

# ON A QUADRATIC COLLATZ TYPE MAP GOVERNED BY PRIME FACTORIZATION. A CONJECTURAL UNIVERSAL CYCLE, TWO REDUCTION THEOREMS AND AN INTRINSIC COMPUTATIONAL BARRIER.

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## Abstract

We study a map  $T$  on the integers defined by  $T(n) = n^2 + 1$  when  $n$  is prime or even and  $T(n) = n/P(n)$  when  $n$  is odd composite, where  $P(n)$  denotes a designated prime factor  $n$ . Two variants arise according to the choice of  $P(n)$ . In Part *I*,  $P(n)$  is the largest prime factor of  $n$  while in Part *II*, it is the smallest.

Informally, the conjecture of this paper asserts that every integer greater than one in absolute value eventually enters the single twelve element cycle

$5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901 \rightarrow 52357 \rightarrow 41 \rightarrow 1682 \rightarrow 2829125 \rightarrow 1625 \rightarrow 125 \rightarrow 25$  and remains there forever.

Formally, we verify this for every integer  $n \in [2, 1000]$  under Part *I* with no exception and no alternate cycle observed and we conjecture it holds for every integer  $n$  in the domain

$\mathbb{Z}^* = \mathbb{Z} \setminus \{-1, 0, 1\}$  but we do not prove it.

Toward this conjecture, we prove two theorems. The first is exact rather than asymptotic. For any odd composite  $m$ , repeated application of the largest prime factor reduction reaches a prime in precisely  $\Omega(m) - 1$  steps where  $\Omega(m)$  counts the prime factors of  $m$  with multiplicity. Each application removes exactly one element from the prime factorization multiset  $m$  so the count decreases by exactly one per step and terminates uniquely at a prime. The second theorem follows from the first. Under the extension of primality to negative integers through  $|n|$ , which is the only convention under which the conjecture is well posed on  $\mathbb{Z}$ , every negative integer reaches a positive value within at most  $\Omega(|n|)$  steps. This reduces the negative integer case of the conjecture entirely to the positive integer case.

We also identify an obstruction to further verification that is structural rather than a matter of computing resources. The reduction step of  $T$  requires the complete factorization of the input which is a problem for which no general sub exponential algorithm is known. Repeated application of the squaring branch can therefore produce integers whose factorization lies beyond any presently known method regardless of computing time available.

Our verification required factoring intermediate values of up to 96 digits and succeeded in every instance though with no guarantee that a harder instance does not arise beyond the tested range. For Part *II*, this obstruction is severe enough to foreclose even a conjecture. Every trajectory examined exceeded 100 digits within fewer than 25 iterations without any value repeating. We are unable to characterize the long-term behavior of Part *II* by any method available to us and as a result, Part *II* is entirely open.

## 1. Introduction

The Collatz conjecture asserts that the map  $C(n) = n/2$  for even  $n$  and  $C(n) = 3n + 1$  for odd  $n$  sends every positive integer eventually to the cycle  $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ . It remains open since its formulation in the 1930s. Its difficulty has motivated two distinct lines of generalization.

The first line establishes fundamental limits on what can be proved about maps of this structural type as a class. Conway showed that a broad family of piecewise linear maps is rich enough to simulate arbitrary computation. A consequence is that no single algorithm can decide for every map in this family whether every orbit eventually enters a cycle [Conway 1972].

The second line studies specific subfamilies analytically. Matthews and Watts conjectured a sharp boundedness criterion for a broad class of affine maps whose governing rule depends only on the residue of the input modulo a fixed integer [Matthews and Watts 1984]. Tao proved in 2022 that for the original Collatz map, almost all positive integers in logarithmic density eventually attain values below any prescribed function that grows however slowly to infinity [Tao 2022]. This falls short of the full conjecture but establishes that divergence is rare in a precise sense. Gonçalves, Greenfeld and Madrid extended Tao's result in 2025 to a general class of maps of the form divide by  $p$  if divisible and otherwise multiply by  $q$  and translate under explicit conditions on  $p$  and  $q$  [Gonçalves, Greenfeld and Madrid 2025].

A structural feature unites every map addressed by these results. The rule applied to  $n$  depends only on  $n$  modulo a fixed integer  $p$ . This periodicity is what makes the class analytically tractable. It is also what makes the class linear in character. Even when the reduction step involves a prime  $p$ , the growth step remains an affine function  $qn + r$ .

This paper studies a map outside that class. We define

$$T(n) = \begin{cases} n^2 + 1 & n \text{ prime or even} \\ n/P(n) & n \text{ odd composite} \end{cases}$$

where  $P(n)$  denotes a designated prime factor of  $n$ . In Part *I*,  $P(n)$  is the largest prime factor while in Part *II*, it is the smallest.

Two features place  $T$  outside the existing toolkit simultaneously. First, the growth branch is quadratic rather than affine. Iterated squaring doubles digit length per growth step rather than increasing it linearly.

Second and more consequentially, the reduction branch depends on the complete factorization of  $n$  rather than on  $n$  modulo any fixed integer. Primality and the largest prime factor are not periodic functions of  $n$  in the sense required by Conway's framework or by the conjecture of Matthews and Watts. This follows from Dirichlet's theorem on primes in arithmetic progressions which guarantees that every residue class modulo every integer contains both prime and composite members. To the best of our knowledge, no existing boundedness or undecidability result applies to  $T$  in either direction.

We report three results.

For Part  $I$  where  $P(n)$  is the largest prime factor of  $n$ , computation verifies that every integer  $n \in [2, 1000]$  converges under iteration of  $T_I$  to the single twelve element cycle

$5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901 \rightarrow 52357 \rightarrow 41 \rightarrow 1682 \rightarrow 2829125 \rightarrow$   
 $1625 \rightarrow 125 \rightarrow 25$

with zero exceptions and zero alternate cycles observed. We denote this cycle  $C^*$ . We conjecture that  $C^*$  is the unique attractor for every integer  $n$  in the domain  $\mathbb{Z}^* = \mathbb{Z} \setminus \{-1, 0, 1\}$  but we do not prove it.

Toward this conjecture, we prove two theorems. The first is exact rather than asymptotic. For any odd composite  $m$ , exactly  $\Omega(m) - 1$  applications of the largest prime factor reduction are required to reach a prime where  $\Omega(m)$  counts the prime factors of  $m$  with multiplicity. Each application removes exactly one element from the prime factorization multiset of  $m$  so the count decreases by exactly one at every step and terminates at the unique index where it equals one. The second theorem follows from the first. Under the extension of primality to negative integers through  $|n|$ , which is the only convention under which the conjecture is well posed on  $\mathbb{Z}^*$ , every negative integer reaches a positive value within at most  $\Omega(|n|)$  steps. This reduces the negative integer case of the conjecture entirely to the positive integer case.

We further identify an obstruction to verification that is one of principle rather than of resources. The reduction step of  $T_I$  requires complete integer factorization which is a problem for which no general sub exponential algorithm is known. Repeated firing of the squaring branch can therefore produce integers whose factorization lies beyond any presently known method regardless of computing resources available. Our verification of the conjecture for Part  $I$  across  $n \in [2, 1000]$  required factoring intermediate values of up to 96 digits and succeeded in every case though without any guarantee that a harder instance does not arise beyond the tested range.

For Part *II*, where  $P(n)$  is the smallest prime factor, this obstruction is severe enough that we report no cycle, no convergence evidence and no conjecture at all. Every trajectory examined exceeded 100 digits within fewer than 25 iterations without any value repeating and we are unable to characterize the long-term behavior of Part *II* by any method available to us. As a result, Part *II* is entirely open.

The remainder of the paper is organized as follows. Section 2 fixes definitions and the precise statement of  $T$  on  $\mathbb{Z}^*$ . Section 3 presents the computational verification methodology and results for Part *I*. Section 4 proves the two reduction theorems. Section 5 documents the factorization obstruction precisely. Section 6 reports the state of knowledge for Part *II*. Section 7 discusses the relationship to existing theory and states open problems. Section 8 concludes.

## 2. Definitions

We fix the domain of  $T$  precisely before stating the rule. The classification used to define  $T$  depends on primality and compositeness which are properties that are conventionally attributed only to positive integers. Two specific integers also require explicit exclusion from the domain for reasons recorded below.

For a positive integer  $m$  greater than one, write  $m = P_1^{a_1} P_2^{a_2} \dots P_k^{a_k}$  for its prime factorization with  $p_1 < p_2 < \dots < p_k$  and each  $a_i$  a positive integer. We write  $\Omega(m)$  for the number of prime factors of  $m$  counted with multiplicity so that  $\Omega(m) = a_1 + a_2 + \dots + a_k$ . The integer  $m$  is prime when  $k = 1$  and  $a_1 = 1$ , equivalently when  $\Omega(m) = 1$ . The integer  $m$  is composite when  $\Omega(m) \geq 2$ . The integer 1 satisfies  $\Omega(1) = 0$  and is therefore neither prime nor composite which is in agreement with the standard convention. For composite  $m$  we write

$P_{\{max\}}(m) = p_k$  for its largest prime factor and  $P_{\{min\}}(m) = p_1$  for its smallest.

### Definition 2.1.

The map  $T_I$  on the positive integers greater than one is defined by

$$T_I(n) = \begin{cases} n^2 + 1 & n \text{ prime or even} \\ n/P(n) & n \text{ odd composite} \end{cases}$$

The map  $T_{II}$  is defined identically except that the second branch uses  $P_{\{min\}}(n)$  in place of  $P_{\{max\}}(n)$ .

Every even number satisfies the first branch regardless of its primality and the only even prime is two. The case distinction in Definition 2.1 is therefore exhaustive and mutually exclusive once  $n = 1$  is excluded. We also exclude  $n = 0$  and record the reason as a remark rather than leaving it implicit.

**Remark 2.2.**

The integer zero is even so  $T_I(0) = T_{II}(0) = 0^2 + 1 = 1$ . Since 1 is excluded from the domain, zero is not a valid starting point for either map. Zero is also the only integer for which the first branch produces the value one and since  $n^2 + 1 = 1$  forces  $n = 0$  over the integers. The second branch likewise never produces one. An odd composite  $n$  has  $\Omega(n) \geq 2$ , so after dividing by either  $P_{\{max\}}(n)$  or  $P_{\{min\}}(n)$  the quotient retains at least one prime factor and is therefore at least a prime and never equal to one. The domain of  $T_I$  and  $T_{II}$  on the positive integers is accordingly taken to be the set of integers greater than one.

We now extend both maps to the negative integers. This step requires an explicit convention because primality and compositeness are conventionally properties of positive integers only.

**Definition 2.3.**

For a negative integer  $n$ , we classify  $n$  according to the prime factorization of  $|n|$ . We say  $n$  is prime when  $|n|$  is prime and composite when  $|n|$  is composite. We extend  $\Omega$ ,  $P_{\{max\}}$  and  $P_{\{min\}}$  to negative arguments by

$\Omega(n) = \Omega(|n|)$ ,  $P_{\{max\}}(n) = P_{\{max\}}(|n|)$  and  $P_{\{min\}}(n) = P_{\{min\}}(|n|)$ . Parity is unaffected by sign and requires no extension. Under this convention, Definition 2.1 extends verbatim to every negative integer except  $-1$  which is excluded for the same reason as 1 since  $|-1| = 1$  is neither prime nor composite.

This convention is the one used in the reduction of the negative case to the positive case proved in Section 4. It is the only convention under which the central conjecture of this paper is well posed on the full integers. Under the stricter convention that reserves primality for positive integers alone, a negative integer such as  $-7$  would satisfy none of the three branches of Definition 2.1 and the map would be undefined there.

We write  $\mathbb{Z}^* = \mathbb{Z} \setminus \{-1, 0, 1\}$  for the common domain of  $T_I$  and  $T_{II}$  fixed by Definitions 2.1 through 2.3.

We turn to the dynamical notions used throughout the paper. For  $n \in \mathbb{Z}^*$  and  $k \geq 0$ , we write  $T^{(k)}(n)$  for the  $k$ th iterate of  $T$  applied to  $n$  with  $T^{(0)}(n) = n$  where this notation refers to  $T_I$  or  $T_{II}$  as determined by context. The orbit of  $n$  is the sequence  $n, T(n), T^{(2)}(n), \dots$  defined for as long as every iterate remains in  $\mathbb{Z}^*$ . A cycle is a finite sequence of distinct integers  $c_0, c_1, \dots, c_{\{L-1\}}$  in  $\mathbb{Z}^*$  such that  $T(c_i) = c_{\{i+1 \bmod L\}}$  for every  $i$  and  $L$  is called the length of the cycle. The orbit of  $n$  enters a cycle  $C$  when  $T^{(k)}(n) \in C$  for some  $k \geq 0$  and the least such  $k$  the transient length of  $n$ .

We write  $C^*$  for the specific cycle of length twelve identified in Section 1 namely

$$C^* = \{5, 26, 677, 458330, 210066388901, 52357, 41, 1682, 2829125, 1625, 125, 25\}$$

with the convention that the listed order is the order of traversal under  $T_I$  so that  $T_I(5) = 26$  and so forth closing with  $T_I(25) = 5$ .

**Conjecture 2.4.**

***For every  $n \in \mathbb{Z}^*$ , the orbit of  $n$  under  $T_I$  enters the cycle  $C^*$  in finitely many steps.***

We make no corresponding conjecture for  $T_{II}$ . The available evidence at present does not support any claim about the long-term behavior of  $T_{II}$ . This includes the claim that its orbits remain bounded at all as discussed in Section 1 and documented in Section 6.

**3. Computational Verification of Part I**

**3.1 Method**

We verified Conjecture 2.4 by iterating  $T_I$  from each integer  $n$  in the range from 2 to 1000 and recording for every starting value either the step at which the orbit first entered  $C^*$  or evidence of failure. An iteration was classified as having failed only when factorization required by the reduction branch of  $T_I$  could not be completed within an allocated time. This possibility and its mathematical significance are discussed at length in Section 5.

Integer factorization was performed using a standard computer algebra system combining trial division with Pollard's rho method and the elliptic curve method for larger factors. Each factorization call was bounded by a timeout of 120 seconds. This bound was found sufficient to resolve every instance encountered in the present search. No iteration was permitted to exceed 400 applications of  $T_I$ , a bound far in excess of any transient length actually observed.

### 3.2 Results

Every one of the 999 integers from 2 to 1000 was found to enter  $C^*$  under iteration of  $T_I$ . No integer in this range entered a different cycle and no factorization required for this range exceeded the allocated timeout. The full search required 69 seconds of computation.

Six starting values namely 5, 25, 26, 41, 125 and 677 are themselves members of  $C^*$  and enter it at step zero. For the remaining 993 starting values, the transient length ranged from a minimum of one step to a maximum of twenty-five steps with a mean of 6.37 steps and a median of six steps. The single longest transient was achieved by  $n = 367$  whose complete orbit we record in full.

367 → 134690 → 18141396101 → 1993 → 3972050 → 15777181202501 → 82500673  
→ 653 → 426410 → 181825488101 → 4157 → 17280650  
→ 298620864422501 → 457 → 208850 → 43618322501 → 397553 → 689  
→ 13 → 170 → 28901 → 835267802 → 697672301057911205 → 8905  
→ 65 → 5 → 26 → 677 → 458330 → 210066388901 → 52357 → 41 → 1682  
→ 2829125 → 1625 → 125 → 25 → 5

Step 0: 367 - prime

Step 1: 134690 - even

Step 2: 18141396101 - odd composite

Step 3: 1993 - prime

Step 4: 3972050 - even

Step 5: 15777181202501 - odd composite

Step 6: 82500673 - odd composite

Step 7: 653 - prime

Step 8: 426410 - even

Step 9: 181825488101 - odd composite

Step 10: 4157 - prime  
Step 11: 17280650 - even  
Step 12: 298620864422501 - odd composite  
Step 13: 457 - prime  
Step 14: 208850 - even  
Step 15: 43618322501 - odd composite  
Step 16: 397553 - odd composite  
Step 17: 689 - odd composite  
Step 18: 13 - prime  
Step 19: 170 - even  
Step 20: 28901 - prime  
Step 21: 835267802 - even  
Step 22: 697672301057911205 - odd composite  
Step 23: 8905 - odd composite  
Step 24: 65 - odd composite  
Step 25: 5 - prime  
Step 26: 26 - even  
Step 27: 677 - prime  
Step 28: 458330 - even  
Step 29: 210066388901 - odd composite  
Step 30: 52357 - odd composite  
Step 31: 41 - prime  
Step 32: 1682 - even  
Step 33: 2829125 - odd composite  
Step 34: 1625 - odd composite  
Step 35: 125 - odd composite

Step 36: 25 - odd composite

Step 37: 5 - prime

This orbit visits seven distinct primes before its final entry into  $C^*$  namely 367, 1993, 653, 4157, 457, 13 and 28901. No two of these excursions share a common length. The orbit does not repeat prior to reaching 5. The transient of twenty-five steps is therefore not an artifact of early cycle detection but the true transient length of  $n = 367$  under  $T_I$ .

The largest intermediate value encountered in the full search arose independently from two starting value  $n = 936$  and  $n = 986$ . Each produced a 96 digit odd composite at its fifth step of its orbit. We record both orbits in full.

For  $n = 936$ :

936 → 876097 → 767545953410 → 589126790596065890628101 →  $T_I^{(4)}(936)$   
→  $T_I^{(5)}(936)$  → 915605 → 745 → 5 → 26 → 677 → 458330  
→ 210066388901 → 52357 → 41 → 1682 → 2829125 → 1625 → 125 → 25  
→ 5

where  $T_I^{(4)}(936)$  is the 48 digit even number

347070375398020869903742997821851263544290866202

and  $T_I^{(5)}(936)$  is the 96 digit odd composite

1204578454789231303090572051942405109732698204049259302919135777875904157392  
07973282923465904805

whose largest prime factor is the 90 digit prime

1315609301815991943131123193890820943237201854565297593306213681528502091395  
39401033113041

division by which produces 915605 and returns the orbit to within three Steps of  $C^*$ .

Step 0: 936 - even

Step 1: 876097 - prime

Step 2: 767545953410 - even

Step 3: 589126790596065890628101 - prime

Step 4: 347070375398020869903742997821851263544290866202 - even

Step 5:

1204578454789231303090572051942405109732698204049259302919135777875904157392  
07973282923465904805 - odd composite

Step 6: 915605 - odd composite

Step 7: 745 - odd composite

Step 8: 5 - prime

Step 9: 26 - even

Step 10: 677 - prime

Step 11: 458330 - even

Step 12: 210066388901 - odd composite

Step 13: 52357 - odd composite

Step 14: 41 - prime

Step 15: 1682 - even

Step 16: 2829125 - odd composite

Step 17: 1625 - odd composite

Step 18: 125 - odd composite

Step 19: 25 - odd composite

Step 20: 5 - prime

For  $n = 986$ :

$986 \rightarrow 972197 \rightarrow 945167006810 \rightarrow 893340670762174586376101 \rightarrow T_I^{(4)}(986)$   
 $\rightarrow T_I^{(5)}(986) \rightarrow 7585 \rightarrow 185 \rightarrow 5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901$   
 $\rightarrow 52357 \rightarrow 41 \rightarrow 1682 \rightarrow 2829125 \rightarrow 1625 \rightarrow 125 \rightarrow 25 \rightarrow 5$

where  $T_I^{(4)}(986)$  is the 48 digit even number

798057554037812011881307978772920644079823962202

and  $T_I^{(5)}(986)$  is the 96 digit odd composite

6368958595568152394290609159629141170531004212919906694639921919371560076619  
74861117626324688805

whose largest prime factor is the 92 digit prime

8396781272996904936441145892721346302611739239182474218378275437536664570362  
2262507267808133

division by which produces 7585 and returns the orbit to within three Steps of  $C^*$ .

Step 0: 986 - even

Step 1: 972197 - prime

Step 2: 945167006810 - even

Step 3: 893340670762174586376101 - prime

Step 4: 798057554037812011881307978772920644079823962202 - even

Step 5:

6368958595568152394290609159629141170531004212919906694639921919371560076619  
74861117626324688805 - odd composite

Step 6: 7585 - odd composite

Step 7: 185 - odd composite

Step 8: 5 - prime

Step 9: 26 - even

Step 10: 677 - prime

Step 11: 458330 - even

Step 12: 210066388901 - odd composite

Step 13: 52357 - odd composite

Step 14: 41 - prime

Step 15: 1682 - even

Step 16: 2829125 - odd composite

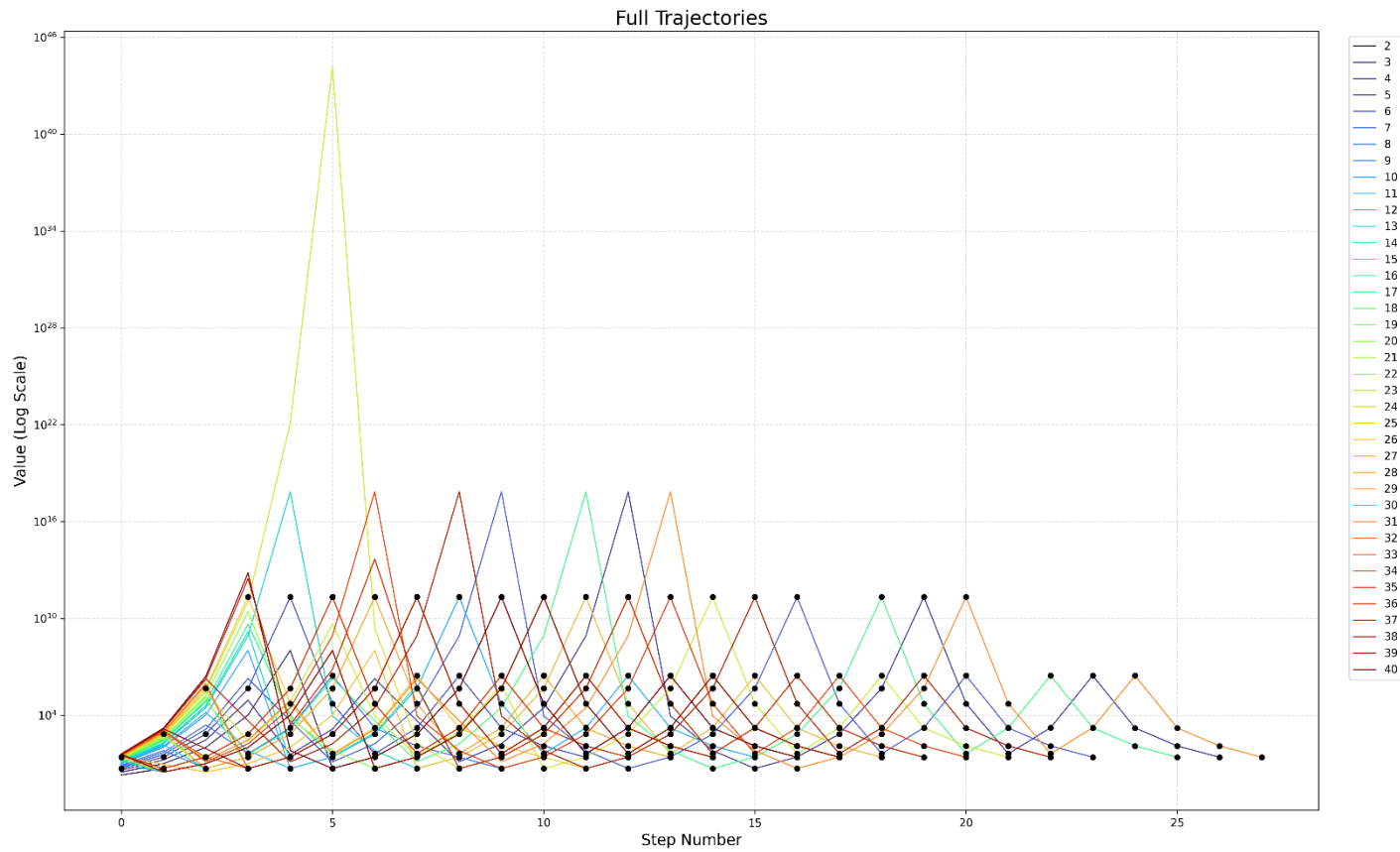
Step 17: 1625 - odd composite

Step 18: 125 - odd composite

Step 19: 25 - odd composite

Step 20: 5 - prime

We note that the structural pattern in both orbits is identical. An even starting value squares to a six-digit prime. The prime squares to a twelve digit even value. That even value squares to a twenty-four-digit prime. That prime squares to a 48 digit even value. That even value squares to a 96 odd digit composite. Division by the largest prime factor then collapses the orbit to a small odd composite within one step and the orbit reaches  $C^*$  within three further steps.



**Figure 1.** Hailstone trajectories of  $T_l$  for  $n \in [2, 40]$  plotted on a logarithmic scale. Each colored line traces the complete orbit of one starting value from its initial step until the orbit first enters  $C^*$ . Black dots mark the steps at which each orbit has entered and is traversing  $C^*$ . The periodic sawtooth formed by the black dots near the bottom of the figure is  $C^*$  itself. The dominant spike reaching  $10^{44}$  arises from  $n = 24$ .

Figure 1 shows that  $n = 24$  produces an intermediate value of 45 digits at step five before collapsing to a ten-digit value at step six. We record the complete orbit of  $n = 24$  as a witness.

$$\begin{aligned}
 24 &\rightarrow 577 \rightarrow 332930 \rightarrow 110842384901 \rightarrow 12286034290541432779802 \rightarrow T_l^{(5)}(24) \\
 &\rightarrow 2384462965 \rightarrow 185 \rightarrow 5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901 \\
 &\rightarrow 52357 \rightarrow 41 \rightarrow 1682 \rightarrow 2829125 \rightarrow 1625 \rightarrow 125 \rightarrow 25 \rightarrow 5
 \end{aligned}$$

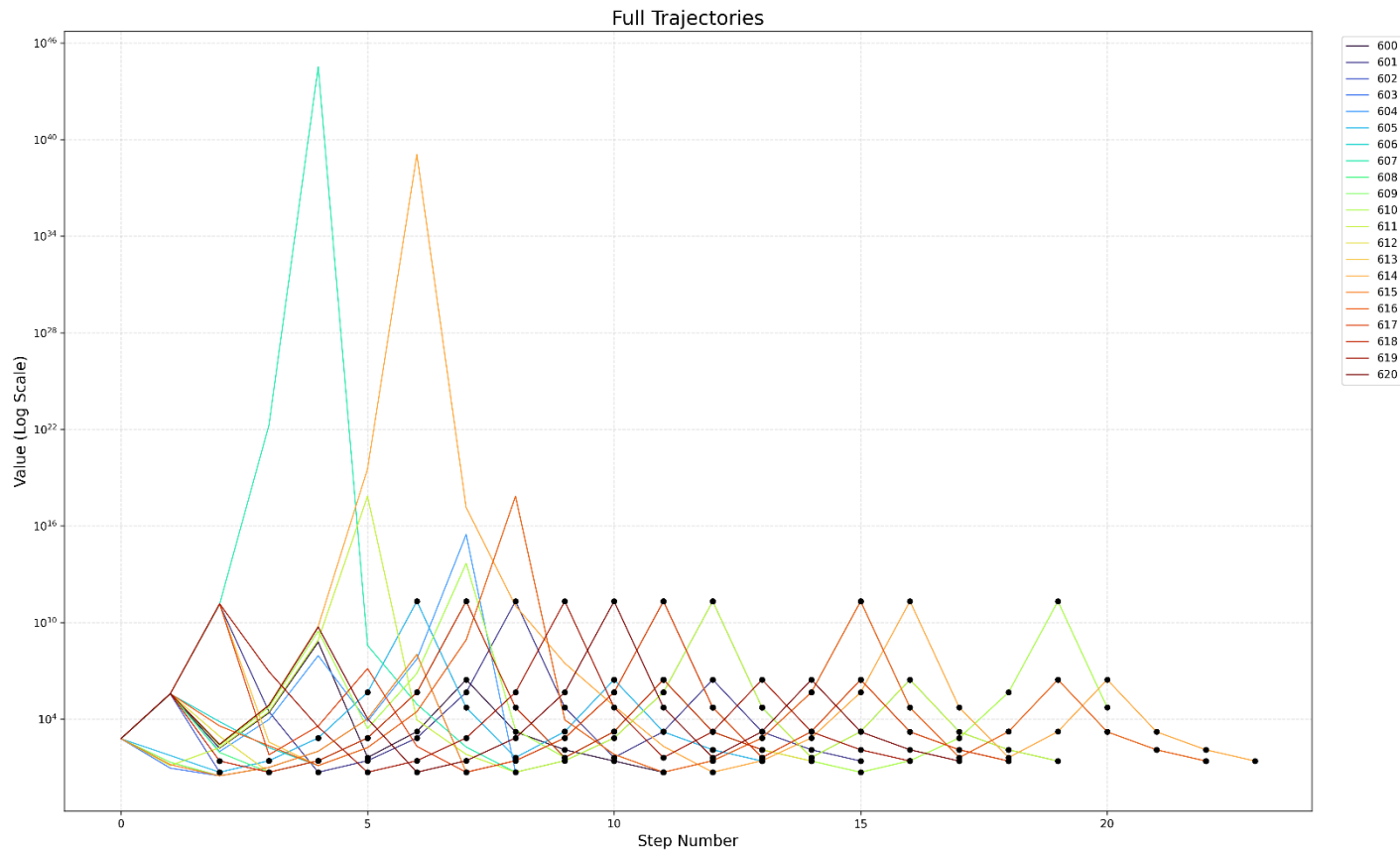
where  $T_l^{(5)}(24)$  is the 45 digit odd composite

$$150946638588359927497047932276204121019159205$$

whose largest prime factor is the 35 digit prime

$$63304249553886414627978058059796337.$$

Division by this factor yields 2384462965 and returns the orbit to a ten-digit value in a single step. The orbit enters  $C^*$  three steps later at  $n = 5$ . Nothing in the trajectory of  $n = 24$  up to step four distinguishes it from its neighbors. The height of the spike is determined entirely by the arithmetic of the value at step five and specifically by how large its largest prime factor is relative to the value itself.



**Figure 2.** Hailstone trajectories of  $T_l$  for  $n \in [600, 620]$  plotted on a logarithmic scale using the same conventions as Figure 1. Two trajectories spike to 45 digits and 40 digits respectively arising from  $n = 607$  and  $n = 614$  while the remaining nineteen starting values in this range converge to  $C^*$  without exceeding twelve digits at any intermediate step. All 21 trajectories enter  $C^*$  without exception.

We record the complete orbit of  $n = 607$ .

$$\begin{aligned}
 607 &\rightarrow 368450 \rightarrow 135755402501 \rightarrow 18429529308208517055002 \rightarrow T_l^{(4)}(607) \\
 &\rightarrow 382162085 \rightarrow 85285 \rightarrow 185 \rightarrow 5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \\
 &\rightarrow 210066388901 \rightarrow 52357 \rightarrow 41 \rightarrow 1682 \rightarrow 2829125 \rightarrow 1625 \rightarrow 125 \rightarrow 25 \\
 &\rightarrow 5
 \end{aligned}$$

where  $T_l^{(4)}(607)$  is the 45 digit odd composite

$$339647550522116701216797893359458707093220005$$

whose largest prime factor is the 36 digit prime

$$888752610092434212087779177150602753.$$

Division by this factor yields 382162085 and returns the orbit to a nine-digit value in a single step. The orbit enters  $C^*$  four steps later at  $n = 5$ .

We record the complete orbit of  $n = 614$ .

$$\begin{aligned}
 614 &\rightarrow 376997 \rightarrow 277 \rightarrow 76730 \rightarrow 5887492901 \rightarrow 34662572659325395802 \rightarrow T_l^{(6)}(614) \\
 &\rightarrow 139301438285214745 \rightarrow 109912834645 \rightarrow 29698145 \rightarrow 64985 \rightarrow 205 \\
 &\rightarrow 5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901 \rightarrow 52357 \rightarrow 41 \rightarrow 1682 \\
 &\rightarrow 2829125 \rightarrow 1625 \rightarrow 125 \rightarrow 25 \rightarrow 5
 \end{aligned}$$

where  $T_l^{(6)}(614)$  is the 40 digit odd composite

$$1201493943363012441540623037463959223205$$

whose largest prime factor is the 22 digit prime

$$8625136668747068396909.$$

Division by this factor yields 139301438285214745 and returns the orbit to an eighteen-digit value in a single step. The orbit enters  $C^*$  six steps later at  $n = 5$ .

The two orbits differ structurally in a way the figure makes visible. The orbit of  $n = 607$  reaches its 45 digit peak at step four after two consecutive squarings of primes and collapses to a single digit value in four further steps. The orbit of  $n = 614$  reaches its 40 digit peak at step six but then requires six reduction steps before entering  $C^*$  because the descent chain is longer.

Figure 1 and 2 display the trajectories on a logarithmic scale. The vertical axis records the value of the iterate at each step and the horizontal axis records the step number. Black dots mark the steps at which each orbit has entered and is traversing  $C^*$ . The periodic floor formed by the black dots is  $C^*$  itself. Its twelve elements alternate between values near 25 and values near 210066388901 and produce the regular sawtooth visible at the bottom of both figures.

The structure of each individual trajectory before convergence is the visual expression of Theorem 4.2. Each rise corresponds to one application of the squaring branch. Each descent corresponds to the finite reduction chain guaranteed by Theorem 4.2 in which  $\Omega(m) - 1$  applications of the reduction branch collapse the orbit back toward small values. The peak height at any given step of a trajectory is determined by the arithmetic of the value encountered at that step, bears no simple relationship and produces no visible pattern relative to the starting value.

These figures serve two purposes in this paper. They make the convergence stated in Conjecture 2.4 visible across ranges of different magnitudes and show that the qualitative behavior is consistent from small starting values to values in the hundreds. They also make the factorization obstruction of Section 5 visible. Every spike represents an intermediate value whose factorization was required to continue the orbit and whose size was not predictable from the starting value alone.

These results constitute the entirety of the computational evidence offered in this paper for Conjecture 2.4. We make no claim beyond the range tested. Whether this evidence extends to a larger range or to all of  $\mathbb{Z}^*$  is genuinely open and is addressed in Section 7.

## 4. Proof of the Two Reduction Theorems

This section proves the two theorems announced in the introduction. We establish a parity lemma first because it underpins both theorems and is what permits the chain length to be computed exactly rather than merely bounded.

### 4.1 A parity lemma

#### Lemma 4.1.

Let  $m$  be an odd composite integer with  $m > 1$ . Then  $m/P_{\{max\}}(m)$  is odd.

#### Proof

Since  $m$  is odd, every prime factor of  $m$  is odd. In particular,  $P_{\{max\}}(m)$  is an odd prime. Write  $m = P_{\{max\}}(m) \cdot k$  where  $k = m/P_{\{max\}}(m)$  is a positive integer. Both  $m$  and  $P_{\{max\}}(m)$  are odd. Since the product of two integers is odd if and only if both factors are odd and  $P_{\{max\}}(m)$  is odd,  $k$  must be odd.

#### Remark

Lemma 4.1 implies that every value produced by a reduction step applied to an odd composite is itself odd. A chain of reduction steps beginning at an odd composite therefore consists entirely of odd values. The squaring branch of  $T_I$  fires at a value  $n$  when  $n$  is prime or even. Since no value in a reduction chain is even, the squaring branch cannot fire until the chain terminates at a prime. This justifies treating reduction chains in complete isolation from the squaring branch in the proof of Theorem 4.2.

#### Example

Let  $m = 315 = 3^2 \times 5 \times 7$ .

Then  $P_{\{max\}}(315) = 7$  and  $315/7 = 45$ .

The value 45 is odd.

Applying the lemma again:

$P_{\{max\}}(45) = 5$  and  $45/5 = 9$ .

The value 9 is odd. Then  $P_{\{max\}}(9) = 3$  and  $9/3 = 3$  which is prime and odd.

## 4.2 The exact length of a reduction chain

### Theorem 4.2.

Let  $m$  be an odd composite integer with  $m > 1$  and define  $m_0 = m$  and

$m_{\{i+1\}} = m_i / P_{\{max\}}(m_i)$  for as long as  $m_i$  remains composite. This sequence is well defined, every  $m_i$  is odd and there is a unique index  $i^* = \Omega(m) - 1$  at which  $m_{\{i^*\}}$  is prime.

### Proof

That every  $m_i$  is odd follows from Lemma 4.1 by induction, the base case  $m_0 = m$  being odd by hypothesis.

We show  $\Omega(m_{\{i+1\}}) = \Omega(m_i) - 1$  whenever  $m_i$  is composite. Write the prime factorization of  $m_i$  with multiplicity as a multiset of  $\Omega(m_i)$  primes. Dividing  $m_i$  by  $P_{\{max\}}(m_i)$  removes exactly one occurrence of the largest element from this multiset regardless of how many times that prime appears in the factorization. The resulting multiset has size  $\Omega(m_i) - 1$  exactly.

By induction,  $\Omega(m_i) = \Omega(m) - i$  for every  $i$  at which  $m_i$  is defined. Since  $m$  is composite  $\Omega(m) \geq 2$ . The value  $m_i$  is composite for exactly those  $i$  satisfying  $\Omega(m) - i \geq 2$  and prime  $i = \Omega(m) - 1$ . This index is unique because  $\Omega(m_i)$  decreases strictly by one at each step so no value is repeated.

### Example using values from the paper.

The cycle  $C^*$  contains odd composite  $210066388901 = 41 \times 1277 \times 4012193$  with  $\Omega(210066388901) = 3$ . Theorem 4.2 predicts the reduction chain reaches a prime in exactly  $3 - 1 = 2$  steps.

Step 0: 210066388901 with  $P_{\{max\}} = 4012193$ . Division gives 52357.

Step 1:  $52357 = 41 \times 1277$  with  $P_{\{max\}} = 1277$ . Division gives 41.

Step 2: 41 is prime. The chain terminated in exactly  $2 = \Omega(210066388901) - 1$  steps as claimed.

Applied to  $T_l$ : starting from any odd composite  $m$  exactly  $\Omega(m) - 1$  consecutive applications of  $T_l$  are required before  $T_l$  first encounters a prime and switches to the squaring branch.

### 4.3 Two lemmas on the squaring branch

#### Lemma 4.3.

For every nonzero integer  $n$ ,  $n^2 + 1 > 0$ .

#### Proof

Since  $|n| \geq 1$ , we have  $n^2 \geq 1$  and therefore  $n^2 + 1 \geq 2 > 0$ .

#### Lemma 4.4.

If  $p$  is an odd prime, then  $p^2 + 1$  is even.

#### Proof

Since  $p$  is odd,  $p^2$  is odd and  $p^2 + 1$  is even.

#### Remark

Lemma 4.4 describes a structural feature of the dynamics that every time the squaring branch fires at an odd prime, the result is even. This means the squaring branch fires again at the very next step. Consecutive pairs of the form prime  $\rightarrow$  even appear throughout the orbits documented in Section 3.

### 4.4 The exact reduction of negative orbits to positive orbits

#### Theorem 4.5.

Let  $n$  be a negative integer with  $n \neq -1$  and let  $k(n)$  denote the least positive integer such that

$T_I^{(k(n))}(n) > 0$  under the extension of  $T_I$  to negative integers fixed in Definition 2.3. Then  $k(n)$  exists and satisfies

$$k(n) = \begin{cases} 1 & n \text{ even} \\ \Omega(|n|) & n \text{ odd.} \end{cases}$$

In particular,  $k(n) \leq \Omega(|n|)$  for every admissible  $n$  with equality if and only if  $n$  is odd.

## Proof

### Case 1: $n$ even.

Since parity is unaffected by sign,  $n$  satisfies the first branch of  $T_I$  directly and

$T_I(n) = n^2 + 1 > 0$  by Lemma 4.3. Therefore,  $k(n) = 1$ . Since  $|n| > 1$ , we have  $\Omega(|n|) \geq 1$  so  $k(n) = 1 \leq \Omega(|n|)$  with equality exactly when  $\Omega(|n|) = 1$ , that is when  $n = -2$ .

### Case 2: $n$ odd with $|n|$ prime.

Then  $n$  is classified as prime by Definition 2.3 and  $T_I(n) = n^2 + 1 > 0$  by Lemma 4.3. Therefore,  $k(n) = 1 = \Omega(|n|)$ .

### Case 3: $n$ odd with $|n|$ composite.

Define  $n_0 = n$  and  $n_{\{i+1\}} = n_i / P_{\{max\}}(|n_i|)$  for as long as  $|n_i|$  remains composite. Since each divisor  $P_{\{max\}}(|n_i|)$  is a positive prime and  $n_0$  is negative, every  $n_i$  in this sequence is negative. By Lemma 4.1, every  $n_i$  is odd. Since  $|n_i|$  is odd and composite, the reduction branch fires at every step and the squaring branch cannot fire until  $|n_i|$  becomes prime.

The sequence of absolute values  $|n_i|$  coincides with the sequence  $m_i$  of Theorem 4.2 applied to  $m = |n|$ . By that theorem, there is a unique index  $i^* = \Omega(|n|) - 1$  at which  $|n_{\{i^*\}}|$  is prime. At step  $i^*$ , the value  $n_{\{i^*\}}$  is a negative prime and  $T_I(n_{\{i^*\}}) = n_{\{i^*\}}^2 + 1 > 0$  by Lemma 4.3. This is the application of  $T_I$  at step  $\Omega(|n|)$  counted from  $n_0$ . Every earlier iterate  $n_i$  is negative so  $k(n) = \Omega(|n|)$ .

## Example

For  $n = -6$ (even):

$T_I(-6) = 36 + 1 = 37 > 0$ . So,  $k(-6) = 1 = 1$  as the formula gives.  $\Omega(6) = 2$  and

$k(-6) = 1 \leq 2$ .

For  $n = -315$  (odd composite,  $315 = 3^2 \times 5 \times 7$ ,  $\Omega(315) = 4$ ): the formula gives

$k(-315) = 4$ . Tracing the orbit:

$$-315 \xrightarrow{\div 7} -45 \xrightarrow{\div 5} -9 \xrightarrow{\div 3} -3 \xrightarrow{(-3)^2+1} 10 > 0.$$

The orbit takes exactly  $4 = \Omega(315)$  steps to become positive.

### Corollary 4.6.

Every negative integer  $n \neq -1$  satisfies  $T_I^{(k)}(n) > 0$  for some  $k \leq \Omega(|n|)$ .

This corollary reduces the negative integer case of Conjecture 2.4 to the positive integer case. Once an orbit becomes positive, it remains positive from that point forward since

$T_I(n) = n^2 + 1 \geq 2$  for all nonzero  $n$  and division of a positive odd composite by a positive prime factor yields a positive result. Theorem 4.5 is strictly sharper than Corollary 4.6 since it gives the exact value of  $k(n)$  rather than an upper bound.

## 5. The Factorization Obstruction

### 5.1 The factorization requirement

The reduction branch of  $T_I$  requires the complete prime factorization of every odd composite input. This is not a consequence of how verification was implemented but is rather a property of the rule itself. The largest prime factor  $P_{\{max\}}(n)$  cannot be identified without factoring  $n$  or solving an equivalent problem. No algorithm that factors a general integer of  $d$  decimal digits in time polynomial in  $d$  is currently known. The most powerful general methods available run in sub exponential time and become infeasible in practice beyond a threshold that currently lies near 230 decimal digits for the hardest instances. An integer at this threshold is one composed of two primes of roughly equal and large size.

Once the squaring branch of  $T_I$  drives an intermediate value past this threshold without small prime factors being present, verification of the trajectory would require solving a problem that has resisted resolution for decades. This is the obstruction we call intrinsic.

## 5.2 The factorization encountered

The verification of Part *I* across  $n \in [2, 1000]$  required factoring all odd composite values encountered along every trajectory. Three starting values produced intermediate values exceeding 90 decimal digits. These were  $n = 686$  at 91 digits and  $n = 936$  and  $n = 986$  each at 96 digits.

The 96 digit value arising from  $n = 936$  at step five is

1204578454789231303090572051942405109732698204049259302919135777875904157392  
07973282923465904805

Its complete factorization is  $5 \times 149 \times 1229 \times q$  where  $q$  is the 90 digit prime

1315609301815991943131123193890820943237201854565297593306213681528502091395  
39401033113041

This factorization required 0.002 seconds. Trial division immediately located the factors 5, 149 and 1229. Once these were removed the remaining 90 digit quotient was prime and required only a primality certificate rather than further factoring. Division by  $q$  yields  $5 \times 149 \times 1229 = 915605$  and returns the orbit of  $n = 936$  to a six-digit value in a single step.

The 96 digit value arising from  $n = 986$  at step five is

6368958595568152394290609159629141170531004212919906694639921919371560076619  
74861117626324688805

Its complete factorization is  $5 \times 37 \times 41 \times r$  where  $r$  is the 92 –digit prime

8396781272996904936441145892721346302611739239182474218378275437536664570362  
2262507267808133

This factorization required 0.001 seconds for the same reason. Trial division found 5, 37 and 41 immediately. Division by  $r$  yields  $5 \times 37 \times 41 = 7585$  and returns the orbit of  $n = 986$  to a four-digit value in a single step.

The 91 digit value arising from  $n = 686$  at step five is

5786107103608017965891980199189078317715564022647216673325690705564497975686  
593433362384805

Its largest prime factor is the 62 digit prime

58114511414845791227553048436672239097587170410484412182614321

and this factorization also required effectively no measurable time due to the presence of small prime factors detected by trial division.

In each of the three cases, the factorization succeeded within milliseconds because the intermediate value carried prime factors small enough for trial division to locate. This is the arithmetic condition that made verification feasible. It was not guaranteed by the rule and was not predictable in advance.

### 5.3 The theoretical boundary

The computations of Section 5.2 succeeded because every large intermediate value produced by the squaring branch happened to carry small prime factors. No property of  $T_I$  guarantees this. The squaring branch applies to both prime and even inputs. When applied to an odd prime  $p$ , it produces  $p^2 + 1$  which is always even and thus itself enters the squaring branch without requiring factorization. When applied to an even number  $m$ , it produces  $m^2 + 1$  which is always odd. Whether  $m^2 + 1$  carries a small prime factor depends entirely on the arithmetic of  $m$  and provides no guarantee. An integer of the form  $m^2 + 1$  that has no prime factor below  $10^{40}$  would be infeasible to factor by any currently known method once  $m$  has more than 25 decimal digits. Nothing prevents the orbits of  $T_I$  from eventually producing such a value.

The hardest instances for factorization algorithms are integers that are the product of exactly two primes of roughly equal and large size. These are called semiprimes. The security of RSA cryptosystem rests on the practical infeasibility of factoring semiprimes of 300 or more digits. The current public record for factoring a general integer stands at 250 digits achieved through a large coordinated computation. A trajectory of  $T_I$  that reached a 300 digit semiprime as an intermediate value would be unverifiable at present regardless of computing resources. The trajectory could be traced up to that point and no further.

We state this without claiming that such a value must arise. We only claim that the verification reported in Section 3 succeeded for a specific arithmetic reason that may not persist at larger scales. Whether the integers produced along orbits of  $T_I$  systematically tend to carry small prime factors is a number theoretic question about the polynomial  $n^2 + 1$  that is entirely open and appears to require techniques well beyond what is currently available.

#### **5.4 The severity of the obstruction for Part II**

For Part II, the reduction step divides by the smallest prime factor of the input. This is always a very modest contraction. If the input has smallest prime factor 3, the value shrinks by a factor of 3 regardless of its size. The squaring branch doubles digit length at every step. The net effect of one squaring followed by one reduction by the smallest prime factor is therefore growth by approximately a factor of  $n/3$  per cycle rather than the dramatic collapses produced by the largest prime factor in Part I.

All Part II trajectories examined exceeded 100 digits within fewer than 25 iterations. The factorization problem arising in Part II is therefore encountered at these digit lengths almost immediately. Unlike the Part I cases where small prime factors made the factorization tractable, Part II values at these lengths produced no corresponding simplification in any trajectory we were able to examine before the computations became infeasible. For Part II, the obstruction identified in this section is not a theoretical boundary that might never be encountered. It is a practical limit reached within the first twenty-five steps from any small starting value.

### **6. Part II: An Open Problem**

#### **6.1 The structural difference from Part I**

Part II replaces the largest prime factor in the reduction branch with the smallest while leaving the squaring branch unchanged. The formal definition is that  $T_{II}(n) = n^2 + 1$  when  $n$  is prime or even and  $T_{II}(n) = n/P_{\{min\}}(n)$  when  $n$  is an odd composite. This single change produces a map that is qualitatively more expansive than  $T_I$  at every step where the reduction branch fires.

The reason is arithmetic. The smallest prime factor of an odd composite is at most its square root and is frequently 3 or 5. Dividing by 3 shrinks a value by a factor of three regardless of its magnitude. The squaring branch doubles digit length at every step it fires.

One squaring followed by one smallest prime factor reduction therefore produces a net growth far in excess of what the same pair of operations produce under  $T_I$ .

Under  $T_I$  the largest prime factor can be nearly as large as the input itself and division by it can collapse a value of hundreds of digits to one of a few digits in a single step as documented for  $n = 936$  and  $n = 986$  in Section 3. No comparable collapse occurs under  $T_{II}$ .

## 6.2 The orbit of $n = 2$ under $T_{II}$

We trace the orbit of  $n = 2$  in full until computation becomes infeasible. The first five steps coincide with the Part  $I$  orbit of  $n = 2$  because those steps produce only primes and even numbers at which both maps apply the identical squaring branch.

$$2 \rightarrow 5 \rightarrow 26 \rightarrow 677 \rightarrow 458330 \rightarrow 210066388901 \rightarrow 5123570461 \rightarrow 4012193 \\ \rightarrow 16097692669250 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11}$$

where  $S_9$  is the 27 digit prime 259135709273625189895562501 and  $S_{10}$  is the 53 digit even number  $S_9^2 + 1$  67151315820744796242563193846279056996907571197375002 and  $S_{11}$  is the 106 digit odd composite  $S_{10}^2 + 1$

4509299216457410367812375434001667228387011675625816355931329006574544945162090251765532678837175414500005 at which factorization timed out and the trajectory could not be continued.

The divergence from the Part  $I$  orbit of  $n = 2$  begins at step five. Both maps arrive at the odd composite 210066388901. Under  $T_I$ , this value is divided by its largest prime factor 4012193 yielding 52357. Under  $T_{II}$ , the same value is divided by its smallest prime factor 41 yielding

5123570461. That value is approximately 98 times larger than 52357 and the orbit then grows rapidly rather than collapsing. The value 4012193 which was the divisor under  $T_I$  appears again in the  $T_{II}$  orbit at step seven but now as a prime input to the squaring branch rather than as a reductive factor. From that point, the orbit grows without interruption to 106 digits within three further steps.

## 6.3 Results across $n = 2$ to 15

We investigated  $T_{II}$  for every starting value from 2 to 15. We traced each orbit until either a value exceeded the feasibility threshold or the trajectory timed out during factorization. In every case, growth to beyond 100 digits occurred before any repetition was observed.

The step at which each orbit first exceeded 100 digits ranged from step 8 for  $n = 14$  to step 23 for  $n = 6$ . The orbit of  $n = 6$  was the most slowly growing among those examined and reached 193 digits at step 23 before the factorization at step 24 timed out.

The orbit of  $n = 5$  reached 106 digits at step 10. No orbit among  $n = 2$  through 15 exhibited any tendency toward reduction or cycle formation within the computationally accessible range. No value was repeated in any trajectory.

#### **6.4 What the evidence establishes and what it does not**

The computations of Section 6.3 establish only that Part *II* orbits grow rapidly and that no small starting value produced a cycle or repeated value within the feasible range. They do not establish divergence. A trajectory could in principle reach very large values and then encounter a long sequence of reductions that return it to a small value from which a cycle emerges. They do not establish unbounded growth and they do not support a conjecture of convergence to any cycle because no cycle was observed.

The obstruction of Section 5 applies with considerably greater force to Part *II* than to Part *I*. Under  $T_I$ , a reduction step can divide by a prime factor nearly as large as the input itself and can collapse a 96 digit value to a six-digit value in one step as shown in Section 3. Under  $T_{II}$ , the reduction step divides by the smallest prime factor which is never larger than the square root of the input. Reductions of the scale produced by  $T_I$  are structurally impossible under  $T_{II}$ .

#### **6.5 An honest statement**

Part *II* is included in this paper because it arises from Part *I* by replacing one word in the definition and because we regard this symmetry as mathematically natural even in the complete absence of a positive result. The conclusion of our investigation of Part *II* is entirely negative. We do not know whether its trajectories are bounded. We do not know whether any orbit is eventually periodic. We do not know whether different starting values lead to fundamentally different behaviors and thus we offer no conjecture. We regard Part *II* as an open problem whose nature is not yet understood well enough to admit a precise conjecture and we state this plainly rather than speculating.

## 7. Relationship to Existing Theory and Open Problems

### 7.1 Placement within the generalized Collatz literature

The study of maps sharing the structural features of the Collatz map has proceeded along two broad lines since Conway's foundational work of 1972. The first line establishes fundamental limits on what can be proved about such maps as a class. Conway showed that a broad family of piecewise linear maps of this kind is rich enough to simulate arbitrary computation. A consequence is that no single algorithm can decide for every map in this family whether every orbit eventually enters a cycle [Conway 1972]. The second line studies specific subfamilies analytically. Matthews and Watts conjectured in 1984 a sharp boundness criterion for a broad class of affine maps whose governing rule depends only on the residue of the input modulo a fixed integer [Matthews and Watts 1984]. Tao proved in 2022 that for the original Collatz map, almost all positive integers in logarithmic density eventually attain values below any prescribed function that grows however slowly to infinity [Tao 2022]. Gonçalves, Greenfeld and Madrid extended Tao's result in 2025 to a general class of maps of the form divide by  $p$  if divisible and otherwise multiply by  $q$  and translate under explicit conditions on  $p$  and  $q$  [Gonçalves, Greenfeld and Madrid 2025].

A structural feature unites every map addressed by these results. The rule applied to  $n$  depends only on  $n$  modulo a fixed integer  $p$ . This periodicity is what makes the class analytically tractable. It is also what makes the class linear in character. Even when the reduction step involves a prime  $p$ , the growth step remains an affine function  $qn + r$ .

The map  $T_I$  studied in this paper does not belong to that class for two independent reasons. The growth branch is quadratic rather than affine. Iterated squaring doubles digit length per growth step rather than increasing it linearly. More consequentially, the reduction branch depends on the complete prime factorization of the input rather than on the input modulo any fixed integer. Primality and the largest prime factor are not periodic functions of  $n$  modulo any fixed integer. This follows from Dirichlet's theorem on primes in arithmetic progressions which guarantees that every residue class modulo every integer contains both prime and composite members. Conway's undecidability result accordingly neither applies to  $T_I$  nor provides any information about whether the behavior of  $T_I$  is decidable. The conjecture of Matthews and Watts was designed for affine residue gated maps and applies to neither the growth branch nor the reduction branch of  $T_I$ . Tao's theorem and result of Gonçalves, Greenfeld and Madrid require the affine structure that  $T_I$  does not possess.

However, there is a body of analytic number theory directly relevant to understanding  $T_I$  more deeply even though it has not previously been applied in a dynamical context of this kind. The largest prime factor of  $n^2 + 1$  as  $n$  ranges over the integers has been the subject of progressively sharper estimates over several decades. Hooley obtained the first quantitative results in 1967 [Hooley 1967]. Deshouillers and Iwaniec improved bounds in 1982 [Deshouillers and Iwaniec 1982]. Merikoski obtained in 2023 the current best unconditional bound showing that the largest prime factor of  $n^2 + 1$  exceeds  $n^{1.279}$  for infinitely many  $n$  [Merikoski 2023]. These results concern the static problem of bounding a single prime factor of a single value of  $n^2 + 1$ . They do not address the dynamical problem of iterating  $T_I$  but they provide the most directly relevant arithmetic context for understanding why the reduction step of  $T_I$  is capable of producing the dramatic collapses documented in Sections 3 and 5. If the largest prime factor of  $n^2 + 1$  is often of order  $n^{1.279}$  or larger, then dividing  $n^2 + 1$  by that factor produces a quotient of order at most  $n^2/n^{1.279} = n^{0.721}$  which is smaller than  $n$  itself. This heuristic is not a proof. The bound on the largest prime factor of  $n^2 + 1$  holds for infinitely many  $n$  but not necessarily for the specific values arising along orbits of  $T_I$ . Making this argument rigorous appears to require techniques substantially beyond the current state of the art in the distribution of prime factors of polynomial values.

## 7.2 The factorization obstruction as a mathematical feature

One contribution of this paper that has no precedent in the existing literature on maps of Collatz type is the identification of an intrinsic connection between the verifiability of the central conjecture and an independently hard problem. For the standard Collatz map and for every map in the Conway and Matthews and Watts frameworks, the next iterate can be computed in time logarithmic in the current value since the rule depends only on a residue. For  $T_I$ , the computation of the next iterate at an odd composite input requires complete integer factorization. The time required for this grows superpolynomially in the digit length of the input by any known algorithm. The cost of tracing a single orbit under  $T_I$  is therefore not controlled by the length of the orbit but by the arithmetic of the values encountered.

This feature appears to be genuinely new in the study of maps of this structural type and we believe it deserves recognition independently of the main conjecture.

### 7.3 Open problems

We state the open problems arising from this work in precise terms.

#### Open problem 1 (central open problem)

The central open problem is Conjecture 2.4. Every integer in the range from 2 to 1000 enters  $C^*$  under  $T_I$  and no alternate cycle has been observed in any computation performed in this paper. Whether every integer in  $\mathbb{Z}^*$  enters  $C^*$  remains entirely open.

A logically prior and separate question is whether  $C^*$  is the unique cycle of  $T_I$  on  $\mathbb{Z}^*$ . A proof that  $C^*$  is the unique cycle would not by itself establish Conjecture 2.4 since it would leave open the possibility that some orbit grows without bound and never enters any cycle. A proof that every orbit is eventually periodic would, combined with the uniqueness of  $C^*$  establish the conjecture but would depend on the claim that no orbit diverges which our computations provide no proof of.

#### Open problem 2

A second open problem concerns the arithmetic of intermediate values arising along orbits of  $T_I$  beyond the tested range. The verification of Section 3 succeeded because every large intermediate value encountered happened to carry small prime factors accessible to trial division. Whether this continues to hold for starting values beyond 1000 is unknown. Whether there exists any starting value whose orbit under  $T_I$  produces an intermediate value with no prime factor below  $10^{50}$  is unknown. If such a value exists, the trajectory from that starting point would be unverifiable by any presently known method.

#### Open problem 3

The third open problem is Part *II* in its entirety. As documented in Section 6, we have no conjecture and no observed cycle. The most basic question of whether any orbit of  $T_{II}$  is eventually periodic is open.

#### Open problem 4

A fourth open problem concerns an analogue of Theorem 4.2 for Part *II*. Theorem 4.2 gives the exact number of reduction steps required to reach a prime from an odd composite under  $T_I$  namely  $\Omega(m) - 1$  steps. An analogue for  $T_{II}$  would require understanding how many smallest prime factor reductions are needed to reach a prime from an odd composite.

This count is not determined by  $\Omega(m)$  alone. It depends on the full recursive prime factor structure of the input because the smallest prime factor of the quotient at each step may differ from that at the previous step. Whether a closed form analogous to Theorem 4.2 exists for this quantity is open.

### **Open problem 5**

The fifth open problem is to make the heuristic of Section 7.1 rigorous. Turning the observation about Merikoski's bound into a proof of any statement about the orbits of  $T_I$  would require at minimum a lower bound on the largest prime factor of  $n^2 + 1$  that holds for almost all  $n$  in a density sense rather than for infinitely many  $n$  together with a propagation argument through the iteration of  $T_I$ . Neither ingredient is currently available. We regard a rigorous version of this heuristic as the most natural direction toward a proof of Conjecture 2.4.

## **8. Conclusion**

This paper has studied a map  $T$  on the integers defined by squaring and adding one at prime or even inputs and dividing by a designated prime factor at odd composite inputs. The map exists in two variants according to whether the designated factor is the largest or the smallest and the two variants have turned out to behave in ways that are structurally unlike each other in every respect that our methods can detect.

For Part *I*, the evidence is as coherent as one could reasonably ask of a result that falls short of a proof. Every integer from 2 to 1000 enters the same twelve element cycle  $C^*$  and no alternate cycle appeared in any computation performed in this paper. The two theorems of Section 4 account precisely and unconditionally for the behavior of negative integers and for the reduction chains that precede each excursion of the squaring branch. The conjecture that  $C^*$  is the unique attractor for all  $\mathbb{Z}^*$  is stated honestly as a conjecture, supported by computation and illuminated by partial results but unproven.

For Part *II*, the situation is qualitatively different and honesty requires saying so plainly at the close as well as in the body. No cycle, no convergence and no conjecture can be offered. The dynamics visible within the computationally feasible range show only rapid unbroken growth. The arithmetic of the map provides no evident mechanism by which this growth could be arrested and Part *II* is open in a sense that is not the ordinary openness of an unproven conjecture but the deeper openness of a problem whose behavior our current tools cannot characterize well enough to guess at.

Two features of the object studied here appear to be genuinely novel in the context of dynamics of Collatz type. The first is the quadratic growth branch. Iterated squaring roughly doubles digit length per step which is a qualitatively faster and considerably less studied growth regime than the linear case. This produces the dramatic amplitude fluctuations visible in the trajectories of Section 3 where a value in the thousands can rise to 96 digits and return to single digits within a handful of steps. The second novel feature is the intrinsic connection between verifying the conjecture and the independently hard problem of integer factorization. For every map previously studied in the Collatz literature, computing the next iterate requires time at most logarithmic in the size of the current value since the rule depends only on a residue. For  $T_I$ , the next iterate at an odd composite input requires complete integer factorization. A trajectory could in principle reach a value whose factorization is beyond any presently known method and the conjecture would then be unverifiable for that starting value not because the mathematics is hard but because a second independent problem that has resisted solution for decades stands in the way.

Whether this connection is merely an obstacle or also an opportunity is a question we leave open. The largest prime factor of  $n^2 + 1$  which governs the reduction step of  $T_I$  is the subject of deep and active research in analytic number theory with contributions from Hooley, Deshouillers, Iwaniec and Merikoski among others as reviewed in Section 7. That body of work has been applied only to static questions about individual values of the polynomial  $n^2 + 1$  and never to a dynamical system built around iteration of the associated map. Bringing those techniques to bear on the dynamics of  $T_I$  is in our view the most natural path toward any future progress on Conjecture 2.4 and the most concrete mathematical legacy of the present work is to have identified that path and explained why it is the right one.

## References

- [1] Conway, J. H. Unpredictable iterations. In Proceedings of the 1972 Number Theory Conference, University of Colorado, Boulder, Colorado, pp. 49-52, 1972. Reprinted in J. C. Lagarias (ed.), *The Ultimate Challenge: The  $3x+1$  Problem*, American Mathematical Society, Providence, RI, 2010.
- [2] Deshouillers, J.-M. and Iwaniec, H. On the greatest prime factor of  $n^2 + 1$ . *Annales de l'Institut Fourier (Grenoble)*, 32(4):1-11,1982.
- [3] Gonçalves, F., Greenfeld, R. and Madrid, J. Generalized Collatz maps with almost bounded orbits. *Indiana University Mathematics Journal*, 74(1):1-46,2025.  
<https://doi.org/10.1512/iumj.2025.74.60034>
- [4] Hooley, C. On the greatest prime factor of a quadratic polynomial. *Acta Mathematica*, 117:281-299, 1967. <https://doi.org/10.1007/BF02395047>
- [5] Lagarias, J. C. The  $3x+1$  problem and its generalizations. *American Mathematical Monthly*, 92(1):3-23, 1985.
- [6] Matthews, K. R. and Watts, A. M. A generalization of Hasse's generalization of the Syracuse algorithm. *Acta Arithmetica*, 43(2):167-175, 1984. <https://doi.org/10.4064/aa-43-2-167-175>
- [7] Matthews, K. R. and Watts, A. M. A Markov approach to the generalized Syracuse algorithm. *Acta Arithmetica*, 45(1):29-42, 1985.
- [8] Merikoski, J. On the largest prime factor of  $n^2 + 1$ . *Journal of the European Mathematical Society*, 25(4):1253-1284, 2023.
- [9] Tao, T. Almost all orbits of the Collatz map attain almost bounded values. *Forum of Mathematics, Pi*, 10, Paper No. e12, 56 pp., 2022. <https://doi.org/10.1017/fmp.2022.8>



