New Polyhedra with Ambo

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Abstract

The study of polyhedra and their classification can be significantly expanded by focusing on their generation through operations such as the Conway.. This approach gives rise to new polyhedra, new classifications, new problems, and new solutions. The present work focuses on a few preliminary observations concerning the simpler operators when applied to lesser-known polyhedra**,** with the hope that these insights may be transferable to others.

Simple ambo operation

There is a significant new trend in the field of polyhedra, initiated by John H. Conway and later disseminated and generalized by George W. Hart [2][3][4]. This trend centers on the use of polyhedral operators—some long known, such as **ambo**, and others more novel, such as Hart's **propellor** operator. A comprehensive study of all these operators would require a full volume.

The **ambo operation** (or *rectification*), as defined within **Conway's polyhedron notation** is a foundational transformation that creates a new polyhedron by truncating a polyhedron to the midpoints of its edges. This results in a polyhedron whose vertices correspond to the original's edge midpoints, and whose faces are a combination of rectified original faces and new faces corresponding to the original vertices

Why "Ambo"?

Given a polyhedron and its dual. If we apply the ambo process only to the faces of each, we obtain a polyhedron lacking the faces corresponding to the truncation of the original. The process can be summarized as follows:

Polyhedron	**Dual**	**Combining Both Face Types**
Ambo of the faces	= Truncation of the dual	**Total Ambo** of the polyhedron and its dual
Truncation of the Polyhedron	= Ambo of the dual's faces	**Total Ambo** of the polyhedron and its dual

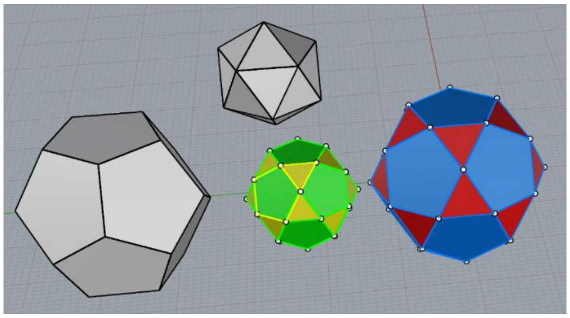


Figure 1: *Dodecahedron, its dual Icosahedron, with her partial ambo and union.*

New Operations

-Symmetrisation (see symmetries in figure 3).

-Definition of partial ambo by faces or partial ambo with help.

Partial ambo by Faces and by Vertices

The ambo of a polygon is equal to the polygon resulting from joining the midpoints of each side of the original polygon. For a square, it will be a rotated square, 90 degrees and smaller.

I hereby define, for the first time, the following **partial ambos**:

Partial ambo by vertex truncation on a polyhedron P is equal to the polygons resulting from joining the midpoints of the edges of a polyhedron, but only for those edges coincident at the same vertex. This is done for all vertices of P. Notation: pav(P).

Partial ambo by faces on a polyhedron P is equal to the polygons resulting from joining the midpoints of the edges of a polyhedron, but only for those points that form a face of P. This is done for all faces. Notation: pac(P).

The **ambo of a polyhedron** is equal to the ambos of the faces (as polygons they are) plus the faces resulting from the truncation passing through the midpoints of the edges that meet at a vertex. This is done for all vertices.

In Conway notation: **a(P)=pac(P)+pav(P).** I will explain the "+" later.

Why this distinction? Because the algorithmic difficulty of computing pac(P) is very low if we have the vertices of the faces of P ordered, in the sense that each face can be computed with a single instruction defining it from the vertices of the face-polygon.

In contrast, for the vertex truncation faces of P, we do not have them well defined. We do not even know which vertices of P form them, or whether those vertices are ordered. Algorithms for computing these faces (except for 3-sided faces, which present no problem) are very time-consuming, often fail, or frequently hang the computer. In general, also in Mathematica, faces with crossed edges appear.

In summary: pac(P) can be simple; pav(P) can be very complicated.

Given this, one solution for computing the ambo is to ask: why ambo? Conway and the books tell us that the ambo is an intermediate polyhedron between P and dual(P). In what sense? They do not tell us.

The explanation I found lies in pavpav and pacpac. It turns out that:

pac(P)=pav(dual(P)), pav(P)=pac(dual(P))

Now then:

a(P)=pac(P)+pav(P)=pav(dual(P))+pac(dual(P))=a(dual(P))

Thus the property **a(P)=a(dual(P)),** which many state but I have not seen proven, is here demonstrated.

Furthermore, by showing how the ambo is the result of a(P)=pac(P)+pav(P)a(P)=pac(P)+pav(P), we can deduce:

a(P)=pac(P)+pac(dual(P)

This gives us an algorithmically easier way to compute the ambo: compute the partial ambo by faces of PP, compute the partial ambo by faces of dual(P) and then join them ("+"). We join them by performing the calculations in the same position (e.g., at the origin (0,0,0)) and uniting the two parts into a single solid. I have verified that this algorithm works perfectly.

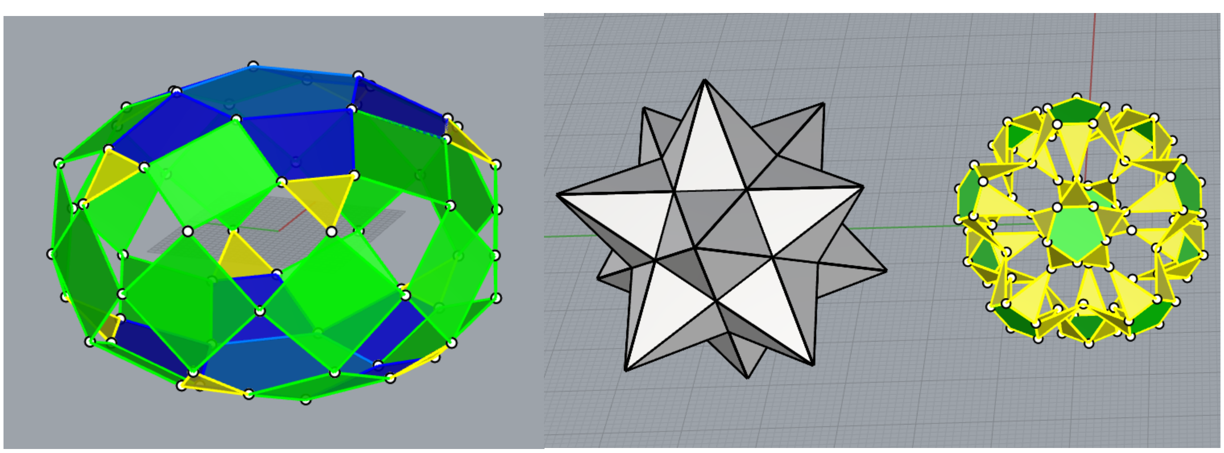
The problem is that it is not always easy to compute the dual of P. And without that datum- which sometimes does not exist for large sets of polyhedra for which we do have other data - the method is not directly applicable.

One question that can be asked is whether these partial ambos serve to find new solids, new polyhedra. A provisional answer is that, in at least 23 cases, exact partial ambos do exist (from an investigation of 418 classical polyhedra), of which we present 3 in the exhibition of Lugo in 2026. And there are many more if the ambo algorithm is extended slightly without completing it fully (an example of which we present in the exhibition and at Bridges 2026).

In very rare cases, for polyhedra generated with multiple copies of a base polyhedron, you can use a partial ambo with exact results. In others, it is necessary complete them, but you can get a better, simpler result than with the full ambo. There are at least 23 polyhedra composed of tetrahedrons, ambo partial of which give a exact polyhedron.

Ambiguity Concave-Convex in the Ambo Operation

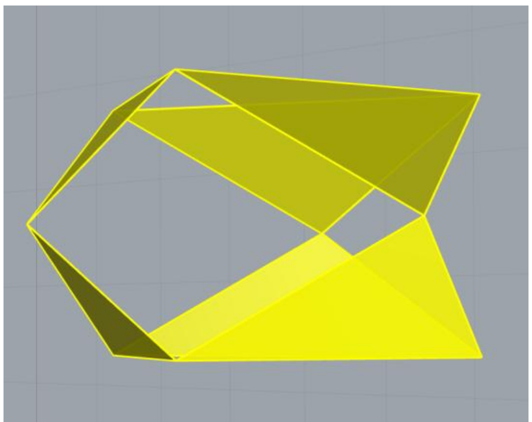
In various cases, applying the ambo operation to a polyhedron creates ambiguity. This occurs because the resulting polyhedron has faces from truncation or from the ambo of faces that are non-planar, requiring a rule for how to handle them.

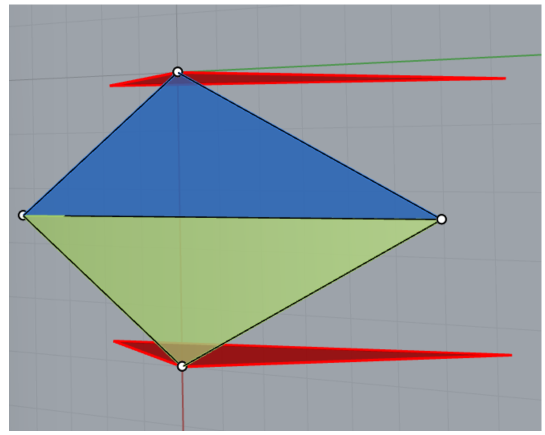


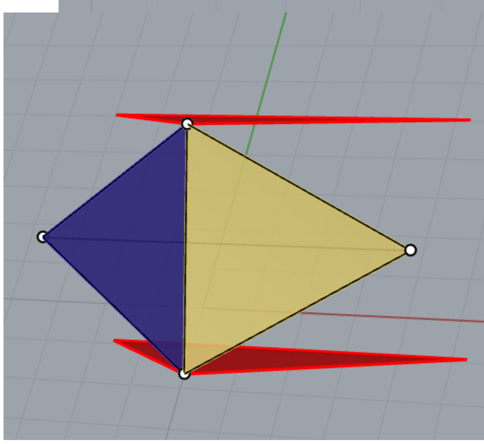
(a) **(b)**

Figure 2: *(a)Ambo of Elongated Square Gyrobicupola and (b) Dodecahedron Stellation 1*

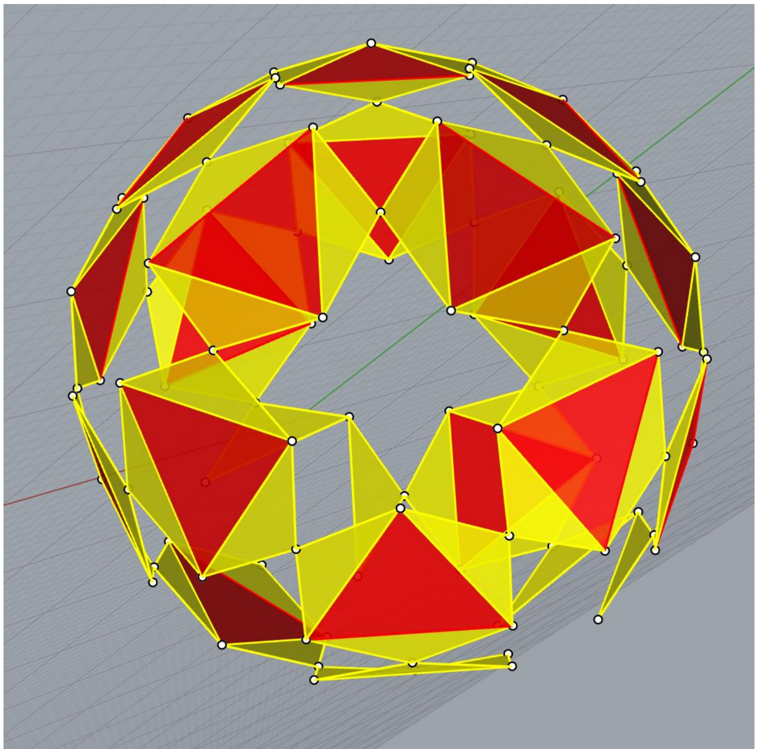
In this and many other examples (Figure 2 and dipyramid in Figure 3), we can choose between the concave or convex option at every instance of this ambiguity. In Figure 2 (a), if we choose triangles with an edge on the "equator," the result will be convex; otherwise, it will be concave. This is akin to known chiral situations If we take the case of the Canonical Dipyramid 3, we have two solutions: one concave and one convex. And that for each of the three ambiguities that the ambo has. In this case we can decide 1) that it has no ambo, 2) that it has 2x2x2 = 8 possible ambos, or 3) we can complete the definition by requiring a convex solution (for each ambiguity) or a concave one, or an alternating one, or any other rule we adopt.

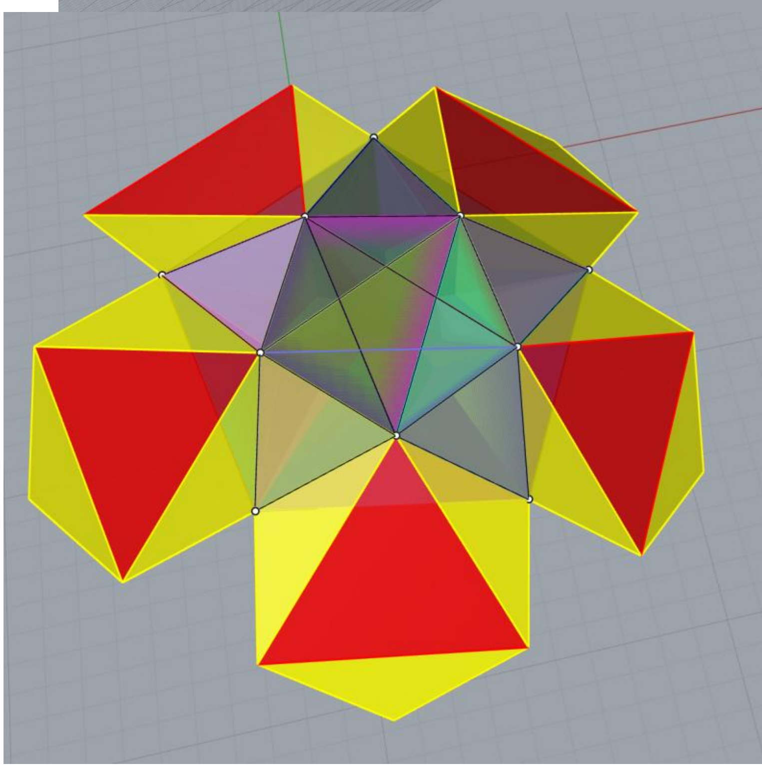






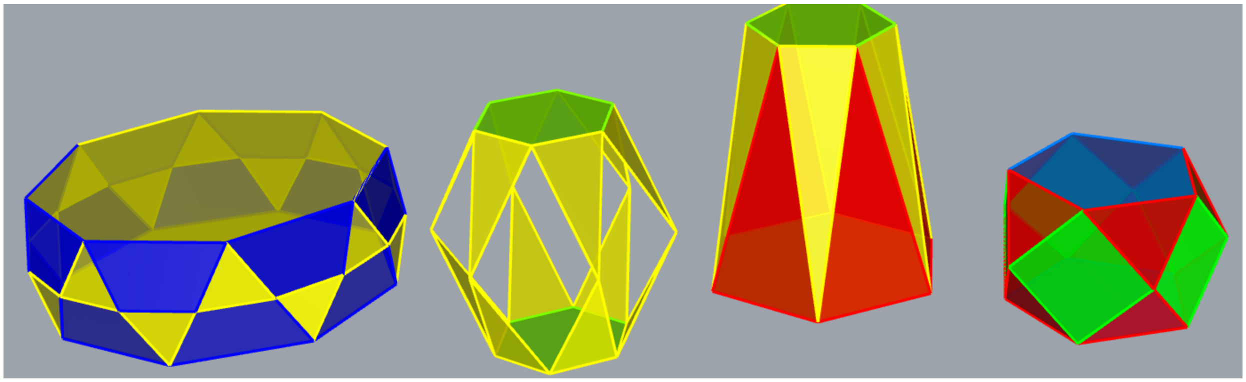
In the case of the Triakisicosahedron, there are ambiguities of 10, which give rise to 120 triangles, so the possible ambos are not 2 as in the previous case, but rather all combinations of some of the 120 triangles that form a closed 'roof' over the hole created by that ambiguity. Predictably, there are many. And also of varied types: concave, convex, and concave-convex combinations. And as if that weren't enough, in this ambo there are 6 of them. There is no single ambo or there are thousands or more possible ambos?





The Case of Pyramids, Prisms, Antiprisms, and Dipyramids

Perhaps the simplest realizations of polyhedra transformed by the ambo operation are with prisms, antiprisms, and pyramids. We can easily imagine a visual rule to generate any of these polyhedra. See the following examples:



(a) (b) (c) (d)

Figure 3: *(a)Antiprism8, (b) Dipyramid6, (c) Pyramid6, (d) Prism5, It is easy to generalize the "ambo" of any regular-based prism, pyramid, dipyramid, antiprism...*

Limits and Reflections

The Limit of Iterating the Ambo Operation is spherization, or the successive approximation of the polyhedron to a sphere (Starting with a regular polyhedron, the pieces originating from the faces become successively smaller and more numerous, and the curvature of the polyhedron- *which should be appropriately defined*- tends toward the curvature of a sphere).

A Reflection. At some point in 2025, I had the impression that my Rhino programs had a utility beyond the composition of a very large number of works (certainly more than any artist has ever produced): a mathematical property. They extend the number of second-level polyhedra, creating a second generation of polyhedra using only the ambo operation and truncation. Each program had its errors, but each error was a new discovery of an unforeseen problem.

Well, this is the description of an experimental mathematics of polyhedra, which could give rise to hundreds of new forms and applications. We can choose from hundreds of tent designs. Do any possess optimal properties? Where does the infinite iteration of ambo or truncation processes lead us?

Some properties of geodesic domes appear in these polyhedra. As in the works of Buckminster Fuller, the fact that the structures use unique or very few distinct pieces facilitates their fabrication.

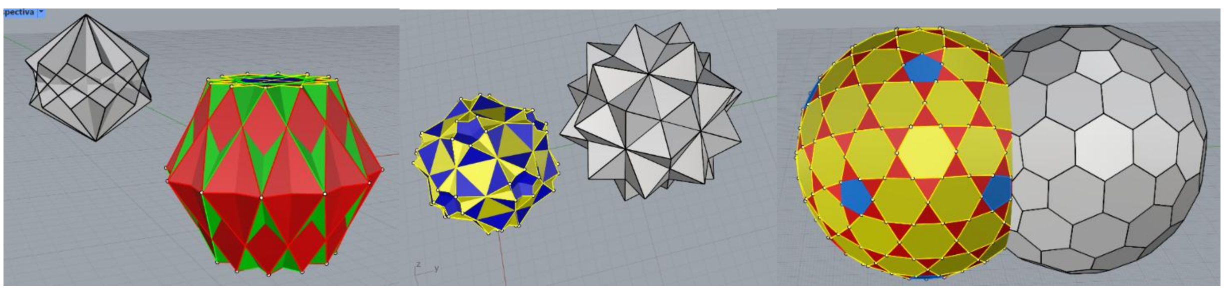
One aspect that solidifies the importance of these discoveries was already present in Kepler: *"Beyond doubt the true form of all these shapes exists eternally in the mind of God the Creator. – Johann Kepler, 1611."* The same reason that led the Islamic world to discover new forms of mosaics, going far beyond Greek or Roman mosaics, was that it strengthened their faith and revealed beauties of God, hidden at first glance. They were discovering, unveiling the power of God.

I have always wondered why those Islamic mosaics are filled with Islamic stars. The reason, as Abas and Salam explain in their book [1], is that the p6m and p4m symmetry groups are the most used in Islamic mosaics. The greater the intrinsic symmetry elements, the greater the beauty. Or, put differently: beauty is related to—is proportional to—the highest number of applied symmetry elements (6-fold or 4-fold rotation) and to the ingenuity of the basic components, especially those that conceal the trace of their creation, appearing as if by magic. Hence, stars appear very frequently in Islamic art.

About Truncation

In my experiments with successive truncation, I observed that results are not always unique when applied to polyhedra with lower symmetry. It is not always feasible for truncation to produce regular polygons at every vertex. In some cases, we may prioritize truncating the largest polygons, or we might choose the most frequently occurring face type in the polyhedron as the "regular" reference. Sometimes, illustrations misleadingly suggest that any truncation is valid, when, in fact, we may only be able to regularize one type of polygon per vertex.

Also, about Beauty



Summary and Conclusions

Building on prior work, I believe it is possible to construct a **genealogy** of polyhedra using generalized operations—or alternatively, a **cladistic classification based on "nobility".** This would allow us to measure how far a given polyhedron is from a regular polyhedron, whether in terms of the number of operational steps required to reach it, the degree of symmetry lost from its regular root, the count of regular faces, or the number of vertex-transitive groups it retains. Such criteria could help classify or contextualize polyhedra under investigation.

In general, the ambo operation tends to reduce a polyhedron's level of nobility: it typically introduces at least one additional irregular polygon and creates new vertices that are not transitive with the existing ones. Thus, while the order of nobility decreases, the aesthetic beauty of the polyhedron does not necessarily diminish.

References

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