The nonlinear Schrödinger equation and de Broglie pilot wave

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Abstract

The Schrödinger equation with the logarithmic nonlinear term $-b(\ln|\Psi|^2)\Psi$ is derived by the natural generalization of the hydrodynamical model of quantum mechanics. The nonlinear term appears to be logically necessary because it enables explanation of the classical motion for the infinite mass limit. The article is the modified version of the articles by author (Pardy, 1993; 2001).

1 Introduction

Many authors have suggested that the quantum mechanics based on linear Schrödinger equation is only an approximation of some more nonlinear theory with the nonlinear Schrödinger equation. The motivation for considering the nonlinear equations is to get some more nonstandard solution in order to get the better understanding of the synergism of wave and particle. The article is the modified articles by author (Pardy, 1993; 2001).

The ambitious program to create nonlinear wave mechanics was elaborated by de Broglie (1960) and his group. Bialynicki-Birula and Mycielski (1976) considered the generalized Schrödinger equation with the additional term $F(|\Psi|^2)\Psi$ where F is some arbitrary function which they later specified to $-b(\ln|\Psi|^2)$, $b > 0$. The nonlinear term was selected by assuming the factorization of the wave function for the composed system.

The most attractive feature of the logarithmic non-linearity is the existence of the lower energy bound and validity of Planck's relation $E = \hbar\omega$. At the same time the Born interpretation of the wave function cannot be changed. In this theory the estimation of b was given by the relation $b < 4 \times 10^{-10} eV$ following from the agreement between theory and the observed $2S - 2P$ Lamb shift in hydrogen. This implies an upper bound to the electron soliton spatial width of 10 μm.

Shimony (1979) proposed an experiment which is based on idea that a phase shift occurs when an absorber is moved from one point to another along the path of one of the coherent split beams in a neutron interferometer. In case of the logarithmic non-linearity Shull at al. (1980) performed the experiment with a two-crystal interferometer. They searched for a phase shift when an attenuator was moved along the neutron propagation direction in one arm of the interferometer. A sheet of Cd, 0.086 mm thick, was used for the absorber.They obtained the upper bound on b of $3.4 \times 10^{-13} eV$ which is more than three orders of magnitude smaller than the bound estimated by Bialynicky-Birula and Mycielski (1976).

The best upper limit on b has been reported by Gähler, Klein and Zeilinger (1981) who has been searched for variations in the free space propagation of neutrons. 20 Å neutrons were diffracted from an abrupt highly absorbing knife edge at the object position. By comparing the experimental results with the solution to the ordinary Schrödinger equation they were able to get the limit $b < 3 \times 10^{-15} eV$, which corresponds to an electron soliton width of 3 mm. The similar results was obtained by the same group from diffraction a 100 μm boron wire.

To our knowledge the Mössbauer effect was not used to determine the constant b although this effect allows to measure energy losses smaller than $10^{-15} eV$. Similarly the Josephson effect has been not applied for the determination of the constant b.

We see that the constant b is very small, nevertheless we cannot it neglect a priori, because we do not know its role in the future physics. The corresponding analog is the Planck constant which is also very small, however, it plays the fundamental role in physics.

The goal of this article is to give the original derivation of the logarithmic non-linearity, to find the solution of the nonlinear Schrödinger equation of the one-dimensional case and to show that in the mass limit $m \to \infty$ we get exactly the delta-function behavior of the probability of finding the particle at point x. It means that there exists the classical motion of a particle with sufficient big mass. In other words, there is no quantum motion of planets, or, Moon. The non-linearity of the Schrödinger equation also solves the collapse of the wave function and the Schrödinger cat paradox. We will start from the hydrodynamical formulation of quantum mechanics. The mathematical generalization of the Euler hydrodynamical equations leads automatically to the logarithmic term with $b > 0$.

2 De Broglie pilot wave

New quantum theory (Sokolov et al., 1979) began with the discovery that light has particle properties in addition to wave properties (characterized by the wavelength λ and the frequency ω). The energy ϵ and momentum π of a quantum of light (photon) were established by Einstein as

$$\epsilon = \hbar\omega = h\nu \tag{1}$$

$$\mathbf{p} = \hbar\mathbf{k} = \frac{h}{\lambda}\mathbf{k^0}. \tag{2}$$

After Analysis of these equations, de Broglie suggested the generalization of these equation to ordinary particles and, in particular, electrons. Or, de Broglie assumed that the wave-particle duality is not an exclusive property of light, but it is also a internal property of electrons and all other particles.

In other words the free electron, whose relative energy E and momentum $\mathbf{p}$ is related to the velocity $\mathbf{v}$ by the equation

$$E = \frac{m_0 c^2}{\sqrt{1-\beta^2}}, \quad \mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}} \tag{3}$$

should also exhibit wave-like properties. The corresponding frequency and wave number were defined by equations similar to the Einstein relations:

$$E = \hbar\omega = h\nu, \quad \mathbf{p} = \hbar\mathbf{k}. \tag{4}$$

This hypothesis was made by de Broglie with a twofold purpose: 1) to provide a physical basis for the Bohr quantization; 2) to explain the first experiments on electron diffraction.

Consequently, de Broglie wavelength of the moving particles is

$$\lambda = \frac{2\pi}{k} = \frac{h}{p}. \tag{5}$$

The de Broglie relations (4) thus generalized Einstein equations (1), (2) derived from the photon theory. These now became equally applicable to the analysis of light in terms of its particles and of moving electrons in terms of their wave properties. It is worth noting that the dual character of particles and light disappears if Planck's constant $\hbar$ is allowed to go to zero (the correspondence principle).

Taking de Broglie's equations (4) we may describe the motion of free particles (along, say, the x axis) by the so-called wave function, which for this particular case is analogous to that of light and represents a plane wave:

$$\psi(x,t) = e^{-i\frac{i}{\hbar}(Et-px)}. \tag{6}$$

From the standpoint of Eq. (6), we can attempt to explain Bohr's postulate of stationary states as follows: the only allowed circular orbits are those which are divisible by an integral number of de Broglie wavelengths, that is,

$$\frac{2\pi r}{\lambda} = n, \tag{7}$$

where n is the natural number.

Indeed, the wave function is single-valued only when this condition is satisfied. Furthermore, since in the nonrelativistic case

$$\lambda = \frac{h}{m_0 v}. \tag{8}$$

Eq. (8) yields the Bohr condition for stationary states

$$p_\varphi = m_0 r v = \hbar n. \tag{9}$$

3 The heuristic derivation of the Schrödinger equation from de Broglie wave

The Schrödinger equation can be obtained simply from Broglie wave. The derivation is not rigorous but heuristic, since it is not, in general, possible to set up a new theory entirely on the basis of old postulates. We shall adopt a presentation which consists essentially of a reasonable generalization of the wave equation from classical electrodynamics or optics

$$\Delta\varphi(\mathbf{x},t) - \frac{1}{u^2}\frac{\partial^2\varphi(\mathbf{x},t)}{\partial t^2} = 0 \tag{10}$$

to the case of de Broglie waves. Here φ is a function describing a wave disturbance propagating with velocity u. If the wave is monochromatic, a solution to Eq. (10) may be sought in the form of de Broglie waves.

$$\varphi(\mathbf{x},t) = e^{-i\omega t}\varphi(\mathbf{x}). \tag{11}$$

After insertion of eq. (11) into eq. (10), we get

$$\Delta\varphi(\mathbf{x}) + \frac{\omega^2}{u^2}\varphi(\mathbf{x}) = 0. \tag{12}$$

In this equation, we can use a single parameter in place of the two parameters ω and u, namely, the wavelength

$$\lambda = \frac{2\pi u}{\omega}. \tag{13}$$

We then have

$$\Delta\varphi(\mathbf{x}) + \frac{4\pi^2}{\lambda^2}\varphi(\mathbf{x}) = 0. \tag{14}$$

From this general equation, we can obtain a wave equation describing the wave motion of electrons by substituting the de Broglie wavelength

$$\lambda = \frac{h}{m_0 v} = \frac{2\pi h}{p}. \tag{15}$$

Using the law of conservation of energy

$$\frac{p^2}{2m_0} + V(\mathbf{x}) = E = const, \tag{16}$$

we have

$$\frac{4\pi^2}{\lambda^2} = \frac{2m_0}{\hbar^2}\left[E - V(\mathbf{x})\right]. \tag{17}$$

Substituting the last expression into eq. (14), we obtain the time-independent (or stationary) Schrödinger equation

$$\Delta\psi(\mathbf{x}) + \frac{2m_0}{\hbar^2}\left[E - V(\mathbf{x})\right]\psi(\mathbf{x}) = 0. \tag{18}$$

Once we have found the space-dependent part $\psi(\mathbf{x})$ of the wave function from (14), we can use Eq. (11), which is valid for all monochromatic waves, to obtain the complete wave function, which depends on both (spatial and time) coordinates. Substituting $\omega = \frac{E}{\hbar}$ we have

$$\varphi(\mathbf{x},t) = e^{-i\frac{E}{\hbar}t}\varphi(\mathbf{x}). \tag{19}$$

The functions ψ which describe the behavior of a particle may be statistically interpreted by means of the Schrödinger theory. In particular, the quantity $\psi^*\psi$, which plays the role of a distribution function, represents the probability density, or probability of finding the particle at any particular region in space. If the probability density differs from zero only in some arbitrarily large, but finite, region of space, Ω, it is accurate enough to say that the particle is located somewhere in this region. In other words, the probability of detecting the particle in the region is unity. Mathematically, this can be expressed by the relationship

$$\int_{\Omega} \psi^* \psi dV = 1, \tag{20}$$

which is called the normalization condition.

Let us remark that the probabilistic interpretation of the wave function is involved in de Broglie pilot wave because the existence of the pilot wave is fundamental and logical necessity of foundation of physics. It follows from the notion called inertia. The explanation is as follows.

Elementary particles takes its inertia from vacuum. Without vacuum is no inertia. Newton's first law is formulated by the following words. A body that is not under external forces does not change its momentum. After application of the Newton law $F = ma$, the force acting on the elementary particle generates the wave called wave function which is de Broglie pilot wave, which is the wave state of vacuum. So, de Broglie wave being inborn in vacuum controls the motion of elementary particle.

In other words. Inertia is inborn in the de Broglie pilot wave of vacuum and de Broglie pilot wave is wave described by the solution of the Schrödinger equation , or, the Dirac equation for spin 1/2. So, to be pedagogically clear - de Broglie pilot wave is the necessary starting point of all future textbooks on classical mechanics. First, de Broglie pilot wave an then, Schrödinger equation, and then classical mechanics as the limit of quantum mechanics by WKB method (Pardy, 1973; 2021).

4 The heuristic derivation of the nonlinear Schrödinger equation

According so called Dirac heuristic principle (Pais, 1986) it is useful to postulate some mathematical requirement in order to get the true information about nature. While the mathematical assumption is a priori, the consequences have the physical interpretation, or, in other words they are physically meaningful. In derivation of the logarithmic nonlinearity we use this Dirac method.

According to Madelung (1926), Bohm and Vigier (1954), Wilhelm (1970), Rosen (1974) and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n} e^{\frac{i}{\hbar} S}, \tag{21}$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \to 0$. The mass density is defined by relation $\varrho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (21) into the original Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi, \tag{22}$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}} \tag{23}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0 \tag{24}$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \tag{25}$$

Equation (23) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}, \tag{26}$$

which is called the quantum Bohm potential and equation (24) is the continuity equation.

After application of operator $\bigtriangledown$ on eq. (23), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla(V + V_q). \tag{27}$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\varrho^{-1}\nabla p$ where p is hydrodynamical pressure. We use here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the eq. (27) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla(V + V_q) - \nabla F, \tag{28}$$

where

$$\nabla F = \frac{1}{n}\nabla p. \tag{29}$$

The equation (28) can be obtained by the Madelung procedure from the following extended Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + F\Psi \tag{30}$$

on the assumption that it is possible to determine F in term of the wave function. From the vector analysis follows that the necessary condition of the existence of F as the solution of the eq. (29) is rot grad F = 0, or,

$$\mathrm{rot}(n^{-1}\nabla p) = 0, \tag{31}$$

which enables to take the linear solution in the form

$$p = -bn = -b|\Psi|^2, \tag{32}$$

where b is some arbitrary constant. We do not consider the more general solution of eq. (31). Then, from eq. (29) i.e. grad $F = \mathbf{a}$ we have:

$$F = \int a_i \, dx_i = -b \int \frac{1}{n} \, dn = -b \ln |\Psi|^2, \tag{33}$$

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which corresponds to the complete Euler equation (28) in the following form:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi - b(\ln |\Psi|^2)\Psi. \tag{34}$$

Let us remark that it was shown by some authors (Lemos, 1983; Castro et al., 2006) and others that the Schrödinger equation with a logarithmic nonlinearity proposed originally by Birula and Mycielski can be derived within the context of stochastic mechanics. The expression for the energy was obtained by means of a stochastic variational principle. The stochastic motion is equivalent to the brownian motion of a particle. However, the derivation is heuristic one and we know that Brownian motion is in no case equivalent to the quantum motion of the elementary particle described by the Schrödinger equation. The statement follows also from the Langevin theory of the Brownian motion. Now, let us approach the solving eq. (34).

5 The soliton-wave solution of the nonlinear Schrödinger equation

Let be $c, (\mathrm{Im}\ c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x,t) = cG(x - vt)e^{ikx - i\omega t} \tag{35}$$

into the one-dimensional equation (34) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \tag{36}$$

and for function G we get the following nonlinear equation (the symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \tag{37}$$

where

$$A = \frac{2m}{\hbar}\omega - k^2 + \frac{2m}{\hbar^2} b \ln c^2 \tag{38}$$

$$B = \frac{4mb}{\hbar^2}. \tag{39}$$

After multiplication of eq. (37) by G' we get:

$$\frac{1}{2}\Big[G'^2\Big]' + \frac{A}{2}\Big[G^2\Big]' + B\left[\frac{G^2}{2}\ln G - \frac{G^2}{4}\right]' = 0, \tag{40}$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2}G^2 + const. \tag{41}$$

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B\ln G}} = d\xi, \tag{42}$$

where

$$a = \frac{B}{2} - A. \tag{43}$$

Eq. (42) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi + d)^2}, \tag{44}$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x,t) = ce^{\frac{a}{B}} e^{-\frac{B}{4}(x - vt + d)^2} e^{ikx - i\omega t}. \tag{45}$$

6 Normalization and the classical limit

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^*\Psi\, dx = 1. \tag{46}$$

Using the Gauss integral

$$\int_0^{\infty} e^{-\lambda^2 x^2}\, dx = \frac{\sqrt{\pi}}{2\lambda}, \tag{47}$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \tag{48}$$

and the density probability $\Psi^*\Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \tag{49}$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \to \infty$ is just the Dirac δ-function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behavior of a particle cannot be obtained in the standard quantum mechanics because the plane wave

$$e^{ikx-i\omega t} \tag{50}$$

corresponds to the free particle with no possibility of localization for $m \to \infty$.

Let us still remark that coefficient c^2 is real and positive number because it is a result of the solution of eq. (48) which can be transformed into equation ($x = c^2$)

$$x^{1-r} = const. \tag{51}$$

7 The principle of superposition

The principle of superposition is in the nonlinear theory broken. Namely, if φ_1 and φ_2 are two different solution of the nonlinear Schrödinger equation then the linear combination $\varphi = a\varphi_1 + b\varphi_2$ where a and b are the arbitrary constants is not the solution of the same equation because of its non-linearity.

In other words the original principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the resolution of the Schrödinger cat paradox (Glauber, 1986).

The proof of the validity, or, non-validity of the principle of superposition can be verified in the two-slit experiment. Let φ_1 is the wave function for an electron going through slit 1. Then the density of probability distribution is $\varrho_1 = \varphi_1^*\varphi_1$. If φ_2 is the wave function for an electron going through slit 2, then the density of probability distribution is $\varrho_2 = \varphi_2^*\varphi_2$. According to the superposition principle, the wave function for an electron going through open slit 1 and slit 2 is $\varphi_{12} = \varphi_1 + \varphi_2$ and the probability distribution for an electron going through slit 1 and at the same time slit 2 is $\varrho_{12} = |\varphi_1 + \varphi_2|^2$. And this formula can be experimentally verified.

In case of the broken superposition principle the non-validity of the formula

$$\varrho_{12} = |\varphi_1 + \varphi_2|^2, \tag{52}$$

can be observed in experiment performed by the well-educated experimenters.

8 Discussion

We have seen that the introduction of the logarithmic non-linearity in the Schrödinger equation was logically supported by the fact that the nonlinear Schrödinger equation gives results which are physically meaningful. We have obtained the correct mass limit of the wave function.

The further strong point of the nonlinear Schrödinger equation (34) is the result (26) which is equivalent to the famous de Broglie relation $\lambda = \frac{h}{p}$, because of $\lambda = 2\pi/k = 2\pi(\hbar/mv) = 2\pi(h/2\pi)(1/p)$ and it means that de Broglie relation is involved in this form of the nonlinear quantum mechanics.

The nonlinear equation (34) has also the normalized plane-wave solution

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{ikx-i\omega t}. \tag{53}$$

After insertion of eq. (53) into eq. (34), we get the following dispersion relation:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + b\ln(2\pi), \tag{54}$$

from which the relations follows:

$$\hbar\omega = b\ln(2\pi); \quad k = 0 \tag{55}$$

and

$$k = \pm i\sqrt{\frac{2m}{\hbar^2} b\ln(2\pi)}; \quad \omega = 0. \tag{56}$$

It is no easy to give the physical interpretation of eqs. (55) and (56) and so we cannot say that the plane-solution of the nonlinear Schrödinger equation is physically meaningful. Only the soliton-wave solution of the nonlinear Schrödinger equation can be taken as relevant. Only this solution is suitable for the physical verification. The possible new tests of the nonlinear quantum mechanics are discussed in the author article (Pardy, 1994).

The generalization to the motion of particle in the electromagnetic field with potentials $\varphi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ can be performed by the standard transformation

$$\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - \left(\frac{e}{c}\right)\mathbf{A}(\mathbf{x}, t) \tag{57}$$

and adding the scalar potential energy $\varphi(x, t)$ in the Schrödinger equation for the free particles. According to (Bialynicky-Birula et al., 1976) the solution of the equation in this case can be taken in the form

$$\Psi(\mathbf{x}, t) = e^{\frac{i}{\hbar}S} G(\mathbf{x} - \mathbf{u}(t)), \tag{58}$$

where function G is necessary to determine. In the similar form the problem was yet solved (Barut, 1990).

Kamesberger and Zeilinger (1998) have given the numerical solution of the original Schrödinger equation and this equation with the nonlinear term $-b(\ln|\Psi|^2)\Psi$ in order to visualize the spreading of the diffraction waves. When comparing the evolution patterns of the nonlinear case with the linear one, one notices that the maxima are more pronounced in the nonlinear solution. It can be understood as a mechanism compressing the wave maxima spatially. In the quantitative comparison of the both cases this enhancement of the maxima and minima can be seen very clearly.

Although we have given reasons for the introducing of the nonlinear Schrödinger equation it is obvious that only the crucial experiments can establish the physical and not only logical necessity of such equation. In case that the nonlinear Schrödinger equation will be confirmed by experiment, then it can be expected that it will influence other parts of theoretical physics.

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