Analyzing a Variant Version of a Fibonacci Polynomial

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Abstract

In this paper, we introduce a new finite polynomial called the *Variant Fibonacci Polynomial*. This polynomial is defined by using the standard Fibonacci sequence as both the coefficients and the exponents of each term. Although the definition is simple, it leads to several interesting mathematical properties.

We begin by defining the polynomial and giving several examples. We then evaluate it at different values of x, identify its guaranteed real root, discuss its complex roots, and derive formulas for both its derivative and antiderivative. We also examine the infinite series version of the polynomial and observe when it converges and when it diverges.

Finally, we discuss whether this polynomial could be used to construct a new type of series expansion for functions and whether such an expansion might eventually be useful for solving differential equations. While it is still too early to draw definite conclusions, the results of this paper suggest that the Variant Fibonacci Polynomial is an interesting mathematical object that deserves further study.

1 Introduction

Fibonacci polynomials have existed since 1883. While Leonardo of Pisa (Fibonacci) introduced the numerical Fibonacci sequence to Western Europe in his book *Liber Abaci* in 1202, the French-Belgian mathematician Eugène Charles Catalan and the German mathematician Ernst Jacobsthal independently introduced and studied Fibonacci polynomials. Catalan is generally credited with introducing the earliest variation of these polynomials. Over the years, research related to Fibonacci polynomials has increased significantly [1–3]. One particular study by Sibel Koparal, Neşe Ömür, Sezer Boz, Khidir Shaib Mohamed, Waseem Ahmad Khan, and Alawia Adam investigated a generalized Fibonacci polynomial and its properties [4].

In this paper, we address a modest and simple question: *"What properties and applications does this variant Fibonacci polynomial possess?"* or in general, *"What does this polynomial have to offer?"* After conducting a mathematical analysis of this polynomial, we also investigate whether it is possible to construct an infinite series expansion of a function using this variant of the Fibonacci polynomial.

1.1 The Mathematical Definition of the Variant Fibonacci Polynomial

This variant of the polynomial is very simple. It is simply a finite power series where both the coefficient and the exponent follow the standard Fibonacci sequence. Let us first look at an example and then present the general term $V_n(x)$.

Consider the standard Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \dots \tag{1}$$

Using these Fibonacci numbers as both coefficients and exponents, we obtain

$$V_1(x) = x,\ V_2(x) = x+x,\ V_3(x) = x+x+2x^2,\ V_4(x) = x+x+2x^2+3x^3,\ V_5(x) = x+x+2x^2+3x^3+5x^5. \tag{2}$$

Continuing this process, we observe that each new term is formed by taking the next Fibonacci number and using it as both the coefficient and the exponent of (x).

We now define the Variant Fibonacci Polynomial.

Definition 1.1. Let F_k denote the k-th Fibonacci number, where

$$F_0 = 0, \qquad F_1 = 1, \tag{3}$$

and

$$F_k = F_{k-1} + F_{k-2} \tag{4}$$

for $k \geq 2$.

The n-th Variant Fibonacci Polynomial is defined by

$$V_n(x) = \sum_{k=1}^{n} F_k x^{F_k}. \tag{5}$$

Expanding the summation gives

$$V_n(x) = F_1 x^{F_1} + F_2 x^{F_2} + F_3 x^{F_3} + \cdots + F_n x^{F_n}. \tag{6}$$

Therefore, the coefficient sequence and the exponent sequence are identical, both being generated by the Fibonacci sequence. This simple construction will serve as the main object of study throughout this paper.

2 Evaluation at some points

To begin our mathematical analysis of this variant of the fibonacci polynomial, we're gonna start off with a simple task: evaluating the polynomial at different points, specifically only 4 points.

2.1 at $x = 1$

We simply let $x = 1$ and the entire finite sum becomes a series of the standard fibonacci sequence:

$$V_n(1) = \sum_{k=1}^{n} F_k = 1 + 1 + 2 + 3 + 5 + 8 + \ldots + F_{n-1} + F_n \tag{7}$$

Using the well-known identity

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1, \tag{8}$$

we obtain

$$V_n(1) = F_{n+2} - 1. \tag{9}$$

Therefore, evaluating the Variant Fibonacci Polynomial at $x = 1$ produces another fibonacci number minus one.

Its infinite series version is

$$\sum_{k=1}^{\infty} F_k, \tag{10}$$

which diverges since the fibonacci sequence grows without bound.

2.2 at $x = -1$

Now let us evaluate the polynomial at $x = -1$:

$$V_n(-1) = \sum_{k=1}^{n} F_k(-1)^{F_k}. \tag{11}$$

The first few terms are

$$V_n(-1) = -1 - 1 + 2 - 3 - 5 + 8 - 13 - 21 + 34 - \dots \tag{12}$$

Notice that every third fibonacci number is even while the remaining fibonacci numbers are odd. Because of this, the signs repeat in the pattern

$$-,,-,,+,,-,,-,,+,\dots \tag{13}$$

Thus, evaluating the polynomial at $x = -1$ produces an alternating fibonacci sum.

Its infinite series version is

$$\sum_{k=1}^{\infty} F_k(-1)^{F_k}. \tag{14}$$

Since the fibonacci numbers themselves become arbitrarily large, the terms do not approach zero. Therefore, the infinite series diverges.

2.3 at $x = 0$

Now let us evaluate the polynomial at $x = 0$:

$$V_n(0) = \sum_{k=1}^{n} F_k 0^{F_k}. \tag{15}$$

Since every fibonacci exponent satisfies

$$F_k \geq 1, \tag{16}$$

every term contains a positive power of zero. Therefore,

$$0^{F_k} = 0 \tag{17}$$

for all $k \geq 1$.

Substituting this into the finite sum gives

$$V_n(0) = 0 + 0 + 0 + \cdots + 0. \tag{18}$$

Hence,

$$V_n(0) = 0. \tag{19}$$

Therefore, every Variant Fibonacci Polynomial passes through the origin.

Its infinite series version is

$$\sum_{k=1}^{\infty} F_k 0^{F_k}, \tag{20}$$

which is simply

$$0 + 0 + 0 + \cdots . \tag{21}$$

Thus,

$$\sum_{k=1}^{\infty} F_k 0^{F_k} = 0, \tag{22}$$

and the infinite series converges.

2.4 at $x = \frac{1}{2}$

Next, let us evaluate the polynomial at

$$x = \frac{1}{2}. \tag{23}$$

Substituting this value into the polynomial gives

$$V_n\left(\frac{1}{2}\right) = \sum_{k=1}^{n} F_k \left(\frac{1}{2}\right)^{F_k}. \tag{24}$$

The first few terms are

$$\frac{1}{2} + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{32} + \frac{8}{256} + \ldots \tag{25}$$

As the fibonacci exponents increase, the powers of $\frac{1}{2}$ become smaller and smaller. This causes the terms of the series to decrease very rapidly.

Therefore, the infinite series

$$\sum_{k=1}^{\infty} F_k \left(\frac{1}{2}\right)^{F_k} \tag{26}$$

converges.

This gives us our first example of a convergent infinite series generated by the Variant Fibonacci Polynomial.

2.5 at $x = -\frac{1}{2}$

Finally, let us evaluate the polynomial at

$$x = -\frac{1}{2}. \tag{27}$$

Substituting into the polynomial yields

$$V_n\left(-\frac{1}{2}\right) = \sum_{k=1}^{n} F_k \left(-\frac{1}{2}\right)^{F_k}. \tag{28}$$

The first few terms are

$$-\frac{1}{2} - \frac{1}{2} + \frac{2}{4} - \frac{3}{8} - \frac{5}{32} + \frac{8}{256} - \dots \tag{29}$$

Again, every third fibonacci number is even while the others are odd. As a result, the signs follow the same repeating pattern that appeared in the evaluation at $x = -1$.

Since the magnitude of

$$\left(-\frac{1}{2}\right)^{F_k} \tag{30}$$

decreases rapidly as the fibonacci exponents increase, the terms become very small.

Therefore, the infinite series

$$\sum_{k=1}^{\infty} F_k \left(-\frac{1}{2}\right)^{F_k} \tag{31}$$

also converges.

Unlike the case $x = \frac{1}{2}$, this series alternates in sign, but it still converges because the magnitude of the terms tends to zero sufficiently fast.

This simple observation suggests that the size of x plays an important role in determining whether the infinite series version of the Variant Fibonacci Polynomial converges or diverges. In the next section, we will investigate this behavior in greater detail.

3 Identifying all the roots of the Variant Fibonacci Polynomial

Now that we have evaluated the Variant Fibonacci Polynomial at several points, we can move on to another basic question: where are its roots? In other words, we want to determine every value of (x) for which

$$V_n(x) = 0. \tag{32}$$

Recall that

$$V_n(x) = \sum_{k=1}^{n} F_k x^{F_k}. \tag{33}$$

Since every coefficient is a positive Fibonacci number, the only way for the polynomial to become zero is if the powers of (x) cancel each other out.

3.1 The root at $x = 0$

From the previous section, we already know that

$$V_n(0) = 0. \tag{34}$$

Therefore,

$$x = 0 \tag{35}$$

is always a root of every Variant Fibonacci Polynomial.

This is true for every positive integer (n).

3.2 Are there any positive roots?

Suppose that

$$x > 0. \tag{36}$$

Since every Fibonacci coefficient is positive and every positive number raised to an integer power is also positive, every term of the polynomial is positive.

That is,

$$F_k x^{F_k} > 0 \tag{37}$$

for every (k).

Therefore,

$$V_n(x) > 0 \tag{38}$$

whenever

$$x > 0. \tag{39}$$

Hence, there are no positive roots.

3.3 Are there any negative roots?

Now suppose that

$$x < 0. \tag{40}$$

Since the Fibonacci exponents alternate between odd and even values, some terms of the polynomial are positive while others are negative.

For example,

$$V_5(x) = x + x + 2x^2 + 3x^3 + 5x^5. \tag{41}$$

When (x) is negative, this becomes

$$-x - x + 2x^2 - 3|x|^3 - 5|x|^5. \tag{42}$$

Therefore, cancellation between positive and negative terms is possible.

However, unlike the positive case, there is no simple argument that immediately tells us whether additional real roots exist. In general, the location of negative roots depends on the value of (n).

For this reason, there is no simple closed formula that describes all negative roots of the Variant Fibonacci Polynomial.

3.4 first conclusion

From the discussion above, we can immediately conclude the following.

Theorem 3.1. *Every Variant Fibonacci Polynomial has the root* [1]

$$x = 0. \tag{43}$$

Furthermore, it has no positive real roots.

The existence of additional negative real roots depends on the degree of the polynomial and must be studied separately for each value of (n). Therefore, (x=0) is the only root that every Variant Fibonacci Polynomial is guaranteed to have.

3.5 Are there any complex roots?

So far, we have only discussed the real roots of the Variant Fibonacci Polynomial. However, every non-constant polynomial with complex coefficients has exactly as many complex roots as its degree, counting multiplicities. Therefore, it is natural to ask whether our polynomial also has complex roots.

Recall that

$$V_n(x) = \sum_{k=1}^{n} F_k x^{F_k}. \tag{44}$$

The degree of this polynomial is simply

$$\deg(V_n) = F_n. \tag{45}$$

Since every Variant Fibonacci Polynomial has finite degree, the Fundamental Theorem of Algebra immediately tells us that it has exactly

$$F_n \tag{46}$$

complex roots, counted according to multiplicity.

One of these roots is already known:

$$x = 0. \tag{47}$$

The remaining roots are generally complex numbers. Unlike the real root at the origin, these complex roots do not appear to follow a simple closed formula.

For example, the polynomial

[1] Throughout this paper, the term *Variant Fibonacci Polynomial* refers only to the polynomial introduced in Definition 1.1. This statement does not apply to other variants or generalizations of Fibonacci polynomials found in the literature.

$$V_4(x) = x + x + 2x^2 + 3x^3 = 2x + 2x^2 + 3x^3 \tag{48}$$

can be written as

$$V_4(x) = x\left(2 + 2x + 3x^2\right). \tag{49}$$

The root

$$x = 0 \tag{50}$$

is immediately visible, while the remaining roots are obtained by solving

$$2 + 2x + 3x^2 = 0. \tag{51}$$

Since

$$\Delta = 2^2 - 4(3)(2) = -20 < 0, \tag{52}$$

the remaining two roots are complex.

Likewise, every higher-degree Variant Fibonacci Polynomial also possesses complex roots. As the degree increases, these roots become increasingly difficult to compute exactly, and numerical methods are generally required to approximate them.

Although finding an explicit formula for all of the complex roots appears to be difficult, the Fundamental Theorem of Algebra guarantees that they always exist. Therefore, every Variant Fibonacci Polynomial of degree F_n has exactly F_n complex roots when multiplicities are counted.

4 The Derivative and the Antiderivative of the Variant Fibonacci Polynomial

Another basic property that we can study is the derivative and the antiderivative of the Variant Fibonacci Polynomial. Since our polynomial is just a finite sum of power functions, we can differentiate and integrate it term by term using the usual rules from elementary calculus.

Recall that

$$V_n(x) = \sum_{k=1}^{n} F_k x^{F_k}. \tag{53}$$

4.1 The derivative

To differentiate the polynomial, we simply apply the power rule to every term.

Since

$$\frac{d}{dx}\left(x^m\right) = mx^{m-1}, \tag{54}$$

we have

$$\frac{d}{dx}\left(F_k x^{F_k}\right) = F_k\left(F_k x^{F_k - 1}\right). \tag{55}$$

Therefore,

$$V_n'(x) = \sum_{k=1}^{n} F_k^2 x^{F_k - 1}. \tag{56}$$

Expanding the first few terms gives

$$V_n'(x) = 1 + 1 + 4x + 9x^2 + 25x^4 + 64x^7 + \cdots + F_{n-1}^2 x^{F_{n-1}-1} + F_n^2 x^{F_n - 1}. \tag{57}$$

An interesting observation is that every coefficient is now the square of a Fibonacci number, while the exponents remain one less than the corresponding Fibonacci numbers.

4.2 The antiderivative

Finding the antiderivative is just as straightforward.

Using the power rule for integration,

$$\int x^m dx = \frac{x^{m+1}}{m+1} + C, \tag{58}$$

we obtain

$$\int F_k x^{F_k} dx = \frac{F_k}{F_k + 1} x^{F_k + 1}. \tag{59}$$

Applying this to every term of the polynomial gives

$$\int V_n(x) dx = \sum_{k=1}^{n} \frac{F_k}{F_k + 1} x^{F_k + 1} + C. \tag{60}$$

Expanding the first few terms, we have

$$\int V_n(x), dx = \frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{5}{6}x^6 + \frac{8}{9}x^9 + \cdots + \frac{F_{n-1}}{F_{n-1}+1}x^{F_{n-1}+1} + \frac{F_n}{F_n + 1}x^{F_n + 1} + C. \tag{61}$$

Notice that each exponent increases by one after integration, while each coefficient becomes

$$\frac{F_k}{F_k + 1}. \tag{62}$$

Unlike the derivative, the coefficients are no longer Fibonacci numbers.

4.3 Summary

The derivative and the antiderivative of the Variant Fibonacci Polynomial are both easy to compute because the polynomial is a finite power series.

Differentiating gives

$$V_n'(x) = \sum_{k=1}^{n} F_k^2 x^{F_k - 1}, \tag{63}$$

while integrating gives

$$\int V_n(x)dx = \sum_{k=1}^{n} \frac{F_k}{F_k + 1} x^{F_k+1} + C. \tag{64}$$

These two formulas will be useful later if we wish to study differential equations, series expansions, or other calculus-related properties involving the Variant Fibonacci Polynomial.

5 The Final Analysis

Throughout this paper, we introduced the Variant Fibonacci Polynomial and studied some of its most basic mathematical properties. We defined the polynomial, evaluated it at several points, identified its guaranteed real root, discussed its complex roots, and computed both its derivative and antiderivative. The next natural question is whether this polynomial has a deeper use beyond being an interesting mathematical object.

One possible direction is to ask the following question:

> Can this kind of polynomial be used to define a brand new series expansion of a function that may eventually become useful for developing new methods for solving differential equations?

At this point of the study, the answer is *maybe*. However, we must be careful not to make claims that our current results cannot support. The purpose of this paper is only to introduce the polynomial and study its elementary properties. Whether it can become a useful tool for differential equations is a much larger question that requires considerably more work.

Nevertheless, the results obtained in this paper suggest that the idea may be worth investigating further.

One reason is that the Variant Fibonacci Polynomial is not an arbitrary polynomial. Both its coefficients and exponents are generated by the same recursive sequence. This gives the polynomial a built-in structure that ordinary power series do not possess.

Another interesting observation is that the exponents become increasingly far apart as the Fibonacci sequence grows. Consequently, the polynomial becomes very sparse while still reaching extremely high degrees. This means that a relatively small number of terms can represent a polynomial of very large degree.

The derivative and the antiderivative also preserve much of this structure. Differentiation changes each coefficient into the square of a Fibonacci number while decreasing each exponent by one. Integration increases each exponent by one while replacing every coefficient with a simple rational expression involving the corresponding Fibonacci number. In both cases, the resulting expressions remain easy to compute term by term.

These observations suggest that the Variant Fibonacci Polynomial behaves in a very predictable way under the basic operations of calculus. Such behavior is often desirable when constructing approximation methods for differential equations.

Another idea that naturally arises is to consider a telescoping series constructed from these polynomials. For example, let

$$T(x) = \sum_{n=1}^{\infty} \left(V_{n+1}(x) - V_n(x) \right). \tag{65}$$

Since

$$V_{n+1}(x) = V_n(x) + F_{n+1} x^{F_{n+1}} \tag{66}$$

every difference simplifies immediately to

$$V_{n+1}(x) - V_n(x) = F_{n+1} x^{F_{n+1}}. \tag{67}$$

Therefore,

$$T(x) = \sum_{n=1}^{\infty} F_{n+1} x^{F_{n+1}}, \tag{68}$$

which is simply the infinite version of the Variant Fibonacci Polynomial.

Although this telescoping construction appears simple, it reveals an interesting property. Every partial sum differs from the next one by exactly one new Fibonacci term. Nothing more needs to be recomputed. Each approximation naturally grows from the previous approximation by adding only a single term.

This incremental behavior may become useful if one wishes to construct successive approximations to a function. Instead of rebuilding an approximation from the beginning, one simply appends the next Fibonacci term.

Whether this idea can actually produce a practical method for solving differential equations is still unknown. At present, there is not enough mathematical evidence to answer that question. The convergence, approximation error, completeness of the resulting basis, and computational efficiency would all need to be studied carefully before such claims could be made.

For this reason, we view the Variant Fibonacci Polynomial as an object that is still in its early stages of development. The results presented in this paper should be regarded as the first mathematical investigation rather than the final one.

It is entirely possible that future work will show that these polynomials have applications in approximation theory, infinite series, or differential equations. It is equally possible that they remain primarily an interesting family of polynomials with unique structural properties. At the present time, both possibilities remain open.

For now, the most reasonable conclusion is that the Variant Fibonacci Polynomial possesses enough interesting mathematical properties to justify further investigation. Whether those properties eventually lead to new analytical techniques is a question left for future research.

6 Conclusion

In this paper, we introduced a new finite polynomial called the Variant Fibonacci Polynomial. Unlike the classical Fibonacci polynomial, this variant is constructed by using the standard Fibonacci sequence as both the coefficients and the exponents of the polynomial. Although its definition is very simple, it gives rise to several interesting mathematical properties.

We began by defining the polynomial and presenting several examples. We then evaluated the polynomial at different values of x, namely $x = 1$, $x = -1$, $x = 0$, $x = \frac{1}{2}$, and $x = -\frac{1}{2}$. These evaluations showed that the corresponding infinite series may either converge or diverge depending on the value of x.

Next, we investigated the roots of the Variant Fibonacci Polynomial. We proved that $x = 0$ is always a root and showed that no positive real roots exist. We also discussed the existence of negative real roots and complex roots, noting that every polynomial in this family has exactly as many complex roots as its degree when multiplicities are counted.

After that, we derived simple formulas for both the derivative and the antiderivative of the polynomial. These formulas showed that the polynomial behaves in a predictable way under differentiation and integration while preserving much of its original Fibonacci structure.

Finally, we discussed the possibility of extending this polynomial into an infinite series and considered whether such a construction might eventually become useful for developing new methods for solving differential equations. At the present stage of this research, it is still too early to answer that question with certainty. Nevertheless, the structural properties of the Variant Fibonacci Polynomial suggest that this direction is worthy of further investigation.

This paper should therefore be viewed as an introductory study of a new polynomial rather than a complete theory. Many questions remain open. For example, future work may investigate recurrence relations, generating functions, orthogonality, approximation theory, convergence properties of the associated infinite series, numerical methods, and possible applications to differential equations.

In conclusion, the Variant Fibonacci Polynomial is a simple mathematical object with several interesting properties. Whether it eventually becomes a useful analytical tool or simply remains an interesting family of polynomials is still unknown. Either way, we hope that this paper provides a useful starting point for future studies on this new variant of the Fibonacci polynomial.

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