Ehrenfest Paradox and Consistent Relativistic Circular Motion

Yang-Ho Choi

Department of Electrical and Electronic Engineering
Kangwon National University
Chuncheon, Gangwon-do 24341, South Korea

Abstract: Some conundrums such as the Ehrenfest paradox have been raised in relation to the relativistic approach to circular motions. Different tangential velocities along the radius of a rotating frame bring about a nonuniform scalar field of potential. Based on the Schwarzschild metric for the potential field, we consistently and comprehensively deal with the problems of relativistic circular motion including the Ehrenfest paradox and the Sagnac effect. The Ehrenfest paradox is readily resolved via a visualization of wave propagation in the field, which shows that the length of radius in the rotating frame is different from the corresponding one seen in the laboratory frame. From the visualization, the anisotropy of the speed of light in inertial frames is also clearly shown. Moreover, a coordinate transformation for the circular motion at a fixed radius is developed based on the metric, which enables us to derive a transformation between inertial frames through limiting operations. The derived inertial transformation becomes the same as the Lorentz transformation if the standard synchronization is introduced. With the transformations, the Michelson-Morley experiment result and the generalized Sagnac effect are consistently explained.

Keywords: Relativistic circular motion, Ehrenfest paradox, Speed of light, Schwarzschild metric, Potential field

1. Introduction

There have been some conundrums, such as the Ehrenfest paradox and the problem of time gap, in the comprehensive understanding of relativistic circular motions [1–12]. The Ehrenfest paradox is concerned with the circumference in the rotating frame in relation to the radius in the laboratory frame [1–8]. According to special relativity, the radius in the former has been considered to be the same as the one in the latter because the tangential velocity is perpendicular to the radial direction. When the standard synchronization is introduced along a closed loop such that the speed of light appears to be the constant c, the problem of time gap [9–12] that multiple times are defined at the same place is caused because the clock synchronization is path-dependent. Though time gaps are inevitable for the standard synchronization, on account of the problem, it is stated in Ref. 9 that "However, synchronization of clocks along a closed contour turns out to be impossible in general."

Since the finding of the Sagnac effect in 1913 [13], the interpretation of it within relativity has a long history of controversy [12,14–23]. The Sagnac effect, which traditionally has been dealt with by employing Galilean transformations in cylindrical coordinates, appears to be in agreement with the nonrelativistic analysis. However, according to Ref. 14, "For uniform rotation in the case of the Sagnac effect one would expect on intuitive grounds that a Galilean rotation (absolute time) might give the correct choice of space-time coordinate transformation. In consideration, however, of well known experiences with electromagnetic theory in the realm of uniform translations where the Galilean translation (absolute time) is not an adequate substitute for a Lorentz translation, it is useful to give special attention to the question of selecting the right transformation for uniform rotations." The famous experiment of Michelson and Morley (MM) [24] showed null results, which might have led to the belief that the speed of light is isotopic in inertial frames. On the contrary, in the experiments of the generalized Sagnac effect [25,26], the Sagnac effect has been discovered in rectilinear motion as well as circular motion, which indicates the anisotropy of the speed of light in inertial frames [27–29]. The generalized Sagnac effect presents another conundrum in the relativity [30–33]. Moreover, recently it has been revealed through relativistic analyses that the Michelson-Gale (MG) experiment also shows the anisotropy [34].

In a rotating frame, a nonuniform scalar field of potential is formed along the radius by the different tangential velocities. Based on the Schwarzschild metric [35], from which the relationship of radius between the rotating and the laboratory frames is found, we consistently and comprehensively deal with the problems of circular motion including the conundrums, the MM experiment result, and the generalized Sagnac effect. The Ehrenfest paradox is readily resolved through a visualization of wave propagation in the field, which shows that the radius in the rotating frame is different from the corresponding one in the laboratory frame. Moreover, the visualization enables us to clearly see the anisotropy of the speed of light in inertial frames.

"The right transformation for uniform rotations" at fixed radius is developed based on the metric. The two-way speed of light is c regardless of direction in the transformation. In addition, it allows us to derive a transformation between inertial frames by making the radius tend to infinity under the constant tangential speed. The derived inertial transformation becomes identical with the Lorentz transformation if the standard synchronization is introduced. With the transformations, the problems of relativistic circular motion are consistently dealt with.

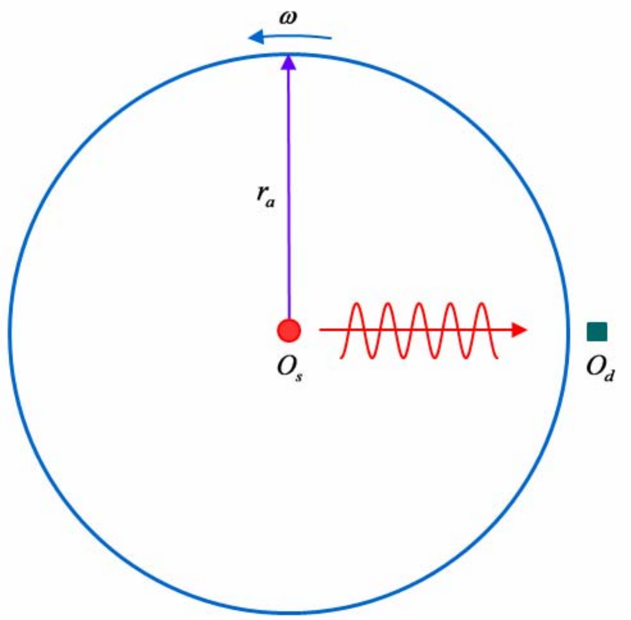

Fig. 1. A disk of radius r_a is rotating with angular velocity ω. A light source O_s located at the center of rotation transmits a light signal toward a detector O_d at rest in a laboratory frame.

2. Rotation and Ehrenfest paradox

When seen in a laboratory frame S, as shown in Fig. 1, a disk of an arbitrary radius r_a is rotating with a constant angular velocity ω and a light source O_s, which is located at the center of rotation, transmits a light signal toward a detector O_d at rest in S. The laboratory frame S during a certain test of very short time can be considered to belong to an inertial frame, which is standard-synchronized so that the speed of light is c with respect to the adjusted time (AT). The coordinate system of the disk is denoted by $\tilde{S}'$. The transformation between the unprimed S and the primed $\tilde{S}'$ is relativistic. The unprimed coordinate system corresponding to the primed $\tilde{S}'$ is $\tilde{S}$. The transformation between $\tilde{S}$ and S is non-relativistic Galilean and $\tilde{S}$ can be viewed as $\tilde{S}'$ seen in the unprimed [27,29]. Similarly S' is the primed coordinate system corresponding to S. The transformation between S and S' is relativistic while that between S' and $\widetilde{S}'$ is Galilean. In the primed, $\widetilde{S}'$ is rotating relative to S'.

The radius in $\widetilde{S}'$ is identical to that in S'. We will first seek the relationship of radius between S and S'. To this end, the coordinates concerned are time and radius, i.e., $[t, r]$ in S and $[t', r']$ in S'. The rotation of $\widetilde{S}'$ leads to acceleration in the radial direction, which brings about a potential field along the radius. The potential φ normalized with respect to c^2 can be expressed as

$$\varphi(r)\,(=\varphi'(r')) = -\frac{1}{2}\left(\frac{r\omega}{c}\right)^2 \tag{1}$$

where the potential is assumed to be zero at $r = 0$. Based on the Schwarzschild metric [35], the transformation between S and S' can be written as

$$dt' = (1+2\varphi(r))^{1/2}\,dt\,,\quad dr' = (1+2\varphi(r))^{-1/2}\,dr\,. \tag{2a,b}$$

As explained in Appendix, the coordinates in the Schwarzschild metric represent those of an inertial frame, not arbitrary ones. If $r = 0$, $r' = 0$. Integrating (2b), we have

$$r' = \frac{r}{\beta}\arcsin\beta \equiv g(r) \tag{3}$$

where $\beta = r\omega/c$. Clearly $g(r)$ is a monotonically increasing function. Given $\varphi(r)$, the expression for $\varphi'(r')$ is discovered by putting $r = g^{-1}(r')$ into $\varphi(r)$, though it may be very complex to find the inverse function. Equation (3) can be rewritten using the Taylor series expansion of $\arcsin\beta$ as $r' = r(1+\beta^2/6+3\beta^4/40+\cdots)$, which, by multiplying both sides by ω/c, results in

$$\beta' = \beta + \frac{\beta^3}{6} + \frac{3\beta^5}{40} + \cdots \tag{4}$$

where $\beta' = r'\omega/c$. The r' can be expressed, in the approximation of $O(\beta^3)$, as $r' = r(1+\beta^2/6)$. It is not difficult to see from (4) that in the approximation of $O(\beta^3)$ or equivalently $O(\beta'^3)$,

$$\beta'^2 = \beta^2\,. \tag{5}$$

Then, replacing β^2 in $r' = r(1+\beta^2/6)$ with β'^2 yields

$$r = r'(1-\beta'^2/6)\,. \tag{6}$$

The radius r is less than r'.

When r or r' is fixed, the other is determined by (3). Given $r = r_a$ or $r' = r'_a$, to develop "the right transformation", we introduce a new coordinate system $\widetilde{S}'_l$ where the potentials are the same as $\varphi(r_a)\,(=\varphi'(r'_a))$ irrespective of radius r'_l, which is related to the radius r by $r'_l = \gamma_a r$ with $\gamma_a = (1-\beta_a^2)^{-1/2}$ and $\beta_a = r_a\omega/c$. The time coordinate of $\widetilde{S}'_l$ is given by $t' = \gamma_a^{-1}t$. In the primed,

$\widetilde{S}'_{/}$ is rotating with an angular velocity $\omega'_{/}$ relative to $S'_{/}$ ($\omega'_{/}$ is found in Appendix) and the transformation between them is Galilean. A rotating observer $\widetilde{O}'$ is located at $r = r_a$ as seen in S. The proper distance between the source and the observer is the rest length r'_a. The corresponding radius in $S'_{/}$ is $r'_{/a} = \gamma_a r_a$, which is expressed using (5) and (6) as

$$r'_{/a} = r'_a(1+\beta'^2_a/3)\,. \tag{7}$$

It is larger than r'_a.

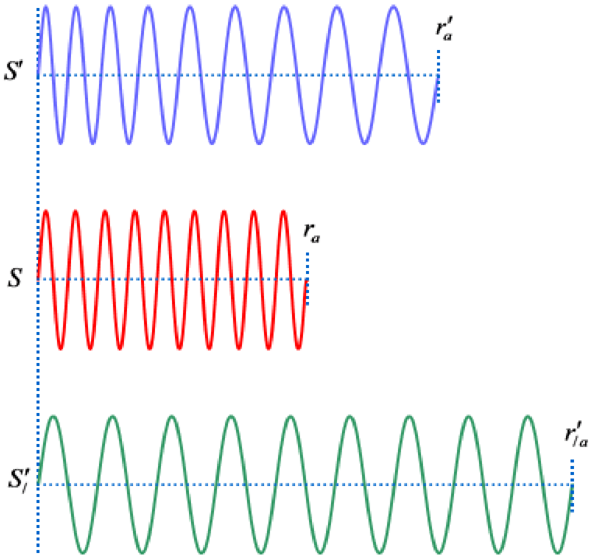

Fig. 2. It illustrates a waveform in S and the corresponding ones in S' and $S'_{/}$. Since the wavelength in S is the smallest, the radius r_a in S is smaller than the rest radius r'_a in S'. The wavelength in $S'_{/}$ is the largest and so $r'_{/a}$ is larger than r'_a.

Fig. 2 illustrates a waveform of the light signal with a wavelength of λ_a at a certain instant in S. In S', according to (2b), the wavelength of the ray is smaller as it is close to the source. In other words, the wavelength is larger as it is close to the detector. And the wavelength in $S'_{/}$ is $\gamma_a\lambda_a$. Reflecting these facts, Fig. 2 depicts the waveforms in S' and $S'_{/}$ corresponding to the one in S. The numbers of the crests of the waves in S, in S', and in $S'_{/}$ are all equal. Even if the source O_s is attached to the axis of rotation so that it belongs to S', it is at the same potential as in S. The wavelength in S is the smallest. Hence the distance r_a that the waveform in S covers becomes less than the proper length r'_a, which can also be seen from (6). In contrast, the wavelength at $r' = r'_a$ in S' is the largest and it is equal to that in $S'_{/}$. Since the wavelength is the largest, the distance $r'_{/a}$ is larger than r'_a, which is also shown in (7).

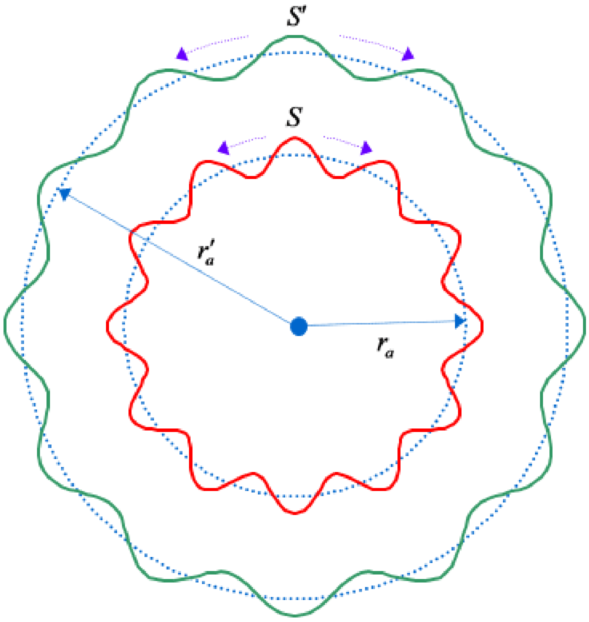

Fig. 3. When the wave in S travels along the circumference clockwise or counterclockwise, the corresponding one in S' does so. The speed of the wave in S is always c and thus that in S' is independent of the direction. Since the transformation between S' and $\widetilde{S}'$ is Galilean, the speed in $\widetilde{S}'$ depends on the direction. When $r_a \to \infty$ subject to the constant β_a, $\widetilde{S}'$ consists of inertial frames. The speed of light is anisotropic in the inertial frames.

The Ehrenfest paradox concerns the primed circumference in relation to the unprimed radius. The spatial coordinates are irrelevant to synchronization schemes. Although the proper radius of the disk is r'_a in S', the circumference of radius r'_a is different from $2\pi r'_a$ though that in S is $l_a = 2\pi r_a$. Consider a waveform along the circumference of radius r_a in S, as illustrated in Fig. 3. The potential is formed by the tangential velocity and the wavelength on the corresponding circumference in S', which is the same as the wavelength in $S'_{/}$, becomes γ_a times larger than the wavelength λ_a in S. The circumference in S' is $l'_a = 2\pi r'_{/a}$, neither $2\pi r'_a$ nor $2\pi r_a$, which resolves the Ehrenfest paradox. Though $r'_a \neq r_a$, $l'_a / r'_a > 2\pi$, as Einstein expected [36]. The spatial spaces of S' and $\widetilde{S}'$ are curved. On the contrary, in $S'_{/}$ and $\widetilde{S}'_{/}$, the ratio $l'_a / r'_{/a}$ is 2π and their spatial spaces are flat.

3. Coordinate transformations

Hereafter, we are only concerned about the coordinates at a fixed radius $r = r_a$ or $r' = r'_a$. Then, the time coordinates according to (2a) are related by $t' = \gamma_a^{-1} t$ with the initial condition of $t = t' = 0$.

Since $\lambda'_a = \gamma_a \lambda_a$ and $r'_{/a} = \gamma_a r_a$ so that $\lambda'_a / r'_{/a} = \lambda_a / r_a$, the azimuth angles ϕ and $\phi'_/$, respectively, in S and $S'_/$ become equal. The *z*-components of the unprimed and the primed are the same. The coordinate transformation between S and $S'_/$ is then given by

$$t' = \gamma_a^{-1} t\,,\ \ r'_/ = \gamma_a r\,,\ \ \phi'_/ = \phi\,,\ \ z' = z\,. \qquad \text{(8a–d)}$$

Since the transformation between $S'_/$ and $\tilde{S}'_/$ is Galilean, the azimuth angle $\tilde{\phi}'_/$ in $\tilde{S}'_/$ is related to ϕ by $\tilde{\phi}'_/ = \phi - \omega t$. The transformation between S and $\tilde{S}'_/$ is written as

$$t' = \gamma_a^{-1} t\,,\ \ r'_/ = \gamma_a r\,,\ \ \tilde{\phi}'_/ = \phi - \omega t\,,\ \ z' = z\,. \qquad \text{(9a–d)}$$

The $\tilde{S}'$ is transformed into $\tilde{S}'_/$ such that the primed space-time appears flat. In the flat space-time, (9) is the right transformation for circular motion at $r = r_a$.

Since the circumferences in $\tilde{S}'$ and $\tilde{S}'_/$ are equal, $r'_a \tilde{\phi}' = r'_{/a} \tilde{\phi}'_/$ and so $\tilde{\phi}' = \alpha_a \tilde{\phi}'_/$ where $\tilde{\phi}'$ is the azimuth angle in $\tilde{S}'$ and $\alpha_a = r'_{/a} / r'_a$. If r_a tends to infinity under the constraint that β_a is constant, ω approaches zero and then $\tilde{S}'$ is composed of inertial frames that move in different directions with the same speed β_a relative to S. Accordingly, a coordinate transformation between S and an inertial frame can be derived from the transformation for circular motion. From (9a–c), $dt' = \gamma_a^{-1} dt$ and $r'_{/a} d\tilde{\phi}'_/ = \gamma_a r_a d\phi - \gamma_a r_a \omega\, dt$. As $r_a \to \infty$ subject to the condition that β_a is constant, the differentials $r'_{/a} d\tilde{\phi}'_/$ and $r_a d\phi$ approach dx' and dx, respectively. Thus,

$$dt' = \gamma_a^{-1} dt\,,\ \ dx' = \gamma_a (dx - \beta_a c\, dt)\,. \qquad \text{(10a,b)}$$

Circular motion can be considered locally inertial, according to which the same inertial transformation can also be derived. As $dt \to 0$ and $d\phi \to 0$, $r'_{/a} d\tilde{\phi}'_/$ approaches dx' and the same equation as (10b) is obtained. The azimuth angles $\tilde{\phi}'$ and $\tilde{\phi}'_/$ are related by $\tilde{\phi}' = \alpha_a \tilde{\phi}'_/$. In differential form, $d\tilde{\phi}' = \alpha_a d\tilde{\phi}'_/$, which leads to $r'_a d\tilde{\phi}' = r'_{/a} d\tilde{\phi}'_/$. In terms of the coordinates of $\tilde{S}'$, the dx' in (10b) represents $r'_a d\tilde{\phi}'$. It is well known [30,37–40] that the two-way speed of light is c in the transformation (10), which results in the Lorentz transformation when the standard synchronization is employed.

The two-way speeds of light in $\tilde{S}'$ and in $\tilde{S}'_/$ are the constant c, which is in agreement with the result of the MM experiment [24]. As a matter of fact, the transformation (9) is identical to the transformation under the constant light speed (TCL) [27,29,34]. Anisotropy of the one-way speed of light in inertial frames can be readily seen from the wave propagation in $\tilde{S}'$ when $r_a \to \infty$ subject to the constant β_a. The speed of light in S is assumed to actually be c irrespective of the direction. In Fig. 3, imagine that the wave in S is traveling clockwise or counterclockwise. Then the corresponding wave in S' propagates accordingly. The speed of light in S is independent of the direction, so is the speed in S'. The transformation between $\tilde{S}'$ and S' is Galilean. Thus the one-way speeds of light in the inertial frames, which constitute $\tilde{S}'$, come to depend on the propagation direction, which has already been observed in the MG experiment [34] as well as in the experiment of the generalized Sagnac effect [25–29].

In Appendix, the speed of light has been derived using the transformations, which indicates that it is anisotropic in inertial frames. Moreover, the average speed of light when it traverses a closed loop in $\tilde{S}'$ or $\tilde{S}'_l$ is shown to be c, which leads to the constant average speed of light in inertial frames via the limiting operation. In the experiment of the generalized Sagnac effect, two light beams traverse a closed loop in opposite directions and the travel time difference is obtained as $\Delta t'_d = 2l'_a\beta_a / c$ where l'_a is the rest length of the loop. Their speeds are given by $\tilde{c}'_\pm = c/(1 \pm \beta_a)$, respectively, where $\tilde{c}'_+$ and $\tilde{c}'_-$ are the speeds of the co- and counter-propagating beams, respectively. The $\tilde{c}'_+$ and $\tilde{c}'_-$ are exact as the average speeds even if S is not actually isotropic. In the experiment of the Sagnac effect that involves only circular motion, the travel time difference is also given by $\Delta t'_d = 2l'_a\beta_a / c$ where l'_a is the rest length of the circular loop. Although the light beams transverse closed paths in $\tilde{S}'$ their average speeds are not equal to c. The reason is because they traverse very special paths, as explained in Appendix. The time gaps in the travel paths are closely connected with the travel time difference, which is just the difference between them. See Appendix for details.

4. Conclusion

The relationship of radius in (2) is different from that in (8). Given $r = r_a$, the proper radius in the primed is found by (2b), but its perimeter is $2\pi r'_{la}$, not $2\pi r'_a$, by (8b). In the Sagnac effect experiment, the travel time difference and the speed of light are events on the primed circumference and they can be discovered conveniently in the flat space-time $\tilde{S}'_l$ or S'_l, instead of the curved one $\tilde{S}'$ or S'. When events at fixed radius are concerned circular motions can be relativistically approached by (8) or (9), which is consistent with the inertial transformation. However, if events in the radial direction are involved, (2) should be employed. With the transformations (2), (8), and (9), consistently without contradiction, the problems of relativistic circular motion can be dealt with.

Many laws of physics have been known in the isotropic space-time. The standard synchronization is very useful. Making a local space-time appear as an isotropic frame, it allows us to use not only the convenient Lorentz transformation but also the known laws of physics. However, the isotropy of the speed of light in inertial frames is not the physical nature. The standard synchronization is a tool to approach problems of physics. Einstein, with great insight, stressed [41], “That light requires the same time to traverse … is in reality neither a supposition nor a hypothesis about the physical nature of light, but a stipulation which I can make of my own freewill in order to arrive at a definition of simultaneity."

Appendix

1. Derivation of the speed of light

Before finding the one-way speed of light in the primed rotating frame, we first calculate the angular velocity of an observer $\widetilde{O}'$, who is at rest in it. The azimuth angle in $S'_{/}$ of $\widetilde{O}'$ can be written as $\phi'_{/}(=\phi)=\omega t+\widetilde{\phi}'_{/a}$ where the constant $\widetilde{\phi}'_{/a}$ is the azimuth angle in $\widetilde{S}'_{/}$ of $\widetilde{O}'$. The angular velocity of $\widetilde{O}'$ relative to $S'_{/}$ is computed as $\omega'_{/}(=d\phi'_{/}/dt')=\gamma_a\omega$. Meanwhile, the angular velocity in S' of $\widetilde{O}'$ is obtained as $\omega'=\alpha_a\gamma_a\omega$. It is easy to see that $\phi'_{/}=\widetilde{\phi}'_{/}+\omega'_{/}t'$ and that $\phi'=\widetilde{\phi}'+\omega't'$ where ϕ' represents the azimuth angle in S'.

In S, the squared line element is expressed as

$$ds^2=-c^2dt^2+\ r_a^2d\phi^2+dz^2\,. \tag{11}$$

Substituting (9) with $\widetilde{\phi}'_{/}=\phi-\omega'_{/}t'$ into (11) yields

$$ds^2=-c^2(\gamma_a dt')^2+r'^2_{/a}(d\widetilde{\phi}'_{/}+\omega'_{/}dt')^2/\gamma_a^2+dz'^2\,. \tag{12}$$

Equation (12) is rewritten using $r'_{/a}\omega'_{/}=\gamma_a^2c\beta_a$ as

$$ds^2=-(cdt')^2+2\beta_a\,r'_{/a}d\widetilde{\phi}'_{/}(cdt')+r'^2_{/a}d\widetilde{\phi}'^2_{/}/\gamma_a^2+dz'^2\,. \tag{13}$$

Setting $ds=0$ for a light signal, we have $cdt'=\beta_a r'_{/a}d\widetilde{\phi}'_{/}+dl'$, which is expressed according to $r'_a d\widetilde{\phi}'=r'_{/a}d\widetilde{\phi}'_{/}$ as

$$cdt'=\beta_a r'_a d\widetilde{\phi}'+dl' \tag{14}$$

where $dl'=(r'^2_a d\widetilde{\phi}'^2+dz'^2)^{1/2}$. Then the speed of light is given by

$$\widetilde{c}'=\frac{dl'}{dt'}=\frac{c}{1+\beta_a\cos\xi'} \tag{15}$$

where $\cos\xi'=r'_a d\widetilde{\phi}'/dl'$. As long as the circumference in $\widetilde{S}'$ is $2\pi\gamma_a r_a$ and $r'_a d\widetilde{\phi}'=r'_{/a}d\widetilde{\phi}'_{/}$, (10) and (15) are valid regardless of the relationship between r'_a and r_a. For example, even if $r'_a=r_a$, they hold. In a round trip, the sign of $d\widetilde{\phi}'$ in one way is opposite to that in the other and the two-way speed of light is c [27,29]. Furthermore, even if a light beam traverses a closed path in $\widetilde{S}'$, its speed becomes c [29]. It is because the integration of the first term $\beta_a r'_a d\widetilde{\phi}'$ in (14) along the path is zero. If the standard synchronization is introduced so that the first term appears to have no effect, the speed becomes c with respect to AT.

When $r_a \to \infty$ subject to the constant β_a, the observer $\widetilde{O}'$ belongs to an inertial frame moving with a constant velocity with its magnitude β_a relative to S and (14) and (15) are valid, by replacing $r'_a d\widetilde{\phi}'$ with dx', for rectilinear motion as well. Then (15) represents the speed of light in the inertial frame, which is a function of direction ξ' and is anisotropic. However, when a light beam traverses a closed path in the inertial frame, its speed becomes c because the integration of $\beta_a dx'$ along the path is zero.

2. Sagnac effect and time gap

In the experiment of the Sagnac effect, the co-rotating b_+ and the counter-rotating light beam b_- emitted at the same time from a light source $\widetilde{O}'_s$, which is at rest in $\widetilde{S}'$, traverse a circular loop of radius r'_a and the effect of the travel time difference is recorded by a detector $\widetilde{O}'_d$ located at the same place as $\widetilde{O}'_s$. The laboratory frame S can be considered to belong to an inertial frame during the test. Since $dz' = 0$, $dl' = r'_a | d\widetilde{\phi}' |$. Integrating (14) along the travel paths of $b_\pm$, we have $ct'_\pm = (1 \pm \beta_a) l'_a$, respectively, where $t'_\pm$ are the travel times of $b_\pm$ and $l'_a = 2\pi r'_{/a}$. The time difference is written as $\mathit{\Delta} t'_d (= t'_+ - t'_-) = 2 l'_a \beta_a / c$. The speeds of $b_\pm$ are obtained by inserting $\cos\xi' = \pm 1$ in (15), which yields $\widetilde{c}'_\pm = c/(1 \pm \beta_a)$.

In the experiment of the generalized Sagnac effect [25,26], the co- and counter-propagating light beams $b_\pm$ traverse a closed loop, whose motion involves linear motion as well as circular motion. It is easy to see by using (10) or (14) with $r'_a d\widetilde{\phi}'$ replaced by dx' that the resulting time difference is also given by $\mathit{\Delta} t'_d = 2 l'_a \beta_a / c$ where l'_a is the rest length of the closed loop. The Sagnac effect results from the first term in (14).

If the standard synchronization is employed along the co- and counter-propagating paths, which causes the time gap problem though, the speeds of $b_\pm$ become c with respect to AT as the effect of the first term is cancelled. The time gaps for the travel paths of $b_\pm$ at the position of the detector are $\pm l'_a \beta_a / c$, respectively. Since the travel time difference takes place due to the first term and the time gaps do so, the former is the same as the difference between the latter gaps. Through the standard synchronization along the closed paths despite time gaps, the same travel times can also be obtained. If the standard synchronization is set up, the travel times of $b_\pm$ when taking no account of time gap are both l'_a / c. We now know why the time gaps occur. They should be added to the travel times, which results in $t'_\pm = (1 \pm \beta_a) l'_a / c$. The same travel times are found, which shows that the standard synchronization is very useful.

3. Instantaneous speed and average speed

Equation (15) represents an instantaneous speed. Unless S is actually isotropic the instantaneous speed is not exact since β_a is not the speed with respect to the isotropic frame. Despite it, the $\tilde{c}'_\pm$ $(= c/(1 \pm \beta_a))$ are exact as the average speeds, as explained in the following. In S, the travel times of $b_\pm$ are given by $t_\pm = l_a / c(1 \mp \beta_a)$ in AT where $l_a = 2\pi r_a$. The elapsed times recorded by the clock of the detector $\tilde{O}'_d$, which are proper times, are $\tilde{t}'_\pm = t_\pm / \gamma_a$ by (9a). Recall $l'_a = \gamma_a l_a$. The average speeds of $b_\pm$ are given by $l'_a / \tilde{t}'_\pm$, which are calculated as $c/(1 \pm \beta_a)$.

In the experiment of the Saganc effect, though the light beams traverse a closed loop, the average speeds are not c, which seems to contradict what was mentioned above. Recall $r'_a d\tilde{\phi}' = r'_{/a} d\tilde{\phi}'_/$ in (14). The loop is exceptionally special because the integration of $d\tilde{\phi}'_/$ along a closed path is 2π or -2π, not 0. The integration of $d\tilde{\phi}'_/$ along a closed loop in $\tilde{S}'_/$ is always 0, if its absolute value is not 2π. Then the average speed becomes c even if the speed of light is not actually isotropic in an inertial frame S.

4. Schwarzschild metric

Let S' represent the space-time with a normalized potential $\varphi(r)$. The Schwarzschild metric is written as

$$ds'^2 = -(1 + 2\varphi(r))(cdt)^2 + (1 + 2\varphi(r))^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) . \qquad (16)$$

.It is clear that if $\varphi(r) = 0$, the space-time S' becomes an inertial frame S and $ds' = ds$ where ds is the line element in S, which is actually isotropic or is standard-synchronized. Accordingly, the coordinates $[t, r, \theta, \phi]$, which are independent of $\varphi(r)$, represent those of S, not arbitrary ones. And the ds'^2 of (16) denotes the squared line element in S'.

We fix the azimuth and elevation angles so that $ds'^2 = -(1 + 2\varphi(r))(cdt)^2 + (1 + 2\varphi(r))^{-1} dr^2$. The relationships between the differentials in S' and in S are given as (2) with the different potential. In case $\varphi(r) = 0$, $dt' = dt$, $dr' = dr$, and $ds' = ds$.

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