

From : -  
B.N.Pathak  
6B/B5, Yashopuram  
Pimpri Chinchwad Link Road,  
Chinchwad, Pune – 411033,  
Maharashtra, India.

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### Proof to Beal's Conjecture.

By: B.N. Pathak.

1. Beal's Conjecture states that if  $x^m + y^n = z^r$  ---- ( 1.0 ), where  $x, y, z$  and  $m, n$  and  $r$  are all positive integers and  $m, n$  and  $r$  are greater than 2, then  $x, y$  and  $z$  must have a common prime factor.
  - 1.1 In fact as an amendment to above, under aforesaid conditions,  $x, y$  and  $z$  shall have a common factor which could as well be a non-prime or a composite number comprising of products of prime numbers.
  - 1.2 Alternately and as also equation (1.0) above, shall have no positive integral solutions in  $x, y$  and  $z$ , with no common factor amongst themselves, i.e. when they are co-prime i.e. when  $\gcd(xy) = \gcd(yz) = \gcd(zx) = 1$ , where  $m, n$  and  $r$  are all positive integers greater than 2.
  - 1.3 We shall also deal with the 'Specific Case of Beal's Conjecture' as per para 2.4 and its sub-paras.

### 2. Proof

The proof of the above statements can easily be derived from the detailed analysis of the very structure of equation (1.0). There will be following three distinct considerations, which will be taken up one by one. Let  $x, y, z, m, n, r$  be all positive integers and  $m, n$  and  $r$  be all greater than 2.

( i ) When  $y = ax$ , where  $a =$  Any positive integer including unity and  $m = n$ , i.e.  $m - n = 0$

( ii ) When  $y = bx$ , where  $b = 1$  and  $m - n = 1$

( iii ) When  $y \neq ax$ , where  $a =$  Any positive integer.

2.1 From equation ( 1.0 ), we have the following at  $y = ax$

$$z^r = x^m + y^n \quad \text{---- ( 1.0 )}$$

$$= x^m + (ax)^n = x^m (1 + a^n x^{(n-m)}) \quad \text{---- (2.0 )}$$

Assuming  $a = 1$ , i.e.  $x = y$  and  $m = n$ , we have :-

$$z^r = x^m (1 + 1)$$

Further assuming  $x = 2^g$ , where  $g = \text{Any positive integer}$ , we have :-

$$z^r = 2^{gm} (1 + 1) = 2^{gm} + 2^{gm} = 2^{gm} (2) = 2^{(gm+1)} \text{ ---- (2.01)}$$

From (2.01), we have  $z = 2$  and  $r = gm + 1$ , whence we have :-

$$\begin{aligned} 2^{(gm+1)} &= 2^{gm} + 2^{gm} = (2^g)^m + (2^g)^m = (2^m)^g + (2^m)^g \\ &= 2^{mg} + 2^{mg} \text{ ---- (2.01.1)} \end{aligned}$$

Giving different values to  $m$  greater than 2, we have following sub-equations from equation (2.01.1) :-

$$2^4 = 2^3 + 2^3 \quad (\text{at } g = 1, m = 3) \quad \text{---- (i)}$$

$$2^5 = 2^4 + 2^4 \quad (\text{at } g = 1, m = 4) \quad \text{----(ii)}$$

$$2^7 = 2^6 + 2^6 = (2^2)^3 + (2^2)^3 = 4^3 + 4^3 \quad (\text{at } g = 2, m = 3) \text{ ---- (iii)}$$

$$2^{16} = 2^{15} + 2^{15} = (2^3)^5 + (2^3)^5 = 8^5 + 8^5 \quad (\text{at } g = 3, m = 5) \text{ ---- (iv)}$$

Or,  $(2^4)^4 = 16^4 = (2^3)^5 + (2^3)^5 = 8^5 + 8^5$

i.e.  $16^4 = 8^5 + 8^5 \quad \text{---- (v)}$

Or,  $16^4 = 8^5 + 2^{15} \quad \text{---- (vi)}$

Or,  $16^4 = (2^5)^3 + (2^5)^3 = 32^3 + 32^3 \quad \text{---- (vii)}$

Also  $16^4 = (2^3)^5 + (2^5)^3 = 8^5 + 32^3 \quad \text{---- (viii)}$

.... and so on, etc.

2.1.1 Comparing the aforesaid sub-equations with original equation (1.0), we see that the common factor amongst  $x, y$  and  $z$  in sub-equations (i), (ii), (iii) is 2, which is a prime number, and common factors amongst  $x, y$  and  $z$  in sub-equations (iv), (v) & (vi) are 2, 8 and 16 respectively. Common factors amongst  $x, y$  and  $z$  in sub-equations (v) & (vii) are 8 & 16 which are non-primes or composite #s, being equal to  $2^3$  &  $2^4$ .

2.2 Again from equation (1.0), we have the following at  $y = ax$  with  $x = a^m + 1$  and  $n = m$

$$z^r = x^m + y^n \quad \text{---- (1.0)}$$

$$= x^m + a^n x^n = x^m (1 + a^n x^{(n-m)})$$

$$= (a^m + 1)^m (1 + a^m), \text{ since } n = m \text{ or, } n - m = 0$$

$$= (a^m + 1)^m + a^m (a^m + 1)^m = (a^m + 1)^{(m+1)}$$

$$\text{Or, } (a^m + 1)^{m+1} = (a^m + 1)^m + \{a(a^m + 1)\}^m \quad \text{---- (2.02)}$$

Giving different values to  $a$  and  $m$ , we have the following :-

$$(2^3 + 1)^4 = (2^3 + 1)^3 + 2^3 (2^3 + 1)^3 \text{ at } a = 2 \text{ \& } m = 3$$

$$9^4 = 9^3 + 2^3 9^3 = 9^3 + 18^3 \quad \text{---- (2.02.1)}$$

$$\text{Also, } (3^3 + 1)^4 = (3^3 + 1)^3 + 3^3 (3^3 + 1)^3 \text{ at } a = 3 \text{ \& } m = 3$$

$$\text{Or, } 28^4 = 28^3 + 3^3 28^3 = 28^3 + 84^3 \quad \text{---- (2.02.2)}$$

2.2.1 Comparing eqn. (2.02) with original equation (1.0), we find that the common factor amongst  $x$ ,  $y$  and  $z$  is  $(a^m + 1)$ .

Similarly in equations (2.02.1) and (2.02.2), the common factors amongst  $x$ ,  $y$  &  $z$  is 9 (i.e.  $3 \times 3$ , a composite number or product or square of two prime numbers) and 28 ( $4 \times 7 = 2 \times 2 \times 7$ , --- a composite number equal to product of three prime numbers) respectively.

2.3 Similarly from equation (1.0), we have the following with  $y = bx$  with  $b = 1$ , &  $x = a^m - 1$ , &  $n - m = 1$ .

$$z^r = x^m + y^n \quad \text{---- (1.0)}$$

$$= x^m + b^n x^n = x^m (1 + b^n x^{(n-m)})$$

$$= (a^m - 1)^m (1 + a^m - 1), \text{ as } b = 1, x = a^m - 1, \\ \text{\& } n - m = 1$$

$$= a^m (a^m - 1)^m = [a(a^m - 1)]^m$$

$$\text{i.e. } [a(a^m - 1)]^m = (a^m - 1)^m (1 + a^m - 1) = (a^m - 1)^m + (a^m - 1)^{m+1} \quad \text{---- (2.03)}$$

From above we have the following :-

$$[2(2^3 - 1)]^3 = (2^3 - 1)^3 + (2^3 - 1)^4 \quad \text{---- at } a = 2 \text{ \& } m = 3$$

$$\text{Or, } 14^3 = 7^3 + 7^4 \quad \text{---- (2.03.1)}$$

$$\text{Also, } [3(3^3 - 1)]^3 = (3^3 - 1)^3 + (3^3 - 1)^4 \quad \text{---- at } a = 3 \text{ \& } m = 3$$

$$\text{Or, } 78^3 = 26^3 + 26^4 \quad \text{---- (2.03.2)}$$

2.3.1 Comparing equation (2.03) with original eqn.(1.0) of Beal's Conjecture, we find that the common factor amongst x, y and z is  $a^m - 1$  or rather strictly  $(a^m - 1)^m$ . Thus in equations (2.03.1) & (2.03.2) the common factors amongst x, y and z are 7 (a prime number) and 26 (i.e. a non-prime or a composite number being equal to product of two primes i.e = 2 x 13 ) respectively.

2.3.2 Thus when x , y and z are positive integers and m, n and r are also positive integers greater than 2, we will have following equations as already derived earlier with common factors amongst x, y and z.

$$2^{gm+1} = 2^{gm} + 2^{gm} = (2^m)^g + (2^m)^g \quad \text{---- (2.01.1)}$$

$$= (2^g)^m + 2^{gm} \quad \text{---- (2.01.1/1)}$$

$$= (2^m)^g + 2^{gm} \quad \text{---- (2.01.1/2)}$$

$$(a^m + 1)^{m+1} = (a^m + 1)^m + [a(a^m + 1)]^m \quad \text{---- (2.02)}$$

$$[a(a^m - 1)]^m = (a^m - 1)^m + (a^m - 1)^{m+1} \quad \text{---- (2.03)}$$

Giving different integral positive values to a, g, and m greater than 2, we shall have innumerable integral equations of Beal's Conjecture  $z^r = x^m + y^n$  with different respective common factors amongst x, y and z.

#### 2.4 Special Case for Beal's Conjecture :-

In view of the fact that following two simple integral equations exist, we can make use of them to form various equations of Beal's Conjecture, where at least one of the values of m, n and r is equal to 2.

$$1 + 2^3 = 3^2 \quad \text{---- (2.04)}$$

$$1 + 3 = 2^2 \quad \text{---- (2.04.1)}$$

2.4.1 Multiplying both sides of equation (2.04) with  $2^{2f}$ , we have :-

$$2^{2f}(1 + 2^3) = 2^{2f} 3^2$$

$$\text{Or, } 2^{2f} + 2^{2f+3} = (2^f 3)^2 \quad \text{---- (2.05)}$$

2.4.2 Giving different values to f, viz., 1, 2, 3, etc., in equation (2.05), we shall have following equations of Beal's Conjecture all with the common factor of  $2^{2f}$  amongst x, y and z .

$$2^2 + 2^5 = (2 \times 3)^2 = 6^2 \quad \text{at } f = 1 \quad \text{---- (2.05.1)}$$

$$2^4 + 2^7 = (2^2 \times 3)^2 = 12^2 \quad \text{at } f = 2 \quad \text{---- (2.05.2)}$$

$$2^6 + 2^9 = (2^3 \times 3)^2 = 24^2 \quad \text{at } f = 3 \quad \text{---- (2.05.3)}$$

2.4.3 Using equation ( 1.0 ) of Beal's Conjecture we have : -

$$z^f = x^m + y^n \quad \text{---- (1.0)}$$

$$= x^m + a^n x^n, \quad \text{assuming } y = ax$$

$$= x^m (1 + a^n x^{n-m})$$

$$= 3^{2f}(1 + 3), \quad \text{putting } a = 1, x = 3, n-m = 1 \text{ \& } m = 2f, \\ \text{where } f = \text{a positive integer.}$$

$$= 3^{2f} + 3^{2f+1} = 3^{2f}(1+3) = 3^{2f} \cdot 2^2 = (3^f \cdot 2)^2,$$

since as per (2.04.1), we have  $1 + 3 = 2^2$

$$\text{i.e. } (3^f \cdot 2)^2 = 3^{2f} + 3^{2f+1} \quad \text{---- (2.06)}$$

2.4.4 Giving different values to f in equation (2.06), we shall have following equations of Beal's Conjecture, all with common factors of  $3^{2f}$  amongst x, y and z .

$$(3 \times 2)^2 = 6^2 = 3^2 + 3^3 \quad \text{---- at } f = 1$$

$$(3^2 \times 2)^2 = 18^2 = 3^4 + 3^5 \quad \text{---- at } f = 2$$

$$(3^3 \times 2^2)^2 = 54^2 = 3^6 + 3^7 \quad \text{---- at } f = 3$$

2.4.5 Thus we have two following equations of special cases of Beal's Conjecture, as derived earlier.

$$(2^f \times 3)^2 = 2^{2f} + 2^{2f+3} \quad \text{---- (2.05)}$$

$$(2 \times 3^f)^2 = 3^{2f} + 3^{2f+1} \quad \text{---- (2.06)}$$

2.4.6 With possible suitable uses of equations (2.04) & (2.04.1) we can have only some limited equations of Beal's Conjecture with no common factor amongst x, y and z in which at least one of the integers amongst m, n and r will be 2.

2.4.7 To elucidate the aforesaid statement, only one following example is being given.

$$\begin{aligned} 3^4 &= (3^2)^2 = (1 + 2^3)^2 \quad \text{---- Since as per (2.04) : } 3^2 = 1 + 2^3 \\ &= 1 + 2 \times 2^3 + 2^6 \\ &= 1 + 2^4 + 2^6 \\ &= 1 + 2^4(1 + 2^2) = 1 + 2^4(2 + 3) \\ &= 1 + 2^5 + 3 \times 2^4 \\ &= 2^5 + 1 + 3 \times 2^4 \\ &= 2^5 + 7^2 \end{aligned}$$

Thus we have :  $2^5 + 7^2 = 3^4 \quad \text{---- (2.07)}$

Other such examples which have already been presented by various famous mathematicians are :

$$7^3 + 13^2 = 2^9$$

$$2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

The other two such simple equations that have already been mentioned earlier are :

$$1 + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4$$

The other larger solutions of Beal's Conjecture with no common factors amongst x, y and z, but with at least one of the values of 2 amongst m, n and r, are as follows:

$$17^7 + 76271^3 = 21063928^2$$

$$1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7$$

$$43^8 + 96222^3 = 30042907^2$$

$$33^8 + 1549034^2 = 15613^3$$

2.5 Beal's conjecture  $x^m + y^n = z^r$  ---- (1.0) can also be represented by the following simple equation:

$$[a(a^m + b^m)]^m + [b(a^m + b^m)]^m = (a^m + b^m)^{m+1}, \text{ where } a, b \text{ \& } m \text{ are all +ve integers.}$$

The above equation shall have the common factor of  $a^m + b^m$  or strictly  $(a^m + b^m)^m$  amongst x, y & z. The above equation shall be:

(i)  $9^3 + 18^3 = 9^4$  at  $a = 1, b = 2 \text{ \& } m = 3$ , the common factor amongst x, y & z being 9 (i.e.  $3 \times 3$ ) or  $9^3$

(ii)  $2^4 \times 97^4 + 3^4 \times 97^4 = 97^5$   
 i.e.  $194^4 + 291^4 = 97^5$   
 for  $a = 2, b = 3 \text{ \& } m = 4$ , the common factor amongst x, y & z being 97 or  $97^4$ .

3.0 We will now provide proof of Beal's Conjecture that equation  $x^m + y^n = z^r$  ---- (1.0), shall not hold good, when m, n and r all are positive integers greater than 2 and positive integers x, y and z shall not have a common factor.

3.1 As we have earlier dealt with the case where  $y = ax$  or  $x$  with  $a = 1$ , let  $x \neq y$  and y be greater than x, whence let  $y = p + x$ , where p is another positive integer. Here  $p \neq x$  or  $p \neq ax$  where  $a =$  another positive integer, because if  $p = ax$  (including  $a = 1$ ), then  $y = p + x = x(a+1) = cx$  (say), which is similar to  $y = ax$ , the case which we have already dealt earlier. Thus here  $y \neq ax$

3.2 From equation ( 1.0 ) of Beal's Conjecture we have :-

$$\begin{aligned} z^r &= x^m + y^n \\ &= x^m + (p + x)^n \end{aligned}$$

$$= p^n + {}^nC_1 p^{n-1} x + {}^nC_2 p^{n-2} x^2 + \dots + {}^nC_{n-1} p x^{n-1} + x^n + x^m \quad \text{---- (3.0)}$$

by putting  $x^m$  at the end.

3.2.1 From binomial theorem we have :-

$$(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$$

Putting  $x = -1$  in above equation we have : -

$$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^{n-1} {}^nC_{n-1} + (-1)^n {}^nC_n$$

Or,

$$1 = {}^nC_1 - {}^nC_2 + {}^nC_3 - {}^nC_4 + \dots - (-1)^{n-1} {}^nC_{n-1} - (-1)^n {}^nC_n \quad \text{---- (3.01)}$$

Using equation (3.01) we can write  $p^n$  in equation (3.0) as follows .

$$p^n = p^n [{}^nC_1 - {}^nC_2 + {}^nC_3 - {}^nC_4 + \dots - (-1)^{n-1} {}^nC_{n-1} - (-1)^n]$$

$$\text{since } {}^nC_n = 1$$

Putting the aforesaid value of  $p^n$  in equation (3.0) and rearranging its various terms we have :-

$$z^r = -(-1)^n p^n + {}^nC_1 p^{n-1} (x+p) + {}^nC_2 p^{n-2} (x^2-p^2) + {}^nC_3 p^{n-3} (x^3+p^3) + \dots \\ \dots + {}^nC_{n-1} p(x^{n-1} - (-1)^{n-1} p^{n-1}) + x^n + x^m$$

Adding and subtracting  $p^n$  at beginning of the above equation and rearranging we have :-

$$z^r = p^n - [1 + (-1)^n] p^n + {}^nC_1 p^{n-1} (x+p) + {}^nC_2 p^{n-2} (x^2-p^2) + {}^nC_3 p^{n-3} (x^3+p^3) + \dots \\ \dots + {}^nC_{n-1} p(x^{n-1} - (-1)^{n-1} p^{n-1}) + x^n + x^m \quad \text{---- (3.02)}$$

Further rearranging aforesaid equation (3.02) we have :-

$$z^r = \underline{p^n + {}^nC_1 p^{n-1} (x+p) + {}^nC_2 p^{n-2} (x^2-p^2) + {}^nC_3 p^{n-3} (x^3+p^3) + \dots}$$

$$\underline{\dots + {}^nC_{n-1} p(x^{n-1} - (-1)^{n-1} p^{n-1})}$$

*(n-1) terms*

$$+ \underline{[x^n + x^m - \{1 + (-1)^n\} p^n]}$$

*(total (n+1) terms)* ----- (3.03)

$$= Q^n \quad (\text{say}) \quad \text{---- (3.03.1)}$$

$$= (p + t)^n \quad (\text{say}) \quad \text{---- (3.03.2)}$$

As the RHS of equation (3.03) has  $n + 1$  terms and first  $n$  terms are associated with their respective coefficients (including unity with first first term i.e.  $p^n$ ) with descending powers of  $p$  starting from  $p^n$ , as also the various terms are respectively associated with  ${}^nC_0$  (with  $p^n$ , as  ${}^nC_0 = 1$ ),  ${}^nC_1, {}^nC_2, {}^nC_3, \dots, {}^nC_{n-1}$  &  ${}^nC_n$  (since again  ${}^nC_n = 1$ ) with the last  $(n+1)^{\text{th}}$  term, thus RHS can be expressed as  $Q^n$  where  $Q$  is a positive quantity. Further since the LHS of equation (3.03) is equal to  $z^r$ , where  $z$  and  $r$  are positive integers, and is equal to  $Q^n$  of RHS, where  $n$  is a positive integer,  $Q^n$  has to be a positive integer, in order to maintain the integral equivalence of both sides.

3.2.2 Since equations (3.03), (3.03.1), (3.03.2) have been derived from equation (3.0), the value of RHS of all equations (3.0) and (3.03), (3.03.1), (3.03.2) which is equal to  $Q^n$  is greater than  $p^n$ , since RHS of equation (3.0) is equal to sum of  $p^n$  and other positive integers comprising of further  $n$  terms. Thus we have  $Q^n$  is greater than  $p^n$  or  $Q$  is greater than  $p$ . Let  $Q = p + t$ .

3.2.2.1 As  $Q^n$  has to be a +ve integer as per 3.2.2, which is equal to  $(p + t)^n$ , where  $p$  is a positive integer,  $t$  shall have to be also an integer, in order to make  $Q$  also an integer, enabling maintenance of perfect integral equivalence (i.e. keeping both  $z$  and  $Q$  as +ve integers) of both sides of equations (3.03), (3.03.1), (3.03.2).

3.2.3 We will however prove just hereinafter that if  $x, y$  and  $z$  are maintained as positive integers, and  $m, n$  and  $r$  are also positive integers all greater than 2, then  $t$  shall not be an integer, whence  $p + t$  i.e.  $Q$  shall not be an integer.

3.2.4.1 This is a contradiction of conclusions of para 3.2.3.1, whence the validities of equations (3.0) & (3.03), (3.03.1), (3.03.2), etc., which are derived from equation (1.0), i.e. the equation for Beal's Conjecture cannot hold good for positive integral values of  $x, y$  and  $z$  and  $m, n$  and  $r$ , where  $m, n$  and  $r$  are all greater than 2. In fact when  $Q$  is not an integer,  $z$  has also to be a non-integer. This is the proof to Beal's Conjecture with no common factor amongst  $x, y$  and  $z$ , since  $y \neq ax$  or  $x$ , as assumed in para 3.1 and  $z$  is not an integer.

3.2.5 From equations (3.03.1) and (3.03.2) we have the following :

$$\begin{aligned}
 z^r = Q^n &= (p + t)^n \\
 &= p^n + {}^nC_1 p^{n-1} t + {}^nC_2 p^{n-2} t^2 + {}^nC_3 p^{n-3} t^3 + \dots \\
 &\quad \dots + {}^nC_{n-1} p t^{n-1} + t^n \quad \text{---- (3.04)}
 \end{aligned}$$

Comparing the coefficients of equal powers of  $p$  in different terms in RHS expressions of equations (3.03) and (3.04) we have :

$$\begin{aligned}
 t &= x + p && \text{---- (i)} \\
 t^2 &= x^2 - p^2 && \text{---- (ii)} \\
 t^3 &= x^3 + p^3 && \text{---- (iii)} \quad \text{---- (3.05)}
 \end{aligned}$$

$$t^{n-1} = x^{n-1} - (-1)^{n-1} p^{n-1} \quad \text{---- (n-1)}$$

Also,

$$t^n = x^n + x^m - \{1 + (-1)^n\} p^n \quad \text{---- (3.06)}$$

3.3 Vertically multiplying the various (n-1) sub-equations of both LHS and RHS of equation (3.05), we have the only following valid and acceptable equation incorporating all the aforesaid (n-1) relationships.

$$t^{1+2+3+\dots+(n-1)} = t^{n(n-1)/2} = \frac{(x+p)(x^2-p^2)(x^3+p^3) \dots [x^{n-1} - (-1)^{n-1} p^{n-1}]}{(n-1) \text{ terms}} \quad \text{---- (4.0)}$$

Or,

$$\begin{aligned}
 t &= [(x+p)(x^2-p^2)(x^3+p^3) \dots \{x^{n-1} - (-1)^{n-1} p^{n-1}\}]^{2/[n(n-1)]} \\
 &= B^{2/[n(n-1)]} \quad \text{(Say)} \quad \text{---- (4.01)}
 \end{aligned}$$

3.3.5 Since as per (3.1), we have  $p \neq x$ , whence  $t \neq 0$ . Also the RHS of equations (4.0) and (4.01) shall have (n-1) terms i.e. only one term when  $n = 2$ , two terms when  $n = 3$ , three terms when  $n = 4$  etc., and so on.

3.4 We will now examine values of  $t$  from equations (4.0), (4.01) and (3.06) and will find when  $t$  becomes integral . From equation (4.0) we find that RHS has only one term at  $n = 2$ , two terms at  $n = 3$  and so on, having  $(n - 1)$  terms in total.

3.5 We will now examine the value of t at n = 2. From equation (4.0), we have the following at n=2, when the RHS shall have only one term.

$$t^{n(n-1)/2} = t = x + p$$

Same value is obtained from equation (4.01), at n=2, whence we have :

$$t = B^{2/n(n-1)} = B = x + p \text{ ---- (having only one term at } n = 2)$$

Thus from both equations (4.0) and (4.01) we have at n=2 :

$$t = x + p \text{ ---- (4.0/2)}$$

$$\text{i.e. } t^2 = (x + p)^2 \text{ ---- (4.02.1)}$$

Also from equation (3.06) we have at n = 2 the following :

$$t^n = x^n + x^m - [1 + (-1)^n] p^n$$

Or,

$$t^2 = x^2 + x^m - 2 p^2 \text{ ---- (3.06.1/2)}$$

Equating equations (4.02.1) and (3.06.1/2) we have at n = 2 .

$$t^2 = (x + p)^2 = x^2 + x^m - 2 p^2$$

Or,

$$3p^2 + 2px - x^m = 0$$

From above quadratic equation we have :

$$\begin{aligned} p &= [-2x \pm (4x^2 + 12 x^m)^{1/2}] / 6 = [-x \pm x(1 + 3x^{m-2})^{1/2}] / 3 \\ &= [x(-1 \pm A)] / 3 \quad (\text{Say}) \end{aligned}$$

$$\text{where } A = (1 + 3 x^{m-2})^{1/2} \text{ ----(4.02.2)}$$

3.6 In equation (4.02.2), if m is also assumed to be equal to 2, we will have there from :

$$p = [x(-1 \pm 2)] / 3 = x / 3 \quad \text{Or } -x \text{ ( inadmissible being negative)}$$

Or,

$$3p = x \text{ considering only positive value of } p, \text{ which has been assumed to be a +ve integer. ----(4.02.3)}$$

Since as per para (3.1)  $y = p + x$  we have :

$$y = 4p \quad \text{---- (4.02.4)}$$

From equations (1.0), (3.03.1) and (3.03.2), we have :

$$\begin{aligned} z^r = Q^n &= (p + t)^n = x^m + y^n \\ &= (p + t)^n = (3p)^2 + (4p)^2 \\ &= (2p + x)^2 = 25 p^2 \\ &\text{since as per eqn (4.02) } t = p + x \text{ at } n = 2 \\ &= (2p + 3p)^2 = 25p^2 \end{aligned}$$

$$\text{i.e. } z^r = (5p)^2 = (3p)^2 + (4p)^2 \quad \text{---- (4.02.5)}$$

From equation (4.02.5), the simplest solution that we can have, will be as follows :

$$z^2 = x^2 + y^2$$

Where  $x = 3p$ ,  $y = 4p$ , &  $z = 5p$  in equation  $z^2 = x^2 + y^2$  (i.e. when  $m = n = r = 2$ ), whence by giving different integral values to  $p$ , we will have following different equations. Incidentally when  $m = n = r = 2$ , the above equation becomes the special case of Fermat's Last Theorem at  $n = 2$ .

$$\begin{aligned} z^2 &= x^2 + y^2 \\ 5^2 &= 3^2 + 4^2 \quad \text{at } p = 1 \\ 10^2 &= 6^2 + 8^2 \quad \text{at } p = 2 \\ 15^2 &= 9^2 + 12^2 \quad \text{at } p = 3 \end{aligned}$$

3.6.1 Still more general values of  $x$ ,  $y$  and  $z$  for the equation  $x^2 + y^2 = z^2$  can be given by following sub-equations :

$$\text{Assuming } x = c [(a + b)^2 - a^2] = cb(2a + b)$$

$$\text{and } z = c[(a + b)^2 + a^2]$$

---- Since any two positive integers can be generally expressed as difference or sum of two integers. The common factor of  $c$  in  $x$ ,  $y$  and  $z$  has been included to get still more general values for them.  $a$ ,  $b$  and  $c$  are any three appropriate positive inetegers.

Since  $y = (z^2 - x^2)^{1/2}$  we have  $y = 2ac(a + b)$

3.6.2 A few examples are given below :-

(i) at  $a = b = c = 1$  we have :

$$x = cb(2a + b) = 3$$

$$y = 2ac(a + b) = 4$$

$$z = c[(a + b)^2 + a^2] = 5$$

whence for  $x^2 + y^2 = z^2$ , we have  $3^2 + 4^2 = 5^2$

(ii) Similarly at  $a = 1, b = 2$  and  $c = 1$ , we have :

$$x = 8, y = 6 \text{ and } z = 10, \text{ whence we have :}$$

$$8^2 + 6^2 = 10^2$$

(iii) at  $a = 2, b = 1$  and  $c = 1$ , we have :

$$x = 5, y = 12 \text{ and } z = 13, \text{ whence we have :}$$

$$5^2 + 12^2 = 13^2$$

(iv) at  $a = 2, b = 1$  and  $c = 2$ , we have :

$$x = 10, y = 24 \text{ and } z = 26, \text{ whence we have :}$$

$$10^2 + 24^2 = 26^2$$

and so on.

By assigning different integral positive values to  $a, b$  and  $c$ , innumerable integral solutions for

the equation  $x^2 + y^2 = z^2$  can be obtained .

3.7 Further dealing the case with  $n = 2$  as per para (3.5), we can visualize other cases when  $m$  could have values other than 2 in equation (4.02.2) in order to make  $A$  or  $(1 + 3x^{m-2})^{1/2}$  as integral to get integral solutions of Beal's Conjecture.

3.7.1 Let us assume  $n = 2$ , and  $m = 3$ , and deal a few cases as follows . From (4.02.2) we have:

$$p = x(-1 \pm A)/3 = x(-1 + A)/3$$

taking only the +ve value.

$$\text{Also as } y = p + x = x(2 + A)/3$$

$$\text{where } A = (1 + 3x^{m-2})^{1/2}$$

By trial we find that at  $x = 5, 16, 21, 33$  etc., all  $x, y$  and  $z$  or  $Q$  are Integral, for  $m = 3$

Thus, when (i)  $x = 5$ , we have :

$$A = (1 + 3x^{m-2})^{1/2} = 4, \text{ for } m = 3$$

$$\text{whence } y = x(2 + A)/3 = 5(2 + 4)/3 = 10$$

Since  $z^r = Q^n = x^m + y^n$ , where  $Q = p + t = 2p + x$ ,

as  $t = p + x$  at  $n = 2$

$$\text{and } p = x(-1 \pm A)/3 = 5(-1 + 4)/3 = 5$$

$$Q = 2p + x = 10 + 5 = 15$$

Thus at  $x = 5$ , we have :

$$x^m + y^n = Q^n, \text{ whence at } m = 3 \text{ \& } n = 2, \text{ we have :-}$$

$$5^3 + 10^2 = 15^2$$

Similarly for (ii)  $x = 16$ , we have :

$$16^3 + 48^2 = 80^2$$

Also at (iii)  $x = 21$ , we have :

$$21^3 + 70^2 = 119^2 \text{ and so on ....}$$

3.7.2 We find that with  $n = 2$ , but with other values of  $m$  equations for Beal's Conjecture also hold good for all integral values of  $x, y, z$ , &  $m, n$  and  $r$ . We will deal just for example cases when (i)  $n = 2$  but  $m = 4$  and when  $n = 2$  but  $m = 6$ , only here.

3.7.3 With  $n = 2$  and  $m = 4$ , we find that  $A$  and  $p$  become integers, at  $x = 4$  &  $15$  etc., whence we have :-

(i) With  $m = 4$ ,  $n = 2$  and  $x = 4$  we have :

$$\begin{aligned} p &= x(-1 + A) / 3 \text{ where } A = (1 + 3x^{m-2})^{1/2} = (1 + 3 \cdot 4^2)^{1/2} \\ &= 4(-1 + 7) / 3 = 8 \end{aligned}$$

$$y = p + x = x(2 + A) / 3 = 4(2 + 7) / 3 = 12$$

whence from above, we have the following, for  $x = 4$ ,  $m = 4$  &  $n = 2$  :

$$\begin{aligned} x^m + y^n &= 4^4 + 12^2 = Q^n = (p + t)^n = (2p + x)^2, \\ &\text{as } t = p + x \text{ at } n = 2 \\ &= 20^2 \end{aligned}$$

$$\text{Thus we have : } 4^2 + 12^2 = 20^2$$

(ii) Similarly with  $m = 4$ ,  $n = 2$  &  $x = 15$ , we have :

$$\begin{aligned} p &= x(-1 + A) / 3 = 15(-1 + 26) / 3, \text{ as } A = (1 + 3x^{m-2})^{1/2} \\ &= (1 + 3 \cdot 15^2)^{1/2} \\ &= 26 \\ &= 125 \end{aligned}$$

whence from above :

$$\begin{aligned} x^m + y^n &= Q^n = (p + t)^n \\ 15^4 + (p + x)^2 &= (p + t)^2 = (2p + x)^2 \\ 15^4 + 140^2 &= 265^2 \end{aligned}$$

3.7.4 Similarly with  $m = 6$ ,  $n = 2$  and  $x = 2$  we have :-

$$\begin{aligned} p &= x(-1 + A) / 3 = 2(-1 + 7) / 3 = 4, \text{ as } A = (1 + 3x^{m-2})^{1/2} \\ &= (1 + 3 \cdot 2^4)^{1/2} \\ &= 7 \end{aligned}$$

whence from :

$$\begin{aligned} x^m + y^n &= Q^n = (p + t)^n = (2p + x)^n \\ 2^6 + 6^2 &= 10^2 \end{aligned}$$

3.8 So far we have dealt equation (1.0) of Beal's Conjecture and its further derivations from equation (3.0) onwards with values of  $n = 2$  as per para (3.5). Now we shall examine the case with values of  $n$  greater than 2. As we have already dealt with the case when  $m = n = 2$ , let  $m$  as also  $r$  be greater than 2.

3.8.1 As already stated in para (3.3.1), the RHS of equations (4.0) & (4.01) shall have  $n - 1$  terms, i.e., one term for  $n = 2$ , and two terms for  $n = 3$ , three terms for  $n = 4$  and so on. Thus for  $n$  greater than 2, the RHS of equations (4.0) & (4.01) shall have more than one term equal to  $n - 1$  terms .

3.9 The index of B or the total bracketed figure of RHS of equation ( 4.01 ) shall always be a fraction less than unity for values of  $n$  greater than 2. The denominator of this fractional index shall always be a multiple of  $n$ , whence  $t$  to be an integer, the first requirement will be that at least the rational/integral  $n$ th root of B could be taken out. This is however just not possible as the RHS of equation ( 4.01 ) has only  $n - 1$  terms, each being a different integer for assumed integral values of  $x$  and  $p$ , and as no two consecutive terms, after the first term, are fully divisible by each other, and as also even though the first term divides all other terms individually but the individual such divisions do not yield each time either the same number/quantity, the individual terms being divided are different numbers, also such divisions do not yield a square or a cube of any number/quantity etc., whence no  $n$  factors of same integer can be equal to the value of B or the total bracketed figure of equation ( 4.01), whence rational/integral  $n$ th root of B is not possible at all. Thus  $t$  will never be an integer for values of  $n$  greater than 2. Since as per para 3.2.3,  $Q = p + t$ , it leads to the conclusion that  $Q = p + t \neq$  An integer for values of  $n$  greater than 2, as  $t$  is not an integer for  $n$  greater than 2. From this it is also concluded that since as per equation (3.03.1) that :

$$z^r = Q^n, \text{ we cannot have } z \text{ as an integer for } r \text{ as an integer for } n \text{ greater than } 2 .$$

4.0 From all these it is concluded, as we have been dealing with the case for  $x$ ,  $y$  and  $z$  not having a common factor as per para 3, with positive integral values of  $x$ ,  $y$

and  $z$ , as also with positive integral values of  $m$ ,  $n$  and  $r$  all greater than 2, that we cannot have all integral values in equation :-

$$z^r = x^m + y^n$$

under these conditions, which is a contradiction. This is because of the fact that as per para 3.1, we have  $y \neq ax$  as also we have proved and confirmed at para (3.9) that for all positive integral values of  $m$ ,  $n$  and  $r$  greater than 2, both  $Q$  and  $z$  cannot be an integer, whence  $x$ ,  $y$  and  $z$  shall not have any common factor.

5, All these prove Beal's Conjecture, as per proposals at para 1 and its sub-paras, in total.

5.1 We will illustrate the aforesaid statements by a few examples only to end this matter. As per equation (3.03.1), where  $z^r = Q^n$ , its simplest solution will be to have  $z = Q$  and  $r = n$ , but there could also be solutions where  $Q$  &  $z$  will be different from each other, similarly  $r$  and  $n$  will be different from each other, yet the equation  $z^r = Q^n$  shall hold good.

(i) Thus  $5^3 = 5^3$ ,  $17^5 = 17^5$ ,  $6^8 = 6^8$  are examples,

where  $z = Q$  and  $r = n$

(ii) Similarly we have :

$$\begin{aligned} 9^6 &= (3^2)^6 \quad \text{or} \quad 9^6 = 3^{12} \\ &= (3^4)^3 = 81^3 \quad \text{or} \quad 9^6 = 81^3 \end{aligned}$$

Here  $z$  and  $Q$  are different as also  $r$  and  $n$  are different from each other

(iii) Also since :

$$15^6 = (15^2)^3 \quad \text{or} \quad 15^6 = 225^3$$

Here also  $z$  &  $Q$  as also  $r$  and  $n$  are different from each other.

5.2 If we assume  $x = 3$ ,  $p = 1$ , then as per equation (4.01), with value of  $n = 3$ , we shall have the following :

$$t = [(x + p)(x^2 - p^2)]^{2/[n(n-1)]}$$

as per equation (4.01)

$$\text{or, } t = (4 \cdot 8)^{1/3} = (32)^{1/3} = 3.175 \quad \text{---- (5.0)}$$

$$\text{i.e. } t^3 = 32 \quad \text{---- (5.01)}$$

From above as per equations (3.03.1) and (3.03.2) we have :

$$z^r = Q^n = (p + t)^n = (1 + 3.175)^3 = (4.175)^3 = 72.77$$

whence with  $r = 3, 5$  or  $7$  etc., we will have :

$$z = 4.175 \quad \text{i.e. } z^r = z^3 = 72.77 \quad \text{at } r = 3$$

$$z = 2.357 \quad \text{i.e. } z^r = z^5 = 72.77 \quad \text{at } r = 5$$

$$z = 1.844 \quad \text{i.e. } z^r = z^7 = 72.77 \quad \text{at } r = 7$$

5.3 A little more probe and analysis of the matter will make things more clear. While as per equation (4.01), we calculated the value of  $t$  at  $n = 3$  for  $x = 3$  and  $p = 1$  at para (5.2), we will have also the value of  $m$  in  $x^m$  from equation (3.06) as follows :

$$t^n = x^n + x^m - [1 + (-1)^n] p^n \quad \text{---- (3.06)}$$

$$t^3 = x^3 + x^m - 0 = 3^3 + 3^m$$

$$\text{or, } 32 = 3^3 + 3^m \quad \text{---- Since } t^3 = 32 \text{ as per equation (5.01)}$$

$$\text{i.e. } 3^m = 5$$

taking logarithms of both sides in the above equation, we have :

$$m \log 3 = \log 5$$

$$\text{or, } m = \log 5 / \log 3 = 0.699 / 0.477 = 1.465$$

5.3.1 In this case for the same value of  $t$  as per equations (4.01) and (3.06) at  $n = 3$  for  $x = 3$  &  $p = 1$ , the value of  $m$  becomes non-integral equal to 1.465 (less than 2), which is contrary to our assumptions of  $m, n$  and  $r$  as integral, which again leads to the inevitable conclusion that Beal's Conjecture does not hold good for integral  $x, y$  &  $z$  with all integral values of  $m, n$  and  $r$  greater than 2, with no common factors amongst amongst  $x, y$  &  $z$ .

5.4 Alternately at  $n = 3$  for  $x = 3$  &  $p = 1$ , we will have different values of  $t$  for integral values of  $m$  greater than 2, i.e at  $m = 3, 4, 5$  etc. for example from equation (3.06) as follows :

$$t^n = x^m + x^n - [1 + (-1)] p \quad \text{---- as per equation (3.06)}$$

or, (i)  $t^3 = 3^3 + 3^3 - 0 = 54$  ---- for  $m = 3$

i.e.  $t = 3.779$  (approx.) ---- ( 5.02 )

Also, (ii)  $t^3 = 3^4 + 3^3 = 108$  ---- for  $m = 4$

i.e.  $t = 4.762$  ---- (5.03)

Similarly (iii)  $t^3 = 3^5 + 3^3 = 270$  ---- for  $m = 5$

i.e.  $t = 6.463$  ---- (5.04)

5.4.1 Thus for the same value of  $n$  (i.e. 3) for same  $x$  (i.e. 3) and same  $p$  (i.e. 1), we will have different non-integral values of  $t$  from equation (4.01) and equation (3.06) for different integral values of  $m$  greater than 2. This also leads to the same conclusion that Beal's Conjecture does not hold good for integral  $x, y$  and  $z$  and all integral values of  $m, n$  and  $r$  greater than 2, having no common factors amongst  $x, y$  and  $z$

5.5 In general from equation (4.01), since we have :

$$t = B^{2/[n(n-1)]} = [(x+p)(x^2-p^2) \text{ ---- } \{x^{n-1} - (-1)^{n-1} p^{n-1}\}]^{2/[n(n-1)]} \text{ ----(4.01)}$$

as also from equation (3.06) we have :

$$t^n = x^n + x^m - [1 + (-1)^n] p^n \text{ ---- (3.06)}$$

$$\text{or, } t^n = \{x^n + x^m - [1 + (-1)^n] p^n\}^{1/n} \text{ ---- (3.06.1)}$$

As per para (3.9) the non-integral values of  $t$  as per equation (4.01) and as per equation (3.06.1), the different values of  $t$  for different integral values of  $m$  greater than 2, shall not tally with each other. Various values of  $t$  for different integral values of  $m$  greater than 2 shall always be non – integral, since as per para ( 3.1 ) we have  $p \neq ax$  or  $x$  (with  $a = 1$ ), whence even for values of  $m = n$ , the RHS of equation (3.06.2) shall never have an integral value. This also leads to the same conclusion that for no common factor amongst integral  $x, y$  and  $z$  and all integral values of  $m, n$  and  $r$  greater than 2, Beal's Conjecture shall not hold good.

This proves Beal's conjecture.

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