivation used from the CFT volume. The Ward identity gives the starting equation. If ϕ_i are primaries of weights h_i, the OPE

$$T(z)\phi_i(w_i) \sim \frac{h_i\phi_i(w_i)}{(z-w_i)^2} + \frac{\partial_{w_i}\phi_i(w_i)}{z-w_i} \tag{14.1}$$

implies, after inserting $T(z)$ in a correlator and summing residues,

$$\left\langle T(z)\prod_{i=1}^{n}\phi_i(w_i)\right\rangle = \sum_{i=1}^{n}\left[\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i}\right]\left\langle\prod_{i=1}^{n}\phi_i(w_i)\right\rangle. \tag{14.2}$$

This is the local reason the stress tensor generates conformal transformations. With

$$L_m = \oint_0 \frac{dz}{2\pi i}\, z^{m+1}T(z), \tag{14.3}$$

the TT OPE gives

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}. \tag{14.4}$$

Thus the algebra used in string spectra is not a postulate: it is the residue algebra of the stress tensor OPE [42, 107].

The contour proof of this residue statement, including the radial-ordering step that converts OPE coefficients into charge commutators, is given in Ref. [4]. Here the result is used in its worldsheet form: Virasoro constraints, current-algebra Ward identities, and BRST commutators are all computed by the same residue operation.

Characters and modular invariance. Radial quantization maps a local operator to a state,

$$|\mathcal{O}\rangle = \lim_{z\to 0}\mathcal{O}(z)|0\rangle, \qquad L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle. \tag{14.5}$$

For a rational theory the Hilbert space decomposes into finitely many representations, whose characters are

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{H}_i} q^{L_0 - c/24}, \qquad q = e^{2\pi i\tau}. \tag{14.6}$$

The torus partition function must be single-valued under large diffeomorphisms of the torus:

$$Z(\tau,\bar{\tau}) = \sum_{i,j} M_{ij}\,\chi_i(\tau)\,\bar{\chi}_j(\bar{\tau}), \qquad MS = SM, \quad MT = TM. \tag{14.7}$$

This is the CFT version of the modular-invariance checks used in one-loop string amplitudes, orbifolds, heterotic lattices, and Gepner models.

Open/closed consistency has recently been recast in operator-algebraic terms. Gui's 2026 Cardy-type construction builds open and closed conformal field theories from conformal nets, not necessarily rational, and interprets Haag-duality statements as Minkowskian versions of modular invariance, Cardy consistency, and Morita equivalence of boundary field algebras [249]. This gives a modern operator-algebraic counterpart of the annulus/cylinder consistency condition used throughout the D-brane chapters. A separate 2026 matrix-model duality constructs a closed-string worldsheet dual for interacting Hermitian one-matrix models, with a B-twisted Landau–Ginzburg model coupled to topological gravity and an explicit dictionary between matrix traces and worldsheet vertex operators [236]. Both works reinforce the same principle: a consistent spectrum is fixed by the compatibility of bulk, boundary, and modular quantities.

14.1 Representations of Virasoro algebra

A Virasoro representation is built from a highest-weight state $|h\rangle$ satisfying

$$L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0 \quad (n>0), \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{14.8}$$

The descendants are $L_{-n_1}\cdots L_{-n_r}|h\rangle$. Whether such a module is unitary, and whether it contains null states, is a property of the pair (c,h) alone, and these are the same restrictions that later control the string partition function. How the allowed representations combine inside correlation functions is the question taken up next. [42, 107].

14.2 The conformal bootstrap

The conformal bootstrap defines a CFT by quantities constrained by crossing symmetry and OPE associativity. Schematically,

$$\sum_{\mathcal{O}} C_{12\mathcal{O}}C_{34\mathcal{O}}\mathcal{F}^{(s)}_{\mathcal{O}} = \sum_{\mathcal{O}} C_{13\mathcal{O}}C_{24\mathcal{O}}\mathcal{F}^{(t)}_{\mathcal{O}}. \tag{14.9}$$

This is the local form of the factorization principle used in string amplitudes, the requirement that two different decompositions of the same correlator agree. The most tightly constrained solutions of these crossing equations are the minimal models. [359, 362].

14.3 Minimal models

Minimal models are useful because they show how severely conformal symmetry can restrict a theory. The unitary Virasoro minimal models have

$$c = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \ldots. \tag{14.10}$$

Their finite spectra make the modular-invariance and operator-product constraints visible in a controlled setting, and the same models reappear later as building blocks of the Gepner and coset constructions. An explicit free-field realization of their representations is given by the Feigin–Fuchs construction. [359, 362].

14.3.1 Feigin–Fuchs representation

The Feigin–Fuchs construction realizes Virasoro representations using a free boson with background charge. In one common convention,

$$T(z) = -\frac{1}{2} :\partial\phi\,\partial\phi: + i\alpha_0\,\partial^2\phi, \qquad c = 1 - 12\alpha_0^2, \qquad h_\alpha = \frac{1}{2}\alpha(\alpha - 2\alpha_0). \tag{14.11}$$

Degenerate representations and screening charges then give an explicit method to build minimal-model correlators, turning the abstract null-vector constraints into calculable Coulomb-gas integrals.[1] [42].

14.4 Current algebras

Affine current algebras are the local CFT mechanism behind many heterotic gauge groups. The basic OPE is

$$J^a(z)J^b(0) \sim \frac{k\,\delta^{ab}}{z^2} + \frac{if^{abc}J^c(0)}{z}, \tag{14.12}$$

and the Sugawara construction gives

$$T(z) = \frac{1}{2(k + h^\vee)} : J^a J^a : (z), \qquad c = \frac{k \dim G}{k + h^\vee}. \tag{14.13}$$

Thus the gauge algebra, central-charge budget, and modular-invariant spectrum are fixed together by one set of consistency conditions, not chosen independently. [39, 138].

14.4.1 Modular invariance of current-algebra spectra

For a theory whose chiral algebra is an affine current algebra, modular invariance becomes a finite problem, because level k admits only finitely many integrable highest weights λ (those with $(\lambda, \theta) \le k$). The characters $\chi_\lambda(\tau) = \mathrm{Tr}_{\mathcal{H}_\lambda} q^{L_0 - c/24}$ close among themselves under the modular group, with the Kac–Peterson S-matrix

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda\mu}\,\chi_\mu(\tau), \qquad S_{\lambda\mu} \propto \sum_{w \in W} \epsilon(w)\, \exp\left[-\frac{2\pi i\,\big(w(\lambda+\rho), \mu+\rho\big)}{k + h^\vee}\right], \tag{14.14}$$

so a modular-invariant partition function is a finite matrix $M_{\lambda\bar\mu}$ commuting with S and T, an algebraic problem with a complete answer. For $SU(2)_k$ the solutions follow an A-D-E pattern: the diagonal invariants exist at every level, while exceptional pairings occur exactly at the levels tied to the D and E Dynkin diagrams. This classification is the technology behind the statement, used in Chapter 9, that a level-one simply laced current algebra has essentially a unique spectrum: the lattice construction and the character-theoretic construction are forced to agree [138, 359].

[1] The signs in Eq. (14.11) change across Euclidean/Lorentzian and real/imaginary background-charge conventions. What matters is not the Coulomb-gas method itself but the consistent choice of conventions for $T = -\frac{1}{2}(\partial\phi)^2 + Q\partial^2\phi$ and for the two-point function of ϕ; once the OPE $\phi(z)\phi(0) \sim -\log z$ or $+\log z$ is fixed, c and h_α follow from that choice.

14.4.2 Strings on group manifolds

Strings on a compact group manifold are described by the Wess–Zumino–Witten model. Its action contains both the sigma-model kinetic term and a quantized three-form term,

$$S_{\mathrm{WZW}} = \frac{k}{8\pi}\int_\Sigma \mathrm{Tr}(g^{-1}\partial_a g\, g^{-1}\partial^a g) + \frac{k}{12\pi}\int_B \mathrm{Tr}(g^{-1}dg)^3, \qquad \partial B = \Sigma. \tag{14.15}$$

The integer k is the affine level in Eq. (14.12), so a group-manifold background is an exact CFT, not a low-energy geometry with uncontrolled α' corrections. [42, 138].

14.5 Coset models

Coset CFTs divide a current algebra by a subalgebra. At the level of stress tensors,

$$T_{G/H} = T_G - T_H, \qquad c_{G/H} = c_G - c_H. \tag{14.16}$$

This construction produces many rational CFTs, and when the gauged WZW model has a geometric interpretation it gives an exact string background, so subtracting gauge redundancy on the worldsheet builds a new target-space geometry. [42, 144].

14.6 Representations of the $N = 1$ superconformal algebra

The $N = 1$ superconformal algebra extends the Virasoro algebra by a dimension-3/2 supercurrent $G(z)$:

$$\begin{aligned} T(z)G(0) &\sim \frac{3G(0)}{2z^2} + \frac{\partial G(0)}{z}, \\ G(z)G(0) &\sim \frac{2c}{3z^3} + \frac{2T(0)}{z}. \end{aligned} \tag{14.17}$$

The mode expansion has Neveu–Schwarz and Ramond sectors, corresponding to half-integer and integer modes of G. These two sectors are the representation-theoretic origin of the spin-structure sums and GSO projections used earlier in the superstring chapters. [42, 121].

14.7 Rational CFT

A rational CFT has finitely many irreducible representations of the chiral algebra. The characters form a finite-dimensional representation of the modular group,

$$\chi_i(-1/\tau) = S_{ij}\chi_j(\tau), \qquad \chi_i(\tau+1) = T_{ij}\chi_j(\tau), \tag{14.18}$$

and the fusion rules are encoded by the Verlinde formula,

$$N_{ij}{}^{k} = \sum_\ell \frac{S_{i\ell}S_{j\ell}S^*_{k\ell}}{S_{0\ell}}. \tag{14.19}$$

Therefore rational CFT is the natural language for minimal models, WZW models, Gepner models, and many exactly solvable string compactifications. [42, 107].

Calderon-Infante and Mohseni study infinite-distance limits on conformal manifolds of four-dimensional $\mathcal{N} = 2$ quiver SCFTs and argue that, in holographic large-N limits, the approach to the overall-free point is dual to a tensionless-string limit in the bulk [266]. The diagnostic is the large-N Hagedorn temperature T_H, extracted from the exponential density of states. This connects three parts of the review: rational or symmetric-product CFT technology organizes the spectrum, the Hagedorn growth of Chapter 6 detects a string tower, and the AdS/CFT map interprets the tower as the ultraviolet completion of bulk gravity.

14.8 Renormalization group flows

A two-dimensional QFT deformed away from a CFT by operators $\mathcal{O}_i$ has

$$S = S_{\rm CFT} + \sum_i \lambda^i \int d^2z\, \mathcal{O}_i(z,\bar{z}), \qquad \beta^i = \frac{d\lambda^i}{d\log\mu}. \tag{14.20}$$

Fixed points $\beta^i = 0$ are conformal theories. In string theory the same condition is a spacetime equation of motion for the background fields, so RG flow on the worldsheet is the origin of target-space dynamics. [10, 42].

14.8.1 Scale invariance and renormalization group flows

Scale invariance requires the trace of the stress tensor to vanish up to improvement terms. Perturbatively,

$$T^a_{\ a} = \sum_i \beta^i \mathcal{O}_i + \partial_a J^a. \tag{14.21}$$

For unitary, Poincare-invariant two-dimensional theories, the improvement ambiguity is tightly constrained, so scale-invariant fixed points are normally treated as conformal fixed points in this setting. In the sigma-model chapters this becomes the statement that Weyl invariance is equivalent to the vanishing of target-space beta functions. [10].

14.8.2 The Zamolodchikov c theorem

Zamolodchikov's theorem gives a function $C(\lambda)$ that decreases along unitary two-dimensional RG flows and equals the central charge at conformal fixed points. In schematic normalization,

$$\frac{dC}{d\log\mu} = -G_{ij}(\lambda)\,\beta^i\beta^j \le 0, \tag{14.22}$$

where G_{ij} is a positive two-point-function metric on coupling space. Relevant perturbations of the minimal models therefore flow from larger to smaller central charge, which is why the central charge measures the number of degrees of freedom in the exact CFT blocks used by string theory.[2] [7].

14.8.3 Conformal perturbation theory

Conformal perturbation theory computes the beta functions and anomalous dimensions created by deformations of a known CFT. For operators with OPE coefficients $C_{ij}{}^k$,

$$\beta^k = (2-\Delta_k)\lambda^k - \pi C_{ij}{}^k \lambda^i\lambda^j + \cdots . \tag{14.23}$$

This is the local tool behind statements such as "a vertex operator is marginal" or "a background solves the string equations of motion". Exact marginality is stronger than $\Delta = 2$ at first order; it requires the full beta function to vanish. [10].

The $J\bar{J}$ deformation gives a higher-genus refinement of Eq. (14.23). Liu rewrites conformal perturbation theory for $J\bar{J}$-deformed CFTs by using the Riemann bilinear identity to convert the integrated deformation into a dressing of operators plus large-cycle integrals; the resulting torus and higher-genus partition functions are kernel integrals that preserve modular invariance or covariance [258]. In practice, an exactly marginal current-current deformation can change conformal weights and charges while keeping the modular consistency constraints of the string partition function under analytic control. This is the deformation-theory cousin of the $O(d,d)$ and TsT examples discussed in the compactification chapters.

[2] The numerical coefficient in Eq. (14.22) depends on the normalization of the stress tensor and of the operator two-point functions. The sign is universal for a unitary two-dimensional RG flow, but the prefactor multiplying $G_{ij}\beta^i\beta^j$ is convention-dependent, so unless one fixes a Zamolodchikov metric and a stress-tensor normalization the equation is read only as a monotonicity statement.

14.9 Statistical Mechanics

Two-dimensional statistical systems at a second-order critical point are described by CFT quantities, the central charge, primary fields, scaling dimensions, and OPE coefficients. For example, the Ising model corresponds to the $c = 1/2$ minimal model with primaries $\mathbf{1}, \sigma, \epsilon$. The same CFT methods carry over to string theory, where the quantities that classify critical spin systems also classify the allowed worldsheet sectors. [42, 107].

14.9.1 Landau–Ginzburg models

In two-dimensional $\mathcal{N} = 2$ theories, Landau–Ginzburg models with superpotential $W(\Phi_i)$ flow in the infrared to superconformal theories when the weights are quasi-homogeneous. The chiral ring is

$$\mathcal{R}_{\text{chiral}} = \frac{\mathbb{C}[\Phi_i]}{(\partial_{\Phi_i} W)}, \tag{14.24}$$

and for charges q_i the central charge is

$$c = 3 \sum_i (1 - 2q_i). \tag{14.25}$$

The same chiral-ring calculation is used in minimal models, Gepner models, and Calabi–Yau compactifications. [195, 208].

Chapter 15

Orbifolds

Toroidal compactification of the heterotic string preserves too much supersymmetry and too large a gauge group to resemble particle physics; this chapter develops a minimal controlled construction that breaks both. An orbifold identifies points of the torus under a discrete group, and the worldsheet requires more than the identification alone: closed strings need only close up to a group element, so twisted sectors localized at the fixed points are obligatory, and we prove it directly, the modular transformations of the torus map the projected untwisted trace into the twisted sectors, so a modular-invariant theory must contain them all. With the general rules established (sector sums, fractional moding, the twist-field weight $\nu(1-\nu)/2$), we apply them to the heterotic string, where the geometric twist must be accompanied by a shift in the gauge lattice, and level matching plus modular invariance sharply restrict the combinations. The conditions for surviving spacetime supersymmetry ($SU(3)$ twists), the resulting untwisted and twisted matter spectra, Wilson lines, discrete torsion, and the threshold corrections to gauge couplings complete a construction kit that produces four-dimensional chiral gauge theories from exactly solvable worldsheets [10, 86].

Why twisted sectors are forced. Let a finite group G act on a parent CFT. A closed string does not need to close in the covering space; it only needs to close up to the quotient action:

$$X(\sigma + 2\pi, \tau) = g\, X(\sigma, \tau), \qquad g \in G. \tag{15.1}$$

This defines a Hilbert space $\mathcal{H}_g$. Quantization of the orbifold theory therefore starts from

$$\mathcal{H}_{\rm orb} = \bigoplus_{g\in G} \mathcal{H}_g, \qquad \mathcal{H}_{\rm phys} = (\mathcal{H}_{\rm orb})^G\,. \tag{15.2}$$

The invariant subspace is obtained by inserting the projector

$$P_G = \frac{1}{|G|} \sum_{h\in G} h. \tag{15.3}$$

Thus an orbifold is not "keep the invariant untwisted fields". It is the invariant part of the sum over all sectors that close only up to a group element.

Torus partition function and modular covariance. On the torus one labels the spatial cycle by g and inserts h along the Euclidean-time cycle. Consistency requires the twists to commute, so

$$Z_{\rm orb}(\tau, \bar\tau) = \frac{1}{|G|} \sum_{\substack{g,h\in G\\ gh=hg}} Z\begin{bmatrix} g \\ h \end{bmatrix}(\tau, \bar\tau). \tag{15.4}$$

The modular generators act by changing the basis of cycles:

$$S:\ Z\begin{bmatrix} g \\ h \end{bmatrix} \mapsto Z\begin{bmatrix} h \\ g^{-1} \end{bmatrix}, \qquad T:\ Z\begin{bmatrix} g \\ h \end{bmatrix} \mapsto Z\begin{bmatrix} g \\ hg \end{bmatrix}. \tag{15.5}$$

The full sum is therefore closed under S and T. This derivation makes precise the usual statement that modular invariance forces twisted sectors [10, 58, 86].

Fractional modes and localized states. For one complex boson twisted by $X(e^{2\pi i} z) = e^{2\pi i \nu} X(z)$, $0 < \nu < 1$, the oscillator modes are shifted:

$$\partial X(z) = \sum_{m\in\mathbb{Z}} \alpha_{m-\nu}\, z^{-m+\nu-1}. \tag{15.6}$$

Zeta-regularizing the shifted oscillator zero-point energy gives the bosonic twist-field weight

$$h_{\text{twist}}^{\text{complex}} = \frac{1}{2}\nu(1-\nu). \tag{15.7}$$

For a real $\mathbb{Z}_2$ boson this is $h = 1/16$. These twist fields are localized at fixed points because the boundary condition has no ordinary translational zero mode in the twisted directions.

15.1 Orbifolds of the heterotic string

Applying the general orbifold rules of Eq. (15.4) to the heterotic string introduces one genuinely new element: the group action must reach into the gauge sector. A purely geometric twist θ acting on (X^m, ψ^m) alone is not modular invariant, because the left-moving gauge lattice contributes an uncompensated phase to the transformation of each twisted block. The standard cure embeds the point group into the gauge degrees of freedom as a shift V of the Γ_{16} momenta,

$$P \longrightarrow P + kV \text{ in the } \theta^k\text{-twisted sector}, \qquad Z\begin{bmatrix}\theta^k \\ \theta^l\end{bmatrix} \propto e^{2\pi i\, kl\,(V^2-v^2)/2} \times (\cdots), \tag{15.8}$$

so that the geometric twist phases (from the eigenvalues $e^{2\pi i v_i}$) and the gauge shift phases cancel between the S- and T-images of each block. For a $\mathbb{Z}_N$ orbifold this cancellation is the level-matching condition $N(V^2 - v^2) \in 2\mathbb{Z}$, the single equation that decides which combinations of geometric twist and gauge embedding define consistent heterotic vacua. Everything phenomenological in this chapter, the surviving gauge group, the untwisted and twisted matter, the anomaly structure, descends from the solutions of this one modular constraint [86, 361].

15.1.1 Modular invariance and level matching

The sector sum of Eq. (15.4) can be tested block by block, and for a heterotic $\mathbb{Z}_N$ orbifold the test reduces to one arithmetic condition. Under $T : \tau \to \tau + 1$ each twisted block picks up the phase of its left–right ground-state energy mismatch,

$$Z\begin{bmatrix}\theta^k \\ \theta^l\end{bmatrix}(\tau+1) = e^{2\pi i\,(E_k - \bar{E}_k)}\, Z\begin{bmatrix}\theta^k \\ \theta^{l+k}\end{bmatrix}(\tau), \qquad E_k - \bar{E}_k = \frac{k^2}{2}\big(V^2 - v^2\big) \pmod{\tfrac{1}{N}}, \tag{15.9}$$

where $v = (v_1, v_2, v_3)$ are the geometric twist angles and V is the gauge shift. Requiring the orbit of blocks generated by S and T to close with mutually consistent phases yields $N(V^2 - v^2) \in 2\mathbb{Z}$, which is the level-matching condition of the previous section, here obtained as a consequence of modular closure. Its content is that twisted-sector states can only satisfy $L_0 - \bar{L}_0 \in \mathbb{Z}/N$ when the ground-state mismatch is itself a multiple of $1/N$; otherwise the projector onto invariant states annihilates the whole sector and no modular completion exists. Wilson lines refine the condition sector by sector, and a heterotic orbifold model is fully specified by the input $(v, V, \text{Wilson lines})$ passing this test [86, 361].

The level-matching phase has an anomaly interpretation. For a cyclic heterotic asymmetric orbifold, the T-transformation of the k-twisted block gives

$$Z_k(\tau+1) = e^{2\pi i (E_k - \bar{E}_k)} Z_k(\tau), \qquad E_k - \bar{E}_k = \frac{k^2}{2}(V^2 - v^2) \pmod{1/N}.$$

Worldsheet Dai–Freed anomaly cancellation sets this exponentiated phase to one, with the same even-m refinements that appear in cyclic asymmetric orbifolds [238]. Bosonization gives the same anomaly in the fermionic and lattice descriptions. Free-field and minimal-model constructions of heterotic Calabi–Yau orbifolds then use the same modular test to select physical vertex operators from the $N = 2$ minimal-model input [246]. Level matching is therefore a global quantum consistency condition.

15.1.2 Other free CFTs

The orbifold construction applies to any CFT with a discrete symmetry, beyond bosons on a torus, and two further free realizations matter in practice. In free-fermionic models the internal degrees of freedom are two-dimensional fermions and the “orbifold group” acts through boundary-condition vectors; the sector sum

$$Z_{\text{orb}} = \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} Z \begin{bmatrix} g \\ h \end{bmatrix} \tag{15.10}$$

becomes a sum over spin-structure assignments, with modular invariance imposing the same bilinear constraints on the boundary-condition vectors that level matching imposes on shifts. Asymmetric orbifolds act differently on left- and right-movers, for instance twisting X_L while leaving X_R fixed, which is only possible at special points of the Narain moduli space where the lattice splits; they trade geometric interpretation for smaller spectra and are the natural home of constructions with few or no moduli. In both cases the rule established for geometric orbifolds is unchanged: the group action, the projection, and the twisted sectors form one modular orbit, and consistency is checked on the partition function, not on the target space [58, 78, 86].

Gkountoumis's 2026 dissertation gives a systematic treatment of freely acting type-IIB orbifolds, especially asymmetric non-geometric examples with $\mathcal{N} = 6, 4, 2$, or 0 supersymmetry in five dimensions [330]. The work compares the duality-twist and lattice descriptions, computes spectra and moduli spaces, and tracks how branes transform in these backgrounds. It extends Eq. (15.10): freely acting orbifolds have no fixed-point twist fields in the geometric sense, but modular invariance still forces the full sector sum and determines which non-geometric compactifications are genuine string vacua.

15.2 Spacetime supersymmetry

In orbifold compactifications, spacetime supersymmetry survives only if the orbifold action preserves the internal spinor and is compatible with the GSO projection. For a geometric twist with angles v_i on the three complex internal planes, the familiar local condition is

$$v_1 + v_2 + v_3 = 0 \pmod 1, \tag{15.11}$$

which says that the twist lies in $SU(3)$ instead of only $SO(6)$. In heterotic models this must be supplemented by the gauge shift and modular-invariance constraints. [86, 91].

15.3 Examples

The examples show how the general rules change an actual spectrum. A standard useful case is the circle orbifold $S^1/\mathbb{Z}_2$, where the generator acts by $X \mapsto -X$. The untwisted sector is projected onto $\mathbb{Z}_2$-even states, while the twisted sectors obey

$$X(\sigma + 2\pi) = -X(\sigma) + 2\pi R m. \tag{15.12}$$

The fixed points then support localized twisted states. The example shows that an orbifold is more than a space obtained from target-space points, because the worldsheet theory also contains sectors that a point-particle identification would miss.

A second useful example is a toroidal heterotic orbifold, where the geometric twist is accompanied by a gauge-lattice shift. Modular invariance and level matching then restrict the allowed shifts and Wilson lines. Such examples are worth keeping because they show how spacetime gauge quantities are constrained by the worldsheet construction. [10, 86].

15.3.1 Connection with grand unification

Heterotic orbifolds are useful for grand-unified model building because the ten-dimensional gauge group can be broken by gauge shifts and Wilson lines while still retaining local enhanced symmetries at fixed points. Schematically, a shift V acts on gauge lattice momenta as

$$P \longmapsto P + V, \qquad P^2 = 2 \quad \text{for roots}, \tag{15.13}$$

so the surviving gauge bosons are the roots satisfying the orbifold projection. The projection shows how one can obtain $SU(5)$, $SO(10)$, or Standard-Model-like subgroups from an underlying

$E_8 \times E_8$ or $SO(32)$ lattice.[1] [86, 94, 125].

15.3.2 Generalizations

The natural generalizations are obtained by adding discrete torsion, Wilson lines, asymmetric actions, or non-abelian orbifold groups. Discrete torsion modifies the sector sum by phases,

$$Z_{\text{orb}} = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h) Z \begin{bmatrix} g \\ h \end{bmatrix}, \tag{15.14}$$

where the phases must obey compatibility conditions so that the partition function still transforms properly under the modular group. Wilson lines play a similar role in heterotic compactifications by embedding the space-group action into the gauge lattice.

These generalizations are not arbitrary additions. Each phase, shift, or asymmetric action changes the projection, the twisted-sector spectrum, or the gauge group. The text must therefore list the extra input and then state the corresponding level-matching or modular-invariance condition. [10, 86].

15.3.3 Worldsheet supersymmetry of orbifold models

An orbifold action must be compatible with the worldsheet supersymmetries that the parent theory carries. In the heterotic string the right-moving $(0,1)$ supercurrent $\tilde{T}_F = i\sqrt{2/\alpha'}\,\tilde{\psi}^\mu \bar{\partial} X_\mu$ must remain a well-defined single field after the projection, so the twist must rotate $\tilde{\psi}^m$ and X^m identically, the orbifold group embeds in the diagonal of the rotation acting on bosons and fermions. When the twist lies in an $SU(3)$ subgroup of the internal $SO(6)$, the surviving internal CFT has accidentally *enhanced* supersymmetry: the standard-embedding orbifold is a $(2,2)$ superconformal theory, with a left-moving $N = 2$ algebra assembled from the gauge fermions that were twisted in parallel with the geometry. This is the exactly solvable limit of the Calabi–Yau statement of Chapter 16, and the $(2,2)$ algebra is what Chapter 17 exploits for chiral rings and mirror symmetry. Twists with a more general gauge embedding produce $(0,2)$ models: still supersymmetric in spacetime, but without the left-moving $N = 2$ structure, which is why their Yukawa couplings are much harder to compute [86, 355].

15.4 Low-energy field theory of orbifold models

Reducing the ten-dimensional action

$$S_{\text{NSNS}} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 + \cdots \right) \tag{15.15}$$

on an orbifold produces a four-dimensional $N = 1$ theory whose field content splits by origin, and the split matters because the two kinds of fields have different couplings. Untwisted fields are ten-dimensional fields with invariant internal indices: the dilaton S, the Kähler and complex-structure moduli T and U of the underlying torus, and the untwisted charged matter, all with Kähler potentials of the no-scale form $-\log(S + \bar{S}) - \sum \log(T + \bar{T})$ inherited from the torus reduction. Twisted fields have no ten-dimensional ancestor: they live at the fixed points, their vacuum expectation values are blow-up modes that deform the orbifold toward a smooth Calabi–Yau, and their couplings are computed from twist-field correlators instead of from dimensional reduction. The orbifold point is thus a special corner of a Calabi–Yau moduli space where the worldsheet is free and everything is calculable; moving away from it is controlled by the twisted-sector effective theory constructed here [86, 355].

[1] Obtaining a realistic gauge group is only the first step of a model. A complete construction lists the massless spectrum sector by sector, untwisted and each twisted sector, verifies the absence of gauge and gravitational anomalies apart from at most one Green–Schwarz $U(1)$, works out the Yukawa couplings allowed by the orbifold space-group selection rules, and estimates the threshold corrections of Eq. (15.19). Any of these stages can remove a candidate — unwanted vector-like matter may fail to decouple, an anomalous $U(1)$ may trigger a destabilizing Fayet–Iliopoulos term, and needed Yukawas may be forbidden by the selection rules. The models cited here survive all of these checks.

15.4.1 Untwisted states

The untwisted spectrum is the parent spectrum filtered by the projector, and the filtering can be written state by state. A parent state carries a geometric phase $e^{2\pi i(\rho-\bar{\rho})\cdot v}$ from its oscillator content and a gauge phase $e^{2\pi i P\cdot V}$ from its lattice momentum, and it survives when the product is one:

$$P \cdot V - (\rho - \bar{\rho}) \cdot v \;\in\; \mathbb{Z}. \tag{15.16}$$

Applied to the ten-dimensional gauge bosons ($\rho = \bar{\rho} = 0$), this keeps exactly the roots with $P \cdot V \in \mathbb{Z}$: the orbifold breaks $E_8 \times E_8$ or $SO(32)$ to the commutant of the shift, which is how $E_6 \times SU(3)$ and its relatives appear. Applied to the internal components of the metric and B-field, it keeps the moduli compatible with the twist, for a $\mathbb{Z}_3$ example, nine untwisted moduli $T^{i\bar{j}}$. Applied to the would-be gravitini, it keeps one for $v \in SU(3)$, which is the $N = 1$ statement of the previous section. Charged untwisted matter comes from the states whose two phases cancel between geometry and gauge lattice, one copy per fixed combination, and the count is completed in the twisted sectors below [86, 361].

15.4.2 T-duality of orbifold models

T-duality survives orbifolding, but it acts on more than a radius. The circle formulas

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}, \qquad N - \tilde{N} + nw = 0 \tag{15.17}$$

still govern each torus direction, and the duality group of the parent Narain lattice descends to the subgroup commuting with the orbifold action, so dualities can permute fixed points, mix Wilson lines with radii, and identify orbifold models whose defining input (v, V) look different. Two consequences are worth separating. First, the moduli space of a given orbifold is the parent moduli space restricted to twist-invariant directions and quotiented by this residual duality group, which is how apparently distinct models in the literature turn out to be the same theory. Second, T-duality can map a symmetric orbifold to an asymmetric one, since the duality acts oppositely on X_L and X_R; the asymmetric constructions of the previous subsection are in this sense not exotic but T-dual images of geometric ones. The invariant content is again the CFT, and the geometric description is one coordinate patch on its moduli space [86, 350, 361].

15.4.3 Twisted states

Twisted states are localized at fixed points or fixed tori of the orbifold action. If $X(e^{2\pi i}z) = gX(z)$, the mode expansion has fractional moding, and the ground-state conformal weight is shifted. For a twist angle $0 < \nu_i < 1$ in a complex plane,

$$h_{\text{twist}} = \frac{1}{2}\sum_i \nu_i(1-\nu_i) \quad \text{for the bosonic coordinates.} \tag{15.18}$$

These states are often the localized matter fields of the four-dimensional theory. [58, 86].

15.4.4 Threshold corrections

String threshold corrections compare the running gauge couplings below the compactification scale with the one-loop string amplitude. A standard heterotic form is

$$\Delta_a = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\mathcal{B}_a(\tau, \bar{\tau}) - b_a\right), \tag{15.19}$$

where $\mathcal{B}_a$ inserts the gauge charge operator in the modular-invariant partition function and b_a subtracts the field-theory infrared divergence. Here modular invariance becomes a quantitative correction to four-dimensional unification. [95, 97].

Chapter 16

Calabi–Yau compactification

The previous chapter used orbifolds, which preserve supersymmetry but are singular at their fixed points. This chapter identifies the smooth internal geometries that keep exactly $N = 1$ supersymmetry in four dimensions, and it does so by following a single chain of implications, each link of which is established in a section below. Requiring one unbroken supercharge forces a covariantly constant spinor on the internal six-manifold (Section 16.1). The existence of such a spinor restricts the Riemannian holonomy to $SU(3)$, which is the same as saying the manifold is Kähler with vanishing first Chern class (Section 16.2). Yau's theorem then guarantees a Ricci-flat metric, and Ricci flatness is exactly what the leading string equations of motion demand. The geometric background needed for these steps, real, complex, and Kähler manifolds and Dolbeault cohomology, is assembled first, each harmonic form becomes a massless four-dimensional field, with $h^{1,1}$ Kähler moduli and $h^{2,1}$ complex-structure moduli. For the heterotic string the geometry is not by itself a vacuum; a gauge bundle satisfying the Hermitian Yang–Mills equations and the anomaly Bianchi identity must be chosen, and the number of chiral generations is then set by an index. The corrections the leading-order analysis omits, the α' and worldsheet-instanton terms and the flux generalizations, are taken up in Chapter 18.

Supersymmetry implies special holonomy. Compactify ten-dimensional supergravity on $\mathbb{R}^{1,3} \times X$ with no flux at leading order. The internal part of the gravitino variation is

$$\delta\psi_m = \nabla_m \eta + \cdots . \tag{16.1}$$

For unbroken four-dimensional supersymmetry there must be a nonzero internal spinor η satisfying

$$\boxed{\nabla_m \eta = 0.} \tag{16.2}$$

The geometric consequence follows from this condition. In the flux-free product ansatz the ten-dimensional supersymmetry parameter factorizes as

$$\epsilon_{10} = \zeta_4 \otimes \eta + \text{c.c.}, \tag{16.3}$$

and the external Killing-spinor equation fixes ζ_4, while the internal equation is precisely Eq. (16.2). Parallel transport of η around any closed curve must return the same spinor, so every holonomy element $h \in \mathrm{Hol}(X)$ obeys

$$h \cdot \eta = \eta. \tag{16.4}$$

Equivalently, the curvature generators of the holonomy algebra must annihilate η. This follows by commuting the two covariant derivatives:

$$0 = [\nabla_m, \nabla_n]\eta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \eta. \tag{16.5}$$

By the Ambrose–Singer theorem, the holonomy algebra is generated by such curvature endomorphisms. Hence the holonomy is contained in the stabilizer of a nonzero six-dimensional chiral spinor. Since

$$\mathrm{Spin}(6) \simeq SU(4), \tag{16.6}$$

and a positive-chirality spinor transforms as the fundamental $\mathbf{4}$ of $SU(4)$, choosing a nonzero spinor selects one complex direction in $\mathbf{4}$. The subgroup preserving that vector is

$$SU(4) \longrightarrow SU(3), \qquad \mathbf{4} \longrightarrow \mathbf{1} \oplus \mathbf{3}, \tag{16.7}$$

so

$$\mathrm{Hol}(X) \subseteq SU(3). \tag{16.8}$$

The same conclusion can be written in terms of spinor bilinears. A unit spinor η_+ defines

$$J_{mn} = -i\eta_+^\dagger \gamma_{mn}\eta_+, \qquad \Omega_{mnp} = \eta_-^\dagger \gamma_{mnp}\eta_+. \tag{16.9}$$

Because the Levi–Civita connection is metric-compatible, $\nabla\gamma_m = 0$. Therefore

$$\nabla_r J_{mn} = -i(\nabla_r \eta_+^\dagger)\gamma_{mn}\eta_+ - i\eta_+^\dagger \gamma_{mn}\nabla_r\eta_+ = 0, \tag{16.10}$$

$$\nabla_r \Omega_{mnp} = (\nabla_r \eta_-^\dagger)\gamma_{mnp}\eta_+ + \eta_-^\dagger \gamma_{mnp}\nabla_r\eta_+ = 0. \tag{16.11}$$

Thus the manifold has a covariantly constant Kahler form J and holomorphic three-form Ω:

$$\nabla J = 0, \qquad \nabla\Omega = 0, \qquad c_1(X) = 0, \qquad \mathrm{Hol}(X) \subseteq SU(3). \tag{16.12}$$

The conditions $\nabla J = 0$ and $\nabla\Omega = 0$ imply $dJ = 0$ and $d\Omega = 0$; hence the metric is Kahler and the canonical bundle is trivialized by the nowhere-vanishing $(3, 0)$-form Ω. Since the curvature is valued in $\mathfrak{su}(3)$, its trace part vanishes, so the Ricci form

$$\rho = iR_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} \tag{16.13}$$

represents a zero first Chern class. Conversely, Yau's theorem states that a compact Kahler manifold with $c_1(X) = 0$ admits a unique Ricci-flat Kahler metric in each Kahler class. This is why the geometric condition and the supersymmetry condition meet in Calabi–Yau compactification [185, 355].

How Hodge numbers become fields. Metric deformations split into Kahler and complex-structure deformations. A Kahler deformation changes

$$J \longmapsto J + \delta J, \qquad \delta J = t^a\omega_a, \qquad \omega_a \in H^{1,1}(X), \tag{16.14}$$

so there are $h^{1,1}$ Kahler moduli before quantum corrections and identifications. A complex-structure deformation changes the holomorphic three-form as

$$\delta\Omega = z^i\chi_i - K_i\Omega, \qquad \chi_i \in H^{2,1}(X), \tag{16.15}$$

so there are $h^{2,1}$ complex-structure moduli. The compact way to remember the type-II assignment on a Calabi–Yau threefold is

	vector multiplets	hypermultiplets
IIA on X	$h^{1,1}(X)$	$h^{2,1}(X) + 1$
IIB on X	$h^{2,1}(X)$	$h^{1,1}(X) + 1$

(16.16)

The extra $+1$ is the universal hypermultiplet. Mirror symmetry exchanges $h^{1,1}$ and $h^{2,1}$, so the table is also the first low-energy sign of mirror symmetry.

Heterotic bundle constraint. For heterotic strings the geometry alone is not a vacuum; one must choose a gauge bundle $V \to X$. The Bianchi identity is

$$dH = \frac{\alpha'}{4}\left(\mathrm{Tr}\, R \wedge R - \mathrm{Tr}\, F \wedge F\right). \tag{16.17}$$

In cohomology this requires $c_2(TX) = c_2(V)$, up to five-brane sources. The standard embedding sets $F = R$, embeds the spin connection in the gauge connection, and breaks E_8 to E_6. More general stable holomorphic bundles give different gauge groups and chiral matter, but the Bianchi identity and slope-stability conditions must still be checked. This is why a phrase like "compactify on a Calabi–Yau" is incomplete unless the bundle, flux, and anomaly-cancellation conditions are specified [86, 355].

16.1 Conditions of $N = 1$ supersymmetry

For a flux-free compactification on $\mathbb{R}^{1,3} \times X$, four-dimensional $N = 1$ supersymmetry means that exactly one internal spinor survives. Decompose a ten-dimensional supersymmetry parameter as

$$\epsilon_{10} = \zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-, \tag{16.18}$$

where ζ is the four-dimensional spinor and η is a spinor on the six-dimensional compact space. The internal gravitino variation gives

$$\nabla_m \eta_+ = 0, \qquad \mathrm{Hol}(X) \subseteq SU(3). \tag{16.19}$$

The second statement follows because $SU(3) \subset SO(6)$ is the stabilizer of a nonzero chiral spinor. In heterotic compactification this is only the geometric part. The gauge bundle must also obey the Hermitian Yang–Mills conditions

$$F_{(0,2)} = F_{(2,0)} = 0, \qquad J^{m\bar{n}} F_{m\bar{n}} = 0, \tag{16.20}$$

and the Bianchi identity in Eq. (16.17). Thus the full condition is not "Calabi–Yau geometry alone"; it is geometry plus bundle input plus anomaly cancellation. [86, 185, 355].

16.2 Calabi–Yau manifolds

A Calabi–Yau threefold is usually defined as a compact Kahler threefold with trivial canonical bundle and, for the strict case used in standard compactifications, $SU(3)$ holonomy. Equivalently it admits a nowhere-vanishing holomorphic three-form Ω and a Ricci-flat Kahler metric:

$$c_1(X) = 0, \qquad \Omega \in H^{3,0}(X), \qquad R_{i\bar{j}} = 0, \qquad \mathrm{Hol}(X) = SU(3) \quad \text{in the strict case.} \tag{16.21}$$

Yau's theorem supplies the Ricci-flat metric once the manifold is compact, Kahler, and $c_1(X) = 0$. The physics uses this theorem in a specific way: Ricci flatness cancels the leading sigma-model beta function, while the covariantly constant spinor preserves four-dimensional supersymmetry. The Hodge diamond its entries count harmonic forms used to expand ten-dimensional fields. [185, 355].

16.2.1 Real manifolds

A real n-manifold is locally modeled on $\mathbb{R}^n$, with transition functions that are smooth on overlaps. The concepts that later become physically important are the metric, topology: differential forms, de Rham cohomology, characteristic classes, and Betti numbers. In compactification language,

$$b_k = \dim H^k_{\mathrm{dR}}(M), \tag{16.22}$$

counts harmonic k-forms after a metric is chosen. These harmonic forms are the geometric origin of many lower-dimensional massless fields. [185].

16.2.2 Complex manifolds

A complex manifold of complex dimension n has holomorphic transition functions and a decomposition of complexified forms into (p, q) type. The Dolbeault cohomology groups

$$H^{p,q}_{\bar{\partial}}(X) = \frac{\ker(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\mathrm{im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})} \tag{16.23}$$

give the Hodge numbers $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}(X)$. These numbers later count moduli and massless fields in Calabi–Yau compactification. [42, 185].

16.2.3 Kahler manifolds

A Kahler manifold is both complex and symplectic in a compatible way. In local complex coordinates the metric has only mixed components, and the Kahler form is

$$J = i\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}, \qquad dJ = 0, \qquad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K. \tag{16.24}$$

The last equality is local and introduces the Kahler potential K. This structure is what lets one split deformations into Kahler and complex-structure classes. It is also why the sigma model has enhanced worldsheet supersymmetry: for a Kahler target, the complex structure is covariantly constant and can act on worldsheet fermions consistently. Calabi–Yau geometry is the special case where this Kahler metric can be chosen Ricci-flat. [185, 195].

16.2.4 Manifolds of $SU(3)$ holonomy

The reduction $SO(6) \to SU(3)$ can be seen explicitly from spinor bilinears. A unit covariantly constant spinor η_+ defines

$$J_{mn} = -i\,\eta_+^\dagger \gamma_{mn}\eta_+, \qquad \Omega_{mnp} = \eta_-^\dagger \gamma_{mnp}\eta_+, \qquad \nabla J = \nabla\Omega = 0. \tag{16.25}$$

Covariant constancy implies $dJ = d\Omega = 0$, and the Ricci form

$$\rho = iR_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}} \tag{16.26}$$

represents $2\pi c_1(X)$. Hence $SU(3)$ holonomy forces $c_1 = 0$, while Yau's theorem supplies the reverse direction after fixing a Kahler class. For string theory the internal spinor is the geometric carrier of the unbroken four-dimensional supercharge. [355].

16.2.5 Examples

The standard example is the quintic hypersurface

$$X_5 = \left\{ [z_0 : \cdots : z_4] \in \mathbb{P}^4 \;\middle|\; \sum_{i=0}^{4} z_i^5 = 0 \right\}. \tag{16.27}$$

The adjunction formula gives

$$K_{X_5} = (K_{\mathbb{P}^4} \otimes \mathcal{O}(5))\big|_{X_5} = \mathcal{O}_{X_5}, \tag{16.28}$$

so $c_1(X_5) = 0$. Its Hodge numbers are

$$h^{1,1}(X_5) = 1, \qquad h^{2,1}(X_5) = 101, \qquad \chi(X_5) = 2(h^{1,1} - h^{2,1}) = -200. \tag{16.29}$$

The single Kahler modulus is inherited from the ambient projective space; the 101 complex-structure moduli are counted by polynomial deformations modulo coordinate redefinitions. The mirror quintic reverses these Hodge numbers after a quotient and resolution, making it the basic example where mirror symmetry turns worldsheet instanton sums into period calculations. [195, 355].

16.2.6 Worldsheet supersymmetry

A nonlinear sigma model with Kahler target has $\mathcal{N} = (2,2)$ worldsheet supersymmetry. For a Calabi–Yau d-fold, the vanishing first Chern class removes the axial $U(1)$ anomaly and gives an internal superconformal theory with

$$c = \bar{c} = 3d. \tag{16.30}$$

For a threefold this gives $c = \bar{c} = 9$, exactly the central charge needed for four-dimensional superstring compactification after the noncompact part is included. [185, 195].

Angius and Volpato define sigma-model Hodge loci for a Calabi–Yau sigma model using generalized symmetries: Ramond–Ramond ground states provide the complex cohomology, $U(1) \times U(1)$ R-charges give the Hodge decomposition, and BPS boundary states supply the rational structure and polarization through the open-string Witten index [286]. Topological defects preserving the $\mathcal{N} = (2,2)$ algebra then detect special loci with extra rational Hodge endomorphisms and complex multiplication. In the notation here, the statement $c = \bar{c} = 9$ is only the first constraint; the exact SCFT also carries arithmetic and defect information that can locate special subvarieties inside Calabi–Yau moduli space.

16.3 Massless spectrum

The massless spectrum is obtained by expanding ten-dimensional fields in harmonic forms, organized by the Hodge diamond of Figure 16.1. For example, metric and B-field deformations along $(1,1)$ forms produce Kahler-type scalars,

$$J + iB = \sum_{a=1}^{h^{1,1}} t^a \omega_a, \qquad \omega_a \in H^{1,1}(X), \tag{16.31}$$

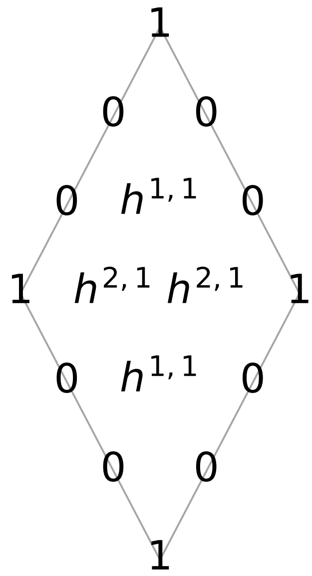


Figure 16.1: The Hodge diamond of a Calabi–Yau threefold. Each entry $h^{p,q}$ counts the harmonic (p,q)-forms, and every such form becomes a massless field in four dimensions. The $SU(3)$ holonomy fixes all entries except $h^{1,1}$, the number of Kähler moduli, and $h^{2,1}$, the number of complex-structure moduli. Mirror symmetry exchanges $h^{1,1}$ and $h^{2,1}$, and with them the two Calabi–Yau geometries.

while complex-structure deformations are represented by $\chi_i \in H^{2,1}(X)$ through

$$\partial_i \Omega = K_i \Omega + \chi_i. \tag{16.32}$$

For heterotic compactifications with an E_8 standard embedding, the charged matter is counted by bundle cohomology. In a common convention,

$$n_{\mathbf{27}} = h^1(X, TX) = h^{2,1}(X), \qquad n_{\overline{\mathbf{27}}} = h^1(X, T^*X) = h^{1,1}(X), \qquad n_{\mathbf{27}} - n_{\overline{\mathbf{27}}} = -\frac{\chi(X)}{2}. \tag{16.33}$$

Conventions may exchange which chirality is called **27**, but the index and the need for a bundle computation are invariant. [129, 134, 355].

16.4 Low energy field theory

The four-dimensional low-energy theory is obtained by expanding ten-dimensional fields in harmonic forms on the Calabi–Yau. For type II compactification on a Calabi–Yau threefold, the geometric moduli are organized by

$$h^{1,1}(X) \ \text{ Kahler moduli}, \qquad h^{2,1}(X) \ \text{ complex-structure moduli}, \tag{16.34}$$

with the precise vector/hypermultiplet assignment depending on type IIA versus type IIB. For heterotic compactification, a gauge bundle must also be chosen, and its cohomology determines charged matter. [129, 134, 185].

Numerical Calabi–Yau metrics add information that Hodge numbers do not determine. Once a Ricci-flat metric $g_{i\bar{j}}$ and Hermitian Yang–Mills bundle connections are approximated, one can compute inner products such as

$$K_{a\bar{b}} \sim \int_X \omega_a \wedge *\omega_{\bar{b}}, \qquad Y^{\text{canon}}_{abc} = \frac{Y_{abc}}{\sqrt{K_{a\bar{a}} K_{b\bar{b}} K_{c\bar{c}}}},$$

as well as Laplacian eigenvalues for Kaluza–Klein spectra, threshold effects, soft terms, and α'-corrected backgrounds [307]. Harmonic-form counting fixes the field content, but the four-dimensional Lagrangian also requires metric-dependent inner products and spectra. This distinction matters for phenomenological compactifications and for quantitative tests of moduli stabilization or swampland-distance claims.

16.5 Higher corrections

Calabi–Yau compactifications receive both perturbative α' corrections and nonperturbative corrections. Perturbative corrections modify the sigma-model beta functions and the effective action, while protected quantities may be computable by topological-string or supersymmetry methods. A schematic expansion is

$$S_{\text{eff}} = S_{\text{SUGRA}} + \sum_{n\geq 1} (\alpha')^n S_n + S_{\text{nonpert}}. \tag{16.35}$$

Therefore the Calabi–Yau condition is the leading geometric constraint, but higher-order corrections and fluxes must still be checked. [188, 195].

16.5.1 Instanton corrections

Worldsheet instantons are holomorphic maps from the string worldsheet into nontrivial two-cycles of the Calabi–Yau. Their contribution is weighted by

$$\exp\left[-\frac{\text{Area}(\Sigma)}{2\pi\alpha'} + i\int_\Sigma B\right]. \tag{16.36}$$

In spacetime, D-brane and NS5-brane instantons can correct hypermultiplet or superpotential terms depending on the compactification. The recurring theme is again charge plus holomorphy: topology determines which instantons can contribute, while supersymmetry controls what they are allowed to correct. [187, 190].

16.6 Generalizations

The strict Calabi–Yau case is the leading approximation. Realistic string vacua often generalize it by adding fluxes, orientifolds, branes, torsion, or non-Kahler geometry. The diagnostic changes from "$dJ = d\Omega = 0$" to differential conditions on an $SU(3)$-structure. Schematically,

$$dJ \neq 0 \text{ or } d\Omega \neq 0 \quad \Longrightarrow \quad \text{intrinsic torsion and flux must enter the SUSY equations.} \tag{16.37}$$

In type IIB orientifold examples, a common controlled limit uses three-form flux

$$G_3 = F_3 - \tau H_3, \qquad W_{\text{flux}} = \int_X G_3 \wedge \Omega, \tag{16.38}$$

with supersymmetric Minkowski conditions roughly selecting primitive $(2,1)$ flux. One caveat is that fluxes backreact, generate tadpole constraints, and can lift moduli; one cannot keep the Calabi–Yau metric unchanged except as an approximation. [187, 190, 355].

Chapter 17

$N = 2$ superconformal methods, mirror symmetry, and string dualities

The final chapter returns to the worldsheet: the geometric statements of Chapters 17 and 18 are rederived as exact statements about the $N = 2$ superconformal algebra, which is what a Calabi–Yau compactification looks like to the worldsheet. From the algebra's $U(1)$ charge come the unitarity bound $h \geq |q|/2$, the chiral primaries that saturate it, spectral flow between the NS and Ramond sectors, and the finite chiral ring, and each of these corresponds to a spacetime quantity, with the Hodge numbers, the Yukawa couplings, and their worldsheet-instanton corrections all read off from ring structure constants. The topological twist isolates exactly this protected sector; Landau–Ginzburg models and Gepner's tensor products of minimal models realize the same algebra with no geometry at all, proving that "compactification" is a property of a CFT, not of a manifold. Mirror symmetry then exchanges the two rings and turns instanton counting into classical period integrals; flops and conifold transitions show spacetime topology changing smoothly once the light wrapped branes are kept; and K3, with its self-dual charge lattice, anchors the six-dimensional duality between heterotic on T^4 and type IIA, completing the duality web and tying the advanced material back to the strong-coupling results of Chapter 13 [195, 208].

The full $N = 2$ superconformal algebra. The holomorphic generators are $T(z)$, a $U(1)$ current $J(z)$, and two supercurrents $G^{\pm}(z)$. The defining singular OPEs are

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{17.1}$$

$$T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \tag{17.2}$$

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2}, \tag{17.3}$$

$$J(z)G^{\pm}(w) \sim \pm\frac{G^{\pm}(w)}{z-w}, \tag{17.4}$$

$$G^{+}(z)G^{-}(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w}. \tag{17.5}$$

All other singular $G^{\pm}G^{\pm}$ OPEs vanish. To see what these OPEs prove, expand the fields in modes,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \qquad G^{\pm}(z) = \sum_{r} G^{\pm}_r z^{-r-3/2}, \tag{17.6}$$

where $r \in \mathbb{Z} + 1/2$ in the NS sector and $r \in \mathbb{Z}$ in the Ramond sector. The modes are contour integrals,

$$L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z), \qquad J_n = \oint_0 \frac{dz}{2\pi i} z^{n} J(z), \qquad G^{\pm}_r = \oint_0 \frac{dz}{2\pi i} z^{r+1/2} G^{\pm}(z). \tag{17.7}$$

The general residue rule is that, if

$$A(z)B(w) \sim \sum_{j\geq 0} \frac{C_j(w)}{(z-w)^{j+1}}, \tag{17.8}$$

then the commutator or anticommutator of modes is obtained by taking the z-contour around w first. Applying this to the first three OPEs gives

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{17.9}$$

$$[L_m, J_n] = -nJ_{m+n}, \tag{17.10}$$

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}. \tag{17.11}$$

Thus T generates conformal transformations, J is a dimension-one current, and the coefficient $c/3$ is the level of the $U(1)$ current algebra. The $JG^{\pm}$ OPE similarly gives

$$[J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm}, \tag{17.12}$$

so G^+ and G^- have $U(1)$ charges $+1$ and -1. The stress-tensor OPE with a dimension-3/2 field gives

$$[L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right) G_{m+r}^{\pm}. \tag{17.13}$$

The most important relation comes from the last OPE. Using

$$G_r^+ = \oint \frac{dz}{2\pi i} z^{r+1/2} G^+(z), \qquad G_s^- = \oint \frac{dw}{2\pi i} w^{s+1/2} G^-(w), \tag{17.14}$$

the triple pole contributes

$$\frac{2c}{3}\binom{r+1/2}{2}\delta_{r+s,0} = \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{17.15}$$

The double pole contributes $2(r+1/2)J_{r+s}$, while the ∂J term in the simple pole contributes $-(r+s+1)J_{r+s}$. Combining them gives

$$\{G_r^+, G_s^-\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{17.16}$$

Since the singular parts of G^+G^+ and G^-G^- vanish, $\{G_r^{\pm}, G_s^{\pm}\} = 0$. This algebra is the local engine behind chiral rings, topological twists, Gepner models, and the Calabi–Yau worldsheet theory [195, 208].

Spectral flow and the chiral-ring bound. In modes, spectral flow by η acts as

$$L_0 \mapsto L_0 + \eta J_0 + \frac{c}{6}\eta^2, \qquad J_0 \mapsto J_0 + \frac{c}{3}\eta. \tag{17.17}$$

The $N=2$ algebra gives a unitarity bound in the NS sector,

$$h \geq \frac{|q|}{2}. \tag{17.18}$$

A chiral primary saturates it:

$$G_{-1/2}^-|\phi\rangle = 0, \qquad h = \frac{q}{2}. \tag{17.19}$$

The OPE of chiral primaries is nonsingular enough to close modulo descendants, giving a finite chiral ring in compact models. Spectral flow by $\eta = -1/2$ maps NS chiral primaries to Ramond ground states; this is why Hodge numbers, Ramond ground states, and marginal deformations can be compared in Calabi–Yau compactification.

Topological twist and BRST operator. The A/B topological twists shift the stress tensor by the $U(1)$ current:

$$T_{\text{top}} = T \pm \frac{1}{2}\partial J. \tag{17.20}$$

The reason this shift changes the theory without changing the local operator algebra is the following. Let $\mathcal{O}$ have conformal weight h and $U(1)$ charge q, so that

$$T(z)\mathcal{O}(w) \sim \frac{h\,\mathcal{O}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}(w)}{z-w}, \qquad J(z)\mathcal{O}(w) \sim \frac{q\,\mathcal{O}(w)}{z-w}. \tag{17.21}$$

Differentiating the second OPE with respect to z gives

$$\partial J(z)\mathcal{O}(w) \sim -\frac{q\,\mathcal{O}(w)}{(z-w)^2}. \tag{17.22}$$

Therefore the twisted stress tensor $T_\alpha = T + \alpha\partial J$ assigns the shifted weight

$$h_{\text{top}} = h - \alpha q, \qquad \alpha = \pm\frac{1}{2}. \tag{17.23}$$

The supercurrents have $(h, q) = (3/2, +1)$ for G^+ and $(3/2, -1)$ for G^-. Hence

$$T_{\text{top}} = T + \frac{1}{2}\partial J: \qquad h_{\text{top}}(G^+) = 1, \qquad h_{\text{top}}(G^-) = 2, \tag{17.24}$$

$$T_{\text{top}} = T - \frac{1}{2}\partial J: \qquad h_{\text{top}}(G^-) = 1, \qquad h_{\text{top}}(G^+) = 2. \tag{17.25}$$

Thus after either twist one supercurrent is a dimension-one current, and its contour integral is dimensionless:

$$Q_{\text{top}} = \oint \frac{dz}{2\pi i} G^+(z) \quad \text{or} \quad \oint \frac{dz}{2\pi i} G^-(z). \tag{17.26}$$

Its nilpotency is not an additional assumption. Since the $N = 2$ algebra has no singular $G^+(z)G^+(w)$ or $G^-(z)G^-(w)$ OPE,

$$Q_{\text{top}}^2 = \frac{1}{2}\oint_w \frac{dw}{2\pi i}\oint_{z=w} \frac{dz}{2\pi i} G^\pm(z)G^\pm(w) = 0. \tag{17.27}$$

Moreover the mixed OPE in Eq. (17.1) implies that the twisted stress tensor is Q_{top}-exact up to the conventional normalization and the sign choice, schematically

$$T_{\text{top}} \sim \{Q_{\text{top}}, G^\mp\}, \tag{17.28}$$

which is why correlation functions of Q_{top}-closed operators are independent of local metric deformations. Physical topological operators are therefore Q_{top}-cohomology classes. The A-model depends on Kahler moduli and counts holomorphic curves; the B-model depends on complex-structure moduli. Mirror symmetry exchanges these two descriptions.

Gepner models and mirror map. An $N = 2$ minimal model at level k has central charge

$$c_k = \frac{3k}{k+2}. \tag{17.29}$$

A Gepner model for a Calabi–Yau threefold tensor-products minimal models so that

$$\sum_i \frac{3k_i}{k_i + 2} = 9. \tag{17.30}$$

The total $c = 9$ is the internal central charge needed for a four-dimensional type-II compactification. A GSO/orbifold projection enforces integral total $U(1)$ charge and spacetime supersymmetry. At special points in moduli space this exactly solvable CFT describes the same physics as a Calabi–Yau sigma model, even though the large-radius geometric picture may be absent.

For a mirror pair (X, Y),

$$h^{1,1}(X) = h^{2,1}(Y), \qquad h^{2,1}(X) = h^{1,1}(Y). \tag{17.31}$$

This exchange has direct computational content: it maps A-model curve-counting problems on X to B-model period computations on Y, often turning hard instanton sums into classical variation-of-Hodge-structure calculations [195, 208, 355].

17.1 The $N = 2$ superconformal algebra

The full OPE algebra in Eq. (17.1) has a useful mode form. Write

$$T(z) = \sum_n L_n z^{-n-2}, \qquad J(z) = \sum_n J_n z^{-n-1}, \qquad G^\pm(z) = \sum_r G_r^\pm z^{-r-3/2}. \tag{17.32}$$

In the NS sector $r \in \mathbb{Z} + 1/2$, while in the Ramond sector $r \in \mathbb{Z}$. Residues of the G^+G^- OPE give

$$\{G_r^+, G_s^-\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{17.33}$$

This equation is the mode-level reason that the $U(1)$ charge controls supersymmetric shortening. It is also the algebraic input for spectral flow, chiral primaries, and the topological twist, so it is the right starting point for Gepner models and Calabi–Yau compactification [195, 208].

17.1.1 Heterotic string vertex operators

A heterotic vertex operator is asymmetric because the right-moving side is superstring-like while the left-moving side carries the bosonic gauge lattice/current algebra. In the NS sector a gauge boson vertex has the schematic zero-picture form

$$V^a_{\epsilon,k} = \epsilon_\mu\, J^a(z)\, \left(\bar\partial X^\mu + ik\cdot\bar\psi\,\bar\psi^\mu\right) e^{ik\cdot X}. \tag{17.34}$$

The left-moving current has weight 1, and the right-moving factor has weight 1 on shell. Thus the integrated vertex is a $(1,1)$ operator precisely when

$$k^2 = 0, \qquad k\cdot\epsilon = 0, \tag{17.35}$$

with $\epsilon_\mu \sim \epsilon_\mu + \lambda k_\mu$ a null-state gauge equivalence. For toroidal compactification, the allowed left/right internal momenta must still lie in an even self-dual Narain lattice,

$$\Gamma^{d+16,d} = \Gamma^{d+16,d\,*}, \qquad p_L^2 - p_R^2 \in 2\mathbb{Z}, \tag{17.36}$$

so gauge vertex operators and modular invariance are two faces of the same left-moving current algebra [10, 86, 350].

17.1.2 Chiral primary fields

The chiral-primary condition is a shortening condition, not a definition by analogy. In the NS sector, positivity of $\|G^-_{-1/2}|\phi\rangle\|^2$ gives

$$0 \le \langle\phi|\{G^+_{1/2}, G^-_{-1/2}\}|\phi\rangle = 2h - q. \tag{17.37}$$

Therefore

$$h \ge \frac{q}{2}, \qquad \text{chiral primary} \Longleftrightarrow h = \frac{q}{2}, \quad G^-_{-1/2}|\phi\rangle = 0. \tag{17.38}$$

The OPE of two chiral primaries cannot produce arbitrary singular terms because charge and dimension are tied by $h = q/2$. Modulo Q-exact descendants after the topological twist, their products define the chiral ring. In Calabi–Yau sigma models this ring becomes the cohomology ring for the A-model or the polyvector/complex-structure ring for the B-model [195, 208].

17.1.3 Spectral flow

Spectral flow is an automorphism of the $\mathcal{N} = 2$ superconformal algebra. With parameter η,

$$h_\eta = h + \eta q + \frac{c}{6}\eta^2, \qquad q_\eta = q + \frac{c}{3}\eta. \tag{17.39}$$

For $\eta = \pm 1/2$ it maps NS-sector chiral primaries to Ramond ground states. The algebraic reason why chiral rings, spacetime supersymmetry, and Calabi–Yau Hodge numbers can be compared. [195, 208].

17.2 Type II strings on Calabi–Yau manifolds

For type-II compactification on a Calabi–Yau threefold X, the worldsheet statement is that the internal CFT has $\mathcal{N} = (2,2)$ and

$$c_{\text{internal}} = \bar c_{\text{internal}} = 9, \qquad c_{\mathbb{R}^{1,3}} = \bar c_{\mathbb{R}^{1,3}} = 6, \qquad c_{\text{matter}} = 15. \tag{17.40}$$

The geometric large-radius limit realizes this internal CFT as a nonlinear sigma model with

$$c_1(X) = 0, \qquad \text{Hol}(X) \subseteq SU(3). \tag{17.41}$$

The two type-II theories assign the same Hodge numbers differently:

$$\begin{array}{c|c|c} & \mathcal{N}=2 \text{ vector multiplets} & \mathcal{N}=2 \text{ hypermultiplets} \\ \hline \text{IIA} & h^{1,1}(X) & h^{2,1}(X)+1 \\ \text{IIB} & h^{2,1}(X) & h^{1,1}(X)+1. \end{array} \tag{17.42}$$

The exchange of the two columns under mirror symmetry is a physical equivalence of SCFTs, beyond a coincidence of Hodge diamonds [195, 355].

17.2.1 Low energy actions

The type-II low-energy action is controlled by special geometry. For vector multiplets the complex scalars X^I are projective coordinates and the prepotential $F(X)$ determines the gauge couplings:

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\,\frac{(\mathrm{Im}F_{IK})X^K(\mathrm{Im}F_{JL})X^L}{X^K(\mathrm{Im}F_{KL})X^L}. \tag{17.43}$$

In type IIA at large radius, the classical part of F is determined by triple intersection numbers,

$$F^{\mathrm{cl}}_{\mathrm{IIA}} = -\frac{1}{6}\frac{\kappa_{abc}t^a t^b t^c}{t^0}, \tag{17.44}$$

while worldsheet instantons correct it. In type IIB, the corresponding vector-multiplet couplings come from periods of the holomorphic three-form,

$$X^I = \int_{A^I}\Omega, \qquad F_I = \int_{B_I}\Omega. \tag{17.45}$$

Thus the four-dimensional action is computed from intersection theory on one side and period integrals on the mirror side [195, 355].

17.2.2 Chiral rings

The chiral ring is the product algebra of chiral primaries modulo descendants. In a large-radius Calabi–Yau sigma model, the A-model ring is the de Rham cohomology ring deformed by worldsheet instantons:

$$\phi_a\cdot\phi_b = C_{ab}{}^{c}(t)\,\phi_c, \qquad C_{abc}(t) = \kappa_{abc} + \sum_{\beta>0} N_\beta\,\frac{\beta_a\beta_b\beta_c\,q^\beta}{1-q^\beta}. \tag{17.46}$$

Here κ_{abc} are classical intersection numbers and N_β are genus-zero curve counts. The B-model chiral ring is instead computed from complex-structure variation of Ω, so its Yukawa couplings are period integrals. Mirror symmetry equates these two rings after the mirror map.

17.2.3 Topological string theory

Topological string theory is obtained by twisting the $\mathcal{N}=2$ algebra so that one supercharge becomes scalar. The genus expansion is

$$F_{\mathrm{top}}(g_s,t) = \sum_{g\geq 0} g_s^{2g-2}F_g(t). \tag{17.47}$$

For the A-model, F_g receives contributions from holomorphic maps of genus g worldsheets into X. For the B-model, F_g depends on complex-structure moduli and obeys holomorphic-anomaly equations instead of being a naive holomorphic function globally. The physics reason this belongs in the string notes is that these protected quantities compute higher-derivative F-terms in the four-dimensional effective action, schematically $F_g\,R_-^2T_-^{2g-2}$ in type-II compactifications [195, 355].

Topological-string methods give a nonperturbative form to Eq. (17.47). The perturbative partition function is

$$Z_{\mathrm{top}}(g_s,t) = \exp\left(\sum_{g\geq 0} g_s^{2g-2}F_g(t)\right),$$

and resurgence organizes it as the asymptotic expansion of a transseries with instanton actions determined by periods of the mirror geometry [293]. In the TS/ST correspondence, a mirror curve is quantized to an operator $\widehat{H}$, and the spectral problem

$$\widehat{H}\,\psi = 0$$

is solved using closed and open topological-string amplitudes; the open sector gives entire off-shell eigenfunctions for the quantized curve [299]. For torus-fibered Calabi–Yau threefolds, $Z_{\rm top}$ transforms as a wave function under relative conifold monodromy, equivalently under a Fourier–Mukai transform, which explains its Jacobi-form behavior [304]. The genus-one term also has a Ray–Singer torsion interpretation and contributes curvature-squared couplings and a moduli-dependent species scale [289]. Periods, worldsheet instantons, BPS invariants, and effective gravitational cutoffs are therefore linked by the same topological-string wave function.

17.3 Heterotic string theories with (2,2) SCFT

A heterotic compactification with $(2,2)$ internal SCFT is the worldsheet version of the standard embedding. In geometric language one embeds the spin connection into one E_8 factor:

$$V = TX, \qquad E_8 \supset E_6 \times SU(3), \qquad \mathbf{248} \to (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}}). \tag{17.48}$$

The $SU(3)$ is the holonomy/gauge-bundle structure group, and its commutant is E_6. The chiral matter is counted by bundle cohomology; in the standard embedding this reduces to Hodge numbers, schematically

$$n_{\mathbf{27}} = h^{2,1}(X), \qquad n_{\overline{\mathbf{27}}} = h^{1,1}(X), \qquad \frac{1}{2}\chi(X) = h^{1,1} - h^{2,1}. \tag{17.49}$$

The $(2,2)$ worldsheet supersymmetry is therefore a strong restriction: it fixes a very special class of heterotic vacua before one moves to more general $(0,2)$ bundles [86, 355].

17.3.1 More on the low energy action

For heterotic or type-II compactifications, the two-derivative four-dimensional action is usually organized by supersymmetry. In $N = 1$ notation the functions are

$$K(\Phi^i, \bar{\Phi}^{\bar{i}}), \qquad W(\Phi^i), \qquad f_{ab}(\Phi^i). \tag{17.50}$$

The Kahler potential gives scalar kinetic terms, the superpotential gives F-term interactions, and the gauge kinetic function gives gauge couplings and theta angles.

String compactification constrains these functions through worldsheet charges, geometry, and duality. In Calabi–Yau compactifications, mirror symmetry exchanges how complex-structure and Kahler moduli appear in type-IIA and type-IIB effective actions. [195, 355].

17.4 $N = 2$ minimal models

The $N = 2$ minimal models are the rational representations of the $N = 2$ superconformal algebra of Eq. (17.1), and they are the building blocks Gepner assembles into compactifications. The unitary discrete series sits at

$$c_k = \frac{3k}{k+2}, \qquad k = 1, 2, 3, \ldots, \tag{17.51}$$

approaching $c = 3$ from below as $k \to \infty$; note the contrast with the Virasoro series $c = 1 - 6/m(m+1)$ of Chapter 14, the extra supercurrents and $U(1)$ current raise the accumulation point from 1 to 3. The model at level k has the coset realization $SU(2)_k \times U(1)_2/U(1)_{k+2}$, primaries labeled (ℓ, m, s) with the weights and charges quoted in Eq. (17.54), and an equivalent Landau–Ginzburg description with superpotential $W = X^{k+2}$, whose quasi-homogeneous charge $q_X = 1/(k+2)$ reproduces c_k exactly via $c = 3(1 - 2q_X)$. Because each factor is rational, tensor products of these models are exactly solvable, which is precisely what makes the Gepner construction of the next section possible [195, 208].

17.4.1 Landau–Ginzburg models

A Landau–Ginzburg model is an $\mathcal{N} = 2$ theory specified by chiral superfields Φ_i and a quasi-homogeneous superpotential W. If

$$W(\lambda^{q_i}\Phi_i) = \lambda W(\Phi_i), \tag{17.52}$$

then the IR fixed point has central charge

$$c = 3\sum_i (1 - 2q_i)\,, \qquad \mathcal{R}_{\text{chiral}} = \frac{\mathbb{C}[\Phi_i]}{(\partial_{\Phi_i} W)}. \tag{17.53}$$

For $W = X^{k+2}$, $q_X = 1/(k+2)$, so $c = 3k/(k+2)$, exactly the $N = 2$ minimal-model central charge. This gives a calculable bridge between minimal models, Gepner tensor products, and Calabi–Yau hypersurface equations [195, 208].

17.5 Gepner models

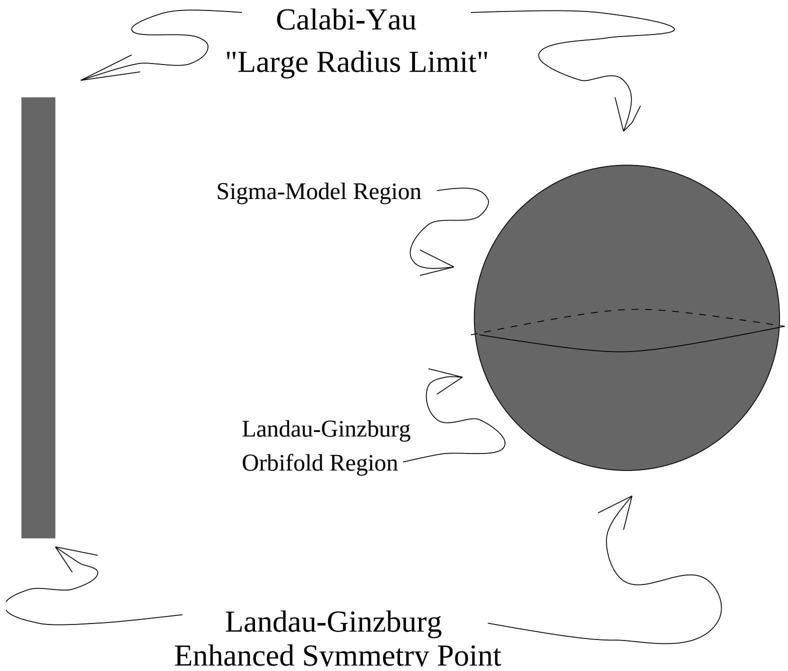


Figure 17.1: The Kähler moduli space of a Calabi–Yau compactification. The large-radius geometric (sigma-model) region and the Landau–Ginzburg orbifold region, joined through an enhanced-symmetry point, are limits of one connected moduli space. A single conformal field theory therefore has both a geometric description and a non-geometric Landau–Ginzburg or Gepner description, which is why "compactification" is a property of the worldsheet theory and not of a fixed manifold. From Ref. [355].

A Gepner model is an exactly solvable internal SCFT built as a tensor product of $N = 2$ minimal models plus a GSO/orbifold projection, and it sits at the Landau–Ginzburg end of the phase diagram in Figure 17.1. The minimal-model labels are usually written (ℓ, m, s), with identifications and selection rules; their conformal weights and $U(1)$ charges are

$$h_{\ell,m,s} = \frac{\ell(\ell+2) - m^2}{4(k+2)} + \frac{s^2}{8} \pmod 1, \qquad q_{\ell,m,s} = \frac{m}{k+2} - \frac{s}{2} \pmod 2. \tag{17.54}$$

The tensor product must satisfy $\sum_i 3k_i/(k_i+2) = 9$, and the projection keeps states of integral total $U(1)$ charge. The projection is not cosmetic: it is what produces spacetime supersymmetry and a modular-invariant partition function. The large-radius Calabi–Yau, when it exists, is a different point in the same SCFT moduli space instead of the starting assumption [185, 208].

17.5.1 Relation to Calabi–Yau compactification

Gepner models realize Calabi–Yau compactifications as exactly solvable tensor products of $\mathcal{N}=2$ minimal models. The minimal-model central charges

$$c_i = \frac{3k_i}{k_i+2} \tag{17.55}$$

are chosen so that $\sum_i c_i = 9$, and a GSO/orbifold projection enforces integral total $U(1)$ charge. The geometric interpretation is that the resulting SCFT often sits at a special point in the moduli space of a Calabi–Yau threefold, not necessarily at a large-radius sigma-model point. [185, 208].

Gepner and Landau–Ginzburg models can be used directly in moduli stabilization because the internal theory is an exact SCFT. Flux stabilization still uses the F-term equations

$$D_i W = 0, \qquad W = \sum_A N_A \Pi_A(\phi),$$

where Π_A are periods or exact CFT periods and N_A are flux integers. In the 2^6 Landau–Ginzburg model, explicit $\mathcal{N}=1$ Minkowski solutions solve these equations with no remaining flat directions, including examples whose flux choices test refined tadpole expectations [290]. The 1^9 and 2^6 models are special because they are mirror to rigid Calabi–Yau threefolds and sit in the interior of moduli space, where an exact Landau–Ginzburg description replaces the large-radius approximation [306]. Orientifolds of such Gepner models can have no Kahler moduli; their complex-structure moduli and tadpole charges can then be computed within the exact CFT [309]. Exact solvability removes geometric moduli that would be present in a large-radius Calabi–Yau description.

17.6 Mirror symmetry and applications

Mirror symmetry is the statement that two different geometries can define equivalent string backgrounds. At the level of Hodge numbers, a mirror pair (X, Y) satisfies

$$h^{1,1}(X) = h^{2,1}(Y), \qquad h^{2,1}(X) = h^{1,1}(Y). \tag{17.56}$$

It is a direct continuation of T-duality: the CFT, not the naive metric picture, decides when two compactifications are the same physical theory. [355].

17.6.1 Moduli spaces

A moduli space is the space of inequivalent exactly marginal deformations of the worldsheet theory, or equivalently the space of massless scalar expectation values that do not generate a potential at the order being considered. On the worldsheet the deformation begins as

$$\delta S = \lambda \int d^2 z\, \mathcal{O}_{(1,1)}(z,\bar{z}), \tag{17.57}$$

but exact marginality requires the full beta function for λ to vanish, the first-order dimension condition alone.

For Calabi–Yau compactifications, the familiar geometric moduli are Kahler and complex-structure deformations. The worldsheet statement is broader: two different geometries can lie on the same CFT moduli space, and singular points may signal additional light states instead of a failure of the full string description. [195, 355].

17.6.2 The flop

A flop occurs when a rational curve $C \simeq \mathbb{P}^1$ shrinks to zero area and the geometry continues by replacing it with a different rational curve C'. Classically this looks singular because a Kahler parameter crosses a wall:

$$t_C = \int_C (J + iB) \longrightarrow 0 \quad \text{and then} \quad t_{C'} = -t_C. \tag{17.58}$$

The stringy statement is subtler. The B-field and worldsheet instanton corrections enlarge the Kahler moduli space so that the conformal field theory can pass through the wall without a physical singularity. The flop is therefore an example of a topology change that is invisible to a point-particle geometer but controlled in the full stringy Kahler moduli space [195, 355].

17.7 The conifold

A conifold is locally the nodal hypersurface

$$\sum_{i=1}^{4} z_i^2 = 0. \tag{17.59}$$

It has two standard smoothings. The deformed conifold replaces the node by an S^3, while the small resolution replaces it by an S^2. In type IIB, if a three-cycle A shrinks, the period

$$X = \int_A \Omega \tag{17.60}$$

goes to zero, and a D3-brane wrapped on A becomes massless with $M \sim |X|/g_s$. The logarithmic singularity in the vector-multiplet metric is then the ordinary one-loop singularity of a charged hypermultiplet that has been integrated out. Including the light state resolves the apparent singularity of the effective theory [354, 355].

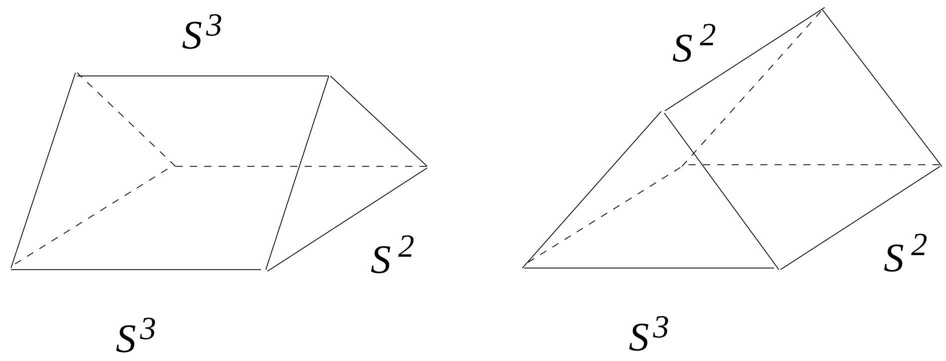


Figure 17.2: The two smoothings of the conifold. The base of the singular cone is $S^2 \times S^3$; the deformation keeps the three-sphere S^3 finite while the small resolution keeps the two-sphere S^2 finite. A conifold transition passes from one smooth geometry to the other through the singular point, and the wrapped brane that becomes massless there is what keeps the effective theory smooth. From Ref. [355].

17.7.1 The conifold transition

A conifold transition changes topology by moving through a conifold point. Schematically,

$$\text{smooth } X \quad \longrightarrow \quad \text{nodal } X_0 \quad \longrightarrow \quad \text{topologically different } Y. \tag{17.61}$$

The Betti numbers change because a collection of three-cycles collapses and two-cycles appear after a small resolution. If N three-spheres shrink with R homology relations, the standard local count is

$$\Delta h^{2,1} = -(N - R), \qquad \Delta h^{1,1} = R. \tag{17.62}$$

The physical conclusion is that topology change is allowed when the extra massless branes at the transition are kept in the spectrum. A low-energy action that integrates them out will look singular exactly where the full string theory is becoming more strongly populated by light states [354, 355].

Dücker, Klemm, and Piribauer analyze variations of Hodge structure, integral monodromy, and restricted moduli loci in type-II/F-theory compactifications, showing that fixed loci of moduli-space symmetries can help stabilize flux vacua but do not by themselves guarantee simultaneous stabilization of several moduli [298]. Hassfeld, Monnee, Weigand, and Wiesner study type-II limits in IIB complex-structure moduli space and argue, via the worldsheet theory of four-dimensional EFT strings, that broad Tyurin-degeneration limits produce a unique critical heterotic string on $T^2 \times K3$ plus Kaluza–Klein towers [300]. The follow-up analysis of K-points identifies limits not captured by ordinary mirror symmetry and relates the decoupled sectors to spacetime-filling heterotic NS5-branes wrapping the heterotic torus [302]. Thus the periods near a conifold or infinite-distance boundary must be read together with the emergent string or brane sector; classical period monodromy alone is not the full physics.

17.8 String theories on $K3$

$K3$ is the Calabi–Yau twofold: it has $SU(2)$ holonomy, a covariantly constant spinor, and

$$h^{0,0} = h^{2,2} = 1, \qquad h^{2,0} = h^{0,2} = 1, \qquad h^{1,1} = 20. \tag{17.63}$$

Equivalently $b_2 = 22$, and the intersection form on $H^2(K3, \mathbb{Z})$ is the even self-dual lattice

$$\Gamma^{3,19} \simeq 3\,\Gamma^{1,1} \oplus 2(-E_8). \tag{17.64}$$

Type IIA on $K3$ gives a six-dimensional theory with sixteen supercharges. Its moduli are organized by a positive four-plane in $\Gamma^{4,20} \otimes \mathbb{R}$, giving the local moduli space

$$\frac{O(4,20)}{O(4) \times O(20)} \tag{17.65}$$

up to the arithmetic identifications preserving the charge lattice. This lattice language is why $K3$ is the natural partner of heterotic T^4 in six-dimensional string duality [10, 355].

17.9 String duality below 10 dimensions

Below ten dimensions, duality is tested by comparing charge lattices and moduli spaces. The most important six-dimensional example is

$$\text{heterotic on } T^4 \quad \longleftrightarrow \quad \text{type IIA on } K3. \tag{17.66}$$

Both theories have sixteen supercharges and the same Narain-type scalar moduli space

$$\frac{O(4,20)}{O(4) \times O(20)} \tag{17.67}$$

with an $O(4, 20; \mathbb{Z})$ charge lattice in the quantum theory. On the heterotic side the lattice comes from momenta, windings, and gauge charges on T^4; on the IIA side it comes from D-branes wrapped on even cycles of $K3$, plus NS charges. Thus the duality is not a vague strong-coupling analogy: it identifies two different geometric origins for the same protected lattice quantities [352, 353, 355].

17.9.1 Heterotic strings in $7 \le d \le 9$

For heterotic compactification on T^{10-d}, the full perturbative moduli are metric, B-field, and Wilson lines. The momenta form an even self-dual Narain lattice

$$\Gamma^{26-d,10-d}, \qquad \mathcal{M}_d^{\text{het}} = O(26-d, 10-d; \mathbb{Z}) \backslash \frac{O(26-d, 10-d)}{O(26-d) \times O(10-d)}. \tag{17.68}$$

Gauge enhancement occurs when a lattice vector satisfies

$$p_R = 0, \qquad p_L^2 = 2, \tag{17.69}$$

because then a left-moving current-like state becomes massless. Thus in $7 \le d \le 9$ the possible enhanced gauge symmetriescan be seen in the Narain lattice [10, 86, 350].

17.9.2 Heterotic–type IIA duality in six dimensions

The six-dimensional heterotic/IIA duality identifies the heterotic fundamental string with a type-IIA NS5-brane wrapped on $K3$. A quick tension check shows why the map is strong/weak.

Since

$$T^{\mathrm{IIA}}_{\mathrm{NS5}} \sim \frac{1}{g^2_{IIA}\ell^6_{IIA}}, \qquad T_{\mathrm{wrapped}} \sim \frac{V^{IIA}_{K3}}{g^2_{IIA}\ell^6_{IIA}}, \tag{17.70}$$

this wrapped object behaves like a six-dimensional string. Matching it to the heterotic fundamental string fixes a relation between the heterotic coupling and the IIA volume/coupling. At the level of protected quantities the statement is

$$\Gamma^{4,20}_{\mathrm{het}}(T^4) \simeq H^{\mathrm{even}}(K3,\mathbb{Z}) \quad \text{with intersection form } \Gamma^{4,20}. \tag{17.71}$$

This identifies heterotic momentum/winding/gauge charges with IIA D0/D2/D4 wrapping charges on $K3$ [353, 355].

17.9.3 Heterotic S-duality in four dimensions

In four-dimensional heterotic compactifications with enough supersymmetry, the axion-dilaton

$$S = a + \frac{i}{g_4^2} \tag{17.72}$$

is expected to transform under an $SL(2,\mathbb{Z})$-type electric-magnetic duality. The perturbative T-duality group acts on the Narain lattice of momentum, winding, and gauge charges; S-duality exchanges electric and magnetic charges. A BPS mass formula must therefore be written in terms of a charge vector $(Q_{\mathrm{e}}, Q_{\mathrm{m}})$ and moduli so that

$$M_{\mathrm{BPS}}(Q_{\mathrm{e}}, Q_{\mathrm{m}}; S, T, \ldots) = M_{\mathrm{BPS}}(aQ_{\mathrm{e}} + bQ_{\mathrm{m}}, cQ_{\mathrm{e}} + dQ_{\mathrm{m}}; \frac{aS+b}{cS+d}, T, \ldots). \tag{17.73}$$

The correct status of this statement is that it is a protected-spectrum and duality-covariance statement, not a perturbative theorem: purely electric perturbative heterotic states must be accompanied by magnetic solitons for the full duality group to act [173, 353].

17.10 Conclusion

The advanced material has one common theme: exact worldsheet structures replace vague geometry. The $N = 2$ algebra gives chiral rings and spectral flow. Gepner and Landau–Ginzburg models give exact non-geometric points. Mirror symmetry turns instanton sums into period computations. Conifold points require extra light branes, and lower-dimensional dualities identify charge lattices across apparently different compactifications. Each claim is therefore backed by an explicit algebra, lattice, protected index, or moduli-space computation.

Chapter 18

Four-dimensional effective physics, fluxes, and quantum-gravity constraints

The constructions of the last three chapters produce four-dimensional theories; this chapter asks what it takes for one of them to describe our world. The method is translation: every quantity a four-dimensional physicist measures must be expressed in compactification quantities. The Planck mass is the internal volume over the squared coupling; gauge couplings are volumes of the cycles branes wrap, or the dilaton for heterotic gauge fields; quantum numbers are lattice vectors or Chan–Paton labels, never assignments made by hand; and the whole two-derivative theory is packaged in the $N = 1$ triple, Kähler potential, superpotential, gauge kinetic function, that supersymmetry allows. We work through the standard checklist in this language: discrete symmetries and the strong-CP problem with its candidate string axions, the hierarchy of mass scales and what "unification" means once thresholds are included, and the conditions under which spacetime supersymmetry survives or breaks, at tree level, in the loop expansion (where a nonzero vacuum energy must always be accompanied by a tadpole and stability analysis), and nonperturbatively through gaugino condensation and brane instantons. The conclusion is that a consistent compactification is a candidate, and phenomenology is the further, separate computation of K, W, and f_{ab} [355, 361].

Dimensional reduction of the gravitational coupling. Start with the ten-dimensional string-frame Einstein term schematically

$$S_{10} \supset \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G_{10}}\,e^{-2\Phi}R_{10}. \tag{18.1}$$

For a direct product with internal volume $V_6 = \int_X d^6y\sqrt{g_6}$ and constant dilaton $g_s = e^{\Phi_0}$, the four-dimensional Einstein term is

$$M_{\rm Pl}^2 = \frac{V_6}{\kappa_{10}^2 g_s^2} \quad \text{up to the convention for } M_{\rm Pl}^2 = (8\pi G_N)^{-1}. \tag{18.2}$$

To derive this coefficient explicitly, take the unwarped product ansatz

$$ds_{10,s}^2 = g^{(s)}_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n, \qquad \Phi(y) = \Phi_0. \tag{18.3}$$

For a direct product,

$$\sqrt{-G_{10}} = \sqrt{-g_4}\sqrt{g_6}, \qquad R_{10} = R_4 + R_6, \tag{18.4}$$

where only the first term contains the four-dimensional graviton kinetic term. Substituting these relations into the ten-dimensional action gives

$$S_{10} \supset \frac{1}{2\kappa_{10}^2}\int d^4x\,d^6y\,\sqrt{-g_4}\sqrt{g_6}\,e^{-2\Phi_0}R_4 + \cdots \tag{18.5}$$

$$= \frac{V_6 e^{-2\Phi_0}}{2\kappa_{10}^2}\int d^4x\sqrt{-g_4}\,R_4 + \cdots. \tag{18.6}$$

The four-dimensional reduced Planck mass is defined by

$$S_4 \supset \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g_4}\,R_4, \qquad M_{\rm Pl}^2 = (8\pi G_N)^{-1}. \tag{18.7}$$

Comparing the two coefficients and using $g_s = e^{\Phi_0}$ gives exactly Eq. (18.2). If one also uses the common string normalization

$$2\kappa_{10}^2 = (2\pi)^7(\alpha')^4, \tag{18.8}$$

then the same relation may be written as

$$M_{\rm Pl}^2 = \frac{2V_6}{(2\pi)^7 g_s^2(\alpha')^4}, \tag{18.9}$$

with the factor of 2 tied to the convention for κ_{10}. This is the first scale relation: large internal volume or weak coupling raises the four-dimensional Planck mass relative to the string scale. In a warped or varying-dilaton background the replacement is not made by hand; it comes from the same coefficient extraction. For example, if

$$ds_{10,s}^2 = e^{2A(y)} g_{\mu\nu}^{(s)}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n, \tag{18.10}$$

then the coefficient multiplying R_4 is proportional to

$$\frac{1}{\kappa_{10}^2}\int_X d^6y\sqrt{g_6}\, e^{-2\Phi(y)}e^{2A(y)}, \tag{18.11}$$

with the precise power of the warp factor depending on the metric convention. Thus warping and varying dilaton replace V_6/g_s^2 by the appropriate weighted internal integral.

Gauge couplings from cycles and branes. For a Dp-brane wrapping an internal $(p-3)$-cycle Σ, the DBI expansion gives

$$S_{\rm DBI} \supset -\frac{T_p(2\pi\alpha')^2}{4g_s}\int_{\mathbb{R}^{1,3}\times\Sigma} d^{p+1}\xi\, F_{\mu\nu}F^{\mu\nu}. \tag{18.12}$$

After integrating over the wrapped cycle,

$$\frac{1}{g_4^2} = \frac{T_p(2\pi\alpha')^2}{g_s}\mathrm{Vol}(\Sigma) = \frac{\mathrm{Vol}(\Sigma)}{(2\pi)^{p-2}g_s(\alpha')^{(p-3)/2}}. \tag{18.13}$$

For heterotic gauge fields the gauge bosons live in ten dimensions, and dimensional reduction instead gives $1/g_4^2 \propto V_6/(g_s^2\alpha'^3)$. Thus four-dimensional gauge couplings are not arbitrary constants: they are functions of dilaton, volumes, and bundle and brane input [355, 361].

Supersymmetric effective-action couplings. At two derivatives, a four-dimensional $N = 1$ effective theory is specified by three holomorphic/geometric objects:

$$K(\Phi^i, \bar\Phi^{\bar j}), \qquad W(\Phi^i), \qquad f_{ab}(\Phi^i). \tag{18.14}$$

The scalar kinetic terms and F-term potential are

$$\mathcal{L}_{\rm kin} = -K_{i\bar j}\,\partial_\mu\Phi^i\partial^\mu\bar\Phi^{\bar j}, \qquad V_F = e^{K/M_{\rm Pl}^2}\left(K^{i\bar j}D_iW\,D_{\bar j}\bar W - \frac{3|W|^2}{M_{\rm Pl}^2}\right). \tag{18.15}$$

The gauge kinetic term is

$$\mathcal{L}_{\rm gauge} = -\frac14\mathrm{Re}\, f_{ab}F^a_{\mu\nu}F^{b\mu\nu} + \frac14\mathrm{Im}\, f_{ab}F^a\wedge F^b. \tag{18.16}$$

This is the practical meaning of the chapter's phenomenology statements: compactification input must be translated into K, W, f, then spectra, anomalies, moduli stabilization, and supersymmetry breaking can be checked.

18.1 Continuous and discrete symmetries

Four-dimensional symmetries descend from three different sources: spacetime isometries, worldsheet current algebras, and topological gauge redundancies of B_2 or R–R potentials. A continuous gauge symmetry is visible in the four-dimensional action through a covariant derivative

$$D_\mu\Phi^i = \partial_\mu\Phi^i + A_\mu^a\, X_a^i(\Phi), \tag{18.17}$$

where X_a is a Killing vector on the scalar moduli space. Discrete symmetries often arise when a continuous symmetry is broken by the charge lattice or by instantons. For an axion $a \sim a + 2\pi$, a coupling

$$\frac{a}{8\pi^2}\int \mathrm{Tr}\, F\wedge F \tag{18.18}$$

is invariant under only those shifts that respect instanton number. Thus string compactification naturally gives both continuous gauge symmetries and exact or approximate discrete remnants, but each candidate symmetry must be checked against the charge lattice and allowed instantons [355, 361].

18.1.1 P,C,T and all that

The discrete symmetries are distinct operations. Spacetime parity reverses an odd number of spatial coordinates, charge conjugation acts on gauge representations, and time reversal is anti-unitary. Worldsheet orientation reversal is a different operation, usually denoted Ω, and acts as

$$\Omega: \quad \sigma \mapsto 2\pi - \sigma, \qquad X_L^\mu \leftrightarrow X_R^\mu. \tag{18.19}$$

Worldsheet orientation reversal is the operation used in orientifold projections and in the type-I construction from type-IIB strings.

The operations P, C, T, and Ω are therefore not interchangeable. Projections by these symmetries change the Hilbert space and introduce unoriented diagrams, crosscaps, orientifold planes, and tadpole constraints. [10, 352].

18.1.2 The strong CP problem

The QCD theta term is

$$S_\theta = \frac{\theta_{\mathrm{QCD}}}{8\pi^2}\int \mathrm{Tr}\, G\wedge G, \qquad \bar\theta = \theta_{\mathrm{QCD}} + \arg\det M_q. \tag{18.20}$$

The strong CP problem is the empirical smallness of $\bar\theta$. String compactifications often contain axions from periods such as

$$a_\Sigma = \int_\Sigma B_2 \quad \text{or} \quad \int_\Sigma C_p. \tag{18.21}$$

A Peccei–Quinn solution requires a coupling

$$\frac{a}{f_a}\frac{1}{8\pi^2}\int \mathrm{Tr}\, G\wedge G \tag{18.22}$$

and a potential dominated by QCD so that the minimum sets $\bar\theta_{\mathrm{eff}} = 0$.[1]

18.1.3 Axionic cosmic strings

The periodicity $a \sim a + 2\pi f_a$ of such an axion has a topological consequence. Write the Peccei–Quinn field as a complex scalar $\Phi = \rho\, e^{ia/f_a}$ with a Mexican-hat potential

$$V(\Phi) = \lambda\left(|\Phi|^2 - \tfrac{1}{2}f_a^2\right)^2, \tag{18.23}$$

whose minima form the circle $|\Phi| = f_a/\sqrt{2}$. The vacuum manifold is therefore

$$\mathcal{M} = U(1) \cong S^1. \tag{18.24}$$

On a large loop C encircling a line in space, the field is a map $C \to \mathcal{M}$, and such maps are classified by the first homotopy group

$$\pi_1(S^1) = \mathbb{Z}, \qquad n = \frac{1}{2\pi}\oint_C d\left(\frac{a}{f_a}\right) \in \mathbb{Z}. \tag{18.25}$$

A configuration with winding $n \neq 0$ cannot be deformed to a constant. The phase a/f_a is undefined wherever $\rho = 0$, so continuity forces Φ to vanish on a codimension-two surface, a line in three spatial dimensions. This line is the cosmic string; its core sits at the symmetric point

[1] String compactifications naturally produce axion-like fields by integrating B_2, C_p, or dual potentials over cycles, but a generic string axion is not automatically the QCD axion. A candidate QCD axion must couple as $\frac{a}{f_a}\,\mathrm{Tr}\, G\wedge G$ and must have a potential whose minimum cancels the effective θ_{QCD}. Having a pseudoscalar, or an imaginary part of f_{ab}, is not enough; one must check the anomaly coefficient, decay constant, Stückelberg mixings, non-QCD instanton corrections, and whether any of these spoil the Peccei–Quinn symmetry.

$\Phi = 0$, at the top of the potential, and the winding cannot be undone without lifting the whole string over the barrier. The defect is topologically stable for this reason.

The energy per unit length follows from the angular gradient $|\nabla a| \sim f_a/r$. For a global (axionic) string this integrates to a logarithmically cut-off tension

$$\mu \simeq \pi f_a^2 \ln \frac{L}{\delta}, \qquad \delta \sim \frac{1}{\sqrt{\lambda}\, f_a}, \tag{18.26}$$

with δ the core size and L the distance to the nearest string; gauging the $U(1)$ screens the gradient and leaves a finite tension $\mu \sim f_a^2$.

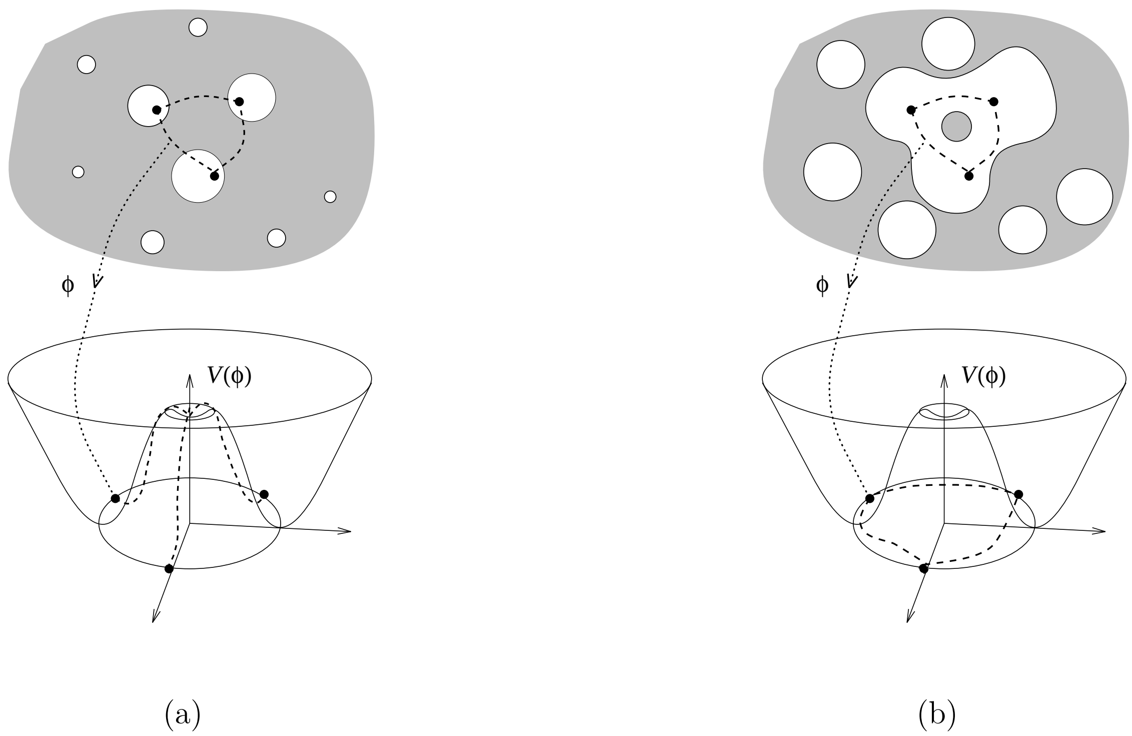


Figure 18.1: Formation of a topological string defect by the Kibble mechanism. In each panel the Mexican-hat potential $V(\Phi)$ of Eq. (18.23) sits below, and the phase of Φ takes uncorrelated values in causally disconnected domains of space above; the map Φ sends each domain to a point of the vacuum circle. (a) As the symmetry breaks, domains of definite phase (white) grow inside the false vacuum (grey). (b) When they merge so that the phase winds through 2π around a point, the field is trapped at the symmetric value $\Phi = 0$ and a string forms there. From Ref. [373].

Such strings form unavoidably by the Kibble mechanism. As the universe cools through the scale f_a, the phase a/f_a settles to independent values in causally disconnected regions, and wherever the phases of adjacent domains run through a full 2π a unit of winding is trapped, as in Figure 18.1. Because the same $U(1)$ can be the phase of a B_2 or C_p period of the kind introduced above, the axionic strings of a string compactification are continuously connected to fundamental and D-strings, and the evolution of their network is a cosmological probe of the compactification [373].

18.2 Gauge symmetries

Gauge bosons arise differently in different corners of string theory. In heterotic compactification they are left-moving current-algebra states, so a root P gives a massless gauge boson when

$$p_R = 0, \qquad p_L^2 = 2. \tag{18.27}$$

In open-string/type-I or D-brane compactifications, gauge bosons are massless open strings with Chan–Paton labels. For N coincident D-branes, the endpoint labels give $U(N)$; orientifold projections can reduce this to $SO(N)$ or $Sp(N/2)$. In type-II closed-string compactifications, abelian gauge fields often come from expanding R–R potentials in harmonic forms, for example

$$C_3 = A^a \wedge \omega_a + \cdots , \qquad \omega_a \in H^{1,1}(X). \tag{18.28}$$

Thus the same word "gauge group" can mean a current algebra, an open-string endpoint algebra, or a cohomological R–R reduction.

18.2.1 Gauge and gravitational couplings

The gravitational coupling follows from integrating the ten-dimensional Einstein term over the internal space, while gauge couplings depend on where the gauge fields live. For a direct product compactification,

$$M_{\rm Pl}^2 \sim \frac{V_6}{g_s^2 \ell_s^8}, \qquad \frac{1}{g_{4,\rm het}^2} \sim \frac{V_6}{g_s^2 \ell_s^6}, \qquad \frac{1}{g_{4,Dp}^2} \sim \frac{\mathrm{Vol}(\Sigma_{p-3})}{g_s \ell_s^{p-3}}. \tag{18.29}$$

The different powers of g_s are physical: closed-string tree-level terms carry $e^{-2\Phi}$, while D-brane disk gauge terms carry $e^{-\Phi}$. This is why comparing Planck scale, string scale, and gauge unification requires the compactification volume and brane and bundle input, more than a single string scale [355, 361].

18.2.2 Gauge quantum numbers

A four-dimensional particle's gauge quantum numbers are the charges appearing in its worldsheet vertex operator. In a heterotic Narain compactification a state carries a lattice vector

$$P = (p_L; p_R) \in \Gamma^{d+16,d}, \qquad Q_a = P \cdot \alpha_a \tag{18.30}$$

for simple roots α_a of the unbroken gauge algebra. In D-brane models, the corresponding labels are Chan–Paton representations: an open string from stack a to stack b transforms as

$$(\mathbf{N}_a, \overline{\mathbf{N}}_b) \tag{18.31}$$

under $U(N_a) \times U(N_b)$. Therefore quantum numbers are computed from either lattice inner products or brane endpoint positions; they are not assigned by hand after compactification.

18.2.3 Right moving gauge symmetries

The phrase "right-moving gauge symmetries" needs care in heterotic conventions. The perturbative nonabelian gauge algebra is normally carried by the bosonic left-moving current algebra; the right-moving side supplies spacetime supersymmetry. The heterotic massless gauge condition is

$$N_R = \frac{1}{2}, \quad p_R = 0, \qquad N_L = 0, \quad p_L^2 = 2. \tag{18.32}$$

The right-moving condition makes the state massless and supersymmetric; the left-moving root p_L identifies the gauge generator. Thus the phrase "right-moving gauge" means "right-moving mass-shell or supersymmetry condition plus left-moving gauge current" unless an asymmetric construction has explicitly moved the current algebra.

18.2.4 Gauge symmetries of type II strings

Perturbative type-II closed strings do not contain ten-dimensional nonabelian Yang–Mills gauge bosons. After compactification, abelian vectors arise from the metric, B-field, and R–R potentials. For example, in type IIA on a Calabi–Yau threefold,

$$C_3 = A^a \wedge \omega_a + \cdots , \qquad a = 1, \ldots, h^{1,1}(X), \tag{18.33}$$

and these A^a sit in vector multiplets. Nonabelian gauge symmetry in type-II compactifications usually appears nonperturbatively: from coincident D-branes, wrapped branes becoming massless at singularities, or ADE singularities in K3/Calabi–Yau geometry. Thus nonabelian enhancement is a statement about extra light branes or singular CFT sectors, not a generic perturbative closed-string oscillator.

18.3 Mass scales

Four-dimensional physics contains several different scales, not a single "string scale". For an approximately isotropic internal radius R,

$$M_s = \frac{1}{\ell_s}, \qquad M^{(n)}_{\mathrm{KK}} \sim \frac{|n|}{R}, \qquad M^{(w)}_{\mathrm{winding}} \sim \frac{|w|R}{\alpha'}, \qquad M_{Dp \text{ on } \Sigma} \sim \frac{\mathrm{Vol}(\Sigma)}{g_s\, \ell_s^{p+1}}. \tag{18.34}$$

The four-dimensional effective theory is reliable below the lightest tower that has been integrated out:

$$E \ll \min(M_s, M_{\mathrm{KK}}, M_{\mathrm{winding}}, M_{\mathrm{wrapped\ brane}}, \ldots). \tag{18.35}$$

T-duality exchanges the second and third entries when $R \leftrightarrow \alpha'/R$, but phenomenology also depends on wrapped branes, warping, fluxes, and threshold corrections. Thus a mass-scale discussion must state which towers are light and which are being integrated out. [350, 355, 361].

18.4 More on unification

Gauge unification in string compactification is a threshold statement. Below the compactification scale, a useful form is

$$\frac{16\pi^2}{g_a^2(\mu)} = 16\pi^2 \,\mathrm{Re}\, f_a b_a \log \frac{M_s^2}{\mu^2} \Delta_a(T, U, \text{bundle/brane input}) \cdots . \tag{18.36}$$

The first term is the tree-level gauge kinetic function, b_a is the field-theory beta-function coefficient of the light spectrum, and Δ_a is the string threshold correction from massive Kaluza–Klein, winding, oscillator, and possibly brane sectors. Therefore equality of couplings at a high scale is not automatic even if a ten-dimensional gauge group was simple. It can be shifted by Wilson lines, split brane volumes, anisotropic compactification, and string thresholds. [95, 97, 361].

18.4.1 Conditions for spacetime supersymmetry

The GSO projection is necessary but not sufficient after compactification. Spacetime supersymmetry survives when the internal CFT has the right spectral-flow operator, or geometrically when the compact space and bundle preserve an internal spinor. In the Calabi–Yau limit this is

$$\nabla_m \eta = 0, \qquad c_1(X) = 0, \qquad F_{(0,2)} = 0, \qquad J^{m\bar{n}} F_{m\bar{n}} = 0. \tag{18.37}$$

The first two conditions are geometric; the last two are the holomorphy and slope-zero conditions for a supersymmetric heterotic bundle or brane gauge field. In an $N = 1$ effective theory the same condition becomes vanishing auxiliary fields,

$$D_i W = 0, \qquad D^a = 0, \tag{18.38}$$

for an unbroken supersymmetric vacuum. Thus spacetime supersymmetry is checked both on the worldsheet/geometric construction and in the resulting low-energy action. [195, 355, 361].

18.5 Low energy actions

The four-dimensional interpretation is encoded in the $N = 1$ functions K, W, f_{ab}. The two-derivative bosonic terms include

$$\begin{aligned} \mathcal{L}_4 \supset &- K_{i\bar{j}}\, \partial_\mu \Phi^i \partial^\mu \bar{\Phi}^{\bar{j}} - V_F - V_D \\ &- \frac{1}{4} \mathrm{Re}\, f_{ab}(\Phi)\, F^a_{\mu\nu} F^{b\mu\nu} + \frac{1}{4} \mathrm{Im}\, f_{ab}(\Phi)\, F^a \wedge F^b. \end{aligned} \tag{18.39}$$

The F-term potential is

$$V_F = e^{K/M_{\mathrm{Pl}}^2}\left(K^{i\bar{j}}D_iW\,D_{\bar{j}}\overline{W} - \frac{3|W|^2}{M_{\mathrm{Pl}}^2}\right), \qquad D_iW = \partial_iW + \frac{1}{M_{\mathrm{Pl}}^2}(\partial_iK)W. \tag{18.40}$$

The function f_{ab} maps compactification input to gauge couplings and theta angles; K maps moduli geometry to scalar kinetic terms; W controls masses, Yukawa couplings, flux effects, and nonperturbative effects. A viable compactification therefore requires an actual computation of these functions, more than a list of massless fields. [86, 355, 361].

18.6 Flux compactifications and moduli stabilization

Flux compactification adds quantized background field strengths on internal cycles. In type-IIB orientifold compactifications the basic example is the three-form

$$G_3 = F_3 - \tau H_3, \qquad \frac{1}{(2\pi)^2\alpha'}\int_{\Sigma_3} F_3,\ \frac{1}{(2\pi)^2\alpha'}\int_{\Sigma_3} H_3 \in \mathbb{Z}. \tag{18.41}$$

The flux induces the Gukov–Vafa–Witten superpotential

$$W_{\text{flux}} = \int_X G_3 \wedge \Omega. \tag{18.42}$$

Thus complex-structure moduli and the axio-dilaton can be fixed by the F-term equations $D_iW_{\text{flux}} = 0$. The same flux also carries D3-brane charge,

$$N_{\text{flux}} = \frac{1}{(2\pi)^4\alpha'^2}\int_X H_3 \wedge F_3, \tag{18.43}$$

so it is constrained by the D3 tadpole condition together with localized D3-branes and orientifold planes. Fluxes therefore do two jobs at once: they generate a potential for moduli, and they enter the same charge-cancellation conditions that have appeared throughout [356, 357].

Time-dependent compactification changes how energy conditions descend to the effective theory. The higher-dimensional null energy condition follows from the worldsheet Virasoro constraint,

$$T_{MN}v^Mv^N \geq 0, \qquad v^2 = 0,$$

before dimensional reduction. Under a general time-dependent compactification, the lower-dimensional Einstein-frame averaged condition can be violated even when the higher-dimensional condition remains satisfied [271]. A bouncing four-dimensional stress tensor can therefore arise from the reduction procedure, not from a violation of the original worldsheet constraint. The energy condition should be derived in the higher-dimensional background before the truncation is interpreted as a four-dimensional cosmology.

The four-dimensional scalar potential following Eqs. (18.42)–(18.43) is

$$V = e^K\left(K^{i\bar{j}}D_iWD_{\bar{j}}\overline{W} - 3|W|^2\right) + V_D + \cdots, \qquad W = \int_X G_3 \wedge \Omega,$$

with the tadpole condition

$$N_{\text{flux}} + N_{\text{D3}} = N_{\text{O3/O7}} + \cdots.$$

The TASI lectures of McAllister and Schachner review how ten-dimensional flux input, four-dimensional supergravity, and quantum corrections enter this system [327]. Compactifications on G_2-structure seven-manifolds lead to three-dimensional $\mathcal{N} = 1$ potentials with explicit dependence on torsion classes, form fluxes, axions, gauge fields, and Stueckelberg couplings [337]. Warping and non-ISD flux can correct the off-shell potential strongly enough to challenge F-term de Sitter uplifting in KKLT-like IIB orientifolds [328]. Large numerical scans and conditional generative models now search the flux lattice for vacua with specified W_0, tadpole, and Kahler-stabilization properties [319, 320]. Heterotic non-geometric constructions and orientifold analyses propose de Sitter or scale-separated mechanisms through R-flux, nonassociative gauge structure, or leading orientifold terms [314, 316]. A controlled vacuum must satisfy the F-terms, tadpoles, warping expansion, moduli-mass hierarchy, and tower cutoff simultaneously.

Kahler moduli are not generally fixed by W_{flux} alone in the simplest IIB no-scale limit. Their stabilization requires additional effects such as Euclidean D3-brane instantons, gaugino condensation on wrapped D7-branes, α' corrections, or D-terms. This is why moduli stabilization is a consistency problem instead of a parameter choice; the proposed vacuum must solve the F- and D-term equations, satisfy tadpole cancellation, avoid uncontrolled backreaction, and keep all integrated-out towers heavier than the four-dimensional scale being studied.

18.7 The warped deformed conifold and confinement without a mass gap

The warped deformed conifold is a precise background in which a four-dimensional gauge theory confines and yet its spectrum is not fully massive. The geometry realizes the renormalization-group flow of an $\mathcal{N}=1$ gauge theory as motion along a warped throat: the ultraviolet region is a nearly conformal quiver theory, while the infrared end of the throat is capped smoothly by a finite three-sphere, and this smooth cap is the gravitational signal of confinement (Figure 18.2). Confinement does not by itself guarantee a mass gap. Here a baryonic symmetry is spontaneously broken, and Goldstone's theorem leaves a massless mode in the spectrum.

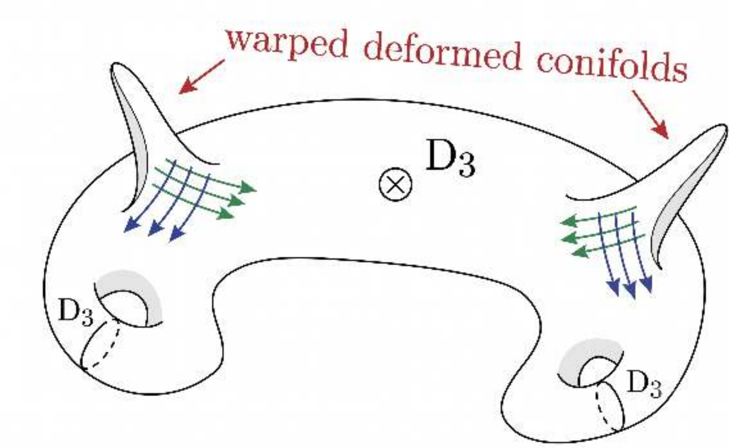


Figure 18.2: A warped deformed conifold throat may be glued into a larger compact space. The throat region carries the redshifted gauge/gravity dynamics, while the bulk determines how the noncompact throat picture is embedded in a full compactification.

Recall that $\mathcal{N}=4$ $SU(N)$ super-Yang–Mills describes N coincident D3-branes in flat ten-dimensional spacetime, the D-branes of Chapter 12, whose near-horizon geometry gives the gauge/gravity correspondence of Chapter 13. A D3-brane fills four spacetime directions, so its worldvolume is $\mathbb{R}^{3,1}$ and the remaining six directions are transverse. Light open strings ending on the stack carry Chan–Paton indices and give a $U(N)$ gauge theory; the overall $U(1)$ decouples from the interacting dynamics, leaving $SU(N)$. Because the ambient type IIB background is flat and maximally supersymmetric, the four-dimensional theory has $\mathcal{N}=4$ supersymmetry.

The stack also sources the self-dual Ramond–Ramond five-form. For an S^5 surrounding the branes in the transverse $\mathbb{R}^6$, the D3-brane charge is

$$N = \frac{1}{(4\pi^2\alpha')^2}\int_{S^5} F_5, \tag{18.44}$$

so the integer N is at once the rank of the gauge group and a quantized flux in the dual geometry (Figure 18.3). This is the first entry in the dictionary between gauge dynamics and geometry.

To reduce the supersymmetry, one replaces the flat transverse $\mathbb{R}^6$ by a Calabi–Yau cone and places the D3-branes at its isolated singularity. The relevant cone is the conifold of Chapter 17, the complex threefold

$$\sum_{a=1}^{4} z_a^2 = 0, \tag{18.45}$$

equivalently $xy-uv=0$, with the conical metric

$$ds_6^2 = dr^2 + r^2\, ds_{T^{1,1}}^2, \tag{18.46}$$

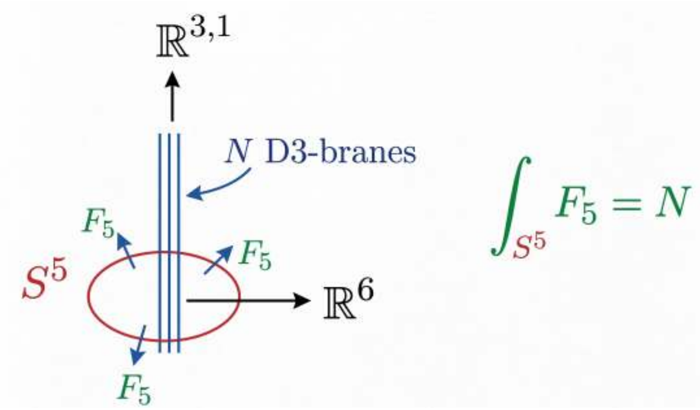


Figure 18.3: For a stack of N D3-branes, an S^5 surrounding the branes measures the five-form flux. The same integer N appears as D3-brane charge in the bulk and as the rank of the gauge theory on the branes.

where

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \simeq S^2 \times S^3. \tag{18.47}$$

At the tip of the singular conifold both the S^2 and the S^3 shrink. A stack of D3-branes placed there gives not the $\mathcal{N} = 4$ theory but the Klebanov–Witten quiver, with gauge group

$$SU(N) \times SU(N), \tag{18.48}$$

bifundamental chiral superfields

$$A_i : (N, \overline{N}), \qquad B_j : (\overline{N}, N), \qquad i, j = 1, 2, \tag{18.49}$$

and superpotential

$$W = h\, \epsilon^{ik} \epsilon^{jl}\, \mathrm{Tr}(A_i B_j A_k B_l). \tag{18.50}$$

Nothing in this geometry sets a scale, which is why the gauge theory is conformal. The singular conifold has no deformation parameter, no resolution parameter, and no finite cycle at its tip; it is a genuine cone, invariant up to rescaling under $r \to \lambda r$, and in the dual field theory this is scale invariance. The equality of the two ranks is essential: with the appropriate anomalous dimensions the two beta functions vanish together, giving an interacting $\mathcal{N} = 1$ superconformal theory.

Adding M fractional D3-branes changes this. A fractional D3-brane at the conifold is a D5-brane wrapped on the collapsed two-cycle, and its effect is to shift the ranks unequally,

$$SU(N) \times SU(N) \quad \longrightarrow \quad SU(N + M) \times SU(N), \tag{18.51}$$

so the theory can no longer be conformal. This follows from the NSVZ beta-function numerators near the Klebanov–Witten anomalous dimension $\gamma \simeq -1/2$. For the larger group $SU(N + M)$ the matter provides effectively $2N$ flavors, so

$$b_1 = 3(N + M) - 2N(1 - \gamma) \simeq 3(N + M) - 3N = 3M, \tag{18.52}$$

while for $SU(N)$,

$$b_2 = 3N - 2(N + M)(1 - \gamma) \simeq 3N - 3(N + M) = -3M. \tag{18.53}$$

One coupling therefore runs to strong coupling in the infrared while the other weakens; the loss of conformality is a direct consequence of the unequal ranks introduced by the fractional branes.

In the string description the fractional branes are Ramond–Ramond three-form flux through the three-sphere of the deformed conifold,

$$\frac{1}{4\pi^2 \alpha'} \int_{S^3} F_3 = M. \tag{18.54}$$

The conifold singularity is smoothed by replacing Eq. (18.45) with the deformed conifold

$$\sum_{a=1}^{4} z_a^2 = \epsilon^2, \tag{18.55}$$

whose parameter ϵ gives the tip S^3 a finite size. This is the geometric origin of the smooth infrared endpoint of the throat, the same deformation of the conifold drawn in Figure 17.2; the three-form flux threading it is shown in Figure 18.4.

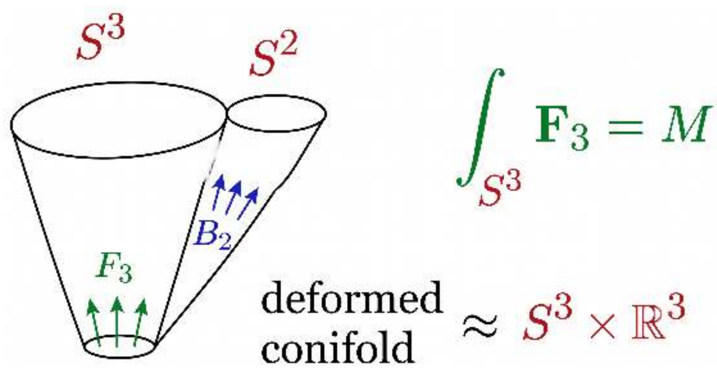


Figure 18.4: The deformed conifold keeps a finite three-sphere at the tip. The Ramond–Ramond flux through this S^3 is quantized, $\int_{S^3} F_3 = M$, and represents the fractional-brane content in the dual gauge theory.

The S^2 enters through the NS–NS two-form period

$$b(r) = \frac{1}{2\pi\alpha'} \int_{S^2} B_2. \tag{18.56}$$

The F_3 flux alone does not give the supersymmetric Klebanov–Strassler background. The combination

$$G_3 = F_3 - \tau_{\mathrm{IIB}} H_3, \qquad H_3 = dB_2, \tag{18.57}$$

must be imaginary self-dual,

$$*_6 G_3 = iG_3, \tag{18.58}$$

so that $\mathcal{N} = 1$ supersymmetry is preserved.

The cascade is encoded in the five-form flux. In type IIB supergravity the five-form obeys the Bianchi identity

$$d\widetilde{F}_5 = H_3 \wedge F_3 + \text{localized D3-brane sources.} \tag{18.59}$$

Since F_3 threads the S^3 and B_2 has a period on the S^2, the effective D3-brane charge measured on $T^{1,1} \simeq S^2 \times S^3$ becomes radius dependent,

$$N_{\text{eff}}(r) = \frac{1}{(4\pi^2\alpha')^2} \int_{S^2 \times S^3} \widetilde{F}_5. \tag{18.60}$$

The B_2 period grows logarithmically with the radial coordinate, so

$$\int_{S^2 \times S^3} F_5 = N_{\text{eff}}(r) = N + \frac{3}{2\pi} g_s M^2 \log(r/r_0). \tag{18.61}$$

This is the supergravity form of the Seiberg cascade. The radial coordinate r is dual to the gauge-theory energy scale, so the logarithmic decrease of $N_{\text{eff}}(r)$ toward the infrared is the geometric version of the repeated reduction of the ranks.

The field-theory cascade is ordinary Seiberg duality. At a typical stage the gauge group is

$$SU(N + M) \times SU(N). \tag{18.62}$$

The larger factor $SU(N + M)$ becomes strongly coupled first, with

$$N_c = N + M, \qquad N_f = 2N, \tag{18.63}$$

because the two A fields and two B fields supply the bifundamental matter. Seiberg duality maps this factor to

$$SU(N_f - N_c) = SU(2N - (N + M)) = SU(N - M), \tag{18.64}$$

so one step of the cascade is

$$SU(N + M) \times SU(N) \longrightarrow SU(N) \times SU(N - M), \tag{18.65}$$

and after relabelling the two factors the same structure repeats with $N \to N - M$, the cascade of Seiberg dualities shown in Figure 18.5.

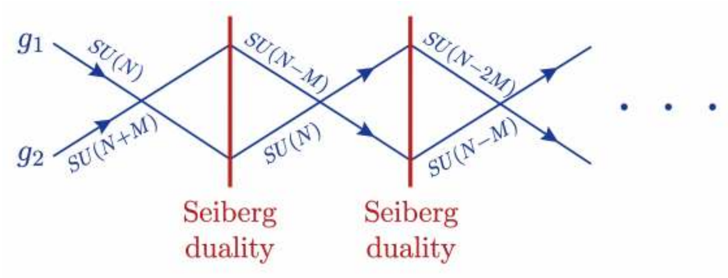


Figure 18.5: The Klebanov–Strassler cascade is a repeated Seiberg duality. Each step lowers the effective ranks while preserving the same quiver structure, until the infrared theory reaches the confining $SU(M)$ sector.

The cascade therefore descends as

$$\cdots \longrightarrow SU(9M)\times SU(8M) \longrightarrow SU(8M)\times SU(7M) \longrightarrow \cdots \longrightarrow SU(2M)\times SU(M) \longrightarrow SU(M), \tag{18.66}$$

ending on an $\mathcal{N}=1$ $SU(M)$ pure-glue theory. Since pure $\mathcal{N}=1$ super-Yang–Mills confines, the smooth cap of the deformed conifold is the gravitational dual of confinement: instead of running into a singularity, the geometry ends on the finite S^3 (Figure 18.6).

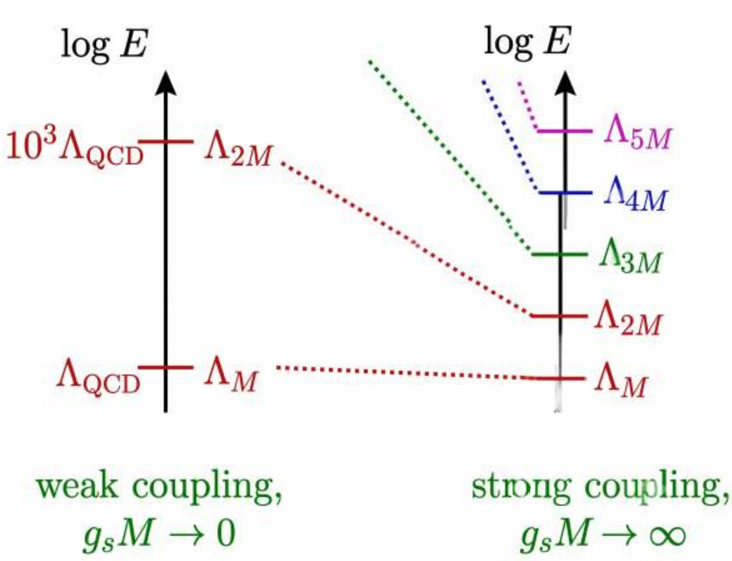


Figure 18.6: Along the cascade the effective number of colors falls by repeated duality steps. In the infrared the throat ends smoothly at the confinement scale, while in the ultraviolet the effective rank grows logarithmically with the radial/energy scale.

One might expect a mass gap, since the cascade ends in a confining pure-glue theory and the dual geometry closes off smoothly (Figure 18.7). That expectation is incomplete. The full noncompact Klebanov–Strassler theory has a baryonic branch on which a continuous baryonic symmetry is spontaneously broken, and Goldstone's theorem then keeps a massless particle in the spectrum.

Consider the penultimate theory,

$$SU(2M)\times SU(M). \tag{18.67}$$

The chiral fields A_i and B_j transform under the global symmetry

$$SU(2)_A \times SU(2)_B, \tag{18.68}$$

which is the field-theory image of the geometric symmetry of the finite three-sphere,

$$SO(4) \simeq SU(2)_A \times SU(2)_B. \tag{18.69}$$

Gauge-invariant baryon-like operators are built by antisymmetrizing the $SU(2M)$ color indices,

$$\begin{aligned} \mathcal{B} &\sim \epsilon_{\alpha_1\cdots\alpha_{2M}} (A_1)^{\alpha_1}_1 \cdots (A_1)^{\alpha_M}_M (A_2)^{\alpha_{M+1}}_1 \cdots (A_2)^{\alpha_{2M}}_M, \\ \overline{\mathcal{B}} &\sim \epsilon^{\alpha_1\cdots\alpha_{2M}} (B_1)^1_{\alpha_1} \cdots (B_1)^M_{\alpha_M} (B_2)^1_{\alpha_{M+1}} \cdots (B_2)^M_{\alpha_{2M}}. \end{aligned} \tag{18.70}$$

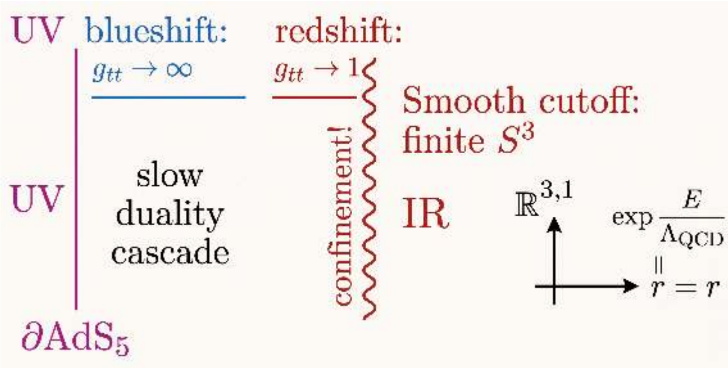


Figure 18.7: The warped throat gives the geometric version of the UV/IR relation. Moving toward the tip redshifts local energies and provides a smooth infrared cutoff through the finite S^3, while the ultraviolet end is connected to the asymptotic compact space.

The epsilon tensor makes these operators neutral under $SU(2M)$, and the remaining indices contract to an $SU(M)$ singlet; they are baryonic in the same sense as QCD baryons, their gauge invariance coming from complete antisymmetrization of color.

The nonperturbative dynamics of the $SU(2M)$ factor produces a baryonic branch obeying

$$\mathcal{B}\overline{\mathcal{B}} = -\Lambda_{2M}^{4M}, \tag{18.71}$$

with Λ_{2M} the strong-coupling scale of the $SU(2M)$ theory. This fixes the product of the two baryonic expectation values but not their common phase, conveniently written as

$$\mathcal{B} = \Lambda_{2M}^{2M} e^{ia/f_a}, \qquad \overline{\mathcal{B}} = -\Lambda_{2M}^{2M} e^{-ia/f_a}. \tag{18.72}$$

Under the baryonic symmetry,

$$U(1)_B : \qquad \mathcal{B} \to e^{i\theta}\mathcal{B}, \qquad \overline{\mathcal{B}} \to e^{-i\theta}\overline{\mathcal{B}}, \tag{18.73}$$

the phase field shifts as

$$a \to a + f_a\theta. \tag{18.74}$$

A vacuum chooses a definite phase, so $U(1)_B$ is spontaneously broken and its Goldstone boson is the massless pseudoscalar $a(x)$,

$$\Box_4 a = 0. \tag{18.75}$$

This is the precise reason the theory confines without a full mass gap. The winding of this baryonic phase is the same structure that produced the axionic cosmic strings of Eq. (18.25): there the broken symmetry was Peccei–Quinn, here it is baryonic.

Holographically, this Goldstone boson must appear as a normalizable zero-mode of the warped deformed conifold. A bulk fluctuation separates as

$$\phi(x, r) = \varphi(x)R(r), \tag{18.76}$$

with the four-dimensional mass set by the radial eigenvalue problem. A massive glueball is a normalizable mode with

$$\Box_4\varphi = m^2\varphi, \tag{18.77}$$

and the Goldstone mode has $m = 0$, so it must be a normalizable radial zero-mode.

Since $a(x)$ is a four-dimensional pseudoscalar, it is natural to dualize it to a three-form field strength,

$$da = *_4 f_3, \tag{18.78}$$

so that

$$df_3 = d(*_4 da) = (\Box_4 a)\,\mathrm{vol}_4, \tag{18.79}$$

and $\Box_4 a = 0$ gives $df_3 = 0$. The simplest guess is therefore

$$\delta F_3 = f_3 = *_4 da. \tag{18.80}$$

This carries the right four-dimensional field but is not by itself a full solution of the linearized type IIB equations, because the background carries nontrivial F_3, H_3, F_5, warp factor, and conifold geometry. The complete fluctuation needs radial corrections,

$$\delta F_3 = *_4 da + f_2(\tau)\, da \wedge dg^5 + f_2'(\tau)\, da \wedge d\tau \wedge g^5, \tag{18.81}$$

and the five-form perturbation is

$$\delta F_5 = (1 + *)\, \delta F_3 \wedge B_2, \tag{18.82}$$

or, for the relevant part,

$$\delta F_5 = \left(*_4 da - \frac{\epsilon^{4/3}}{6K(\tau)^2} h(\tau)\, da \wedge d\tau \wedge g^5 \right) \wedge B_2 + \text{ten-dimensional dual.} \tag{18.83}$$

The extra pieces are exactly what makes the fluctuation satisfy the linearized equations of motion and the Bianchi identities in the warped deformed conifold background.

The radial function obeys

$$f_2(\tau) \propto -\frac{2}{K(\tau)^2 \sinh^2 \tau} \int_0^\tau dx\, h(x) \sinh^2 x, \tag{18.84}$$

which also shows the mode is normalizable. Near the smooth tip, $\tau \to 0$,

$$\sinh \tau \sim \tau, \qquad \int_0^\tau dx\, h(x) \sinh^2 x \sim h(0) \int_0^\tau x^2 dx \sim \tau^3, \tag{18.85}$$

and with $\sinh^2 \tau \sim \tau^2$,

$$f_2(\tau) \sim \tau \quad \text{as} \quad \tau \to 0, \qquad \text{so} \quad f_2(\tau) \to 0 \quad \text{in the infrared.} \tag{18.86}$$

In the ultraviolet the warped-throat asymptotics suppress the same profile, so $f_2(\tau) \to 0$ as $\tau \to \infty$. The fluctuation is thus localized in the throat and gives a finite four-dimensional kinetic term,

$$S_{\text{eff}} = -\frac{1}{2} f_a^2 \int d^4x\, \partial_\mu a\, \partial^\mu a, \qquad f_a^2 < \infty. \tag{18.87}$$

This finite norm is the supergravity statement that the Goldstone boson is a physical four-dimensional state.

The same fluctuation fixes the meaning of a D1-brane in this background. A D1-brane couples electrically to the Ramond–Ramond two-form,

$$S_{\text{D1}} \supset \mu_1 \int_{\Sigma_2} C_2, \qquad F_3 = dC_2, \tag{18.88}$$

so it is an electric source for F_3. Because the four-dimensional Goldstone field is dual to f_3 through $da = *_4 f_3$, an electric source for f_3 is a magnetic source for a, and around the D1-string the phase has a monodromy

$$\oint da = 2\pi n f_a. \tag{18.89}$$

The D1-brane is therefore not a Wilson line but a solitonic vortex string, an Abrikosov–Nielsen–Olesen string carrying the winding of the baryonic Goldstone phase. It is stable because that winding is topological, even though it is not a BPS object; it is the string-theory realization of the global cosmic strings estimated in Eq. (18.26).

Supersymmetry supplies a scalar partner for the Goldstone boson. Because the background preserves $\mathcal{N} = 1$ supersymmetry, the pseudoscalar sits in a chiral multiplet, and its scalar partner must be a modulus, a normalizable deformation of the warped deformed conifold. The deformation should preserve the $SO(4)$ symmetry of the tip S^3, involve NS–NS-sector fields, and break the interchange $A \leftrightarrow B$. A natural ansatz with these properties is

$$\delta B_2 = \chi(\tau)\, dg^5, \tag{18.90}$$

together with the metric perturbation

$$\delta(ds_{10}^2) = m(\tau) \left(g^1 g^3 + g^2 g^4 \right), \tag{18.91}$$

with $\chi(\tau)$ and $m(\tau)$ chosen so that the mode is normalizable. Turning on $m(\tau)$ has the geometric character of resolving the conifold: it introduces a resolution-like deformation while remaining in the warped deformed conifold family, and at linear order the six-dimensional space stays Ricci-flat as supersymmetry requires, though it is no longer the original Calabi–Yau conifold.

One subtlety remains. In the noncompact throat $U(1)_B$ acts as a global symmetry, and its spontaneous breaking gives a physical Goldstone boson. A compact string compactification is not expected to have exact global symmetries, so the baryonic $U(1)_B$ is gauged, and the Goldstone boson is then eaten by the corresponding gauge field through an abelian Higgs or Stückelberg mechanism,

$$A_\mu \to A_\mu + \frac{1}{m_A}\partial_\mu a, \tag{18.92}$$

the same mechanism used for gauged axionic shift symmetries earlier in this chapter. Supersymmetry then also lifts the scalar partner. If

$$K = \frac{1}{4\pi^2\alpha'}\int H_3 \tag{18.93}$$

is the NS–NS flux on the dual three-cycle, this mass is expected to depend on the flux as a power law,

$$\frac{m_H}{\Lambda_{\rm QCD}} \sim (g_s M)^p K^q, \tag{18.94}$$

with the powers fixed by the compact embedding.

The conclusion is that the finite S^3 of the deformed conifold is a smooth infrared cutoff and realizes confinement in the dual gauge theory, and the logarithmic running of the five-form flux in Eq. (18.61) is the gravitational encoding of the Seiberg duality cascade, whose ranks fall step by step until the theory reaches $\mathcal{N} = 1$ $SU(M)$ pure glue. The full noncompact theory is nevertheless not fully gapped, because the baryonic branch breaks $U(1)_B$ and produces a normalizable Goldstone boson. Confinement and a mass gap are related but not identical: a theory can confine and still contain a massless color-singlet mode whenever a continuous global symmetry is spontaneously broken.

18.8 Supersymmetry breaking in perturbation theory

Perturbative supersymmetry breaking must be defined by a modular-invariant string background or by a consistent deformation such as Scherk–Schwarz boundary conditions. It is not enough to "choose the opposite GSO sign" in one sector, because modular transformations mix sectors. The one-loop vacuum energy is the first diagnostic:

$$\Lambda_{1-\text{loop}} = -\frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\mathrm{Tr}_{\mathcal{H}}\Big[(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\Big]_{\text{consistent sectors}}. \tag{18.95}$$

In a supersymmetric vacuum the boson-fermion cancellation makes this vanish after the full sector sum. If supersymmetry is broken perturbatively, the same integral also tests for tachyons and uncanceled tadpoles. A nonzero tadpole means the assumed background is not a solution until backreaction or a Fischler–Susskind-type correction is included. [10, 355, 361].

18.8.1 Supersymmetry breaking at tree level

Tree-level supersymmetry breaking is stated most directly in the four-dimensional effective action. A vacuum preserves $N = 1$ supersymmetry when all auxiliary fields vanish; it breaks supersymmetry when at least one order parameter is nonzero:

$$F^i = -e^{K/2M_{\rm Pl}^2}K^{i\bar{j}}D_{\bar{j}}\overline{W}, \qquad D^a = -\left(\mathrm{Re}\, f\right)^{-1\,ab} K_i X_b^i, \qquad F^i \neq 0 \text{ or } D^a \neq 0. \tag{18.96}$$

The gravitino mass is

$$m_{3/2} = e^{K/2M_{\rm Pl}^2}\frac{|W|}{M_{\rm Pl}^2}. \tag{18.97}$$

In string compactifications, tree-level breaking can come from fluxes, brane angles, Scherk–Schwarz twists, or non-supersymmetric classical backgrounds, but every case must still solve the worldsheet or supergravity consistency conditions. Otherwise the "breaking" is only an inconsistent expansion point. [190, 355, 361].

18.8.2 Supersymmetry breaking in the loop expansion

Loop effects measure the failure of bose-fermi degeneracy after the full projection. In field-theory language this appears as a supertrace expansion,

$$\Lambda_{1-\text{loop}} \sim \frac{1}{64\pi^2}\mathrm{Str}\left[M^4 \log \frac{M^2}{\mu^2}\right], \qquad \mathrm{Str}\, M^{2k} = \sum_s (-1)^{2s}(2s+1)M_s^{2k}. \tag{18.98}$$

String theory replaces the ultraviolet-divergent field-theory trace by a modular integral over the torus. That improvement does not remove the need to inspect infrared physics: tachyons, dilaton tadpoles, or uncanceled NS–NS tadpoles signal that the background wants to roll. Thus loop-induced supersymmetry breaking must be accompanied by a stability and tadpole analysis, more than a nonzero vacuum-energy formula. [10, 355].

18.9 Supersymmetry beyond perturbation theory

Nonperturbative effects can preserve or break supersymmetry depending on how they enter W, K, and the D-terms. Typical contributions are exponential in a modulus because they come from Euclidean branes or strong gauge dynamics:

$$W_{\text{np}} = \sum_\Sigma A_\Sigma(\Phi)\, \exp\left[-\frac{2\pi}{n_\Sigma}T_\Sigma\right], \qquad T_\Sigma = \frac{1}{g_s}\mathrm{Vol}(\Sigma) i \int_\Sigma C. \tag{18.99}$$

These terms can stabilize moduli and generate $D_iW \neq 0$, but selection rules, zero-mode counting, Freed–Witten consistency, and charge conservation decide whether the instanton contribution is actually present. BPS indices help because they are protected under continuous changes of coupling; ordinary nonprotected masses and potentials are not. [187, 190, 361].

18.10 Landscape, swampland, and quantum-gravity constraints

The word landscape refers to the large set of effective theories obtained by choosing topology, flux integers, branes, bundles, and orientifold input. The word swampland refers to the complementary problem: many apparently consistent low-energy effective theories cannot arise from a theory of quantum gravity. A four-dimensional action is admissible only if it descends from a complete set of worldsheet or spacetime consistency conditions: modular invariance, BRST consistency, factorization, tadpole and anomaly cancellation, charge quantization, and a controlled tower of states at the cutoff.

Several quantum-gravity constraints have direct string-theory expressions. Charge completeness is the statement that allowed gauge charges must be represented by states or branes in the spectrum. Infinite-distance limits in moduli space bring down towers of Kaluza–Klein, winding, wrapped-brane, or string states, so the four-dimensional effective description has a moving cutoff. Flux choices are discrete, but not arbitrary: they obey tadpole equations such as Eq. (18.43). Thus the final test of a proposed four-dimensional model is not only whether its low-energy Lagrangian is anomaly-free, but whether it can be embedded in a compactification whose full charge lattice, moduli limits, and tadpoles are consistent [161, 358].

Worldsheet-based swampland arguments make the cutoff dependence quantitative. Modular invariance relates the infrared vacuum energy and gauge couplings to higher-derivative Wilson coefficients through one-loop integrals of the form

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \mathcal{Z}(\tau,\bar\tau)\, \mathcal{O}(\tau,\bar\tau).$$

In species limits, where many states become light, these integrals imply parametric UV/IR relations between the vacuum energy, gauge couplings, and higher-derivative terms [272]. The same scaling viewpoint organizes low-energy phases of string theory beyond what is visible in a finite EFT truncation [280]. Quantum corrections can also obstruct classical infinite-distance limits in type-II and F-theory compactifications; the worldsheet theory of the candidate EFT string tests whether the expected tower remains present after Kahler-sector corrections [273,281]. Thus a geodesic that is long in the classical metric is not sufficient evidence for an infinite-distance string limit; the tower, charge spectrum, and modular scaling must survive quantum corrections.

Several moduli-space analyses state the distance and cutoff constraints explicitly. At an infinite-distance boundary, the Swampland Distance Conjecture predicts a tower

$$m_{\text{tower}}(\varphi) \sim m_0\, e^{-\lambda d(\varphi,\varphi_0)}.$$

For locally symmetric moduli spaces, infinite-distance limits are characterized by rational parabolic subgroups; the tower charges and decay rates can be read from the associated weight polytopes [324, 338]. In six dimensions, the RCFT of the hyperplane string gives a non-geometric test for candidate supergravities with one-dimensional string-charge lattice [317]. Type-II Calabi–Yau rigid limits separate vector multiplets into gravitationally coupled and rigid sectors; curvature divergences and towers of extremal BPS states determine which subsector can decouple from gravity [329]. Black-hole scalar flows probe moduli stabilization because large charge can drive fields over long distances in moduli space [313]. The species scale,

$$\Lambda_{\text{sp}} \sim \frac{M_{\text{Pl}}}{\sqrt{N(\Lambda_{\text{sp}})}},$$

then bounds the EFT cutoff when many tower states lie below Λ_{sp} [312]. Holography imposes a further condition on scale-separated AdS vacua: the scalar potential and quantum-gravity cutoff must allow a decoupled dual CFT sector [325, 326]. Local anomaly cancellation and tadpole cancellation are necessary, but the moduli-boundary tower and cutoff tests are independent requirements.

18.10.1 An example

A standard perturbative example is Scherk–Schwarz breaking on a circle. Fields are allowed to return to themselves only up to an R-symmetry or fermion-number twist:

$$\Phi(y + 2\pi R) = e^{2\pi i\alpha Q_R}\Phi(y), \qquad m_n = \frac{|n + \alpha Q_R|}{R}. \tag{18.100}$$

Bosons and fermions then acquire different Kaluza–Klein spectra. For the standard half-period fermion twist, the gravitino mass is $m_{3/2} \sim 1/(2R)$. The example shows the pattern directly: the breaking disappears as $R \to \infty$, the scale is geometric, and modular invariance requires the full tower of shifted momentum and winding sectors in the string partition function. [10, 355].

18.10.2 Another example

A second example does not repeat the first with different names. A useful contrast is between a geometric modulus and a gauge-bundle modulus. In heterotic compactification, a Wilson line shifts gauge-lattice momenta schematically as

$$p_L \longrightarrow p_L + A \cdot q, \tag{18.101}$$

and can break a non-abelian gauge group to a subgroup without changing the number of spacetime dimensions. A Wilson-line modulus is therefore different from a radius modulus, even though both appear as scalar fields in the lower-dimensional theory. The massless gauge generators are those whose lattice charges remain invariant under the shift, while the remaining gauge bosons acquire masses proportional to the Wilson-line displacement. [10, 86].

18.10.3 Discussion

The discussion collects the consequences of the preceding examples without introducing a new formalism. A useful summary is

$$\text{worldsheet input} \quad \longrightarrow \quad \{\text{massless fields, } K,\ W,\ f_{ab}, \text{ selection rules}\}. \tag{18.102}$$

Here K is the Kahler potential, W the superpotential, and f_{ab} the gauge kinetic function of a four-dimensional supersymmetric effective theory.

An allowed compactification can produce a possible four-dimensional theory, but additional work is needed to obtain the Standard Model spectrum, stabilize moduli, specify supersymmetry breaking, and satisfy cosmological constraints. [355, 361].

Appendix A

Spinors and supersymmetry in various dimensions

The superstring chapters lean constantly on dimension-dependent spinor facts, when Majorana and Weyl conditions can be imposed together, how product representations decompose, which central charges a supersymmetry algebra admits, and this appendix derives all of them from the Clifford algebra with no steps omitted. We construct the spinor representation explicitly in the s_a eigenbasis, build the chirality, conjugation, and charge-conjugation matrices Γ, $B_{1,2}$, C, and obtain the mod-8 pattern of allowed reality conditions; the Majorana–Weyl spinors possible only for $d = 2 \bmod 8$ are exactly what the ten-dimensional superstring needs. The supersymmetry half then works out $d = 4$, $N = 1$ representations from the oscillator construction, extends to N supercharges with the helicity towers and the BPS shortening bound $M \geq |z_r|$ that Chapters 13 and 14 use as their main quantitative input, and closes with the maximal, half-maximal, and quarter-maximal supergravities, eleven-dimensional supergravity, the type II theories, and the multiplet counting behind every low-energy action quoted in the text [5, 6].

A.1 Spinors in various dimensions

Suppose that we are working in even dimensions i.e. $d = 2k + 2$ (where $k \in \{0, 1, 2, \ldots\}$. Let Γ^μ be the gamma matrices in d dimensions. Then, the Clifford algebra is satisfied;

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{21.1.1}$$

Now, we can make creation and annihilation operators from these matrices as follows:

$$\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right)$$
$$\Gamma^{a\pm} = \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right) \qquad \text{where } a = 1, \ldots, k \tag{21.1.2}$$

where $\Gamma^{a\pm}$ satisfy the following anti commutation relations;

$$\left\{\Gamma^{a+}, \Gamma^{b+}\right\} = \frac{1}{4}\left\{\Gamma^{2a} + i\Gamma^{2a+1}, \Gamma^{2b} + i\Gamma^{2b+1}\right\} = 0$$
$$\left\{\Gamma^{a-}, \Gamma^{b-}\right\} = \frac{1}{4}\left\{\Gamma^{2a} - i\Gamma^{2a+1}, \Gamma^{2b} - i\Gamma^{2b+1}\right\} = 0$$
$$\left\{\Gamma^{a+}, \Gamma^{b-}\right\} = \frac{1}{4}\left\{\Gamma^{2a} + i\Gamma^{2a+1}, \Gamma^{2b} - i\Gamma^{2b+1}\right\} = \delta^{ab} \tag{21.1.3}$$

The anti-commutation relations involving $\Gamma^{0\pm}$ are as follows:

$$\left\{\Gamma^{0\pm}, \Gamma^{a+}\right\} = \left\{\Gamma^{0\pm}, \Gamma^{a-}\right\} = \left\{\Gamma^{0+}, \Gamma^{0+}\right\} = \left\{\Gamma^{0-}, \Gamma^{0-}\right\} = 0$$
$$\left\{\Gamma^{0+}, \Gamma^{0-}\right\} = \frac{1}{4}\left\{\Gamma^0 + \Gamma^1, -\Gamma^0 + \Gamma^1\right\} = 1 + \frac{1}{2}\Gamma^0\Gamma^1 \tag{21.1.4}$$

21.1.3) implies that $(\Gamma^{a+})^2 = (\Gamma^{a-})^2 = (\Gamma^{0+})^2 = (\Gamma^{0-})^2 = 0$. Now, let ξ be a spinor that doesn't contain any factor of Γ^{a-} in its expression, then we can make a spinor ζ as follows:

$$\zeta = \Gamma^{k-}\Gamma^{(k-1)-}\dots\Gamma^{0-}\xi \tag{21.1.5}$$

So, ζ is a spinor annihilated by all Γ^{a-}. We can now make a spinor basis by applying Γ^{a+} on ζ (one Γ^{a+} is applied at most once). A member of this spinor basis is written as follows:

$$\zeta^{\mathbf{s}} = \left(\Gamma^{k+}\right)^{s_k+1/2}\dots.\left(\Gamma^{0+}\right)^{s_0+1/2}\zeta \tag{21.1.6}$$

where $s_a = -1/2$ means that the corresponding Γ^{a+} has been applied and $s_a = 1/2$ means that the corresponding Γ^{a+} is applied. Moreover, s is the following k-tuple;

$$\mathbf{s} = (s_0, \dots, s_k)$$

Lorentz generators in the spinor representation are given as follows:

$$\Sigma^{\mu\nu} = -\frac{i}{4}\left[\Gamma^\mu, \Gamma^\nu\right] \tag{21.1.7}$$

which satisfy the Lorentz algebra;

$$i\left[\Sigma^{\mu\nu}, \Sigma^{\alpha\beta}\right] = \eta^{\mu\alpha}\Sigma^{\nu\beta} - \eta^{\mu\beta}\Sigma^{\nu\alpha} - \eta^{\nu\alpha}\Sigma^{\mu\beta} + \eta^{\nu\beta}\Sigma^{\mu\alpha} \tag{21.1.8}$$

We can take out a set of mutually commuting Σ 's as follows:

$$\left[\Sigma^{2a,2a+1}, \Sigma^{2b,2b+1}\right] = i\eta^{2a,2b}\Sigma^{2a+1,2b+1} = i\delta^{2a,2b}\Sigma^{2a+1,2b+1} = 0$$

These Σ 's are calculated as follows:

$$\Sigma^{0,1} = -\frac{i}{4}\left[\Gamma^0, \Gamma^1\right] = -\frac{i}{2}\Gamma^0\Gamma^1 = -i\left(\Gamma^{0+}\Gamma^{0-} - \frac{1}{2}\right)$$

$$\Sigma^{2a,2a+1} = -\frac{i}{4}\left[\Gamma^{2a}, \Gamma^{2a+1}\right] = -\frac{i}{2}\Gamma^{2a}\Gamma^{2a+1} = \Gamma^{(2a)+}\Gamma^{(2a+1)-} - \frac{1}{2}$$

So, we can define the following convenient operator;

$$S_a = \begin{cases} \Sigma^{2a,2a+1} \text{ for } a = 1, \dots, k \\ i\Sigma^{0,1} \text{ for } a = 0 \end{cases} = i^{a,0}\Sigma^{2a,2a+1} = \Gamma^{(2a)+}\Gamma^{(2a+1)-} - \frac{1}{2} \tag{21.1.9}$$

Now, we calculate $S_a\zeta^{\mathrm{s}}$ as follows:

$$\begin{aligned} S_a\zeta^{\mathbf{s}} &= \left(\Gamma^{a+}\Gamma^{a-} - \frac{1}{2}\right)\left(\Gamma^{k+}\right)^{s_k+1/2}\dots\left(\Gamma^{a+}\right)^{s_a+1/2}\dots\left(\Gamma^{0+}\right)^{s_0+1/2}\zeta^{\mathbf{s}} \\ &= \left(\Gamma^{k+}\right)^{s_k+1/2}\dots\Gamma^{a+}\Gamma^{a-}\left(\Gamma^{a+}\right)^{s_a+1/2}\dots\left(\Gamma^{0+}\right)^{s_0+1/2}\zeta^{\mathbf{s}} - \frac{1}{2}\zeta^{\mathbf{s}} \end{aligned} \tag{21.1.10}$$

Now, we need to evaluate $\Gamma^{a-}\left(\Gamma^{a+}\right)^{s_a+1/2}$ for different values of s_a. Here it is;

$$\begin{cases} \Gamma^{a-}\left(\Gamma^{a+}\right)^{s_a+1/2} = \Gamma^{a-} \quad \text{if } s_a = -\frac{1}{2} \\ \Gamma^{a-}\left(\Gamma^{a+}\right)^{s_a+1/2} = \Gamma^{a-}\Gamma^{a+} = 1 - \Gamma^{a+}\Gamma^{a-} \quad \text{if } s_a = \frac{1}{2} \end{cases} \tag{21.1.11}$$

Using 21.1.11 in 21.1.10, we have;

$$S_a\zeta^{\mathbf{s}} = \begin{cases} -\frac{1}{2}\zeta^{\mathbf{s}} \text{ if } s_a = -\frac{1}{2} \\ \zeta^{\mathbf{s}} - \frac{1}{2}\zeta^{\mathbf{s}} = \frac{1}{2}\zeta^{\mathbf{s}} \text{ if } s_a = \frac{1}{2} \end{cases} \Rightarrow S_a\zeta^{\mathbf{s}} = s_a\zeta^{\mathbf{s}}$$

where we used the fact that $\Gamma^{a-}\zeta = 0$. So, ζ^{s} is a simultaneous eigenstate of S_a. It also means that ζ^{s} with even/odd number of non-zero s_a 's don't mix.

We now define the chirality matrix Γ as follows:

$$\Gamma = i^{-k}\Gamma^0\dots\Gamma^{d-1} \tag{21.1.12}$$

This matrix has some useful properties that we now prove.

$$\Gamma^2 = i^{-2k}\Gamma^0\Gamma^1\dots\Gamma^{d-1}\Gamma^0\Gamma^1\dots\Gamma^{d-1} = (-1)^k\left[(-1)^{d-1}(-1)\right](-1)^{d-2}\dots(-1)^1(-1)^0$$

$$= (-1)^P \text{ where } P = k+1+\sum_{j=0}^{d-1} j = k+1+\frac{(d-1)d}{2} = k+1+\frac{(2k+1)(2k+2)}{2} = 2(k+1)^2$$

$$\Rightarrow (-1)^P = 1 \Rightarrow \Gamma^2 = 1 \tag{21.1.13}$$

where the powers of -1 in the first line of 21.1.13 comes as follows. The $d-2$ power comes due to anti-commuting Γ^0 all the way through to other Γ^0 and the other -1 in the square bracket comes because Γ^0 squares to -1 . The rest of the powers after that come due to anti-commuting gamma matrices. We now prove another property;

$$\{\Gamma, \Gamma^\mu\} = i^{-k}\left\{\Gamma^0 \dots \Gamma^{d-1}, \Gamma^\mu\right\} = i^{-k}\left(\Gamma^0 \dots \Gamma^\mu \dots \Gamma^{d-1}\Gamma^\mu + \Gamma^\mu\Gamma^0 \dots \Gamma^\mu \dots \Gamma^{d-1}\right)$$

$$= i^{-k}\left((-1)^{d-\mu-1}\Gamma^0 \dots \Gamma^{\mu-1}\Gamma^{\mu+1} \dots \Gamma^{d-1} + (-1)^\mu\Gamma^0 \dots \Gamma^{\mu-1}\Gamma^{\mu+1} \dots \Gamma^{d-1}\right)$$

$$= i^{-k}\left((-1)^{d-\mu-1} + (-1)^\mu\right)\Gamma^0 \dots \Gamma^{\mu-1}\Gamma^{\mu+1} \dots \Gamma^{d-1} = i^{-k}\left((-1)^{d+\mu-1} + (-1)^\mu\right)\Gamma^0 \dots \Gamma^{\mu-1}\Gamma^{\mu+1} \dots \Gamma^{d-1}$$

$$= i^{-k}(-1)^\mu\left((-1)^{d-1} + 1\right)\Gamma^0 \dots \Gamma^{\mu-1}\Gamma^{\mu+1} \dots \Gamma^{d-1} = 0$$

where in the last step, we used the fact that $(-1)^{d-1} = -1$ because d is even. So, we have shown that;

$$[\Gamma, \Gamma^\mu] = 0 \tag{21.1.14}$$

This property readily implies the following property;

$$[\Gamma, \Sigma^{\mu\nu}] = -\frac{i}{4}[\Gamma, [\Gamma^\mu, \Gamma^\nu]] = -\frac{i}{4}[\Gamma[\Gamma^\mu, \Gamma^\nu] - [\Gamma^\mu, \Gamma^\nu]\Gamma] = -\frac{i}{4}[\Gamma\Gamma^\mu\Gamma^\nu - \Gamma\Gamma^\nu\Gamma^\mu - \Gamma^\mu\Gamma^\nu\Gamma + \Gamma^\nu\Gamma^\mu\Gamma]$$

$$= -\frac{i}{4}[\Gamma\Gamma^\mu\Gamma^\nu - \Gamma\Gamma^\nu\Gamma^\mu - \Gamma\Gamma^\mu\Gamma^\nu + \Gamma\Gamma^\nu\Gamma^\mu] = 0 \tag{21.1.15}$$

In the last step, we used 21.1.14. Final property is derived as follows:

$$\Gamma = i^{-k}\Gamma^0 \dots \Gamma^{d-1} = 2^{k+1}\left(\frac{1}{2}\Gamma^0\Gamma^1\right)\left(-\frac{i}{2}\Gamma^2\Gamma^3\right)\dots\left(-\frac{i}{2}\Gamma^{2k}\Gamma^{2k+1}\right) = 2^{k+1}S_0 \dots S_k \tag{21.1.16}$$

which implies that Γ is diagonal and it is easy to see that if even number of s_a 's are 1/2, then the diagonal entry is 1/2 and if odd number of s_a 's are 1/2, then the diagonal entry is $-1/2$. Thus, the representation breaks into two representations with chirality +1 and -1 respectively. For odd dimensions i.e. $d = 2k+3$, add Γ as an additional gamma matrix i.e. $\Gamma^{2k+3} = \Gamma$ and we get an irreducible representation of Lorentz algebra.

A.1.1 Majorana Spinors

Let s be the basis where the matrix elements of Γ^{a+} are real. This implies that $\Gamma^3, \dots \Gamma^{d-1} = \Gamma^{2k+1}$ are purely imaginary and $\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^4 \dots \Gamma^d = \Gamma^{2k+2}$ are real. Moreover, since $\Gamma^\mu, \Gamma^{*\mu}$ and $-\Gamma^{*\mu}$ satisfy the Clifford algebra, they should be related by a similarity transformation. We will find this similarity transformation. We define two matrices as follows:

$$B_1 = \Gamma^3..\Gamma^{2k+1}, \quad B_2 = \Gamma B_1 \tag{21.1.17}$$

We now derive the following useful property. For even μ, we have;

$$B_1\Gamma^\mu B_1^{-1} = \Gamma^3..\Gamma^{2k+1}\Gamma^\mu\Gamma^{2k+1}..\Gamma^3 = (-1)^k\Gamma^\mu = (-1)^k\Gamma^{*\mu}$$

Moreover, for odd μ, we have;

$$B_1\Gamma^\mu B_1^{-1} = B_1\Gamma^{2\nu+1}B_1^{-1} = \Gamma^3..\Gamma^{2\nu+1} \dots \Gamma^{2k+1}\Gamma^\mu\Gamma^{2k+1}..\Gamma^{2\nu+1} \dots \Gamma^3$$

$$= (-1)^{k-\nu}\Gamma^3..\Gamma^{2\nu-1}\Gamma^{2\nu+3} \dots \Gamma^{2k+1}\Gamma^{2k+1}..\Gamma^{2\nu+3}\Gamma^{2\nu+1}\Gamma^{2\nu-1} \dots \Gamma^3$$

$$= (-1)^{k-\nu}(-1)^{\nu-1}\Gamma^3..\Gamma^{2\nu-1}\Gamma^{2\nu+3} \dots \Gamma^{2k+1}\Gamma^{2k+1}..\Gamma^{2\nu+3}\Gamma^{2\nu-1} \dots \Gamma^3 = (-1)^k(-\Gamma^\mu) = (-1)^k\Gamma^{*\mu}$$

So, we see that for all μ, we have;

$$B_1\Gamma^\mu B_1^{-1} = (-1)^k\Gamma^{*\mu} \tag{21.1.18}$$

A similar property for B_2 is derived as follows:

$$B_2\Gamma^\mu B_2^{-1} = \Gamma B_1\Gamma^\mu B_1^{-1}\Gamma = (-1)^k\Gamma\Gamma^{*\mu}\Gamma = (-1)^{k+1}\Gamma^{*\mu}\Gamma^2 = (-1)^{k+1}\Gamma^{*\mu} \tag{21.1.19}$$

Anther property for B_1 and B_2 is derived as follows:

$$\begin{aligned}
B_1\Sigma^{\mu\nu}B_1^{-1} &= -\frac{i}{4}B_1[\Gamma^\mu,\Gamma^\nu]B_1^{-1} \\
&= -\frac{i}{4}\left(B_1\Gamma^\mu B_1^{-1}B_1\Gamma^\nu B_1^{-1} - B_1\Gamma^\nu B_1^{-1}B_1\Gamma^\mu B_1^{-1}\right) \\
&= -(-1)^{2k}\frac{i}{4}[\Gamma^\mu,\Gamma^\nu] = -\Sigma^{*\mu\nu}, \\
B_2\Sigma^{\mu\nu}B_2^{-1} &= -\frac{i}{4}B_2[\Gamma^\mu,\Gamma^\nu]B_2^{-1} \\
&= -\frac{i}{4}\left(B_2\Gamma^\mu B_2^{-1}B_2\Gamma^\nu B_2^{-1} - B_2\Gamma^\nu B_2^{-1}B_2\Gamma^\mu B_2^{-1}\right) \\
&= -(-1)^{2k+2}\frac{i}{4}[\Gamma^\mu,\Gamma^\nu] = -\Sigma^{*\mu\nu}.
\end{aligned} \tag{21.1.20}$$

We see that ζ and $B^{-1}\zeta^*$ transforms in the same way under lorentz transformations;

$$\zeta \to \left(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\zeta \tag{21.1.21}$$

$$\begin{aligned}
B^{-1}\zeta^\star &\to B^{-1}\left(1 - i\omega_{\mu\nu}\Sigma^{\star\mu\nu}\right)\zeta^\star \\
&= B^{-1}\zeta^\star - i\omega_{\mu\nu}B^{-1}\Sigma^{\star\mu\nu}B\,B^{-1}\zeta^\star \\
&= B^{-1}\zeta^\star + i\omega_{\mu\nu}\Sigma^{\mu\nu}B^{-1}\zeta^\star \\
&= \left(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu}\right)B^{-1}\zeta^\star.
\end{aligned} \tag{21.1.22}$$

where B stands for B_1 and B_2 collectively and $\omega_{\mu\nu}$ are real parameters. So, Dirac representation is self-conjugate. Now, we prove the following identities;

$$\begin{aligned}
B_1\Gamma B_1^{-1} &= (i)^{-k}\Gamma^3\Gamma^5\dots\Gamma^{d-1}\left(\Gamma^0\Gamma^1\dots\Gamma^{d-1}\right)\Gamma^{d-1}\dots\Gamma^3 = (i)^{-k}(-1)^{k(d-1)}\Gamma^0\Gamma^1\dots\Gamma^{d-1} = i^k\Gamma^0\Gamma^1\dots\Gamma^{d-1} \\
&= i^k(-1)^k\Gamma^0\Gamma^1\Gamma^2\Gamma^{*3}\dots\Gamma^{*(d-1)} = (-1)^k\Gamma^*
\end{aligned}$$

Before proving the next identity, we notice that $B_1\Gamma B_1^{-1}$ is also $i^{2k}\Gamma$ and thus, $i^{2k}\Gamma = (-1)^k\Gamma^*$ - The second identity is as follows:

$$B_2\Gamma B_2^{-1} = \Gamma B_1\Gamma B_1^{-1}\Gamma = i^{2k}\Gamma\Gamma\Gamma = (-1)^k\Gamma = (-1)^k\Gamma^*$$

Now, suppose that ξ is a spinor whose chirality is α i.e.;

$$\Gamma\xi = \alpha\xi$$

then, the chirality of the conjugate representation is as follows:

$$\Gamma\left(B_1^{-1}\xi^\star\right) = B_1^{-1}B_1\Gamma B_1^{-1}\xi^\star = (-1)^k B_1^{-1}(\Gamma\xi)^\star = \alpha(-1)^k B_1^{-1}\xi^\star$$

So, the chirality of a spinor is changed if k is odd (which corresponds to $d = 0 \bmod 4$e.g. $d = 4, 8$) and it doesn't change if k is even (which corresponds to $d = 2 \bmod 4$ e.g. $d = 2, 6, 10$). Thus, Weyl representation is self conjugate for $d = 2 \bmod 4$ and it is not self conjugate for $d = 0 \bmod 4$. Now, we want to impose a Majorana condition i.e. a condition that relates a spinor ζ to it's conjugate ζ^*. Such a condition can be written as follows:

$$\zeta^* = B\zeta \tag{21.1.23}$$

where $B = B_1$ or B_2. We see that 21.1 .23 is consistent with Lorentz transformations using 21.1.21 by showing that both sides of 21.1.23 transform in the same way;

$$\zeta^* \to \left(1 - i\omega_{\mu\nu}\Sigma^{*\mu\nu}\right)\zeta^*$$

$$B\zeta \to B\left(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\zeta = B\zeta + i\omega_{\mu\nu}B\Sigma^{\mu\nu}B^{-1}B\zeta = B\zeta - i\omega_{\mu\nu}\Sigma^{*\mu\nu}B\zeta = \left(1 - i\omega_{\mu\nu}\Sigma^{*\mu\nu}\right)B\zeta$$

Taking the conjugate of 21.1.23, we get;

$$\zeta = B^*\zeta^* = B^*B\zeta \Rightarrow B^*B = 1 \tag{21.1.24}$$

Now, we see that;

$$\begin{aligned} B_1^*B_1 &= \Gamma^{*3}\Gamma^{*5}\dots\Gamma^{*(d-1)}\Gamma^3\Gamma^5\dots\Gamma^{d-1} \\ &= (-1)^k\Gamma^3\Gamma^5\dots\Gamma^{d-1}\Gamma^3\Gamma^5\dots\Gamma^{d-1} \\ &= (-1)^k(-1)^{k-1}(-1)^{k-2}\dots(-1)^0 \\ &= (-1)^{k+(k-1)+(k-2)+\ldots+1} = (-1)^{k+\frac{k(k-1)}{2}} = (-1)^{\frac{k(k+1)}{2}}. \end{aligned}$$

Similarly, moving Γ through B_1 changes the parity of the product by one unit and gives

$$B_2^*B_2 = \Gamma^*B_1^*\Gamma B_1 = (-1)^{\frac{k(k-1)}{2}}$$

So, B_1 can be used for Majorana condition only if;

$$\begin{aligned} \frac{k(k+1)}{2} &= 2nn \in \mathbb{Z}_0^+ \\ \Rightarrow k(k+1) &= 4n \end{aligned}$$

Since k and $k+1$ differ by 1, one of them have to be odd and thus, either k is a multiple of 4 or $k+1$ is a multiple of 4 . So, we get;

$$\begin{aligned} k = 0 \bmod 4 &\text{ or } k = 3 \bmod 4 \\ \Rightarrow d = 2 \bmod 8 &\text{ or } d = 0 \bmod 8 \,(\text{ for } B_1) \end{aligned} \tag{21.1.25}$$

Similarly, B_2 can be used for Majorana condition only if;

$$\begin{aligned} \frac{k(k-1)}{2} &= 2nn \in \mathbb{Z}_0^+ \\ \Rightarrow k(k-1) &= 4n \end{aligned}$$

Since k and $k-1$ differ by 1, one of them have to be odd and thus, either k is a multiple of 4 or $k-1$ is a multiple of 4 . So, we get;

$$\begin{aligned} k = 0 \bmod 4 &\text{ or } k = 1 \bmod 4 \\ \Rightarrow d = 2 \bmod 8 &\text{ or } d = 4 \bmod 8 \,(\text{ for } B_2) \end{aligned} \tag{21.1.26}$$

Now, 21.1.23 implies $\zeta = B^{-1}\zeta^*$ and thus, if we want to impose Majorana condition on a Weyl spinor, then the Weyl spinor have to be self conjugate. Thus, the condition of Weyl self-conjugacy (i.e. $d = 2 \bmod 4$) have to be consistent with either 21.1.25 or 21.1.26. We see that the only case that is consistent with Weyl self-conjugacy is $d = 2 \bmod 8$. Since $\Gamma^{\mu T}$ and $-\Gamma^{\mu T}$ satisfy the Clifford algebra as well, they should be related to Γ^μ by a similarity transformation. We will now find that similarity transformation. Define the charge conjugation matrix as follows:

$$C\Gamma^\mu C^{-1} = -\Gamma^{\mu T} \tag{21.1.27}$$

this implies the following;

$$\left(C\Gamma^0\right)\Gamma^\mu\left(C\Gamma^0\right)^{-1} = \Gamma^{*\mu} \tag{21.1.28}$$

Comparing 21.1.28 with 21.1.18 and 21.1.19, we see that;

$$\begin{aligned} C\Gamma^0 = B_1 \Rightarrow C = -B_1\Gamma^0 &\quad \text{if } k = 0 \bmod 2 \Rightarrow d = 2 \bmod 4 \\ C\Gamma^0 = B_2 \Rightarrow C = -B_2\Gamma^0 &\quad \text{if } k+1 = 0 \bmod 2 \Rightarrow d = 0 \bmod 4 \end{aligned}$$

We also derive a useful property for C as follows:

$$C\Sigma^{\mu\nu}C^{-1} = -\frac{i}{4}\left(C\Gamma^\mu C^{-1}C\Gamma^\nu C^{-1} - C\Gamma^\nu C^{-1}C\Gamma^\mu C^{-1}\right) = \frac{i}{4}\left[\Gamma^\mu,\Gamma^\nu\right]^T = -\Sigma^{\mu\nu T} \tag{21.1.29}$$

A.1.2 Product representations

We first prove that the spinor $\bar{\xi}$ and the spinor $\xi^T C$ transform the same way as follows:

$$\xi \to \left(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\xi$$
$$\Rightarrow \bar{\xi} \to \bar{\xi}\left(1 + i\omega_{\mu\nu}\Gamma^0\Sigma^{\dagger\mu\nu}\Gamma^0\right) \tag{21.1.30}$$

To continue, we calculate the following

$$\Gamma^0\Sigma^{\dagger\mu\nu}\Gamma^0 = \frac{i}{4}\Gamma^0\left(\Gamma^0\Gamma^\nu\Gamma^0\Gamma^0\Gamma^\mu\Gamma^0 - \Gamma^0\Gamma^\mu\Gamma^0\Gamma^0\Gamma^\nu\Gamma^0\right)\Gamma^0 = -\Sigma^{\mu\nu}$$

So, 21.1.30 becomes;

$$\bar{\xi} \to \bar{\xi}\left(1 - i\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \tag{21.1.31}$$

Moreover, we see that;

$$\xi^T C \to \left(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu T}\right)C = \xi^T C\left(1 + i\omega_{\mu\nu}C^{-1}\Sigma^{\mu\nu T}C\right) = \xi^T C\left(1 - i\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \tag{21.1.32}$$

where in the last step, we used 21.1.29. Comparing 21.1.31 with 21.1.31, we see that the spinor $\bar{\xi}$ and the spinor $\xi^T C$ transform the same way.
For even d we use the standard duality identity

$$\Gamma^{\mu_1\cdots\mu_s}\Gamma = \frac{i^{-k+s(s-1)}}{(d-s)!}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d} \quad s < d \tag{21.1.33}$$

which implies that in odd dimensions (where $\Gamma^d = \Gamma$), say $d+1$ dimensions, (where d is the same as in (21.1.33)), an $s+1$ form is related to $d-s$ form which is the same as $d+1-(s+1)$ form. Now, let $d = 2k+3$ be the number of odd dimensions. Then, the independent forms have the following number of indices;

$$0, 1, \ldots \frac{d-1}{2} \text{ or } 0, 1, \ldots k+1$$

This leads us to the following forms;

$$[0], [1], \ldots [k+1]$$
$$\Rightarrow 2^{\mathrm{k+1}} \times 2^{\mathrm{k+1}} = [0] + [1] + \ldots + [k+1] \quad \text{odd } d \tag{21.1.34}$$

where the square brackets indicate the corresponding form. In even dimensions, the linear relation mentioned above doesn't exist and thus, the allowed forms are as follows:

$$[0], [1], \ldots [d] = [0], [1], \ldots [2k+2]$$

But since $[m]$ can be related to $[d-m]$ via Γ, we get the following forms ($[k+1]$ is related to itself and hence, we count it only once).

$$[0]^2, [1]^2, \ldots [k]^2, [k+1]$$
$$\Rightarrow 2^{k+1}_{\text{Dirac}} \times 2^{k+1}_{\text{Dirac}} = [0]^2, [1]^2 + \ldots + [k]^2 + [k+1] \text{ even } d \tag{21.1.35}$$

Now, we prove the following identity;

$$\zeta^T C\Gamma^{\mu_1\mu_2\cdots\mu_m}\Gamma\chi = (-1)^{k+m+1}(\Gamma\zeta)^T C\Gamma^{\mu_1\cdots\mu_m}\chi \tag{21.1.36}$$

where ζ and χ are arbitrary spinors. The RHS is manipulated as follows:

$$\zeta^T C\Gamma^{\mu_1\mu_2\cdots\mu_m}\Gamma\chi = \frac{(-1)^m}{m!}\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}\zeta^T C\Gamma\Gamma^{\mu_1\mu_2\cdots\mu_m}\chi = \frac{(-1)^{k+m+1}}{m!}\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}(\Gamma\zeta)^T C\Gamma^{\mu_1\mu_2\cdots\mu_m}\chi$$
$$= (-1)^{k+m+1}(\Gamma\zeta)^T C\Gamma^{\mu_1\mu_2\cdots\mu_m}\chi(\text{ Shown })$$

using the identity

$$\begin{aligned}
\zeta^T C\Gamma &= \zeta^T i^{-k} C\Gamma^0 \dots \Gamma^{d-1} \\
&= \zeta^T i^{-k} C\Gamma^0 C^{-1} \dots C\Gamma^{d-1} C^{-1} C \\
&= \zeta^T i^{-k} (-1)^d \Gamma^{0T} \dots \Gamma^{(d-1)T} C \\
&= \left(i^{-k} \Gamma^{d-1} \dots \Gamma^0 \zeta\right)^T C \\
&= (-1)^{d-1} \dots (-1)^1 \left(i^{-k} \Gamma^0 \dots \Gamma^{d-1} \zeta\right)^T C \\
&= (-1)^{k+1} (\Gamma\zeta)^T C.
\end{aligned}$$

So, we see that $\zeta^T C\Gamma^{\mu_1\mu_2\dots\mu_m}\chi$ is non vanishing only if $k+m$ is odd and ζ, χ have the same chirality or when $k+m$ is even and ζ, χ have the opposite chirality. So, we get the following;

$$2^{\mathrm{k}}_{\mathrm{Weyl}} \times 2^{\mathrm{k}}_{\mathrm{Weyl}} = \begin{cases} [1] + [3] + \dots [k+1]_+ & \text{even } k \\ [0] + [2] + \dots [k+1]_+ & \text{odd } k \end{cases}$$

$$2^{\mathrm{k}'}_{\mathrm{Weyl}} \times 2^{\mathrm{k}'}_{\mathrm{Weyl}} = \begin{cases} [1] + [3] + \dots [k+1]_- & \text{even } k \\ [0] + [2] + \dots [k+1]_- & \text{odd } k \end{cases}$$

$$2^{\mathrm{k}}_{\mathrm{Weyl}} \times 2^{\mathrm{k}'}_{\mathrm{Weyl}} = \begin{cases} [0] + [2] + \dots [k] & \text{even } k \\ [1] + [3] + \dots [k] & \text{odd } k \end{cases} \tag{21.1.37}$$

where the subscripts on $[k+1]$ are there because the middle-rank form is related to itself by 21.1.33 and therefore splits into self-dual and anti-self-dual parts.
Several of these facts can be re-derived directly from the s_a eigenvalues [2]; we prove the following identities first.

$$S_0\zeta = s_0\zeta \Rightarrow S_0 B^{-1}\zeta^* = s_0 B^{-1}\zeta^*, \quad S_a\zeta = s_a\zeta \Rightarrow S_a B^{-1}\zeta^* = -s_a B^{-1}\zeta^* \quad a = 1, \dots k \tag{21.1.38}$$

The first identity is derived as follows:

$$S_0 B^{-1}\zeta^\star = \frac{1}{2}\Gamma^0\Gamma^1 B^{-1}\zeta^\star = \frac{1}{2}B^{-1}B\Gamma^0 B^{-1}B\Gamma^1 B^{-1}\zeta^\star = \frac{1}{2}B^{-1}\Gamma^{\star 0}\Gamma^{\star 1}\zeta^\star = B^{-1}\left(S_0\zeta\right)^\star = s_0 B^{-1}\zeta^\star$$

where the last line uses the fact that s_0 is real, since Γ^0, Γ^1 and the Weyl spinors are real in this convention. The second identity is proved similarly:
$S_a B^{-1}\zeta^\star = \frac{-i}{2}\Gamma^{2a}\Gamma^{2a+1}B^{-1}\zeta^\star = \frac{-i}{2}B^{-1}B\Gamma^{2a}B^{-1}B\Gamma^{2a+1}B^{-1}\zeta^\star = \frac{-i}{2}B^{-1}\Gamma^{*2a}\Gamma^{*2a+1}\zeta^\star = -B^{-1}\left(S_a\zeta\right)^\star = -s_a B^{-1}\zeta^\star$
So, under conjugation, $s_1, \dots, s_k$ are flipped. For even k, this is an even number of flips and hence, the chirality is unchanged (see (21.1.16). For odd k, this is an odd number of flips and hence, the chirality changes. Thus, self-conjugate Weyl reps can be in even k (or for $d = 2 \bmod 4$) only as we derived before.

A.1.3 Spinors of $SO(N)$

The results for $SO(N)$ with $N = 2l$ follow from the same Clifford-algebra decomposition as for $SO(N+1,1)$ after removing the two light-cone directions. The breakdowns are as follows:

$$2^{1-1} \times 2^{1-1} = \begin{cases} [0] + [2] + \dots [l]_+ & \text{even } k \\ [1] + [3] + \dots [l]_+ & \text{odd } k \end{cases}$$

$$2^{1-1'} \times 2^{1-1'} = \begin{cases} [0] + [2] + \dots [l]_- & \text{even } k \\ [1] + [3] + \dots [l]_- & \text{odd } k \end{cases}$$

$$2^{1-1} \times 2^{1-1'} = \begin{cases} [1] + [3] + \dots [l-1] & \text{even } k \\ [0] + [2] + \dots [l-1] & \text{odd } k \end{cases} \tag{21.1.39}$$

A.1.4 Decomposition under subgroups

If we consider the following decomposition;

$$SO(2k+1) \to SO(2l+1) \times SO(2k-2l)$$

then the Weyl representation 2^{k} of $SO(2k+1,1)$ decomposes according to the Clifford-algebra tensor product as

$$2^{\mathrm{k}} \to \left(2^{\mathrm{l}}, 2^{\mathrm{k-l-1}}\right) \oplus \left(2'\mathrm{l}, 2'^{\mathrm{k-l-1}}\right) \tag{21.1.40}$$

Derive the SO(2 N) decomposition to SU(N)

A.2 Introduction to supersymmetry: $d = 4$

This appendix fixes the spinor and supersymmetry conventions used in the superstring chapters, so the later BPS bounds and supergravity multiplets can be read without changing notation.

A.2.1 $d = 4, N = 1$ supersymmetry

IN $d = 4$, the smallest spinor representation has $2^{4/2} = 4$ degrees of freedom. Since 4 can't be written as $2k+8$ with k a non-negative integer, we can't place a Majorana and Weyl condition on the spinor simultaneously. The Majorana condition can be placed but the Weyl representation is not self-conjugate. So, the four degrees of freedom can be realized in one of the following ways;

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \psi^1, \psi^2 \in \mathbb{C} \text{ or } \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} \psi^1, \psi^2, \psi^3, \psi^4 \in \mathbb{R}$$

The first way corresponds to a Weyl spinor and the second corresponds to a Majorana spinor. In the text, the second way (i.e. Majorana) is used. So, the supercharges are denoted as Q_α with $\alpha \in \{0, 1, 2, 3\}$ and due to the Majorana condition, we have;

$$\bar{Q}_\alpha = Q_\alpha^\dagger \Gamma^0 = (Q_\alpha^*)^T \Gamma^0 = (B_1 Q_\alpha)^T \Gamma^0 = Q_\alpha^T B_1^T \Gamma^0 = Q_\alpha^T C.$$

The last step is the definition of the charge-conjugation matrix in this convention, $C = B_1^T \Gamma^0$. We use B_1 because $d = 4 = 0 \pmod 4$, so a Majorana condition can be imposed on the four-real-component spinor but a simultaneous Weyl-Majorana condition cannot.

Now, the supersymmetry algebra for using only one Majorana spinor (which is $N = 1$ supersymmetry with 4 supercharges) contains the following important commutators and anticommutators;

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \Gamma^\mu_{\alpha\beta} \tag{21.2.1}$$

$$[P^\mu, Q_\alpha] = 0. \tag{21.2.2}$$

In conventions with metric signature $(+1, -1, -1, -1)$, or with Weyl in place of Majorana spinors, the sign in the $\{Q, \bar{Q}\}$ anticommutator is correspondingly changed. Since $4 = 2(1) + 2$, the notation of the previous section has $k = 1$, and spinors are labelled by the (s_0, s_1) tuple. The representations of Eq. (21.2.1) are then obtained as follows. If the representation is massless, then the P_μ vector can be written as follows:

$$\begin{aligned} P_\mu = \left(-k^0, k^0, 0, 0\right) \Rightarrow -2P_\mu \Gamma^\mu &= -2\left(-k^0\Gamma^0 + k^0\Gamma^1\right) = 2k_0\left(\Gamma^0 - \Gamma^1\right) = 2k^0\Gamma^0\left(-1 - \Gamma^0\Gamma^1\right) \\ &= 2k^0\Gamma^0\left(1 + \Gamma^0\Gamma^1\right) = 2k^0\Gamma^0\left(1 + 2S_0\right) \end{aligned}$$

where we used the definition of S_0 from previous section and the fact that $\left(\Gamma^0\right)^2 = -1$. So, 21.2.1 becomes;

$$\begin{aligned}\{Q_\alpha, \bar{Q}_\beta\} &= 2k^0\Gamma^0_{\alpha\gamma}(1+2S_0)_{\gamma\beta}\\ \Rightarrow Q_\alpha Q^\dagger_\gamma \Gamma^0_{\gamma\beta} + Q^\dagger_\gamma \Gamma^0_{\gamma\beta} Q_\alpha &= 2k^0\Gamma^0_{\alpha\gamma}(1+2S_0)_{\gamma\beta}\\ \Rightarrow \left\{Q_\alpha, Q^\dagger_\beta\right\} &= 2k^0(1+2S_0)_{\alpha\beta}\,.\end{aligned}$$

The second implication is obtained by multiplying the equation on the right by $(\Gamma^0)^{-1} = -\Gamma^0$ and using $(\Gamma^0)^2 = -1$; the colored γ index is the contracted spinor slot that disappears.

So, we see that the anti-commutator vanishes if $s_0 = -1/2$. It means that (labeling the spinors by the (s_0, s_1) tuple);
$\|Q_{-1/2,s_1}|\psi\rangle\,\|^2 + \|\,Q^\dagger_{-1/2,s_1}|\psi\rangle\|^2 = \langle\psi|Q^\dagger_{-1/2,s_1}Q_{-1/2,s_1}|\psi\rangle + \langle\psi|Q_{-1/2,s_1}Q^\dagger_{-1/2,s_1}|\psi\rangle = \langle\psi|\left\{Q_{-1/2,s_1}, Q^\dagger_{-1/2,s_1}\right\}|\psi\rangle = 0$
where $|\psi\rangle$ is any state in the Hilbert space. Due to the positivity of the Hilbert space, we then require that $Q_{-1/2,s_1}$ vanish identically. From 21.1 .38, we see that if

$$S_0 Q = s_0 Q, \qquad S_1 Q = s_1 Q.$$

then,

$$S_0 B^{-1}Q^* = s_0 B^{-1}Q^*, \qquad S_1 B^{-1}Q^* = -s_1 B^{-1}Q^*.$$

So, the conjugation flips the s_1 eigenvalue. Thus, the Majorana condition becomes;

$$Q^\dagger_{s_0,s_1} = Q_{s_0,-s_1}.$$

This is the Majorana reality condition translated into the simultaneous S_0, S_1 eigenbasis: complex conjugation preserves s_0 but flips the spatial spin label s_1. In the s basis, the anticommutator becomes;

$$\left\{Q_{s_0',s_1'}, Q^\dagger_{s_0,s_1}\right\} = 4k^0\,\delta_{s_0',1/2}\,\delta_{s_0,1/2}\,\delta_{s_1',s_1}.$$

where we used the fact that this anti-commutator is zero if $s_0 = -1/2$ and thus, we put in $S_0 = 1/2$ in the expression of the anti-commutator to get a factor of 4 . Moreover, $s_1' = s_1$ because if $s_1' = -s_1$ then

$$Q^\dagger_{1/2,s_1} = Q^\dagger_{1/2,-s_1'} = Q_{1/2,s_1'}$$

and thus, the anticommutator vanishes because we will have terms like $Q^2_{1/2,s_1'}$. Therefore, we have a factor of $\delta_{s,s}$, in the anticommutator. The nonzero anticommutator is thus;

$$\left\{Q_{1/2,-1/2}, Q^\dagger_{1/2,-1/2}\right\} = \left\{Q_{1/2,-1/2}, Q_{1/2,1/2}\right\} = 4k^0 \Rightarrow \left\{b, b^\dagger\right\} = 1.$$

where

$$b = \frac{1}{2\sqrt{k^0}}Q_{1/2,-1/2}, b^\dagger = \frac{1}{2\sqrt{k^0}}Q_{1/2,1/2}$$

It also implies

$$b^2 = b^{\dagger 2} = 0$$

which gives a pair of states starting from any state, called the Clifford vacuum:

$$S_1|\Omega\rangle = \Omega|\Omega\rangle; \quad b|\Omega\rangle = \left|\Omega + \frac{1}{2}\right\rangle$$

where the state is labelled by its S_1 eigenvalue. For the chosen lightlike momentum along the x^1-direction, the little group is the rotation group in the transverse $(2,3)$-plane, and $S_1 = -\frac{i}{2}\Gamma^2\Gamma^3$ is precisely the spin generator of that transverse rotation. Thus S_1 is the helicity operator in this frame. The relevant massless representations are as follows:

- $\Omega = 0$: We have the following multiplet

$$\left(|0\rangle + \left|\frac{1}{2}\right\rangle\right) \oplus \left(\left|-\frac{1}{2}\right\rangle + |0\rangle\right)$$

This object is called the chiral multiplet.

A.2.2 Actions with $d = 4, N = 1$ SUSY

The basic $d = 4$, $\mathcal{N} = 1$ action is built from chiral multiplets Φ^i, vector multiplets V^a, a Kahler potential K, a holomorphic superpotential W, and gauge kinetic functions f_{ab}. In component language the scalar potential has the schematic form

$$V = K^{i\bar{j}} F_i \bar{F}_{\bar{j}} + \frac{1}{2}(\mathrm{Re}\, f)^{-1}_{ab} D^a D^b, \qquad F_i = \partial_i W + \cdots . \tag{A.1}$$

The four-dimensional language needed later for compactification moduli, supersymmetry breaking, and protected superpotential terms. [5, 6].

A.2.3 Spontaneous symmetry breaking

In global $\mathcal{N} = 1$ supersymmetry, spontaneous breaking occurs when the vacuum energy is positive because some auxiliary field has a nonzero expectation value:

$$\langle V \rangle = \sum_i |\langle F_i \rangle|^2 + \frac{1}{2} \sum_a \langle D^a \rangle^2 > 0. \tag{A.2}$$

The broken supercharge produces a massless goldstino in global supersymmetry; in supergravity it is eaten by the gravitino through the super-Higgs mechanism. This distinction matters when translating string compactification input into four-dimensional effective actions. [5, 6].

A.2.4 Higher corrections and supergravity

Supergravity is the local version of supersymmetry, so the supersymmetry parameter is spacetime dependent and the gravitino is the corresponding gauge field. In an effective action obtained from string theory, corrections are organized by both the string length and the string coupling:

$$S_{\text{eff}} = S^{(2)}_{\text{SUGRA}} + \sum_{n>0} (\alpha')^n S_n^{\text{higher deriv.}} + \sum_{g>0} g_s^{2g-2} S_g^{\text{loop}} + S_{\text{nonpert}}. \tag{A.3}$$

The two-derivative term is fixed by local supersymmetry and field content. Higher-derivative terms such as R^4, threshold corrections, and instanton effects are constrained but not generally absent. In compactification, preserving supersymmetry means solving the corrected Killing-spinor and auxiliary-field equations order by order. [5, 157, 162].

A.2.5 Extended supersymmetry in $d = 4$

Let $A = 1, \ldots, N$ label independent Majorana supercharges. For a massless particle with momentum $P_\mu = (-E, E, 0, 0)$, the same calculation as in the $N = 1$ case leaves only half of the supercharges acting nontrivially. After a conventional normalization one obtains N fermionic creation-annihilation pairs,

$$\{b^A, b^{B\dagger}\} = \delta^{AB}, \qquad \{b^A, b^B\} = 0, \qquad [h, b^{A\dagger}] = \frac{1}{2} b^{A\dagger}. \tag{A.4}$$

Thus a Clifford vacuum of helicity λ generates states

$$\lambda, \quad \lambda + \frac{1}{2}, \quad \lambda + 1, \quad \ldots, \quad \lambda + \frac{N}{2}, \qquad \text{multiplicity at level } p = \binom{N}{p}. \tag{A.5}$$

CPT must be checked separately: if the set of helicities is not symmetric under $h \mapsto -h$, one must add the CPT-conjugate multiplet. This is the origin of the familiar multiplets:

theory	choose λ	helicities
$N=2$ hyper	$-\frac{1}{2}$	$\left(-\frac{1}{2},0,0,\frac{1}{2}\right)$ plus CPT as needed
$N=2$ vector	-1	$\left(-1,-\frac{1}{2},-\frac{1}{2},0\right) \oplus$ CPT
$N=4$ vector	-1	$\left(-1,-\frac{1}{2}^{4},0^{6},\frac{1}{2}^{4},1\right)$
$N=8$ gravity	-2	$\left(-2,-\frac{3}{2}^{8},-1^{28},-\frac{1}{2}^{56},0^{70},\frac{1}{2}^{56},1^{28},\frac{3}{2}^{8},2\right)$

(A.6)

The $N=4$ vector multiplet is already CPT self-conjugate: the six real scalars transform as the antisymmetric representation of the $SU(4)_R \simeq SO(6)_R$ R-symmetry, and the helicities pair internally. The $N=8$ multiplet is the maximal massless supergravity multiplet with no helicity above 2. Starting with $\lambda > -2$ for $N=8$, or adding too many creation operators in a theory containing gravity, would produce massless states of spin greater than 2; such interacting theories are not part of ordinary local supergravity. [5, 162].

Central charges enter the massive extended supersymmetry algebra. In the rest frame, the antisymmetric complex central-charge matrix $Z^{AB} = -Z^{BA}$ can be brought by a unitary R-symmetry rotation to skew-diagonal form,

$$Z^{AB} \longrightarrow \bigoplus_{r=1}^{\lfloor N/2 \rfloor} \begin{pmatrix} 0 & z_r \\ -z_r & 0 \end{pmatrix}. \tag{A.7}$$

For each block, positivity of the anticommutator gives

$$M \geq |z_r| \qquad \text{for every} \quad r. \tag{A.8}$$

If no equality holds, all creation operators act and the representation is long. If one or more equalities hold, some combinations of supercharges annihilate the state and the multiplet shortens. This is the algebraic origin of BPS protection: the mass is tied to a quantized charge, so it can be compared across weak and strong coupling. The D-brane BPS bounds and the $N=2$ formula $M_{\text{BPS}} = |q_I X^I - p^I \mathcal{F}_I|$ used later are realizations of this same positivity argument. [5, 167, 174].

A.2.6 Supersymmetry in $d=2$

In two dimensions a Majorana spinor can also be Weyl. In light-cone coordinates $x^{\pm} = x^0 \pm x^1$, supersymmetry naturally splits into left-moving and right-moving charges. For $(N, \widetilde{N})$ supersymmetry,

$$\{Q_+^A, Q_+^B\} = 2\delta^{AB} P_+, \qquad \{Q_-^{\dot{A}}, Q_-^{\dot{B}}\} = 2\delta^{\dot{A}\dot{B}} P_-, \qquad \{Q_+^A, Q_-^{\dot{B}}\} = Z^{A\dot{B}}. \tag{A.9}$$

The mixed central term is not an antisymmetric matrix in one set of identical indices; it carries one left index and one right index. For real supercharges the anticommutator is symmetric as an operator product, so the allowed central extensions are fixed by Lorentz weights and by the internal R-symmetry representations. In worldsheet applications, these chiral algebras are the spacetime analogues of the left/right split used throughout the CFT chapters. [5, 10].

A.3 Differential forms and generalized gauge fields

A p-form gauge potential A_p has field strength $F_{p+1} = dA_p$ and gauge symmetry

$$A_p \longmapsto A_p + d\Lambda_{p-1}. \tag{A.10}$$

It couples electrically to a $(p-1)$-brane through $\int_{\Sigma_p} A_p$ and magnetically to a $(D-p-3)$-brane through the dual field strength. Dirac quantization is the statement that electric and magnetic charges pair integrally,

$$q_e q_m \in 2\pi\mathbb{Z}. \tag{A.11}$$

The appendix language behind Ramond–Ramond charges, D-brane couplings, and the earlier BPS charge algebra. [101].

A.4 Thirty two supersymmetries

Thirty-two real supercharges are maximal for ordinary supergravity with spins no higher than two. The massless little-group construction above explains why: too many creation operators would force helicities beyond ± 2. The maximal theories that appear in string/M-theory are organized as

dimension	theory	origin
11	M theory low energy	unique $D = 11$ supergravity
10	type IIA	opposite chiralities, reduction of $D = 11$
10	type IIB	same chiralities, $SL(2, \mathbb{Z})$ duality

(A.12)

String perturbation theory chooses these spectra by GSO projection, but the supergravity multiplet itself is fixed by the number and chirality of the supercharges. [5, 162, 353].

A.4.1 $d = 11$ supergravity

The unique eleven-dimensional supergravity multiplet contains a metric g_{MN}, a three-form C_{MNP}, and a Majorana gravitino ψ_M. Counting physical polarizations with the massless little group $SO(9)$ gives

$$\text{bosons}: \quad \left(\frac{9 \cdot 10}{2} - 1\right) + \binom{9}{3} = 44 + 84 = 128, \qquad \text{fermions}: \quad 9 \cdot 16 - 16 = 128. \tag{A.13}$$

The subtraction in the graviton count removes the trace of a symmetric $SO(9)$ tensor; the subtraction in the gravitino count removes the gamma-trace/gauge redundancy. The bosonic action has the characteristic Chern–Simons term

$$S_{11} \supset \frac{1}{2\kappa_{11}^2} \int \sqrt{-g} \left(R - \frac{1}{2} |G_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int C_3 \wedge G_4 \wedge G_4. \tag{A.14}$$

Reducing this theory on a circle gives type IIA supergravity, which is why strong-coupling IIA reveals an eleventh dimension. [162, 353].

A.4.2 $d = 10$ II A supergravity

Type IIA has two ten-dimensional Majorana–Weyl supersymmetry parameters of opposite chirality. Reducing eleven-dimensional fields along a circle coordinate y gives the NS–NS and R–R fields

$$g_{MN} \to (g_{\mu\nu}, A_\mu^{\mathrm{RR}}, \Phi), \qquad C_{MNP} \to C_{\mu\nu\rho}^{\mathrm{RR}} \oplus B_{\mu\nu}^{\mathrm{NS}}. \tag{A.15}$$

Thus the IIA R–R potentials are odd forms C_1, C_3 in the democratic language, while the fundamental string couples to B_2. The opposite chirality of the two supercharges is the spacetime counterpart of the left-right GSO choice in perturbative type IIA. [352, 353].

A.4.3 $d = 10$ II B supergravity

Type IIB has two ten-dimensional Majorana–Weyl supersymmetry parameters of the same chirality. Its bosonic fields are

$$\mathrm{NS{-}NS}: \quad g_{\mu\nu}, B_2, \Phi, \qquad \mathrm{R{-}R}: \quad C_0, C_2, C_4, \qquad F_5 = *F_5. \tag{A.16}$$

The self-duality of F_5 halves the apparent degrees of freedom of C_4, making the bosonic and fermionic counts match. The scalar $\tau = C_0 + ie^{-\Phi}$ is the field acted on by the nonperturbative $SL(2, \mathbb{Z})$ duality used in Chapter 13. [167, 352].

A.4.4 $d < 10$ supergravity

Lower-dimensional supergravities arise by compactifying ten- or eleven-dimensional supergravity and dualizing p-form fields when useful. Scalars parametrize moduli spaces, vectors come from metric and form components on internal cycles, and duality groups act on the resulting charge lattice. Schematically,

$$\text{metric/form fields on internal cycles} \quad\longrightarrow\quad \text{scalars, vectors, and higher forms in } d<10. \tag{A.17}$$

The effective-field-theory counterpart of the U-duality table in Eq. (13.28). [157, 162].

A.5 Sixteen supersymmetries

Sixteen supercharges are "half-maximal." They occur in ten-dimensional $N = 1$ supergravity coupled to Yang–Mills, and in compactifications such as heterotic on T^d or type II on $K3$. The protected content is a gravity multiplet, vector multiplets, and a charge lattice:

$$\Gamma^{d+16,d}_{\text{het}} \quad \text{or} \quad H^{\text{even}}(K3,\mathbb{Z}) \simeq \Gamma^{4,20} \tag{A.18}$$

depending on the duality frame. This is why half-maximal compactifications are ideal for testing duality: many quantities are constrained by supersymmetry, while the theory is still rich enough to contain nonabelian enhancement and brane charges. [353, 355].

A.5.1 $d = 10, N = 1$ (type I) supergravity

Ten-dimensional $N = 1$ supergravity by itself is chiral, so coupling it to Yang–Mills is constrained by anomaly cancellation. The Green–Schwarz mechanism requires the gauge group and trace identities to make the twelve-form anomaly polynomial factorize:

$$I_{12} = X_4X_8, \qquad X_4 = \operatorname{tr} R^2 - \frac{1}{30}\operatorname{Tr} F^2, \qquad G = SO(32) \text{ or } E_8\times E_8. \tag{A.19}$$

The type-I string realizes the $SO(32)$ case with open strings and orientifolds; the heterotic string realizes both allowed gauge groups through a left-moving current algebra. This is the supergravity version of the trace-convention warnings in the heterotic anomaly section. [167,352].

A.5.2 $d < 10$ supergravity

Lower-dimensional half-maximal supergravity is obtained by compactifying ten-dimensional fields and then dualizing forms when convenient. On a torus, the scalar moduli take the Narain form

$$\mathcal{M}^{16Q}_d = O(10-d, 26-d;\mathbb{Z})\backslash \frac{O(10-d,26-d)}{O(10-d)\times O(26-d)} \times \mathbb{R}^+, \tag{A.20}$$

where $\mathbb{R}^+$ is the lower-dimensional dilaton. The vectors come from the metric, B-field, and gauge fields with one internal index. This is the effective-field-theory counterpart of the heterotic Narain lattice discussion: the same scalar moduli determine the masses and electric/magnetic charges of BPS states. [350, 355].

A.5.3 $d = 6, N = 2$ supersymmetry

Six-dimensional notation is convention-dependent: what some string texts call $N = 2$ may mean sixteen supercharges, either nonchiral $(1, 1)$ or chiral $(2, 0)$. The unambiguous invariant is the chirality split:

$$(1,1):\ 8_L + 8_R \text{ supercharges}, \qquad (2,0):\ 16_L \text{ supercharges}. \tag{A.21}$$

Type IIA on $K3$ gives a nonchiral six-dimensional theory with sixteen supercharges, while type IIB on $K3$ gives a chiral theory. Chiral six-dimensional theories require anomaly checks, so the label $N = 2$ should always be accompanied by the (p, q) chirality notation. [162, 355].

A.5.4 $d = 4, N = 4$ supersymmetry

Four-dimensional $\mathcal{N} = 4$ supersymmetry has one vector multiplet containing

$$(A_\mu,\ 4 \text{ gauginos},\ 6 \text{ real scalars}), \tag{A.22}$$

all in the adjoint representation for Yang–Mills theory. The field theory living on a stack of D3-branes at low energy, and its electric-magnetic duality is the worldvolume form of type IIB S-duality discussed in Eq. (13.27). [5, 167].

A.6 Eight supersymmetries

Eight supercharges are the amount of supersymmetry in four-dimensional $N = 2$, five-dimensional $N = 1$, and six-dimensional $(1, 0)$ theories. This amount is small enough to allow nontrivial hypermultiplet geometry and chiral six-dimensional matter, but large enough to protect vector-multiplet special geometry:

$$4d\ N = 2: \quad \text{vector multiplet couplings} \leftrightarrow \mathcal{F}(X), \qquad M_{\text{BPS}} = |q_I X^I - p^I \mathcal{F}_I|. \tag{A.23}$$

This is the formal reason conifold singularities, Seiberg–Witten periods, and wrapped-brane central charges are computable in many compactifications. [5, 355].

A.6.1 $d = 6, N = 1$ supersymmetry

Six-dimensional $N = 1$ usually means $(1, 0)$, i.e. eight real supercharges of one chirality. This theory is chiral, so gravitational, gauge, and mixed anomalies are central consistency conditions. For a single tensor multiplet, the familiar gravitational anomaly count is

$$H - V + 29T = 273. \tag{A.24}$$

Here H, V, T count hypermultiplets, vector multiplets, and tensor multiplets. String compactifications to six dimensions must satisfy this equation together with factorization of the full anomaly polynomial. Thus the multiplet count is not cosmetic; it is a hard consistency check on any proposed spectrum. [155, 162].

A.6.2 $d = 4, N = 2$ supersymmetry

Four-dimensional $\mathcal{N} = 2$ theories separate vector multiplets from hypermultiplets. The vector-multiplet couplings are controlled by a holomorphic prepotential $\mathcal{F}(X)$, and BPS particle masses are determined by a central charge:

$$M_{\text{BPS}} = |Z|, \qquad Z = q_I X^I - p^I \mathcal{F}_I. \tag{A.25}$$

The effective-language version of the protected charge/mass relations used throughout the D-brane and duality chapters. [5, 6, 167].

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