Expansion for harmonic numbers inspired by Ramanujan

by **©PAYAM DANESH**

Email: Payamdanesh71@gmail.com

Abstract

In this work, we developed a direct formulation based on the variable $X = n(n+1) + h$, which h is a real parameter. This method starts from the digamma representation of H_n, recenters the expansion at $n + \frac{1}{2}$, and then converts the result into descending powers of X. This actually gives an explicit coefficient formula in terms of Bernoulli polynomials evaluated at $1/2$. The special value $h = \frac{1}{3}$ is shown to cancel the first correction term, giving a sharper expansion whose error begins at a higher order. Numerical comparisons showed that the shifted form improves substantially over the standard Euler truncation and the unshifted pronic expansion for small and moderate n and this gives transparent route to Ramanujan-type harmonic-number approximations.

Keywords: harmonic numbers, Ramanujan expansion, asymptotic expansion, Bernoulli polynomials

---

1. Introduction

The harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad (1)$$

form one of the simplest divergent sequences in analysis. Their growth is logarithmic and the constant term in this growth is the Euler–Mascheroni constant:

$$\gamma = \lim_{n\to\infty} (H_n - \log\, n) \qquad (2)$$

The main problem is to approximate H_n accurately while keeping the approximation simple. Euler's summation formula gives the classical asymptotic expansion of H_n in powers of $1/n$ [1]. A different line of idea appeared in the work of Lodge (often discussed in Ramanujan's work), who studied symmetric approximations to partial sums of the harmonic series and Villarino gave a detailed treatment of Ramanujan's expansion [2]. Later, DeTemple and Wang developed half-integer approximations, showing that the point $n + \frac{1}{2}$ is often a better center than n itself [3]. This midpoint idea is closely connected with Ramanujan's expansion, recorded in his notebook and later edited by Berndt [4]. Ramanujan expressed the correction terms using the triangular number

$$m = \frac{n(n+1)}{2} \qquad (3)$$

which is more symmetric than n^2. Further modifications and related Ramanujan–Lodge expansions were studied also by Mortici [5], Chen and Cheng [6] and Feng and Wang [7].

In compare to previous studies, our goal in this paper is to present a clean shifted-pronic derivation of a Ramanujan-type expansion. We use the pronic number

$$q = n(n+1) \qquad (4)$$

and introduce a shift h, so that the main variable becomes

$$X = q + h \qquad (5)$$

Here h controls the first correction term. A specific value, $h = \frac{1}{3}$, cancels that term completely and improves the order of the approximation.

In this work, instead of starting from Ramanujan's coefficients and transforming them, we start from the centered digamma expansion, convert it into the variable X, and obtain the coefficients explicitly. This gives a transparent path from the classical theory to the shifted Ramanujan-type expansion.

2. Preliminaries and notation

To begin, the digamma function is defined by

$$\psi(x) = \frac{d}{dx}\log \Gamma(x) \qquad (6)$$

and the harmonic numbers fit

$$H_n = \gamma + \psi(n+1) \qquad (7)$$

We intend to use Bernoulli polynomials $B_k(x)$ which is defined by the generating relation

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k\,(x)\frac{t^k}{k!} \qquad (8)$$

The point $x = \frac{1}{2}$ is important since

$$B_{2j+1}\left(\frac{1}{2}\right) = 0 \text{ for all } j \geq 0 \qquad (9)$$

This vanishing removes all odd correction powers in the centered expansion. That is the reason the midpoint $n + \frac{1}{2}$ is mathematically natural. We set

$$z = n + \frac{1}{2} \qquad (10)$$

Then

$$q = n(n+1) = z^2 - \frac{1}{4} \qquad (11)$$

For a fixed real parameter h, we define

$$\beta = h - \frac{1}{4} \qquad (12)$$

So Eqs (5), (11) and (12) give the simple identity

$$z^2 = X - \beta \qquad (13)$$

This identity is the bridge between the half-integer expansion and the shifted-pronic expansion.

Next, Lemma 1 is the analytic starting point.

Lemma 1. As $z \to \infty$,

$$\psi\left(z+\frac{1}{2}\right) \sim \log z - \sum_{j=1}^{\infty} \frac{B_{2j}\left(\frac{1}{2}\right)}{2jz^{2j}} \qquad (14)$$

Proof. The standard asymptotic expansion of the digamma function can be written in Bernoulli-polynomial form:

$$\psi(z+a) \sim \log z + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_k(a)}{kz^k} \qquad (15)$$

Taking $a = \frac{1}{2}$, the term $B_1(\frac{1}{2})$ is zero and all higher odd Bernoulli polynomial values at ½ also vanish. Only even indices remain, which gives (14).

Since $H_n = \gamma + \psi(n+1)$ and $n + 1 = z + \frac{1}{2}$, Lemma 1 gives an expansion for H_n centered at $z = n + \frac{1}{2}$. This is the precise center for the Ramanujan-type formulation since z^2 differs from $n(n+1)$ only by the constant $\frac{1}{4}$. We now convert this Lemma 1 into an expansion in descending powers of $X = n(n+1) + h$.

For $k \geq 1$, we can define

$$A_k(h) = -\frac{\left(h-\frac{1}{4}\right)^k}{2k} - \sum_{j=1}^{k} \frac{B_{2j}\left(\frac{1}{2}\right)}{2j} \binom{k-1}{j-1} \left(h-\frac{1}{4}\right)^{k-j} \qquad (16)$$

Theorem 1. Let $h \in \mathbb{R}$ be fixed. As $n \to \infty$,

$$H_n \sim \gamma + \frac{1}{2}\log X + \sum_{k=1}^{\infty} \frac{A_k(h)}{X^k}, X = n(n+1) + h \tag{17}$$

Proof of Theorem 1. From Lemma 1 and Eq (13),

$$H_n \sim \gamma + \frac{1}{2}\log (X - \beta) - \sum_{j=1}^{\infty} \frac{B_{2j}\left(\frac{1}{2}\right)}{2j} (X - \beta)^{-j} \tag{18}$$

For large X,

$$\frac{1}{2}\log (X - \beta) = \frac{1}{2}\log X - \sum_{k=1}^{\infty} \frac{\beta^k}{2kX^k} \tag{19}$$

Also,

$$(X - \beta)^{-j} = X^{-j} \sum_{\ell=0}^{\infty} \binom{j + \ell - 1}{\ell} \frac{\beta^\ell}{X^\ell} \tag{20}$$

The coefficient of X^{-k} in (18) is the sum of the logarithmic contribution from (19) and the Bernoulli-polynomial contributions from (20). By substituting $\beta = h - \frac{1}{4}$ gives exactly (16).

Remark 1. The symbol $\sim$ is used in the standard asymptotic sense: after shortening Eq (17) at $k = N$, the omitted part is of the next asymptotic order in X^{-1}, provided h is fixed and $n \to \infty$.

3. Ramanujan's expansion

The unshifted pronic case is obtained by taking $h = 0$. The first coefficients from (16) are

$$A_1(0) = \frac{1}{6}, A_2(0) = -\frac{1}{30}, A_3(0) = \frac{4}{315}, A_4(0) = -\frac{1}{105}, A_5(0) = \frac{16}{1155} \tag{21}$$

Since $q = 2m$, equation (17) gives

$$H_n \sim \gamma + \frac{1}{2}\log (2m) + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} + \cdots \tag{22}$$

This is the familiar beginning of Ramanujan's harmonic number expansion. The derivation in the top shows why the triangular number enters naturally; since it is the square of the midpoint $n+\frac{1}{2}$, up to a constant shift. From (16)

$$A_1(h) = \frac{1-3h}{6} \qquad (23)$$

Therefore

$$h = \frac{1}{3} \qquad (24)$$

removes the X^{-1} term. This is the main structural improvement. With

$$X = n(n+1) + \frac{1}{3} \qquad (25)$$

the first non-zero correction terms are

$$H_n \sim \gamma + \frac{1}{2}\log X - \frac{1}{180X^2} + \frac{8}{2835X^3} - \frac{5}{1512X^4} + \frac{592}{93555X^5} - \frac{796801}{43783740X^6} + \cdots \qquad (26)$$

This expansion starts at order X^{-2}, while the unshifted pronic expansion starts at order X^{-1}. Since $X \sim n^2$, this means the first correction after the logarithm has moved from size n^{-2} to size n^{-4}. That is the practical reason the shift $h = \frac{1}{3}$ gives a sharper approximation.

For numerical testing, we define the trimmed shifted-pronic approximation

$$P_N(n;h) = \gamma + \frac{1}{2}\log\left(n(n+1)+h\right) + \sum_{k=1}^{N} \frac{A_k(h)}{(n(n+1)+h)^k} \qquad (27)$$

Here we compare three approximations

$$E(n) = \gamma + \log n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} \qquad (28)$$
$$R(n) = P_4(n;0), \qquad (29)$$

and

$$S(n) = P_6\left(n;\frac{1}{3}\right) \qquad (30)$$

In Figure 1 we plot the base-10 logarithm of the absolute error for $1 \le n \le 50$. The shifted expansion with $h = \frac{1}{3}$ stays below the other two curves over the full range.

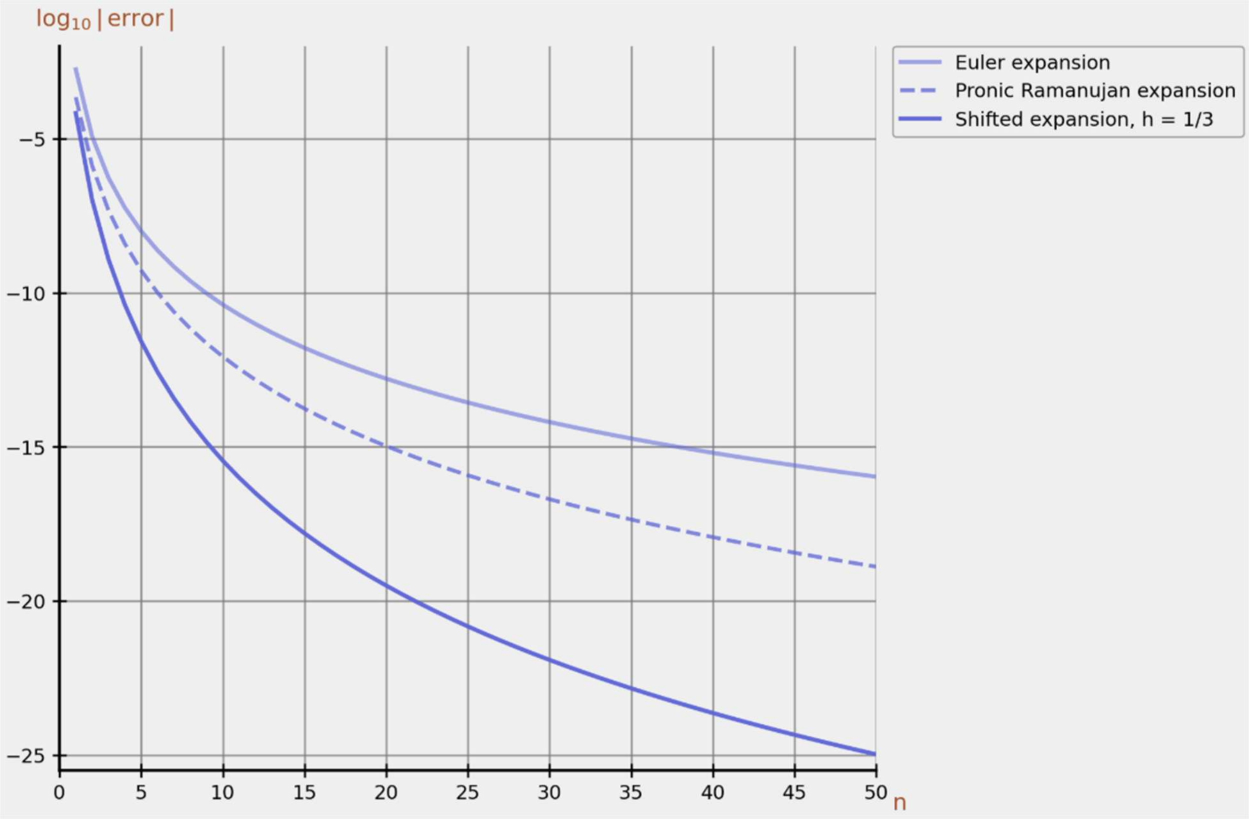

Figure 1. Error comparison

4. Conclusion

The expansion of Ramanujan's harmonic number is particularly simple when considered from the view of the midpoint $n+\frac{1}{2}$. This midpoint converts $n(n+1)$ into the natural quadratic variable and makes clear why triangular and pronic numbers occur in the approximation. Specifically, the shifted-pronic expansion defined above proceeds in a straightforward manner which 1-Represent H_n through the digamma function, 2-Consider the centered expansion about $n+\frac{1}{2}$, 3-Introduce $X = n(n+1)+h$ and 4-Clearly calculate the coefficients via Bernoulli polynomials. This formula for the coefficients is concise and fully computable. The special value of the shift h=1/3 is that it removes the first corrective term. This provides an improved order of approximation, which is numerically verified.

Dedicated to…

To the most brilliant mathematician, dear ***Srinivasa Ramanujan***, who devoted his life to mathematics with purest imagination and faith in hidden order of numbers. And to my beautiful daughter ***Artemisia***, whose lovely eyes gave me courage in hard hours and reminded me why I must keep moving forward.

References

[1] L. Euler and A. Aycock, “Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum,” May 2019, Accessed: Jun. 23, 2026. [Online]. Available: https://arxiv.org/pdf/1905.10438

[2] M. B. Villarino, “Ramanujan’s Approximation to the n th Partial Sum of the Harmonic Series,” 2005.

[3] D. W. DeTemple and S. H. Wang, “Half integer approximations for the partial sums of the harmonic series,” *J. Math. Anal. Appl.*, vol. 160, no. 1, pp. 149–156, Sep. 1991, doi: 10.1016/0022-247X(91)90296-C.

[4] B. C. Berndt, “Ramanujan’s Notebooks,” *Ramanujan's Notebooks*, 1998, doi: 10.1007/978-1-4612-1624-7.

[5] C. Mortici and C. P. Chen, “On the harmonic number expansion by Ramanujan,” *Journal of Inequalities and Applications 2013 2013:1*, vol. 2013, no. 1, pp. 222-, May 2013, doi: 10.1186/1029-242X-2013-222.

[6] C. P. Chen and J. X. Cheng, “Ramanujan’s asymptotic expansion for the harmonic numbers,” *The Ramanujan Journal 2014 38:1*, vol. 38, no. 1, pp. 123–128, Jul. 2014, doi: 10.1007/S11139-014-9575-6.

[7] L. Feng and W. Wang, “Riordan Array Approach to the Coefficients of Ramanujan’s Harmonic Number Expansion,” *Results in Mathematics 2016 71:3*, vol. 71, no. 3, pp. 1413–1419, Sep. 2016, doi: 10.1007/S00025-016-0597-9.

About the author

Payam Danesh is a PhD researcher in Energy, working independently in number theory. He is curious in pure mathematics especially Bernoulli numbers, Ramanujan works and Riemann zeta function.