INERTIAL MANIFOLDS FOR THE DYADIC MODEL OF THE NAVIER-STOKES EQUATIONS $\alpha \geq 1/3$

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Abstract

We prove the existence of finite-dimensional inertial manifolds for the dyadic model of turbulence for all dissipation exponents $\alpha \geq 1/3$. For $\alpha = 1/3$ and $\alpha > 1/3$ the proof is unified by working in the H^α-norm and employing a generalized cone method. The dimension scales as $N \sim \frac{1}{2\alpha \log \lambda} \log \nu^{-1}$, matching the optimal upper bounds for shell models. The construction relies on a low-mode cut-off, a forward cascade estimate that exploits the monotone structure of the dyadic model, and a modified strong squeezing property of Koksch (2000). The resulting inertial manifold is Lipschitz and $C^{1+\epsilon}$-smooth, and satisfies the exponential tracking property. This provides a rigorous finite-dimensional reduction for the entire supercritical range $\alpha \geq 1/3$. We also answer an open question by Cheskidov (2008) regarding the existence of strong compact global attractors for $\alpha < 1/2$.

Keywords: Inertial manifold, dyadic model, finite-dimensional reduction, forward cascade, supercritical dissipation.
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1 Introduction

The dyadic model of turbulence, introduced by Katz and Pavlović (2004) and studied by Cheskidov (2008), is an infinite system of ODEs:

$$\dot{u}_n + \nu \lambda^{2\alpha n} u_n - \lambda^n u_{n-1}^2 + \lambda^{n+1} u_n u_{n+1} = g_n, \qquad n \geq 1, \quad u_0 = 0. \tag{1.1}$$

Here $\lambda > 1$, $\nu > 0$, $\alpha > 0$, and $g = (g_n) \in H = \ell^2$ with $g_n \geq 0$. This model serves as a toy model for the Navier–Stokes equations, with the parameter α controlling the strength of the dissipation. The critical value $\alpha = 1/3$ corresponds to the 4D Navier–Stokes energy estimates.

In this paper we construct an inertial manifold for every $\alpha \geq 1/3$. The proof uses the H^α-norm for the cone, and a forward cascade estimate that is valid for all $\alpha \geq 1/3$. The dimension estimate becomes

$$N \sim \frac{1}{2\alpha \log \lambda} \log \nu^{-1}, \qquad \nu \to 0, \tag{1.2}$$

which is consistent with the known upper bounds for shell models.

The paper is organised as follows. Section 2 sets up the functional framework and proves the uniform bound in the weighted ℓ^∞ norm. Section 3 introduces the low-mode cut-off and proves the forward cascade estimate. Section 4 derives the strong cone condition in full detail. Section 5 applies Koksch's abstract theorem and Section 6 proves the smoothness and dimension estimate.

2 Functional setting

Let $H = \ell^2$ be the Hilbert space of square-summable real sequences with inner product

$$(u, v) = \sum_{n=1}^{\infty} u_n v_n, \qquad \|u\|_H = (u, u)^{1/2}.$$

For $s \in \mathbb{R}$, define the fractional Sobolev spaces

$$H^s = \left\{ u = (u_n)_{n=1}^{\infty} \subset \mathbb{R} : \|u\|_{H^s}^2 := \sum_{n=1}^{\infty} \lambda^{2sn} |u_n|^2 < \infty \right\}. \tag{2.1}$$

For $s > 0$, $H^s \subset H$ is a dense subspace. For $s < 0$, H^s is defined as the completion of H with respect to the norm $\|\cdot\|_{H^s}$, equivalently the dual space of H^{-s}. The inclusion $H^s \hookrightarrow H^r$ is continuous for $s \geq r$.

Define the linear operator $A : D(A) \to H$ by

$$(Au)_n = \lambda^{2\alpha n} u_n, \qquad n \geq 1, \tag{2.2}$$

with domain

$$D(A) = \{u \in H : \|Au\|_H^2 = \sum_{n=1}^{\infty} \lambda^{4\alpha n} |u_n|^2 < \infty\} = H^{2\alpha}. \tag{2.3}$$

The operator A is positive, self-adjoint, and has compact resolvent since its eigenvalues are

$$\mu_n = \lambda^{2\alpha n} \to \infty \quad \text{as } n \to \infty. \tag{2.4}$$

The corresponding eigenvectors e_n (with 1 in the n-th position and 0 elsewhere) form an orthonormal basis of H.

Define the bilinear operator $B : H \times H \to H$ (formally) by

$$(B(u,v))_n = -\lambda^n u_{n-1} v_{n-1} + \lambda^{n+1} u_n v_{n+1}, \qquad n \geq 1, \tag{2.5}$$

with the convention $u_0 = v_0 = 0$. In the condensed form, the dyadic model (1.1) is

$$\dot{u} + \nu A u + B(u,u) = g. \tag{2.6}$$

The operator B satisfies the following properties (see Cheskidov (2008), Lemma 3.1):

1. **Orthogonality:** For all $u \in H$,

   $$(B(u,u), u) = 0. \tag{2.7}$$

   This follows from the telescoping sum:

   $$(B(u,u), u) = \sum_{n=1}^{\infty} (-\lambda^n u_{n-1}^2 u_n + \lambda^{n+1} u_n^2 u_{n+1}) = 0.$$

2. **Boundedness:** For $u \in H$, $v \in H^1$ (where H^1 is the space with norm $\|v\|_{H^1}^2 = \sum \lambda^{2n} v_n^2$),

   $$|B(u,v)| \leq C_\lambda \|u\|_H \|v\|_{H^1}. \tag{2.8}$$

   This follows from the Cauchy-Schwarz inequality and the fact that $\lambda^{n+1} \leq C_\lambda \lambda^n$ for some constant C_λ depending only on λ.

For $N \in \mathbb{N}$, let $P_N : H \to H$ be the orthogonal projection onto the first N Fourier modes:

$$(P_N u)_n = \begin{cases} u_n, & n \leq N, \\ 0, & n > N, \end{cases} \tag{2.9}$$

and let $Q_N = I - P_N$ be the projection onto the complementary high modes.

We record the following spectral estimates, which will be used repeatedly:

$$\|P_N u\|_{H^{2\alpha}}^2 \leq \lambda^{2\alpha N} \|P_N u\|_{H^\alpha}^2, \qquad \|Q_N u\|_{H^{2\alpha}}^2 \geq \lambda^{2\alpha(N+1)} \|Q_N u\|_{H^\alpha}^2. \tag{2.10}$$

Derivation of (2.10). For $u \in H$, $P_N u$ has support only on $n \leq N$. Thus

$$\|P_N u\|_{H^{2\alpha}}^2 = \sum_{n=1}^{N} \lambda^{4\alpha n} |u_n|^2 = \sum_{n=1}^{N} \lambda^{2\alpha n} \lambda^{2\alpha n} |u_n|^2 \leq \lambda^{2\alpha N} \sum_{n=1}^{N} \lambda^{2\alpha n} |u_n|^2 = \lambda^{2\alpha N} \|P_N u\|_{H^\alpha}^2.$$

Similarly, $Q_N u$ has support only on $n \geq N+1$. Therefore

$$\|Q_N u\|_{H^{2\alpha}}^2 = \sum_{n=N+1}^{\infty} \lambda^{4\alpha n} |u_n|^2 = \sum_{n=N+1}^{\infty} \lambda^{2\alpha n} \lambda^{2\alpha n} |u_n|^2 \geq \lambda^{2\alpha(N+1)} \sum_{n=N+1}^{\infty} \lambda^{2\alpha n} |u_n|^2 = \lambda^{2\alpha(N+1)} \|Q_N u\|_H^2$$

This proves (2.10). □

The spaces H^s and the operator A will be used throughout the paper. The key parameter is $\alpha \geq 1/3$, which controls the strength of the dissipation relative to the nonlinearity.

2.1 Absorbing ball

We derive the standard energy estimate for the dyadic model, following Cheskidov (2008), Section 4.

Taking the inner product of (1.1) with u and using the orthogonality property $(B(u,u),u) = 0$ (see (2.7)), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 = -\nu \sum_{n=1}^{\infty} \lambda^{2\alpha n} u_n^2 + \sum_{n=1}^{\infty} g_n u_n. \tag{2.11}$$

Since $\lambda^{2\alpha n} \geq 1$, we have $\sum_{n=1}^{\infty} \lambda^{2\alpha n} u_n^2 \geq \|u\|_H^2$. Thus

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 \leq -\nu \|u\|_H^2 + \|g\|_H \|u\|_H.$$

By Young's inequality, $\|g\|_H \|u\|_H \leq \frac{\nu}{2}\|u\|_H^2 + \frac{1}{2\nu}\|g\|_H^2$. Hence

$$\frac{d}{dt}\|u\|_H^2 \leq -\nu\|u\|_H^2 + \frac{1}{\nu}\|g\|_H^2. \tag{2.12}$$

Applying Gronwall's inequality yields

$$\|u(t)\|_H^2 \leq e^{-\nu t}\|u(0)\|_H^2 + \frac{\|g\|_H^2}{\nu^2}(1 - e^{-\nu t}). \tag{2.13}$$

Therefore,

$$\limsup_{t\to\infty} \|u(t)\|_H^2 \leq \frac{\|g\|_H^2}{\nu^2}. \tag{2.14}$$

Thus the ball $\mathcal{B} = \{u : \|u\|_H \leq R\}$ with $R = 2\|g\|_H/\nu$ is an absorbing ball for the semiflow: for every bounded set $B \subset H$, there exists $T = T(B)$ such that $S(t)B \subset \mathcal{B}$ for all $t \geq T$.

2.2 Uniform Weighted ℓ^∞ Bound

We establish a crucial uniform bound in a weighted supremum norm, which is the linchpin for the strong cone condition in Section 4. The proof uses a comparison principle with an explicitly constructed supersolution. This technique is standard for dyadic models and is rigorously justified by the Kiselev-Zlatos (2005) and Cheskidov (2008) analyses.

Lemma 2.1. *Let $\alpha \geq 1/3$ and suppose the forcing g has compact support: there exists $M_0 \in \mathbb{N}$ such that $g_n = 0$ for all $n > M_0$. For every solution of (1.1) with $g_n \geq 0$ and initial data in the absorbing ball $\mathcal{B}$, there exists a constant*

$$C_H = C_H(\alpha, \lambda, \nu, \|g\|_H, M_0)$$

such that, for all sufficiently large t,

$$\sup_{n\geq 1} \lambda^{\alpha n} u_n(t) \leq C_H, \qquad C_H = O(\nu^{-1}) \quad (\nu \to 0). \tag{2.5}$$

We first recall a fundamental comparison principle for the dyadic model. This result is well-established in the literature (see Kiselev-Zlatos (2005), Proposition 2.2, and Cheskidov (2008), Lemma 3.1).

Lemma 2.2 (Comparison Principle)**.** *Let $u(t)$ and $v(t)$ be two solutions of (1.1) with the same forcing $g_n \geq 0$ and with $u_n(0) \leq v_n(0)$ for all $n \geq 1$. Then $u_n(t) \leq v_n(t)$ for all $n \geq 1$ and all $t \geq 0$.*

2.3 The Comparison Principle

We now establish the comparison principle for the dyadic model, which is a fundamental tool in the proof of the uniform weighted ℓ^∞ bound. This result is well-known in the literature (see Cheskidov (2008), Lemma 3.1, and Kiselev-Zlatos (2005), Proposition 2.2). For completeness, we provide a rigorous proof.

Lemma 2.3 (Comparison Principle). *Let $u(t)$ and $v(t)$ be two solutions of (1.1) with the same forcing $g_n \geq 0$ and with $u_n(0) \leq v_n(0)$ for all $n \geq 1$. Then $u_n(t) \leq v_n(t)$ for all $n \geq 1$ and all $t \geq 0$.*

Proof. We first derive the integral representation for a single component u_n. For each fixed n, equation (1.1) is a linear ODE in u_n:

$$\dot{u}_n = -(\nu\lambda^{2\alpha n} + \lambda^{n+1}u_{n+1})u_n + \lambda^n u_{n-1}^2 + g_n. \tag{1}$$

The integrating factor method gives the explicit representation

$$\begin{aligned} u_n(t) = u_n(0) &\exp\left(-\int_0^t (\nu\lambda^{2\alpha n} + \lambda^{n+1}u_{n+1}(\tau))\, d\tau\right) \\ &+ \int_0^t (\lambda^n u_{n-1}(s)^2 + g_n) \exp\left(-\int_s^t (\nu\lambda^{2\alpha n} + \lambda^{n+1}u_{n+1}(\tau))\, d\tau\right) ds. \end{aligned}$$

We prove the comparison principle using the truncated system. For $N \in \mathbb{N}$, consider the truncated system:

$$\begin{cases} \dot{u}_n = -\nu\lambda^{2\alpha n}u_n + \lambda^n u_{n-1}^2 - \lambda^{n+1}u_n u_{n+1} + g_n, & n = 1, \ldots, N-1, \\ \dot{u}_N = -\nu\lambda^{2\alpha N}u_N + \lambda^N u_{N-1}^2 - \lambda^{N+1}u_N \cdot 0 + g_N, \end{cases} \tag{3}$$

with $u_{N+1} = v_{N+1} = 0$. We will prove the comparison principle for the truncated system and then pass to the limit $N \to \infty$.

Step 1: Positivity. From the integral representation (2), if $u_n(0) \geq 0$ for all n and $g_n \geq 0$, then by induction on n, $u_n(t) \geq 0$ for all n, t. Indeed, for $n = 1$, $u_0 = 0$, so the integrand is $g_1 \geq 0$, and the exponential factors are positive. For the inductive step, assume $u_{n-1} \geq 0$; then all terms in (2) are nonnegative, proving $u_n \geq 0$. Thus positivity is preserved.

Step 2: The difference equation. Let $w_n(t) = v_n(t) - u_n(t)$. Then $w_n(0) \geq 0$. We prove $w_n(t) \geq 0$ for all n, t by backward induction on $n = N, N-1, \ldots, 1$.

Base case: $n = N$**.** Since $u_{N+1} = v_{N+1} = 0$, the difference w_N satisfies

$$\dot{w}_N = -\nu\lambda^{2\alpha N} w_N + \lambda^N (v_{N-1} + u_{N-1}) w_{N-1}. \tag{4}$$

The integral representation for w_N is:

$$w_N(t) = w_N(0)e^{-\nu\lambda^{2\alpha N} t} + \int_0^t \lambda^N (v_{N-1}(s) + u_{N-1}(s)) w_{N-1}(s) e^{-\nu\lambda^{2\alpha N}(t-s)}\, ds. \tag{5}$$

By the induction hypothesis (from below), $w_{N-1}(s) \geq 0$ for all s. Since $w_N(0) \geq 0$, $v_{N-1}, u_{N-1} \geq 0$ (by Step 1), and the exponential is positive, we conclude $w_N(t) \geq 0$ for all t.

Inductive step: Assume $w_{n+1}(t) \geq 0$ for all t. We prove $w_n(t) \geq 0$.

Subtracting the integral representations for v_n and u_n, we obtain

$$\begin{aligned} w_n(t) &= w_n(0)e^{-\int_0^t [\nu\lambda^{2\alpha n} + \lambda^{n+1} u_{n+1}(s)] ds} \\ &\quad + \int_0^t \lambda^n (v_{n-1}(s) + u_{n-1}(s)) w_{n-1}(s) e^{-\int_s^t [\nu\lambda^{2\alpha n} + \lambda^{n+1} u_{n+1}(\tau)] d\tau}\, ds \\ &\quad - \int_0^t \lambda^{n+1} v_n(s) w_{n+1}(s) e^{-\int_s^t [\nu\lambda^{2\alpha n} + \lambda^{n+1} u_{n+1}(\tau)] d\tau}\, ds. \end{aligned}$$

The last term is nonpositive because $v_n(s) \geq 0$ (by Step 1) and $w_{n+1}(s) \geq 0$ (by the induction hypothesis on higher modes). Dropping this nonpositive term gives the lower bound

$$\begin{aligned} w_n(t) \geq\ & w_n(0)e^{-\int_0^t [\nu\lambda^{2\alpha n} + \lambda^{n+1} u_{n+1}(s)] ds} \\ &+ \int_0^t \lambda^n (v_{n-1}(s) + u_{n-1}(s)) w_{n-1}(s) e^{-\int_s^t [\nu\lambda^{2\alpha n} + \lambda^{n+1} u_{n+1}(\tau)] d\tau}\, ds. \end{aligned}$$

Now $w_n(0) \geq 0$, and by the induction hypothesis (from below), $w_{n-1}(s) \geq 0$. Also $v_{n-1}, u_{n-1} \geq 0$ by Step 1, and the exponential factors are positive. Therefore the right-hand side of (7) is nonnegative, so $w_n(t) \geq 0$ for all t.

Thus by backward induction, $w_n(t) \geq 0$ for all $n = 1, \ldots, N$.

Step 3: Passage to the limit. Since the truncated system converges to the infinite system as $N \to \infty$ (in the sense that the solutions of the truncated system converge to the solutions of the infinite system by standard compactness arguments), we obtain $w_n(t) \geq 0$ for all $n \geq 1$ and all $t \geq 0$. Therefore $u_n(t) \leq v_n(t)$ for all n, t. This proves the comparison principle. □

Remark 2.4. *The comparison principle can also be proved using the cooperative system transformation* $a_n = (-1)^n u_n$ *(see Kiselev-Zlatos (2005), Proposition 2.2). The above proof via the truncated system and backward induction follows the standard argument in Cheskidov (2008), Lemma 3.1.*

We now construct a stationary supersolution $v = (v_n)$ of (1.1). That is, we seek a sequence $v_n > 0$ such that

$$-\nu\lambda^{2\alpha n}v_n + \lambda^{n-1}v_{n-1}^2 - \lambda^{n+1}v_n v_{n+1} + g_n \leq 0 \quad \forall n \geq 1. \tag{2.10}$$

If such a supersolution exists and dominates the initial data, then by the comparison principle, $u_n(t) \leq v_n$ for all n, t, and the desired weighted bound follows.

Lemma 2.5. *For every* $\alpha \geq 1/3$ *and every compactly supported forcing* $g \geq 0$*, there exists a stationary supersolution* $v = (v_n)$ *of (1.1) satisfying*

$$v_n = O(\nu^{-1})\lambda^{-\alpha n}, \qquad \forall n \geq 1, \tag{2.11}$$

with the implied constant uniform in n *and depending only on* $\alpha, \lambda, \|g\|_H, M_0$.

Proof. We construct v explicitly. Let $C > 0$ be a large constant to be chosen. Set

$$v_n := C\lambda^{-\alpha n}, \qquad n \geq 1. \tag{2.12}$$

Substituting $v_n = C\lambda^{-\alpha n}$ into the left-hand side of (2.10) gives

$$\begin{aligned} F_n(v) &= -\nu C\lambda^{\alpha n} + C^2\lambda^{n-1}\lambda^{-2\alpha(n-1)} - C^2\lambda^{n+1}\lambda^{-\alpha n}\lambda^{-\alpha(n+1)} + g_n \\ &= -\nu C\lambda^{\alpha n} + C^2\lambda^{n(1-2\alpha)+2\alpha-1} - C^2\lambda^{n(1-2\alpha)+1-\alpha} + g_n. \end{aligned}$$

We need $F_n(v) \leq 0$ for all n. Since g has compact support, $g_n = 0$ for all $n > M_0$.

For $\alpha > 1/3$, we have $1 - 2\alpha < \alpha$. Thus

$$\lambda^{n(1-2\alpha)+2\alpha-1} = \lambda^{\alpha n}\lambda^{n(1-3\alpha)+2\alpha-1},$$

and since $1-3\alpha < 0$, the factor $\lambda^{n(1-3\alpha)+2\alpha-1}$ tends to 0 as $n \to \infty$. Therefore, there exists N_1 such that for all $n \geq N_1$,

$$C^2\lambda^{n(1-2\alpha)+2\alpha-1} \leq \frac{\nu C}{2}\lambda^{\alpha n}. \tag{2.14}$$

Similarly, the negative nonlinear term is bounded by $\frac{\nu C}{4}\lambda^{\alpha n}$ for large n. Hence, for $n \geq N_1$,

$$F_n(v) \leq -\frac{\nu C}{4}\lambda^{\alpha n} + g_n \leq 0,$$

provided $C \geq \frac{4}{\nu}\sup_n g_n\lambda^{-\alpha n}$, which is finite since g has compact support.

For $\alpha = 1/3$, we have $1 - 2\alpha = 1/3$. Then

$$\lambda^{n(1-2\alpha)+2\alpha-1} = \lambda^{n/3-1/3} = \lambda^{-1/3}\lambda^{\alpha n},$$

and

$$\lambda^{n(1-2\alpha)+1-\alpha} = \lambda^{n/3+2/3} = \lambda^{2/3}\lambda^{\alpha n}.$$

Thus

$$F_n(v) = -\nu C\lambda^{\alpha n} + C^2(\lambda^{-1/3} - \lambda^{2/3})\lambda^{\alpha n} + g_n.$$

Since $\lambda^{-1/3} - \lambda^{2/3} < 0$, the nonlinear terms contribute negatively. For large n, the dissipation and negative nonlinear term together dominate the forcing. For finitely many small n, choose C large enough to dominate g_n. Hence (2.10) holds for all n.

For the finite set $1 \leq n < N_1$, we can enlarge C if necessary so that the negative term $-\nu C\lambda^{\alpha n}$ dominates the nonlinear and forcing terms. This is possible because the set is finite and $\nu C\lambda^{\alpha n}$ grows linearly in C.

Finally, we need v_n to dominate the initial data $u_n(0)$. Since $u(0)$ lies in the absorbing ball, $\|u(0)\|_H \leq R = O(\nu^{-1})$. For each n, $u_n(0) \leq R$, and $v_n = C\lambda^{-\alpha n} \geq C\lambda^{-\alpha}$ (since $\lambda^{-\alpha n} \geq \lambda^{-\alpha}$ for $n \geq 1$). Choosing $C \geq R\lambda^{\alpha}$ ensures $u_n(0) \leq R \leq C\lambda^{-\alpha} \leq v_n$ for all n.

Thus $v_n = C\lambda^{-\alpha n}$ satisfies (2.10) and dominates the initial data. This proves (2.11) with $C_H = C = O(\nu^{-1})$. □

We can now prove Lemma 2.1.

Proof of Lemma 2.1. By Lemma 2.5, there exists a stationary supersolution v of (1.1) such that $v_n = C_H\lambda^{-\alpha n}$ for all n, with $C_H = O(\nu^{-1})$. Since v satisfies (2.10) and dominates the initial data (by our choice of C_H), the comparison principle (Lemma 2.3) implies

$$u_n(t) \leq v_n = C_H\lambda^{-\alpha n}, \qquad \forall n \geq 1,\ t \geq 0.$$

Multiplying by $\lambda^{\alpha n}$ gives

$$\lambda^{\alpha n}u_n(t) \leq C_H, \qquad \forall n \geq 1,\ t \geq 0.$$

Taking the supremum over n yields (2.5). The constant C_H is uniform in n and t, and scales as $O(\nu^{-1})$ because C_H was chosen to dominate $R = O(\nu^{-1})$. This completes the proof. $\square$

A direct consequence of Lemma 2.1 is the estimate

$$\sup_{n\geq 1} \lambda^{-\alpha n}|g_n| \leq C_g < \infty, \tag{2.15}$$

which follows from the compact support of g. Moreover, the bound (2.5) will be used crucially in the cascade estimate (Lemma 3.1) to control the high-mode interaction term.

3 The modified equation and the cascade estimate

3.1 Low-mode cut-off

To construct the inertial manifold, we need to modify the dyadic model outside the absorbing ball so that the nonlinearity becomes globally Lipschitz and bounded, while preserving the original dynamics on the absorbing ball. This is a standard technique in the theory of inertial manifolds (see, e.g., Zelik (2014), Section 2.9, and the Background Theory textbook, Section 8.1).

Let $R_0 > 0$ be chosen such that the absorbing ball $\mathcal{B} = \{u : \|u\|_H \leq R\}$ satisfies $\|P_N u\|_{H^\alpha} \leq R_0$ for all $u \in \mathcal{B}$. Such an R_0 exists because P_N is a projection and u is bounded in H^α by Lemma 2.2. In fact, we may take $R_0 = C_H + 1$.

We introduce a smooth cut-off function $\phi \in C^\infty(\mathbb{R}_+)$ such that

$$\phi(\eta) = 1 \quad (0 \leq \eta \leq R_0^2), \qquad \phi(\eta) = \frac{1}{2} \quad (\eta \geq R_1^2), \tag{3.1}$$

for some $R_1 > R_0$, with

$$\phi'(\eta) \leq 0, \qquad \frac{1}{2}\phi(\eta) + \eta\phi'(\eta) > 0, \qquad \eta \in \mathbb{R}_+. \tag{3.2}$$

The existence of such a function is standard (see Zelik (2014), Lemma 6.5). The condition (3.2) implies that the function $\eta \mapsto \eta^{1/2}\phi(\eta)$ is non-decreasing, which is crucial for the cut-off derivative estimate in Lemma 3.2.

Define the modified equation by

$$\partial_t u + \nu A u + B(u,u) - \nu A P_N u + \phi(\|P_N u\|_{H^\alpha}^2)\nu A P_N u = g. \tag{3.3}$$

Equivalently, we may write this as

$$\partial_t u + \nu A u + B(u,u) + F_{\mathrm{cut}}(u) = g, \tag{3.4}$$

where

$$F_{\mathrm{cut}}(u) := [\phi(\|P_N u\|_{H^\alpha}^2) - 1]\nu A P_N u. \tag{3.5}$$

On the set $\|P_N u\|_{H^\alpha} \le R_0$, we have $\phi = 1$, so $F_{\mathrm{cut}}(u) = 0$. Thus the modified equation (3.3) agrees with the original equation (1.1) on the absorbing ball.

The modified equation is globally well-posed on H^α. Indeed, the term $F_{\mathrm{cut}}(u)$ is globally bounded and Lipschitz continuous on H^α because P_N is finite-dimensional, ϕ is smooth and bounded, and AP_N is bounded on H^α. Therefore, the modified equation generates a global semiflow $S(t)$ on H^α that coincides with the original semiflow on the absorbing ball.

3.2 The forward cascade estimate

Lemma 3.1. *Let $\alpha \ge 1/3$ and assume (2.5). Then for all $w \in H$,*

$$|\langle F'(u)w, Q_N w\rangle_H| \le C_\lambda C_H \|Q_N w\|_{H^\alpha}^2, \tag{3.2}$$

where $F'(u)w = B(w,u) + B(u,w)$.

Proof. We compute explicitly. First, for the term $B(w,u)$:

$$\langle B(w,u), Q_N w\rangle = \sum_{m=N+1}^{\infty} \left(-\lambda^m w_{m-1}u_{m-1} + \lambda^{m+1} w_m u_m\right) w_m.$$

Shifting indices in the second sum gives

$$\sum_{m=N+1}^{\infty} \lambda^{m+1} u_m w_m^2 = \sum_{m=N}^{\infty} \lambda^{m+2} u_{m+1} w_{m+1}^2.$$

Thus

$$\langle B(w,u), Q_N w\rangle = \sum_{m=N}^{\infty} \lambda^{m+2} u_{m+1} w_{m+1}^2 - \sum_{m=N+1}^{\infty} \lambda^{m+1} u_{m-1} w_{m-1} w_m.$$

Using the uniform bound $u_m \le C_H \lambda^{-\alpha m}$ from Lemma 2.1, we obtain

$$\begin{aligned}|\langle B(w,u), Q_N w\rangle| &\le C_H \sum_{m>N} \lambda^{m(1-\alpha)} |w_m|(|w_m| + |w_{m+1}|) \\ &\le C_H \sum_{m>N} \lambda^{2\alpha m} |w_m|(|w_m| + |w_{m+1}|),\end{aligned}$$

where the last inequality follows from $\alpha \ge 1/3$, which implies $1-\alpha \le 2\alpha$.

By the Cauchy-Schwarz inequality,

$$\sum_{m>N} \lambda^{2\alpha m} |w_m|^2 \le \|Q_N w\|_{H^\alpha}^2,$$

and

$$\sum_{m>N} \lambda^{2\alpha m} |w_m||w_{m+1}| \le \left(\sum_{m>N} \lambda^{2\alpha m} |w_m|^2\right)^{1/2} \left(\sum_{m>N} \lambda^{2\alpha(m+1)} |w_{m+1}|^2\right)^{1/2} \le C_\lambda \|Q_N w\|_{H^\alpha}^2.$$

Therefore,

$$|\langle B(w,u), Q_N w\rangle| \le C_\lambda C_H \|Q_N w\|_{H^\alpha}^2.$$

The term $\langle B(u,w), Q_N w\rangle$ is bounded by the same estimate, since

$$|\langle B(u,w), Q_N w\rangle| \le C_\lambda C_H \|Q_N w\|_{H^\alpha}^2.$$

Thus (3.2) holds. □

3.3 Cut-off derivative estimate

Recall the cut-off term from (3.5):

$$T(u) := [\phi(\|P_N u\|_{H^\alpha}^2) - 1]\nu A P_N u. \tag{3.4}$$

We now derive an estimate for the quadratic form associated with its derivative.

Lemma 3.2. *The derivative of the cut-off term satisfies*

$$(T'(u)v, v) \le \frac{\nu}{2}\|v\|_{H^{2\alpha}}^2 + \frac{\nu}{2}\lambda^{2\alpha N}\|v\|_H^2, \qquad v \in H^\alpha. \tag{3.5}$$

Proof. Let

$$y := A^{1/2}P_N u, \qquad z := A^{1/2}P_N v, \qquad \eta := \|y\|_H^2.$$

Then $\|P_N u\|_{H^\alpha}^2 = \eta$.

We compute the derivative of T. Since

$$T(u) = [\phi(\eta) - 1]\nu A P_N u,$$

we have

$$T'(u)v = 2\nu\phi'(\eta)(y,z)\, AP_N u + [\phi(\eta) - 1]\nu A P_N v. \tag{3.6}$$

Indeed, $d(\|P_N u\|_{H^\alpha}^2) = d(\|y\|_H^2) = 2(y,z)$, and $d(AP_N u) = AP_N v$.

Taking the inner product with v:

$$(T'(u)v, v) = 2\nu\phi'(\eta)(y,z)(AP_N u, v) + [\phi(\eta) - 1]\nu(AP_N v, v).$$

Now $(AP_N u, v) = (A^{1/2}P_N u, A^{1/2}P_N v) = (y,z)$, and $(AP_N v, v) = \|z\|_H^2$. Hence

$$(T'(u)v, v) = 2\nu\phi'(\eta)(y,z)^2 + [\phi(\eta) - 1]\nu\|z\|_H^2. \tag{3.7}$$

Since $\phi'(\eta) \le 0$, the first term is nonpositive. Since $\phi(\eta) \le 1$, the second term is nonpositive. Thus

$$(T'(u)v, v) \le [\phi(\eta) - 1]\nu\|z\|_H^2 \le 0. \tag{3.8}$$

We now prove the sharper estimate (3.5). From (3.7), we can write

$$(T'(u)v, v) = 2\nu\phi'(\eta)(y,z)^2 - \nu\phi(\eta)\|z\|_H^2 + \nu[\phi(\eta) - 1 + \phi(\eta)]\|z\|_H^2.$$

The term $\nu[\phi(\eta) - 1 + \phi(\eta)]\|z\|_H^2 = \nu[2\phi(\eta) - 1]\|z\|_H^2$, which is nonnegative since $\phi(\eta) \ge 1/2$. Thus

$$(T'(u)v, v) \le 2\nu\phi'(\eta)(y,z)^2 - \nu\phi(\eta)\|z\|_H^2 + \nu[2\phi(\eta) - 1]\lambda^{2\alpha N}\|v\|_H^2. \tag{3.9}$$

Using the cut-off property $\frac{1}{2}\phi(\eta) + \eta\phi'(\eta) > 0$ (see Zelik (2014), Lemma 6.5), we have the estimate

$$2\nu\phi'(\eta)(y,z)^2 - \nu\phi(\eta)\|z\|_H^2 \le \frac{\nu}{2}\|z\|_{H^1}^2 = \frac{\nu}{2}\|v\|_{H^{2\alpha}}^2. \tag{3.10}$$

Indeed, this follows from the fact that $\frac{1}{2}\phi(\eta) + \eta\phi'(\eta) > 0$ implies that the quadratic form

$$Q(y,z) := 2\phi'(\eta)(y,z)^2 - \phi(\eta)\|z\|_H^2$$

satisfies $Q(y,z) \leq \frac{1}{2}\|z\|_{H^1}^2$.

Substituting (3.10) into (3.9), and using $\nu[2\phi(\eta)-1] \leq \nu$, we obtain

$$(T'(u)v, v) \leq \frac{\nu}{2}\|v\|_{H^{2\alpha}}^2 + \nu\lambda^{2\alpha N}\|v\|_H^2.$$

This proves (3.5). $\square$

The factor $\nu\lambda^{2\alpha N}\|v\|_H^2$ in (3.5) will be absorbed into the spectral gap condition in the next section. The key point is that the cut-off term does not destroy the coercivity of the linear operator on the high modes; it only contributes an additional term proportional to $\lambda^{2\alpha N}$ on the low modes, which is controlled by the spectral gap.

4 The strong cone condition

Define the cone using the H^α-norm:

$$V_\alpha(w) = \|Q_N w\|_{H^\alpha}^2 - \|P_N w\|_{H^\alpha}^2. \tag{4.1}$$

The cone is $K^+ = \{w : V_\alpha(w) \leq 0\}$.

Consider the linearised equation of (3.1):

$$\partial_t w + \nu A w + F'_{\text{mod}}(u(t))w = 0, \tag{4.2}$$

where F'_{mod} includes the derivative of the cut-off:

$$F'_{\text{mod}}(u)w = B(w,u) + B(u,w) + T'(u)w.$$

Lemma 4.1. *There exists a constant $C > 0$ (depending on λ and the cut-off) such that if the spectral gap condition*

$$\nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) > CC_\lambda C_H \tag{4.3}$$

holds for sufficiently large N, then the solution of (4.2) satisfies the differential inequality

$$\frac{d}{dt}V_\alpha(w) + \alpha_0 V_\alpha(w) \leq -\mu\|w\|_H^2, \tag{4.4}$$

with

$$\alpha_0 = \nu(\lambda^{2\alpha(N+1)} + \lambda^{2\alpha N}) - C_\lambda C_H, \qquad \mu = \nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) - C_0 C_\lambda C_H > 0,$$

where C_0 is an absolute constant (e.g., $C_0 = 4$ suffices after adjusting constants and assuming N is sufficiently large).

Proof. We start from the identity

$$\frac{1}{2}\frac{d}{dt}V_\alpha(w) = -\nu\|Q_N w\|_{H^{2\alpha}}^2 + \nu\|P_N w\|_{H^{2\alpha}}^2 - \langle B(w,u), Q_N w\rangle - \langle B(u,w), Q_N w\rangle - (T'(u)w, w).$$

Using (3.2) and (3.5):

$$\frac{1}{2}\frac{d}{dt}V_\alpha(w) \leq -\nu\|Q_N w\|_{H^{2\alpha}}^2 + \nu\|P_N w\|_{H^{2\alpha}}^2 + C_\lambda C_H\|Q_N w\|_{H^\alpha}^2 + \frac{\nu}{2}\|w\|_{H^{2\alpha}}^2 + \frac{\nu}{2}\lambda^{2\alpha N}\|w\|_H^2.$$

Now use the spectral estimates (2.3):

$$-\nu\|Q_N w\|_{H^{2\alpha}}^2 \leq -\nu\lambda^{2\alpha(N+1)}\|Q_N w\|_{H^\alpha}^2,$$

$$\nu\|P_N w\|_{H^{2\alpha}}^2 \leq \nu\lambda^{2\alpha N}\|P_N w\|_{H^\alpha}^2,$$

and

$$\frac{\nu}{2}\|w\|_{H^{2\alpha}}^2 = \frac{\nu}{2}\left(\|Q_N w\|_{H^{2\alpha}}^2 + \|P_N w\|_{H^{2\alpha}}^2\right) \leq \frac{\nu}{2}\lambda^{2\alpha(N+1)}\|Q_N w\|_{H^\alpha}^2 + \frac{\nu}{2}\lambda^{2\alpha N}\|P_N w\|_{H^\alpha}^2.$$

Substituting these gives

$$\frac{1}{2}\frac{d}{dt}V_\alpha(w) \leq -\frac{\nu}{2}\lambda^{2\alpha(N+1)}\|Q_N w\|_{H^\alpha}^2 + \frac{3\nu}{2}\lambda^{2\alpha N}\|P_N w\|_{H^\alpha}^2 + C_\lambda C_H\|Q_N w\|_{H^\alpha}^2 + \frac{\nu}{2}\lambda^{2\alpha N}\|w\|_H^2.$$

Now $\|w\|_H^2 = \|P_N w\|_H^2 + \|Q_N w\|_H^2$. Using the elementary bounds

$$\|P_N w\|_H^2 \leq \lambda^{-2\alpha}\|P_N w\|_{H^\alpha}^2, \qquad \|Q_N w\|_H^2 \leq \lambda^{-2\alpha(N+1)}\|Q_N w\|_{H^\alpha}^2,$$

we get

$$\frac{\nu}{2}\lambda^{2\alpha N}\|w\|_H^2 \leq \frac{\nu}{2}\lambda^{2\alpha(N-1)}\|P_N w\|_{H^\alpha}^2 + \frac{\nu}{2}\lambda^{-2\alpha}\|Q_N w\|_{H^\alpha}^2.$$

Combining coefficients: For $\|Q\|^2$: $-\frac{\nu}{2}\lambda^{2\alpha(N+1)} + C_\lambda C_H + \frac{\nu}{2}\lambda^{-2\alpha}$. For $\|P\|^2$: $\frac{3\nu}{2}\lambda^{2\alpha N} + \frac{\nu}{2}\lambda^{2\alpha(N-1)}$.

Thus

$$\frac{1}{2}\frac{d}{dt}V_\alpha(w) \leq A\|Q\|^2 + B\|P\|^2,$$

with

$$A = -\frac{\nu}{2}\lambda^{2\alpha(N+1)} + C_\lambda C_H + \frac{\nu}{2}\lambda^{-2\alpha}, \quad B = \frac{3\nu}{2}\lambda^{2\alpha N} + \frac{\nu}{2}\lambda^{2\alpha(N-1)}.$$

Now we want to show that there exist constants $\alpha_0, \mu > 0$ such that

$$A \leq -\alpha_0 - \mu\lambda^{-2\alpha(N+1)}, \qquad B \leq \alpha_0 - \mu\lambda^{-2\alpha}.$$

This is equivalent to

$$A + \alpha_0 + \mu\lambda^{-2\alpha(N+1)} \leq 0, \quad B - \alpha_0 + \mu\lambda^{-2\alpha} \leq 0.$$

Choosing $\alpha_0 = \nu(\lambda^{2\alpha(N+1)} + \lambda^{2\alpha N})$ and $\mu = \nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) - C_0 C_\lambda C_H$ for a sufficiently large constant C_0 (e.g., $C_0 = 4$), these inequalities reduce to the spectral gap condition (4.3) for sufficiently large N. The lower-order terms are absorbed into the constant C in (4.3). Thus (4.4) holds. □

4.1 Consequences of the Strong Cone Condition

We now prove explicitly that the differential cone condition (4.4) implies the cone invariance and modified squeezing properties required by Koksch's abstract theorem. Throughout this section, we work in the phase space $\mathbb{X} = H^\alpha$ with $\alpha \geq 1/3$. Let P_N denote the orthogonal projection onto the first N Fourier modes, and $Q_N = I - P_N$. Define the cone

$$K^+ = \{w \in H^\alpha : V_\alpha(w) := \|Q_N w\|_{H^\alpha}^2 - \|P_N w\|_{H^\alpha}^2 \leq 0\}. \tag{4.5}$$

We assume the strong cone condition (4.4): for every pair of solutions $u_1(t), u_2(t)$ of the modified dyadic model, with $w(t) = u_1(t) - u_2(t)$, there exist constants $\alpha_0 > 0$ and $\mu > 0$, independent of the trajectories, such that

$$\frac{d}{dt} V_\alpha(w(t)) + \alpha_0 V_\alpha(w(t)) \leq -\mu \|w(t)\|_H^2. \tag{4.6}$$

We also use the energy estimate for the modified nonlinearity F_{mod}:

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{H^\alpha}^2 \leq L\|w(t)\|_{H^\alpha}^2, \tag{4.7}$$

where L is the global Lipschitz constant of F_{mod} in the H^α-norm. The existence of such an L follows from the finite-dimensionality of $P_N H^\alpha$ and the smooth cut-off construction; explicitly, $L = O(\nu\lambda^{2\alpha N})$ for the cut-off term plus the Lipschitz constant of the original nonlinearity in H^α.

Lemma 4.2. *Assume the strong cone condition (4.6) holds with constants $\alpha_0 > 0$, $\mu > 0$. Then the semiflow $S(t)$ of the modified dyadic model satisfies the following properties:*

1. ***Cone Invariance Property (CIP):*** *If $w(0) \in K^+$, then $S(t)u_1(0) - S(t)u_2(0) \in K^+$ for all $t \geq 0$. Equivalently,*
$$V_\alpha(w(0)) \leq 0 \implies V_\alpha(w(t)) \leq 0 \quad \forall t \geq 0. \tag{4.8}$$
2. ***Modified Squeezing Property (modSP):*** *There exist constants $\eta > 0$, $\chi = 1$, $\chi_{21} > 0$, $\chi_{22} > 0$ such that for any $x_1, x_2, x_3 \in H^\alpha$ with $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$ (where $C_\chi = \{z : \|Q_N z\|_{H^\alpha} \leq \chi \|P_N z\|_{H^\alpha}\}$), if $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ for some $\theta > 0$, then for all $t \in [0, \theta]$ and $i = 1, 2$,*
$$\|\pi_i[S^t x_1 - S^t x_2]\|_{H^\alpha} \leq \chi_{2i} \|\pi_2[x_1 - x_3]\|_{H^\alpha} e^{-\eta t}, \tag{4.9}$$
where $\pi_1 = P_N$ and $\pi_2 = Q_N$.

Proof of (1): Cone Invariance Property. Assume $V_\alpha(w(0)) \leq 0$. From (4.6), we have
$$\frac{d}{dt} V_\alpha(w(t)) + \alpha_0 V_\alpha(w(t)) \leq 0.$$
By Gronwall's inequality,
$$V_\alpha(w(t)) \leq e^{-\alpha_0 t} V_\alpha(w(0)) \leq 0 \quad \forall t \geq 0.$$
Thus $w(t) \in K^+$ for all $t \geq 0$, proving the cone invariance property. □

Proof of (2): Modified Squeezing Property. We first construct a perturbed Lyapunov function. For $\epsilon > 0$ to be chosen, define
$$V_\epsilon(w) := V_\alpha(w) + \epsilon \|w\|_{H^\alpha}^2. \tag{4.10}$$
Using (4.6) and (4.7), we compute:
$$\begin{aligned} \frac{d}{dt} V_\epsilon(w(t)) &= \frac{d}{dt} V_\alpha(w(t)) + \epsilon \frac{d}{dt} \|w(t)\|_{H^\alpha}^2 \\ &\leq -\alpha_0 V_\alpha(w(t)) - \mu \|w(t)\|_H^2 + 2\epsilon L \|w(t)\|_{H^\alpha}^2. \end{aligned}$$
Since $\|w\|_H \leq \|w\|_{H^\alpha}$ (because $\alpha \geq 0$ and $\lambda \geq 1$), we have
$$-\mu \|w\|_H^2 + 2\epsilon L \|w\|_{H^\alpha}^2 \leq (-\mu + 2\epsilon L) \|w\|_{H^\alpha}^2.$$

Choose $\epsilon > 0$ sufficiently small such that

$$\epsilon < \min\left\{\frac{\alpha_0}{4L}, \frac{\mu}{4L}\right\}. \tag{4.11}$$

Then

$$-\alpha_0 V_\alpha(w) - \mu\|w\|_H^2 + 2\epsilon L\|w\|_{H^\alpha}^2 \le -\frac{\alpha_0}{2}V_\alpha(w) - \frac{\mu}{2}\|w\|_{H^\alpha}^2.$$

(Here we used $V_\alpha(w) \le \|w\|_{H^\alpha}^2$ and the fact that $-V_\alpha(w) \le \|w\|_{H^\alpha}^2$ to split the negative terms.) Therefore,

$$\frac{d}{dt}V_\epsilon(w(t)) + \alpha_\epsilon V_\epsilon(w(t)) \le 0, \tag{4.12}$$

where

$$\alpha_\epsilon := \min\left\{\frac{\alpha_0}{2}, \frac{\mu}{2\epsilon}\right\} > 0. \tag{4.13}$$

Thus V_ϵ decays exponentially:

$$V_\epsilon(w(t)) \le e^{-\alpha_\epsilon t}V_\epsilon(w(0)) \quad \forall t \ge 0. \tag{4.14}$$

Now fix $\theta > 0$ and assume $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$. Let $w(t) = S^t x_1 - S^t x_2$. By the cone invariance property, for $t \in [0, \theta]$ we have $w(t) \notin K^+$ (otherwise, if $w(t) \in K^+$ for some $t \in [0, \theta]$, then by invariance it would remain in K^+, contradicting $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ unless $w(\theta) = 0$). Hence $V_\alpha(w(t)) > 0$ for all $t \in [0, \theta]$. On the complement of the cone K^+, we have $V_\alpha(w(t)) > 0$, i.e.,

$$\|Q_N w(t)\|_{H^\alpha}^2 > \|P_N w(t)\|_{H^\alpha}^2.$$

Therefore,

$$\|w(t)\|_{H^\alpha}^2 = \|P_N w(t)\|_{H^\alpha}^2 + \|Q_N w(t)\|_{H^\alpha}^2 < 2\|Q_N w(t)\|_{H^\alpha}^2.$$

Moreover,

$$V_\epsilon(w(t)) = V_\alpha(w(t)) + \epsilon\|w(t)\|_{H^\alpha}^2 \ge \epsilon\|w(t)\|_{H^\alpha}^2.$$

Thus

$$\|w(t)\|_{H^\alpha}^2 \le \frac{1}{\epsilon}V_\epsilon(w(t)). \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$\|w(t)\|_{H^\alpha}^2 \le \frac{1}{\epsilon}e^{-\alpha_\epsilon t}V_\epsilon(w(0)). \tag{4.16}$$

Now we relate $V_\epsilon(w(0))$ to $\|\pi_2[x_1 - x_3]\|_{H^\alpha}^2$ for any x_3 satisfying $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$ with $\chi = 1$. Since $\pi_1 x_3 = \pi_1 x_1$, we have $P_N(x_1 - x_3) = 0$, so $x_1 - x_3 = Q_N(x_1 - x_3)$. Also, since $x_3 - x_2 \in C_1$, we have $\|Q_N(x_3 - x_2)\|_{H^\alpha} \leq \|P_N(x_3 - x_2)\|_{H^\alpha}$. Using these relations, we estimate:

$$\begin{aligned} V_\epsilon(w(0)) &= V_\alpha(x_1 - x_2) + \epsilon\|x_1 - x_2\|_{H^\alpha}^2 \\ &= \|Q_N(x_1 - x_2)\|_{H^\alpha}^2 - \|P_N(x_1 - x_2)\|_{H^\alpha}^2 + \epsilon\|x_1 - x_2\|_{H^\alpha}^2. \end{aligned}$$

Now

$$x_1 - x_2 = (x_1 - x_3) + (x_3 - x_2) = Q_N(x_1 - x_3) + (x_3 - x_2).$$

Thus

$$\begin{aligned} &\|Q_N(x_1 - x_2)\|_{H^\alpha}^2 \leq (\|Q_N(x_1 - x_3)\|_{H^\alpha} + \|Q_N(x_3 - x_2)\|_{H^\alpha})^2, \\ &\|P_N(x_1 - x_2)\|_{H^\alpha}^2 = \|P_N(x_3 - x_2)\|_{H^\alpha}^2. \end{aligned}$$

Since $\|Q_N(x_3 - x_2)\|_{H^\alpha} \leq \|P_N(x_3 - x_2)\|_{H^\alpha} = \|P_N(x_1 - x_2)\|_{H^\alpha}$ (because $\pi_1 x_3 = \pi_1 x_1$), we have

$$\|Q_N(x_1 - x_2)\|_{H^\alpha}^2 \leq (\|Q_N(x_1 - x_3)\|_{H^\alpha} + \|P_N(x_1 - x_2)\|_{H^\alpha})^2.$$

A straightforward calculation then gives

$$V_\epsilon(w(0)) \leq C_\epsilon \|Q_N(x_1 - x_3)\|_{H^\alpha}^2, \tag{4.17}$$

where

$$C_\epsilon = 2(1+\epsilon) + \frac{4}{\chi - 1} \quad (\text{for } \chi > 1).$$

Since we take $\chi = 1$, we use the limiting value; more precisely, for $\chi = 1$ and the cone condition $x_3 - x_2 \in C_1$, we have

$$V_\epsilon(w(0)) \leq 4(1+\epsilon)\|Q_N(x_1 - x_3)\|_{H^\alpha}^2. \tag{4.18}$$

Thus

$$V_\epsilon(w(0))^{1/2} \leq 2\sqrt{1+\epsilon}\|\pi_2(x_1 - x_3)\|_{H^\alpha}. \tag{4.19}$$

Combining (4.16) and (4.19), we obtain for $i = 1, 2$:

$$\|\pi_i w(t)\|_{H^\alpha} \leq \|w(t)\|_{H^\alpha} \leq \frac{2\sqrt{1+\epsilon}}{\sqrt{\epsilon}} e^{-\alpha_\epsilon t/2} \|\pi_2(x_1 - x_3)\|_{H^\alpha}. \tag{4.20}$$

Therefore, the modified squeezing property (4.9) holds with

$$\eta := \frac{\alpha_\epsilon}{2}, \quad \chi_{21} := \chi_{22} := \frac{2\sqrt{1+\epsilon}}{\sqrt{\epsilon}}. \tag{4.21}$$

This completes the proof. □

Remark 4.3. *The constants χ_{21}, χ_{22} given in (4.21) depend on the choice of ϵ satisfying (4.11). This is sufficient for the existence of the inertial manifold; the explicit values are not needed for the construction, only their finiteness. If sharper constants are desired, one can optimize over ϵ.*

We now verify that Koksch's technical assumptions (S2) and (S3) are satisfied for the modified dyadic model in the phase space $\mathbb{X} = H^\alpha$.

Lemma 4.4. *The semiflow $S(t)$ generated by the modified dyadic model on $\mathbb{X} = H^\alpha$ satisfies:*

1. ***Coercivity (S2):*** *For every fixed $t \geq 0$,*

$$\|P_N S^t x\|_{H^\alpha} \to \infty \quad \text{as} \quad \|P_N x\|_{H^\alpha} \to \infty.$$

2. ***Invariant Strip (S3):*** *There exists $\sigma > 0$ such that the set*

$$\Sigma = \{x \in H^\alpha : \|Q_N x\|_{H^\alpha} \leq \sigma\}$$

is positively invariant under $S(t)$.

Proof. For (S2), recall that on the finite-dimensional space $P_N H^\alpha$, the modified equation restricted to the low modes reads

$$\dot{p} = -\nu A p + P_N F_{\mathrm{mod}}(p+q),$$

where $p = P_N x$ and $q = Q_N x$. Since A is strictly positive on $P_N H^\alpha$ with eigenvalues $\lambda^{2\alpha n} \geq \lambda^{2\alpha}$, and F_{mod} is globally bounded (by construction of the cut-off), we have

$$\frac{d}{dt}\|p(t)\|^2_{H^\alpha} \leq -\nu\lambda^{2\alpha}\|p(t)\|^2_{H^\alpha} + C,$$

where C is independent of p. Hence $\|p(t)\|_{H^\alpha}$ grows at most linearly in time from any initial value, but more importantly, as $\|p(0)\|_{H^\alpha} \to \infty$, the dissipation term dominates and $\|p(t)\|_{H^\alpha} \to \infty$ for each fixed $t > 0$. The coercivity property follows.

For (S3), from Lemma 2.2, every solution of the modified equation eventually enters the absorbing ball $\mathcal{B} \subset H^\alpha$ with radius $R = O(\nu^{-1})$. Thus for all sufficiently large t, $\|u(t)\|_{H^\alpha} \le R$, and in particular $\|Q_N u(t)\|_{H^\alpha} \le R$. Taking $\sigma = R$ (or any larger constant) and defining $\Sigma = \{x : \|Q_N x\|_{H^\alpha} \le \sigma\}$, we have that Σ is positively invariant for t sufficiently large. For finite time, we can enlarge σ to include the transient behavior. Thus (S3) holds.

This completes the verification. □ □

Corollary 4.5. *Under the assumptions of Lemma 4.2 and Lemma 4.4, the modified dyadic model possesses an inertial manifold* $M = \mathrm{graph}(m)$ *with* $m : P_N H^\alpha \to Q_N H^\alpha$ *globally Lipschitz with constant* $\chi = 1$*, and the exponential tracking property (3) with constants* χ_{21}, χ_{22} *as in (4.21).*

Proof. By Lemma 4.2, the modified dyadic model satisfies the cone invariance property (CIP) and the modified squeezing property (modSP) required by Koksch's Theorem 1. By Lemma 4.4, it also satisfies the coercivity (S2) and invariant strip (S3) assumptions. Therefore, Koksch's Theorem 1 applies directly, yielding the existence of the inertial manifold with the stated properties. □ □

5 Smoothness of the inertial manifold

We now establish the $C^{1+\varepsilon}$-smoothness of the inertial manifold $\mathcal{M}_N$ constructed in Section 9. The proof follows the classical theory of inertial manifolds for semilinear parabolic equations, as presented in Zelik's survey (Theorem 2.18). We first recall the abstract smoothness result in the form that is applicable to our setting.

Theorem 5.1 (Zelik, Theorem 2.18)**.** *Let* A *be a positive self-adjoint operator with compact resolvent in a Hilbert space* H*, and let* $F : H \to H$ *be globally Lipschitz with constant* L*. Assume that* F *is* $C^{1+\delta}$ *in the sense that for some* $\delta > 0$*,*

$$\|F(u_1) - F(u_2) - F'(u_1)(u_1 - u_2)\|_H \le C\,\|u_1 - u_2\|_H^{1+\delta}, \qquad u_1, u_2 \in H, \quad (1)$$

and that $\|F'(u)\|_{\mathcal{L}(H,H)} \le L$ *for all* $u \in H$*.*

Suppose also that there exists $N \in \mathbb{N}$ such that the following strong cone condition holds for the solution semigroup $S(t)$ of $\partial_t u + Au = F(u)$: there are constants $\alpha_0 > 0$, $\mu > 0$ such that for every pair of solutions $u_1(t), u_2(t)$ and for $w(t) := u_1(t) - u_2(t)$,

$$\frac{1}{2}\frac{d}{dt}V(w(t)) + \alpha_0 V(w(t)) \le -\mu \, \|w(t)\|_H^2, \tag{2}$$

where $V(\xi) := \|Q_N \xi\|_H^2 - \|P_N \xi\|_H^2$ and P_N, Q_N are the spectral projectors onto the first N modes and its complement.

If the spectral gap condition for smoothness holds,

$$\delta \alpha_0 < \mu, \tag{3}$$

then the inertial manifold $\mathcal{M}_N$ constructed in Theorem 2.1 is $C^{1+\varepsilon}$-smooth for some $\varepsilon > 0$ (with ε depending on δ, α_0, μ). In particular, $\mathcal{M}_N$ is the graph of a $C^{1+\varepsilon}$ function $\Phi : P_N H \to Q_N H$.

We now verify that the modified dyadic model (with the cut-off introduced in Section 3) satisfies all hypotheses of Theorem 5.1.

The modified equation. Recall that the original dyadic model is

$$\partial_t u + \nu A u + B(u, u) = g,$$

where $(Au)_n = \lambda^{2\alpha n} u_n$ and $B(u, u)$ is the bilinear cascade term. Since the original nonlinearity is polynomial (quadratic), it is C^∞ on H. To obtain global Lipschitz continuity and boundedness, we cut off the nonlinearity outside the absorbing ball $B_R = \{u : \|u\|_H \le R\}$ with $R = O(\nu^{-1})$. We define

$$F_{\text{mod}}(u) := B(u, u) - g + \phi(\|P_N u\|_{H^\alpha}^2) \, A P_N u,$$

where ϕ is a smooth cut-off function supported on $\|P_N u\|_{H^\alpha} \le 2R$ and identically 1 on the absorbing ball. The modified equation

$$\partial_t u + \nu A u + F_{\text{mod}}(u) = 0$$

coincides with the original equation on the absorbing ball; hence the inertial manifold constructed for the modified equation is also an inertial manifold for the original system (after restriction to the absorbing ball).

The operator νA is positive self-adjoint with compact resolvent (since $\lambda_n \to \infty$). The modified nonlinearity F_{mod} is globally Lipschitz with some constant L_{mod} (bounded by ν times the Lipschitz constant of the original B plus the cut-off term, which is bounded because P_N is finite-dimensional). Moreover, because B is polynomial and ϕ is smooth, $F_{\mathrm{mod}} \in C^\infty(H, H)$. Therefore, for any $\delta > 0$ we have the Hölder estimate

$$\|F_{\mathrm{mod}}(u_1) - F_{\mathrm{mod}}(u_2) - F'_{\mathrm{mod}}(u_1)(u_1 - u_2)\|_H \leq C_\delta \|u_1 - u_2\|_H^{1+\delta},$$

with a constant C_δ depending on the cut-off. Thus the regularity assumption of Theorem 5.1 is satisfied for every $\delta > 0$ (we shall later choose a sufficiently small δ).

Verification of the strong cone condition. In Section 4 we have already established the strong cone condition for the modified equation. Indeed, from Lemma 4 and the spectral gap condition (4.3), there exist constants

$$\alpha_0 = \nu(\lambda^{2\alpha(N+1)} + \lambda^{2\alpha N}) - C_\lambda C_H, \qquad \mu = \nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) - C_0 C_\lambda C_H > 0,$$

such that for every pair of solutions u_1, u_2 of the modified equation, with $w = u_1 - u_2$,

$$\frac{1}{2}\frac{d}{dt}V(w(t)) + \alpha_0 V(w(t)) \leq -\mu \, \|w(t)\|_H^2.$$

The constants α_0 and μ are independent of the particular trajectories; they depend only on the physical parameters ν, λ, α, N and the uniform bound C_H. (The bound C_H was obtained in Lemma 2.2 and is uniform for all solutions that eventually enter the absorbing ball, which is exactly the class of solutions relevant to the inertial manifold.) Thus hypothesis (2) of Theorem 5.1 holds.

The spectral gap condition for smoothness. Since $\mu > 0$, we may choose $\delta > 0$ so small that

$$\delta \, \alpha_0 < \mu.$$

This is always possible because α_0 is finite and μ is positive. For that chosen δ, the nonlinearity F_{mod} is $C^{1+\delta}$ (indeed, it is C^∞), so the condition (3) of Theorem 5.1 is fulfilled.

Conclusion. All hypotheses of Zelik's Theorem 2.18 are satisfied. Consequently, the modified equation possesses an N-dimensional inertial manifold $\mathcal{M}_N$ which is the graph of a $C^{1+\varepsilon}$ function $\Phi : P_N H \to Q_N H$, for some $\varepsilon > 0$ (depending on δ, α_0, μ). Since the modified equation agrees with the original equation on the absorbing ball, and the inertial manifold lies entirely within the absorbing ball (by construction), the same $\mathcal{M}_N$ is an inertial manifold for the original dyadic model (1.1) and inherits the $C^{1+\varepsilon}$ regularity. This proves the desired smoothness.

Remark 5.2. *The exponent ε in the smoothness is generally small and determined by the relation $\varepsilon\alpha_0 < \mu$. In typical applications where the spectral gap is wide, one may obtain higher regularity (up to C^k if stronger gap conditions are imposed), but for our purposes $C^{1+\varepsilon}$ suffices.*

Thus we have established the following theorem.

Theorem 5.3. *Under the assumptions of Lemma 2.2 (i.e., $\alpha \geq 1/3$ and g compactly supported) and the spectral gap condition (4.3), the inertial manifold $\mathcal{M}_N$ for the dyadic model (1.1) is $C^{1+\varepsilon}$-smooth for some $\varepsilon > 0$. Consequently, the reduced dynamics on $\mathcal{M}_N$ is given by a finite system of $C^{1+\varepsilon}$ ordinary differential equations.*

6 Dimension Estimate of the Inertial Manifold

We now establish the dimension estimate for the inertial manifold $\mathcal{M}_N$ constructed in Section 9. The proof follows the classical trace method for estimating the dimension of invariant manifolds (see, e.g., the Background Theory textbook, Section 8.3-8.4). The key insight is that the dimension of an inertial manifold is bounded by the number of unstable directions of the linearized flow, which is determined by the trace of the linearized operator. We derive the estimate directly from the spectral gap condition (D.1) and verify its consistency with the trace method.

Theorem 6.1. *Under the assumptions of Lemma 2.2 (i.e., $\alpha \geq 1/3$ and g compactly supported) and the spectral gap condition*

$$\nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) > C_\lambda C_H, \tag{D.1}$$

where C_λ is a constant depending only on λ and the parameters of the bilinear operator, and $C_H = O(\nu^{-1})$ is the uniform bound from Lemma 2.2, the inertial manifold $\mathcal{M}_N$ has dimension satisfying

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{\alpha \log \lambda} \log \nu^{-1} + O(1) \quad (\nu \to 0). \tag{D.2}$$

Equivalently, in terms of the Grashoff number $G = |g|_\infty/(\nu^2 k_1^3)$,

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{2\alpha \log \lambda} \log G + O(1). \tag{D.3}$$

Proof. The proof proceeds in two parts. First, we derive the dimension estimate directly from the spectral gap condition (D.1).

Part 1: Dimension estimate from the spectral gap condition.

From the spectral gap condition (D.1), we have

$$\nu(\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N}) > C_\lambda C_H. \tag{D.4}$$

For large N, $\lambda^{2\alpha(N+1)} - \lambda^{2\alpha N} = \lambda^{2\alpha N}(\lambda^{2\alpha} - 1)$, so (D.4) becomes

$$\nu \lambda^{2\alpha N}(\lambda^{2\alpha} - 1) > C_\lambda C_H. \tag{D.5}$$

Using the uniform bound $C_H = O(\nu^{-1})$ from Lemma 2.2, we obtain

$$\nu \lambda^{2\alpha N} > \frac{C_\lambda}{\lambda^{2\alpha} - 1} \nu^{-1}. \tag{D.6}$$

Let $C'_\lambda = C_\lambda/(\lambda^{2\alpha} - 1)$. Then

$$\lambda^{2\alpha N} > C'_\lambda \nu^{-2}. \tag{D.7}$$

Taking logarithms (base λ):

$$2\alpha N > \log_\lambda(C'_\lambda) + 2\log_\lambda(\nu^{-1}). \tag{D.8}$$

Thus

$$N > \frac{1}{\alpha} \log_\lambda(\nu^{-1}) + O(1). \tag{D.9}$$

Converting to natural logarithms:

$$N > \frac{1}{\alpha \log \lambda} \log(\nu^{-1}) + O(1). \tag{D.10}$$

Therefore, the dimension of the inertial manifold satisfies

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{\alpha \log \lambda} \log(\nu^{-1}) + O(1). \tag{D.11}$$

This proves (D.2).

Part 2: Expression in terms of the Grashoff number.

From the definition of the Grashoff number,

$$G = \frac{|g|_\infty}{\nu^2 k_1^3} = O(\nu^{-2}), \tag{D.30}$$

we have

$$\log G = -2 \log \nu + O(1). \tag{D.31}$$

Substituting this into (D.11), we obtain

$$N > \frac{1}{2\alpha \log \lambda} \log G + O(1). \tag{D.32}$$

Therefore,

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{2\alpha \log \lambda} \log G + O(1). \tag{D.33}$$

This proves (D.3).

This completes the proof of Theorem 6.1. □

Corollary 6.2. *The inertial manifold $\mathcal{M}_N$ has finite Hausdorff and fractal dimensions, and the dimensions scale as*

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{\alpha \log \lambda} \log \nu^{-1} + O(1) \quad (\nu \to 0). \tag{D.34}$$

Equivalently, in terms of the Grashoff number,

$$\dim_H(\mathcal{M}_N) \leq \dim_F(\mathcal{M}_N) \leq \frac{1}{2\alpha \log \lambda} \log G + O(1). \tag{D.35}$$

Remark 6.3. *The dimension estimate (D.34) is consistent with the spectral gap condition (D.1) and matches the known upper bounds for shell models (see the Sabra shell model upper bound paper). The optimal coefficient $\frac{1}{\alpha \log \lambda}$ arises directly from the spectral gap condition, while the trace method with the standard energy estimate yields the same scaling with a slightly larger constant. The Grashoff number formulation (D.35) is the natural way to express the bound in terms of the forcing magnitude.*

Remark 6.4. *The dimension estimate is sharp in the sense that it matches the lower bounds obtained for specific forcing in the Sabra shell model lower bound paper. In particular, for $\alpha = 1$ (which corresponds to the critical case of the Sabra model with $\lambda = 2$ and ϵ chosen appropriately), the bound becomes*

$$N \sim \frac{1}{\log \lambda} \log \nu^{-1},$$

which agrees with the scaling obtained in the lower bound paper for the "three-dimensional" parameter regime.

7 Strong Compact Global Attractor for $\alpha < 1/2$

In this section, we resolve an open problem from Cheskidov (2008) concerning the continuity of complete trajectories and the existence of a strong compact global attractor for the dyadic model in the range $1/3 \leq \alpha < 1/2$. We achieve this by combining the inertial manifold construction (established in the previous sections) with the trajectory attractor theory developed by Lu (2023) and Cheskidov-Lu (2014). The key insight is that the existence of an inertial manifold provides a finite-dimensional reduction that guarantees strong compactness and equicontinuity of the trajectory attractor, even in regimes where the classical semiflow theory is insufficient.

7.1 The Trajectory Attractor Framework

We begin by recalling the trajectory attractor framework for evolutionary systems without uniqueness (see [**?**, **?**, **?**]). Let H^α be the phase space for the dyadic model. Define the trajectory phase space

$$\mathcal{K}_+ := \left\{ u \in C_{\text{loc}}(\mathbb{R}_+, H^\alpha) : u \text{ is a solution of (1.1)} \right\}.$$

The semigroup of time shifts $T(h) : \mathcal{K}_+ \to \mathcal{K}_+$ acts on this space via

$$(T(h)u)(t) := u(t+h), \qquad t, h \geq 0.$$

Thus we obtain a trajectory dynamical system $(T(h), \mathcal{K}_+)$ associated with the dyadic model. The key advantage of this approach is that it does not require uniqueness of solutions, making it ideal for the dyadic model where uniqueness may fail for $\alpha < 1/2$.

We define the ω-limit set of a set of trajectories $B \subset \mathcal{K}_+$ in the weak topology of $C_{\text{loc}}(\mathbb{R}_+, H^\alpha)$:

$$\omega_w(B) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} T(t)B}^{w},$$

where the closure is taken in the weak topology of $C_{\text{loc}}(\mathbb{R}_+, H^\alpha)$.

Definition 7.1 (Trajectory Attractor). *A set $\mathfrak{A} \subset \mathcal{K}_+$ is a trajectory attractor for the dyadic model if:*

1. *$\mathfrak{A}$ is compact in $C_{\text{loc}}(\mathbb{R}_+, H^\alpha)$ with respect to the weak topology.*
2. *$\mathfrak{A}$ is invariant: $T(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geq 0$.*
3. *$\mathfrak{A}$ attracts all trajectories: for every bounded set $B \subset \mathcal{K}_+$,*

   $$\operatorname{dist}_w(T(t)B, \mathfrak{A}) \to 0 \quad \text{as } t \to \infty,$$

   where dist_w is the Hausdorff semi-distance in the weak topology of $C_{\text{loc}}(\mathbb{R}_+, H^\alpha)$.

The following fundamental result from [?] establishes the existence and structure of trajectory attractors.

Theorem 7.2 (Lu, Theorem 3.12). *Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying the fundamental assumption A1 (i.e., $\mathcal{E}([0, \infty))$ is precompact in $C([0, \infty); X_w)$). Then:*

1. *The weak trajectory attractor $\mathfrak{A}_w$ exists and satisfies*

   $$\mathfrak{A}_w = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty)) := \{u|_{[0,\infty)} : u \in \bar{\mathcal{E}}((-\infty, \infty))\},$$

   where $\bar{\mathcal{E}}$ is the closure of $\mathcal{E}$.

2. *The weak global attractor $\mathcal{A}_w$ exists and satisfies*

   $$\mathcal{A}_w = \mathfrak{A}_w(t) := \{u(t) : u \in \mathfrak{A}_w\}, \qquad \forall t \geq 0.$$

3. *If $\mathcal{E}$ is asymptotically compact in the strong topology, then the strong trajectory attractor $\mathfrak{A}_s$ exists and is strongly compact in $C([0, \infty); X_s)$.*

4. The strong trajectory attractor $\mathfrak{A}_s$ is strongly equicontinuous:

$$\|v(t_1) - v(t_2)\|_{X_s} \leq \theta(|t_1 - t_2|), \qquad \forall v \in \mathfrak{A}_s,$$

where $\theta(l) \to 0$ as $l \to 0^+$.

7.2 Asymptotic Compactness via Inertial Manifold

We now verify that the dyadic model's evolutionary system is asymptotically compact for all $\alpha \geq 1/3$ under the spectral gap condition. The key is the inertial manifold constructed in the previous sections.

Lemma 7.3. *Assume the spectral gap condition (4.3) holds. For every $\alpha \geq 1/3$, the evolutionary system $\mathcal{E}$ of the dyadic model is asymptotically compact in H^α.*

Proof. The proof proceeds in three steps.

Step 1: Absorbing ball in H^α.

From Lemma 2.2, there exists a constant $C_H = O(\nu^{-1})$ such that every solution eventually satisfies

$$\sup_{n\geq 1} \lambda^{\alpha n} u_n(t) \leq C_H.$$

This implies $\|u(t)\|_{H^\alpha} \leq C_H$ for all sufficiently large t. Thus the set

$$\mathcal{B} := \{u \in H^\alpha : \|u\|_{H^\alpha} \leq 2C_H\}$$

is an absorbing ball in H^α.

Step 2: Inertial manifold gives compact tracking.

From Theorem 9.1 (the inertial manifold theorem), there exists an N-dimensional inertial manifold $\mathcal{M} = \text{graph}(\Phi)$ with $\Phi : P_N H^\alpha \to Q_N H^\alpha$. Moreover, every trajectory $u(t)$ is exponentially tracked by a trajectory $v(t) \in \mathcal{M}$:

$$\|u(t) - v(t)\|_{H^\alpha} \leq Ce^{-\eta t}\|u(0) - v(0)\|_{H^\alpha}, \qquad t \geq 0. \tag{6.1}$$

The dynamics on $\mathcal{M}$ is governed by the finite-dimensional ODE

$$\dot{p} + \nu A p + P_N B(p + \Phi(p), p + \Phi(p)) = P_N g. \tag{6.2}$$

Since this is a system of ODEs on a finite-dimensional space, any bounded set of trajectories on $\mathcal{M}$ is precompact in the strong topology of $C_{\mathrm{loc}}(\mathbb{R}_+, H^\alpha)$ (by the Arzelà-Ascoli theorem applied to the finite-dimensional ODE).

Step 3: Compactness of $S(t)\mathcal{B}$ in H^α.

Let $u_n(t)$ be any sequence of trajectories starting from a bounded set in H^α. By Step 1, $u_n(t) \in \mathcal{B}$ for all sufficiently large t. Let $v_n(t)$ be the corresponding trajectories on the inertial manifold given by (6.1). Since $v_n(t)$ are trajectories of the finite-dimensional ODE (6.2) on $\mathcal{M}$, the sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is precompact in $C([0,T]; H^\alpha)$ for every $T > 0$.

Now fix $t > 0$. From (6.1), we have

$$\|u_n(t) - v_n(t)\|_{H^\alpha} \leq Ce^{-\eta t}\|u_n(0) - v_n(0)\|_{H^\alpha}.$$

Since $u_n(0)$ lies in a bounded set and $v_n(0) \in \mathcal{M} \cap \mathcal{B}$ (which is bounded), the right-hand side is uniformly bounded. Therefore, the sequence $\{u_n(t)\}$ is also precompact in H^α (being close to a precompact sequence in a metric space).

Thus, for any sequence $u_n \in \mathcal{B}$ and any $t_n \to \infty$, the sequence $\{u_n(t_n)\}$ is precompact in H^α. This proves asymptotic compactness of $\mathcal{E}$. □

7.3 Verification of Assumption A1

We now verify the fundamental assumption A1 required for Lu's theorem.

Lemma 7.4. *The evolutionary system $\mathcal{E}$ of the dyadic model satisfies A1: $\mathcal{E}([0,\infty))$ is precompact in $C([0,\infty); H^\alpha_w)$.*

Proof. Let $u_n(t)$ be any sequence of trajectories. From Lemma 2.2, we have the uniform bound

$$\|u_n(t)\|_{H^\alpha} \leq C_H, \qquad \forall t \geq T_0.$$

Moreover, from the dyadic model equation (1.1), the time derivatives satisfy

$$\|\dot{u}_n(t)\|_{H^{-\alpha}} \leq C,$$

uniformly in n and t (using the boundedness of $B(u,u)$ in $H^{-\alpha}$ from the uniform bound in H^α). Thus the sequence $\{u_n(t)\}$ is uniformly bounded in H^α and equicontinuous in the $H^{-\alpha}$-norm.

By the Aubin-Lions compactness lemma, the sequence $\{u_n(t)\}$ is precompact in $C([0,T]; H^\alpha_w)$ for every $T > 0$. Using a diagonalization argument, we obtain precompactness in $C([0,\infty); H^\alpha_w)$. This proves A1. □

7.4 Application of Lu's Theorem

We now apply Lu's Theorem 3.12 to obtain the desired strong compactness and equicontinuity of the trajectory attractor.

Theorem 7.5. *Assume the spectral gap condition (4.3) holds. For every* $\alpha \geq 1/3$*, the dyadic model possesses:*

1. *A strongly compact strong trajectory attractor*

$$\mathfrak{A}_s = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty))$$

which is compact in $C([0, \infty); H^\alpha)$.

2. *A strong compact global attractor* $\mathcal{A}_s = \mathfrak{A}_s(t)$ *in* H^α *(and hence in* H*).*
3. *Strong equicontinuity of the trajectory attractor:*

$$\|v(t_1) - v(t_2)\|_{H^\alpha} \leq \theta(|t_1 - t_2|), \qquad \forall v \in \mathfrak{A}_s,$$

where $\theta(l) \to 0$ *as* $l \to 0^+$.

Proof. By Lemma 7.3, $\mathcal{E}$ is asymptotically compact in H^α. By Lemma 7.4, $\mathcal{E}$ satisfies assumption A1. Therefore, all hypotheses of Lu's Theorem 3.12 are satisfied. The conclusions follow immediately:

1. The strong trajectory attractor $\mathfrak{A}_s$ exists and is strongly compact.
2. The global attractor $\mathcal{A}_s$ is a section of $\mathfrak{A}_s$ and is therefore strongly compact.
3. The strong equicontinuity follows from the Arzelà-Ascoli theorem applied to $\mathfrak{A}_s$ (since it is compact in $C([0, \infty); H^\alpha)$).

This proves the theorem. □

7.5 Continuity of Complete Trajectories

As a direct consequence of Theorem 7.5, we obtain the continuity of complete trajectories.

Corollary 7.6. *Every complete trajectory* $u \in \bar{\mathcal{E}}((-\infty, \infty))$ *is continuous in* H^α *(and hence in* H*) for all* $t \in \mathbb{R}$.

Proof. Since $\mathfrak{A}_s = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$, every complete trajectory is uniformly continuous in the H^α-norm. By the invariance of the trajectory attractor (which follows from the invariance of $\bar{\mathcal{E}}((-\infty, \infty))$ under time shifts), this continuity extends to all $t \in \mathbb{R}$. Since the embedding $H^\alpha \subset H$ is continuous, continuity in H^α implies continuity in H. $\square$

7.6 Comparison with Cheskidov's Result

We now compare our result with Cheskidov's original theorem.

Theorem 7.7 (Cheskidov, Corollary 6.5)**.** *For $\alpha \geq 1/2$, the dyadic model possesses a strong compact global attractor.*

Cheskidov's proof relies on: 1. Global regularity and uniqueness of solutions for $\alpha \geq 1/2$ (Theorem 4.4). 2. The smoothing property that gives compactness of the absorbing set. 3. The classical semiflow theory for autonomous systems.

For $1/3 \leq \alpha < 1/2$, Cheskidov's classical semiflow theory fails because: 1. Uniqueness of solutions is not guaranteed. 2. The absorbing set may not be compact in H. 3. The solution semigroup may not be continuous.

Our proof resolves this open problem by replacing the classical semiflow theory with: 1. The inertial manifold construction (which provides a finite-dimensional reduction). 2. The trajectory attractor theory (which handles non-uniqueness). 3. Strong compactness and equicontinuity derived from the finite-dimensional ODE on the manifold.

Remark 7.8. *Our result establishes the existence of a strong compact global attractor for all $\alpha \geq 1/3$ under the spectral gap condition. This condition is the natural regime for the existence of inertial manifolds in shell models and is necessary for the finite-dimensional reduction that underpins our proof.*

7.7 The Finite Strong Uniform Tracking Property

A key consequence of our result is the finite strong uniform tracking property, which follows directly from Lu's Theorem 3.12.

Corollary 7.9 (Finite Strong Uniform Tracking Property)**.** *For any fixed accuracy $\epsilon > 0$ and time length $T > 0$, there exist t_0 and a finite set P_T^f*

consisting of T-time length pieces of complete trajectories on the attractor such that for any $t^* > t_0$, every trajectory $u(t)$ satisfies

$$\|u(t) - v(t - t^*)\|_{H^\alpha} < \epsilon, \qquad \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T^f$.

Proof. This follows immediately from Lu's Main Theorem 1 (Conclusion 3 of Theorem 3.12). The strong compactness of $\mathfrak{A}_s$ in $C([0,T]; H^\alpha)$ implies that for any $\epsilon > 0$, the set $\mathfrak{A}_s|_{[0,T]}$ can be covered by finitely many ϵ-balls in the strong topology. The centers of these balls form the finite set P_T^f. □

7.8 Connection to the Open Problem

We now state the resolution of Cheskidov's open problem explicitly.

Theorem 7.10 (Resolution of Cheskidov's Open Problem)**.** *Let $1/3 \leq \alpha < 1/2$ and assume the spectral gap condition (4.3) holds. Then:*

1. *Every complete trajectory of the dyadic model is continuous in H (and in H^α).*
2. *The dyadic model possesses a strong compact global attractor $\mathcal{A}_s$ in H (and in H^α).*
3. *The trajectory attractor $\mathfrak{A}_s$ satisfies the finite strong uniform tracking property.*

Proof. The proof combines: 1. The inertial manifold theorem (Theorem 9.1), which reduces the dynamics to a finite-dimensional ODE on the manifold. 2. The trajectory attractor theory (Lu's Theorem 3.12), which establishes strong compactness and equicontinuity. 3. The verification of asymptotic compactness (Lemma 7.3) via the inertial manifold. 4. The verification of A1 (Lemma 7.4) via the uniform bound and smoothing property.

The continuity of complete trajectories follows from the equicontinuity of the trajectory attractor. The existence of the strong compact global attractor follows from the compactness of the trajectory attractor and the fact that $\mathcal{A}_s = \mathfrak{A}_s(t)$. The finite strong uniform tracking property follows from the strong compactness of $\mathfrak{A}_s$. □

Remark 7.11. *For $\alpha \geq 1/2$, our result reduces to Cheskidov's original theorem, but with the extra tool of the inertial manifold providing a more detailed description of the attractor. For $1/3 \leq \alpha < 1/2$, our result provides the first rigorous proof of strong compactness and continuity for the global attractor in the literature.*

Remark 7.12. *The spectral gap condition (4.3) is necessary for our construction. In the case where the spectral gap condition fails, the dyadic model may still possess a strong compact global attractor, but the proof would require different techniques. This remains an interesting open question.*

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