Towards a Floer theory for Mars
II - Floer Hessian field almost extends

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Abstract

In part I, [FW26a], we showed that collisional periodic orbits of twisted Zeeman systems can be detected variationally by a non-local Hamiltonian action functional.

In this part II we show that the linearized gradient flow of this non-local functional is a Fredholm operator and prove a non-local elliptic regularity result.

These results are obtained with the theory of almost extendability of weak Hessian fields introduced in [FW26c].

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1 Introduction

In [FW26a] we introduce the notion of twisted Zeeman system. This system models an electron in the plane attracted by a proton fixed in the origin and subject to additional forces: an electric force, a Lorentz force, and an Euler force. All these additional forces are allowed to depend periodically on time.

A motivating example of such a system is the restricted elliptic three-body-problem. In the example of the restricted elliptic three-body-problem the proton can be thought of as the planet Mars, the electron is a space station in the vicinity of Mars. The Lorentz force is given by the Coriolis force of the rotating frame. The electric force is a combination of the gravitational force of the sun and the centrifugal force. The Euler force is an additional fictitious force, beside the Coriolis force and the centrifugal force, due to acceleration and deceleration of the rotating frame, which is due to the eccentricity of the elliptic orbit of Mars.

There is the danger that the electron collides with the proton. However two-body-collisions can be regularized. For systems depending periodically on time a new regularization technique was discovered by Barutello-Ortega-Verzini [BOV21] which blows up the loop space. In part I, [FW26a], we showed that this new regularization technique can be applied to twisted Zeeman systems and gives rise to a variational approach to collisional periodic orbits of a twisted Zeeman system. In fact, we showed that there are two action functionals to detect these periodic collisional orbits, a Lagrangian one and a Hamiltonian one, related by a non-local Legendre transformation. Since blowing up the loop space is a non-local technique, both of these functionals are non-local.

1.1 Main results

In this article we prove a Fredholm result for the linearized gradient flow of the Hamiltonian functional for regularized twisted Zeeman systems.
Given such a system, the second derivative of the regularized Hamiltonian functional gives rise to a weak Hessian field $u \mapsto A^u$, roughly as follows. Namely, for the Hilbert space triple of maps

$$(H_0, H_1, H_0) = \big(L^2(\mathbb{S}^1, \mathbb{C}^2), W^{1,2}(\mathbb{S}^1, \mathbb{C}^2), W^{2,2}(\mathbb{S}^1, \mathbb{C}^2)\big)$$

there exists an open subset $U_1 \subset H_1$ such that for any $u \in U_2 := U_1 \cap H_2$ the Hessian operator $A^u \colon H_1 \to H_0$ is a Fredholm operator of index zero which restricts to a Fredholm operator $A_2^u \colon H_2 \to H_1$ also of index zero.
Given $u_-, u_+ \in U_2$ such that A^{u_-} and A^{u_+} are invertible as maps $H_1 \to H_0$, we are considering a connecting path $u \colon \mathbb{R} \to U_2$ from u_- to u_+ (Definition 3.4). This gives rise to two linear operators

$$\begin{aligned} \mathbb{D}^u &= \partial_s + A^u \colon W^{1,2}(\mathbb{R}, H_0) \cap L^2(\mathbb{R}, H_1) \to L^2(\mathbb{R}, H_0) \\ \mathbb{D}_2^u &= \partial_s + A_2^u \colon W^{1,2}(\mathbb{R}, H_1) \cap L^2(\mathbb{R}, H_2) \to L^2(\mathbb{R}, H_1). \end{aligned}$$

In the present article we show furthermore

Theorem A. *Both $\mathbb{D}^u$ and $\mathbb{D}_2^u$ are Fredholm and their Fredholm indices agree.*

The proof of Theorem A uses a technique we developed in [FW26c]. In that article we introduced the notion of an *almost extendable weak Hessian field.*

Theorem B. *The weak Hessian field $A = \{A^u\}_{u \in U_1}$ is almost extendable.*

Theorem A then follows from Theorem B in view of the abstract result [FW26c, Thm. 6.11]; cf. § 4.4.

Non-local elliptic regularity. Theorem A can be thought of as a non-local elliptic regularity result. Since blowing up the loop space is non-local, elements of the kernel and cokernel of the operator $\mathbb{D}^u$ cannot be thought of as solutions of an elliptic PDE.[1] But in view of Theorem A one might interpret them as solutions of an 'elliptic' delay equation.

[1] If the solutions in the kernel of $\mathbb{D}^u$ would satisfy an elliptic PDE, by elliptic regularity they would automatically lie in the kernel of $\mathbb{D}_2^u$, and therefore the kernel and cokernel of $\mathbb{D}^u$ could be identified with the kernel and cokernel of $\mathbb{D}_2^u$, so that both operators would have the same index.

In this paper we consider the Fredholm property in the Hamiltonian setup. The functional $\mathcal{A}$ considered in this paper can be interpreted as the Legendre transform of a Lagrangian one, as we explained in the introduction, and therefore the Fredholm property can also be asked for the Lagrangian functional $\mathcal{B}$. In Appendix B.2 we show that in the Kepler case (no Coriolis/magnetic contribution) the Hessian of the Lagrangian functional is almost extendable as well.

The relation between the Fredholm indices in the Lagrangian case and in the Hamiltonian case for these non-local functionals we plan to discuss in a forthcoming part III [FW26e].

1.2 Outline

Throughout we allow for twisted-periodic one forms (vector potentials) and also for twisted loops, as defined in § 2.1.

Section 2 summarizes notions and results of [FW26d] on which the present article is based.

Section 3 summarizes notions and results of [FW26c] which are needed to understand the statement in Theorem B and to see how Theorem A follows from Theorem B.

Section 4 is the main part of this article. We prove Theorems A and B.

Section A discusses the symmetry of weak Hessians from an abstract point of view.

Appendix B deals with the Hessian field of the regularized Lagrangian action functional. The Hessian is calculated in B.1. The magnetic contributions also play a crucial role in the main Section 4 of this article. Having all the machinery in place we show in B.2 that in the Kepler case (no magnetic field) the Lagrangian Hessian field almost extends (while the Hamiltonian one in § 4.1.1 even extended). The difference comes from the Hamiltonian equations being first order versus second order of the Lagrangian ones. Unfortunately, in the general Lagrangian case (including magnetic contributions) doing the scale Lipschitz estimate seems practically hopeless, because second order leads to an explosion of the number of terms, even in comparison to the already very long calculation in the Hamiltonian scenario § 4.2.3.

Section C carries out the technical estimates.

Notation. Throughout $\langle \cdot, \cdot \rangle$ denotes L^2 inner products with induced norm $\|\cdot\|$. The induced norm of a Hilbert space H_k is often denoted by $|\cdot|_{H_k}$ or $|\cdot|_k$.

Working with functions on function spaces easily triggers excesses of parentheses, which harms legibility. Therefore we often write variables either as subscripts $\mathcal{M}_z$ or in the form $\mathcal{M}|_z$, as opposed to $\mathcal{M}(z)$. For time-dependence subscript has priority, for example if $t \mapsto q(t)$ is a loop then $\theta_t|_{q_t}\dot{q}_t$ denotes a time-dependent 1-form at time t and at the spatial point $q(t)$ evaluated on the velocity vector $\dot{q}(t)$.

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2 Regularized twisted Zeeman systems

2.1 Let's twist again

Euclidean plane – two models $\mathbb{C}$ and $\mathbb{R}^2$

Configuration spaces in this article are open subsets of the Euclidean plane containing the origin. There are two natural models for the plane, the set $\mathbb{C}$ of complex numbers and the set $\mathbb{R}^2$ of pairs of real numbers, each one endowed with the natural structure of a vector space over the real numbers $\mathbb{R}$. The natural isomorphism $\mathbb{C} \to \mathbb{R}^2$, $x + iy \mapsto (x, y)$, identifies multiplication by i viewed as linear map on $\mathbb{C}$ with the matrix $j_0 \colon \mathbb{R}^2 \to \mathbb{R}^2$ of counter-clockwise rotation by $\pi/2$. As each model has its advantage, *we freely change from one to the other*. The complex side is extremely effective with respect to notation, complex multiplication $zw = (z+iy)(u+iv)$ encodes composition of linear maps as well as application of a linear map z to a vector w, on the $\mathbb{R}^2$-side this is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} xu - yv & -(yu + xv) \\ yu + xv & xu - yv \end{pmatrix}, \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} xu - yv \\ yu + xv \end{pmatrix}.$$

Instead of dealing with inverse matrices one just multiplies and divides by a real number, namely $zw^{-1} = \frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}$ where $w\bar{w} = u^2 + v^2 =: |w|^2$. The $\mathbb{R}^2$-side might appeal more to ones geometric intuition, or just ones customs.

Concerning the Euclidean inner product, we feel free to write

$$\langle z, w \rangle_0 = xu + yv = \mathrm{Re}(\bar{z}w)$$

using on the left pairs (x, y), (u, v) and on the right sums $x + iy$ and $u + iv$.

Complex squaring map and sign involution

Throughout $0 \in \mathfrak{Q} \subset \mathbb{C}$ is an open subset which contains the origin, also called singularity or collision locus. To exclude the origin we write $\mathfrak{Q}^\times := \mathfrak{Q} \setminus \{0\}$.

Remark 2.1 (complex square root is not continuous)**.** The complex squaring map $\varsigma \colon \mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^2 = (-z)^2$, is not a bijection, but only 2:1 due to the sign ambiguity. Reverting the perspective, let us define the complex square root $\sqrt{re^{i\varphi}}$, in analogy to the real case, as that one of the two candidates $\pm\sqrt{r}e^{i\phi/2}$ which lies in the upper half plane $\mathbb{H}^+$ or on the positve half axis $\mathbb{R}^+$. One gets to a bijection, see (2.2), by dividing out the sign $\pm$. Unfortunately, the square root definition is not continuous, as illustrated in Figure 1: while the unit circle elements w and 1 are close, their square roots are not.

The pre-image $\mathfrak{Z}^\times := \varsigma^{-1}(\mathfrak{Q})$ is invariant under sign involution and complex squaring is a double cover and invariant under sign involution, in symbols

$$\mathfrak{i} \colon \mathfrak{Z}^\times \to \mathfrak{Z}^\times,\ z \mapsto -z, \qquad \varsigma \colon \mathfrak{Z}^\times \xrightarrow{2:1} \mathfrak{Q}^\times,\ z \mapsto z^2, \qquad \varsigma \circ \mathfrak{i} = \varsigma. \tag{2.1}$$

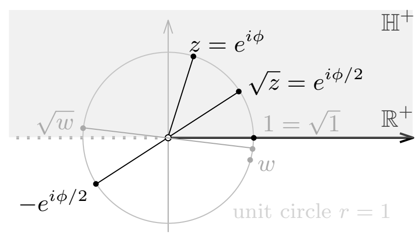

Figure 1: Square root $\sqrt{z} \in \mathbb{H}^+ \cap \mathbb{R}^+$ of non-zero $z \in \mathbb{C}$ – not continuous

Dividing out by sign involution, complex squaring becomes a bijection whose inverse is the complex square root, in symbols

$$
\begin{array}{ccc}
\overline{\mathfrak{Z}^\times} = \mathfrak{Z}^\times/\mathfrak{i} & \underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle 1:1}} & \mathfrak{Q}^\times \\
\pm z & \xmapsto{\varsigma} & z^2 \\
\pm\sqrt{r}e^{i\phi/2} & \xleftarrow{\sqrt{\cdot}} & re^{i\phi} = q
\end{array}
\tag{2.2}
$$

To ease notation we use for space and quotient space elements the same notation.

Twisted loops

While loops $q\colon \mathbb{S}^1 \to \mathfrak{Q}^\times \subset \mathbb{C}$ of even winding number around the origin lift to loops $(+)$ in $\mathfrak{Z}^\times$, loops of odd winding number lift to twisted $(-)$ loops in $\mathfrak{Z}^\times$. This is illustrated in Figure 2. A map γ with domain $\mathbb{R}$ such that $\gamma_{\tau+1} = \gamma_\tau$ for all τ is called **periodic**, domain notation $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

The spaces of **(periodic) loops** $(+)$ and **twisted loops** $(-)$, still called loops,

$$
\mathcal{L}^\times_\pm \mathfrak{Z} := \{z \in C^\infty(\mathbb{R}, \mathfrak{Z}) \mid \forall \tau \in \mathbb{R}\colon z_{\tau+1} = \pm z_\tau\} \setminus \{0\},
\tag{2.3}
$$

are disjoint and invariant under **sign involution** $I\colon \mathcal{L}_\pm\mathfrak{Z} \to \mathcal{L}_\pm\mathfrak{Z}$, $z \mapsto -z$ which acts freely. Set $\mathcal{L}^\times\mathfrak{Z} = \mathcal{L}^\times_+\mathfrak{Z} \cup \mathcal{L}^\times_-\mathfrak{Z}$. The elements τ_* of the set $z^{-1}(0)$ are called **collision times** or simply **collisions**.

The two cotangent bundles are disjoint

$$
\begin{aligned}
T^*\mathcal{L}^\times_\pm\mathfrak{Z} :&= \{(z,\eta) \in C^\infty(\mathbb{R}, \mathfrak{Z} \times \mathbb{C}) \mid z \not\equiv 0,\, (z_{\tau+1}, \eta_{\tau+1}) = \pm\,(z_\tau, \eta_\tau)\,, \forall\tau\} \\
&= \mathcal{L}^\times_\pm\mathfrak{Z} \times \mathcal{L}_\pm\mathbb{C}
\end{aligned}
\tag{2.4}
$$

and invariant under the **sign involution** $T^*I(z,\eta) = -(z,\eta)$ which acts freely. The **base point projection**

$$
\pi\colon T^*\mathcal{L}^\times_\pm\mathfrak{Z} \to \mathcal{L}^\times_\pm\mathfrak{Z}, \quad \Upsilon = (z,\eta) \mapsto z, \qquad \pi \circ T^*I = I \circ \pi,
$$

is sign involution equivariant. The tangent spaces $T\mathcal{L}^\times_\pm\mathfrak{Z}$ are given by the same formulas (since Euclidean $\mathbb{C}$ and its dual space are canonically isomorphic). The uppercase greek letter $\Xi = (z,\xi)$ is a “Xi” and $\Upsilon = (z,\eta)$ is an “Upsilon”.

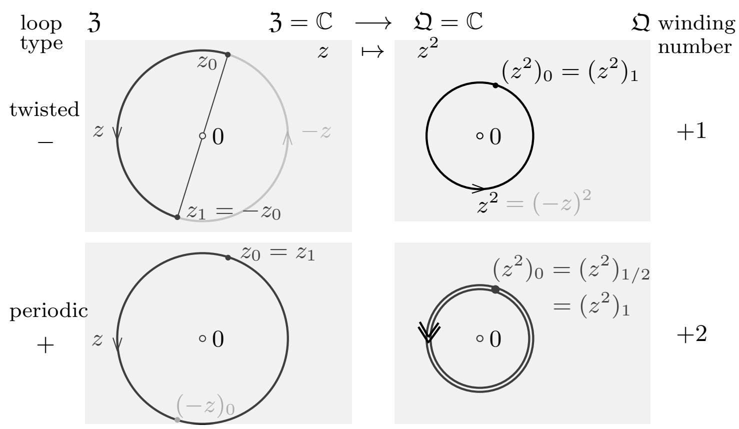

Figure 2: Loop correspondence – ± loops z in $\mathfrak{Z}$ and loops $q = z^2$ in $\mathfrak{Q}$

Remark 2.2. While in part I, [FW26a], it was important to build a model for the paths in $\mathfrak{Q}$, in particular to have a $1:1$ correspondence, and therefore we had to quotient out by sign involution $\pm$, the present part II is of analytic nature and the analysis is done on representatives anyway. Hence, for simplicity, in the present text we work directly on loop spaces, not on quotient spaces.

Twisted-periodic one-forms

If one allows the 1-form to be twisted-periodic and not just periodic, then additional (external) electric forces for an electric potential depending periodically on time can be modeled by the twist, see [FW26b, § 3.3].

Definition 2.3 (twisted-periodic)**.** A **twisted-periodic 1-form** θ is a smooth family of 1-forms $\{\theta_t\}_{t\in\mathbb{R}}$ on $\mathfrak{Q}$ such that a) the time-derivative is periodic and b) $\forall t$ $\exists$ a smooth **twist function** $f_t\colon \mathfrak{Q}\to\mathbb{R}$, that is

$$\text{a) } \dot\theta_{t+1} = \dot\theta_t, \quad \text{b) } \theta_{t+1} = \theta_t + df_t, \qquad \text{c) } \dot{\boldsymbol{A}}_{t+1} \overset{\text{a)}}{=} \dot{\boldsymbol{A}}_t, \quad \text{d) } d\theta_{t+1} \overset{\text{b)}}{=} d\theta_t. \tag{2.5}$$

The time-dependent vector field

$$\boldsymbol{A}_t := \left(A_t^1, A_t^2\right) : \mathfrak{Q} \to \mathbb{R}^2, \qquad \{\theta_t = A_t^1\, dq_1 + A_t^2\, dq_2\}_{t\in\mathbb{R}},$$

is called a **vector potential** of the magnetic field $d\theta$.

Remark 2.4. Let θ be twisted-periodic along $\mathfrak{Q}$ with twist function f. Then the following is true by [FW26b, §5]$_{\text{(i–ii)}}$ and [FW26d, §4.1]$_{\text{(iii)}}$.

(i) The time-slice $f := f_0\colon \mathfrak{Q}\to\mathbb{R}$ is a twist function.

(ii) In the periodic case ($\theta_{t+1} = \theta_t$, $\forall t$) the twist $f = 0$ vanishes.

(iii) The pull-back 1-form $\vartheta := \varsigma^*\theta$ under complex squaring (2.1) is twisted-periodic along $\mathfrak{Z} = \varsigma^{-1}(\mathfrak{Q})$ and $\varsigma^* f$ is a twist function.
The vector potential $\boldsymbol{a}$ of ϑ along $\mathfrak{Z}$, notation

$$\boldsymbol{a}_t := \left(a_t^1, a_t^2\right) : \mathfrak{Z} \to \mathbb{R}^2, \qquad \{\vartheta_t = a_t^1\, dx + a_t^2\, dy\}_{t\in\mathbb{R}},$$

and $\boldsymbol{A}$ of θ along $\mathfrak{Q}$ satisfy, at any point $z \in \mathfrak{Z}$, the identities

$$\begin{aligned} \operatorname{rot} \boldsymbol{a}_t|_z : &= (\partial_x a_t^2 - \partial_y a_t^1)_z = 4\,|z|^2 \operatorname{rot} \boldsymbol{A}_t|_{z^2} \\ (d\vartheta_t)_z &= (\operatorname{rot} \boldsymbol{a}_t|_z)\ \langle j_0\cdot, \cdot\rangle_0 = 4\,|z|^2 \,(\operatorname{rot} \boldsymbol{A}_t|_{z^2})\ \langle j_0\cdot, \cdot\rangle_0 \,. \end{aligned}$$

Barutello-Ortega-Verzini reparametrization

Given $z \in \mathcal{L}^\times\mathfrak{Z}$, the variable τ of $z\colon \mathbb{R} \to \mathfrak{Z}^\times$ is **regularized time**, **classical time** are the values of the map $t_z\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by (cf. App. C.1.2)

$$\forall \tau \in \mathbb{R}\colon \quad t_z(\tau) := \frac{\int_0^\tau |z(s)|^2\, ds}{\|z\|^2}, \quad t_z(0) = 0, \quad t_z(1) = 1. \tag{2.6}$$

2.2 Lagrangian action functional $\mathcal{B}$

For a twisted-periodic 1-form θ along $\mathfrak{Q}$ with twist function f, see (2.5), we defined in [FW26d, §4.2] the **non-local Lagrangian action functional**

$$\begin{aligned} &\mathcal{B}\colon \mathcal{L}^\times\mathfrak{Z} \to \mathbb{R} \qquad\qquad \mathcal{B} = \mathcal{K} - \mathcal{U} + \mathcal{M} \\ &z \mapsto \underbrace{2\|z\|^2\|z'\|^2}_{\mathcal{K}(z)} - \underbrace{\frac{-1}{\|z\|^2}}_{\mathcal{U}(z)} + \underbrace{\int_0^1 \vartheta_{t_z(\tau)}|_{z_\tau}\, z'_\tau\, d\tau - f(z_0^2)}_{\mathcal{M}(z)} . \end{aligned} \tag{2.7}$$

Here $\vartheta := \varsigma^*\theta$ is twisted-periodic along $\mathfrak{Z}$ with twist function $f^*\sigma$ and $\varsigma\colon \mathfrak{Z} \to \mathfrak{Q}$ is the complex squaring map (2.1). Observe that $\mathcal{B} = \mathcal{K} - \mathcal{U} + \mathcal{M}$ is the sum of three terms, kinetic, potential, and magnetic non-local action.

L^2-gradient

We showed in [FW26d, §4.2] that the L^2-gradient is given by the formula

$$\begin{aligned} \operatorname{grad} \mathcal{B}|_z &= \operatorname{grad} \mathcal{K}|_z - \operatorname{grad} \mathcal{U}|_z + \operatorname{grad} \mathcal{M}|_z \\ &= 4\|z'\|^2 z - 4\|z\|^2 z'' - \frac{2z}{\|z\|^4} \\ &\quad - \frac{2z}{\|z\|^4} \int_0^1 \int_0^\sigma |z_\rho|^2\, d\rho \cdot \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma - \frac{|z|^2}{\|z\|^2} \dot{\boldsymbol{a}}_{t_z}|_z \\ &\quad + \frac{2z}{\|z\|^2} \int_{\sigma=\tau}^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma - \left(\operatorname{rot} \boldsymbol{a}_{t_z}|_z\right) j_0 z' \end{aligned} \tag{2.8}$$

whenever $z \in \mathcal{L}^\times\mathfrak{Z}$. Note that all integrands are periodic due to (2.5) and so is, using Remark 2.4 (iii), the final summand $\left(\operatorname{rot} \boldsymbol{a}_{t_z}|_z\right) \langle j_0 z', \cdot\rangle_0 = d\vartheta_{t_z}|_z(z', \cdot)$

2.3 Hamiltonian action functional $\mathcal{A}$

On the cotangent bundle $T^*\mathcal{L}^\times\mathfrak{Z}$, see (2.4), define the **mechanic Hamiltonian**

$$\mathcal{H} =: T^*\mathcal{L}^\times\mathfrak{Z} \to \mathbb{R}, \quad (z,\eta) \mapsto \tfrac{1}{2}\langle\eta,\eta\rangle^z - \tfrac{1}{\|z\|^2} = \underbrace{\tfrac{\|\eta\|^2}{8\|z\|^2}}_{\mathcal{K}^*(z,\eta)} \underbrace{-\tfrac{1}{\|z\|^2}}_{+\,\mathcal{U}(z)}. \tag{2.9}$$

In [FW26d, §5.6] the Hamiltonian is obtained as the Legendre dual of the natural extension of $\mathcal{B}$ to the tangent bundle. The **L^2-gradient of $\mathcal{H}$** is determined by

$$d\mathcal{H}|_{(z,\eta)} = \langle \operatorname{grad}\mathcal{H}|_{(z,\eta)}, \cdot\rangle .$$

Lemma 2.5 (L^2-gradient)**.** *At $(z,\eta) \in T^*\mathcal{L}^\times\mathfrak{Z}$ the value of* $\operatorname{grad}(\mathcal{K}^* + \pi^*\mathcal{U})$ *is*

$$\operatorname{grad}\mathcal{H}|_{(z,\eta)} = \tfrac{1}{4\|z\|^2}\begin{pmatrix} \frac{8-\|\eta\|^2}{\|z\|^2} z \\ \eta \end{pmatrix} = -\tfrac{1}{4\|z\|^2}\begin{pmatrix} \frac{\|\eta\|^2}{\|z\|^2} z \\ \eta \end{pmatrix} + \begin{pmatrix} \frac{2z}{\|z\|^4} \\ 0 \end{pmatrix}. \tag{2.10}$$

Proof. The gradient is the pair $(\partial_z\mathcal{H}, \partial_\eta\mathcal{H})$. Component one is obvious. Observe that $\partial_z(-\|z\|^{-2}) = -\partial_z(\|z\|^2)^{-1} = -(-1)(\|z\|^2)^{-2}2z = \frac{2z}{\|z\|^4} = \operatorname{grad}\mathcal{U}_z$. □

We define the symplectic action ad-hoc[2] in Definition 2.6 below where $\mathcal{V}(z,\eta) := (z',\eta')$ is the canonical loop space vector field and Λ arises by integrating the Liouville form λ associated to $T^*\mathfrak{Z}$; for details see [FW26d, §5].

Definition 2.6. By definition $\mathcal{A} := i_\mathcal{V}\Lambda + \pi^*\mathcal{M} - \mathcal{K}^* - \pi^*\mathcal{U}$, that is

$$\begin{aligned} \mathcal{A}\colon T^*\mathcal{L}^\times\mathfrak{Z} &\to \mathbb{R} \\ (z,\eta) &\mapsto \langle\eta, z'\rangle + \mathcal{M}(z) - \tfrac{\|\eta\|^2}{8\|z\|^2} + \tfrac{1}{\|z\|^2} \\ &= -\langle\eta', z\rangle + \mathcal{M}(z) - \tfrac{\|\eta\|^2}{8\|z\|^2} + \tfrac{1}{\|z\|^2} \end{aligned} \tag{2.11}$$

where equality is integration by parts and the magnetic term $\mathcal{M}$ is given by (2.7).

Lemma 2.7 (L^2-gradient)**.** *At $(z,\eta) \in T^*\mathcal{L}^\times\mathfrak{Z}$ the L^2-gradient is given by*

$$\begin{aligned} \operatorname{grad}\mathcal{A}|_{(z,\eta)} &= \begin{pmatrix} -\eta' \\ z' \end{pmatrix} + \begin{pmatrix} \operatorname{grad}\mathcal{M}|_z - \operatorname{grad}\mathcal{U}|_z \\ 0 \end{pmatrix} + \tfrac{1}{4\|z\|^2}\begin{pmatrix} \frac{\|\eta\|^2}{\|z\|^2} z \\ -\eta \end{pmatrix} \\ &= \begin{pmatrix} -\eta' + \operatorname{grad}\mathcal{M}|_z - \frac{2z}{\|z\|^4} + \frac{\|\eta\|^2}{4\|z\|^4} z \\ z' - \frac{1}{4\|z\|^2}\eta \end{pmatrix}. \end{aligned} \tag{2.12}$$

Proof. Use (2.10) and integrate by parts $\langle z,\eta'\rangle = \underline{0} - \langle z',\eta\rangle$. Here the boundary terms $\langle z_1,\eta_1\rangle_0 - \langle z_0,\eta_0\rangle_0 = \langle \pm z_0, \pm\eta_0\rangle_0 - \langle z_0,\eta_0\rangle_0 = \big((\pm 1)^2 - 1\big)\langle z_0,\eta_0\rangle_0 = \underline{0}$ vanish, since indeed $(z_1,\eta_1) = \pm(z_0,\eta_0) = (\pm z_0, \pm\eta_0)$ by (2.4). □

[2] Abstractly one would twist Λ by adding the pull-back of a 1-form Θ on loop space. This works in the periodic case in which such Θ exists as discussed in [FW26d, App. C]. Calculating $d\Theta$ the formula obtained makes sense in the general, twisted-periodic, case defining a 2-form Σ on loop space [FW26d, App. B], just not an exact one. Similarly, while $\mathcal{M}$ is not defined on loop space, since ϑ is not necessarily periodic, its gradient (2.8) is, hence (2.12) is.

3 Weak Hessian fields and almost extendability

Consider an abstract Hilbert space triple (H_0, H_1, H_2). Let $U_1 \subset H_1$ be open. Set $U_2 := U_1 \cap H_2$. The following definitions are from the article [FW26c].

3.1 Almost extendability

Definition 3.1 (Weak Hessian field)**.** A **weak Hessian field** on U_1 is a continuous map $A \in C^0(U_1, \mathcal{L}(H_1, H_0)) \cap C^0(U_2, \mathcal{L}(H_2, H_1))$, notation $u \mapsto A^u$ and $A_2^u := A^u|_{H_2} \colon H_2 \to H_1$, satisfying the following two conditions:

`(Symmetry)` At any point $u \in U_1$ there is H_0-symmetry in the sense that

$$\forall x, y \in H_1 \colon \quad \langle A^u x, y \rangle_{H_0} = \langle x, A^u y \rangle_{H_0} . \tag{3.13}$$

`(Fredholm)` $\forall u \in U_1 \colon\ A^u \colon H_1 \to H_0$ is Fredholm of index zero.
$\forall u \in U_2 \colon\ A_2^u \colon H_2 \to H_1$ is Fredholm of index zero.

Definition 3.2 (Extendability)**.** We say that a weak Hessian field A on U_1 **extends** if A extends to a continuous map $U_1 \to \mathcal{L}(H_1, H_0) \cap \overline{\mathcal{L}(H_2, H_1)}$, still denoted by $u \mapsto A^u$, such that the restriction $A_2^u := A^u|_{H_2} \colon H_2 \to H_1$ is Fredholm of index zero at every point u of $\underline{U_1}$, and not only of U_2.

In general, the extendability condition is too strong; see [FW26c, §6.1]. A way out is to decompose the operator family A into two summands. The idea why and how this should be done is detailed in [FW26c, §5.2].

Definition 3.3 (Almost extendability)**.** (i) We say that a weak Hessian field A on U_1 **almost extends** if there exists a decomposition

$$A = F + C \tag{3.14}$$

with

$$F \in C^0(U_1, \mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)) \tag{3.15}$$

and

$$\exists r \in [0,1) \colon \quad C \in C^0(U_1, \mathcal{L}(\underline{H_r}, H_0)) \cap C^0(U_2, \mathcal{L}(\underline{H_1}, H_1)) \tag{3.16}$$

such that the following two axioms hold.

`(F)` $\forall u \in \underline{U_1} \colon\ F_2^u := F^u|_{H_2} \colon H_2 \to H_1$ is Fredholm of index zero.

`(C)` $\forall u \in U_1$ there exists an H_1-open neighborhood V_u of u and a constant κ such that for all $v, w \in V_u \cap H_2$ it holds the *scale Lipschitz estimate*

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa \Big(|v - w|_{H_2} + \min\{|v|_{H_2}, |w|_{H_2}\} \cdot |v - w|_{H_1} \Big). \tag{3.17}$$

(ii) If A almost extends we call the pair (F, C) a **decomposition of *A***.

3.2 A Fredholm result

Definition 3.4 (Connecting paths)**.** Fix two points $u_-, u_+ \in U_2 := U_1 \cap H_2$. Fix a **basic path** $\hat{u}$ from u_- to u_+ (see [FW25]), i.e. $\hat{u} \in C^2(\mathbb{R}, U_2)$ with the property that there exists $T > 0$ such that $\hat{u}(s) = u_-$ whenever $s \le -T$ and $\hat{u}(s) = u_+$ whenever $s \ge T$. A **connecting path from $\boldsymbol{u_-}$ to $\boldsymbol{u_+}$** is a continuous map $u \colon \mathbb{R} \to U_1$ such that the difference $u - \hat{u}$ lies in the intersection Hilbert space $W^{1,2}(\mathbb{R}, H_1) \cap L^2(\mathbb{R}, H_2)$, i.e.

$$u \in C^0(\mathbb{R}, U_1), \qquad u - \hat{u} \in W^{1,2}(\mathbb{R}, H_1) \cap L^2(\mathbb{R}, H_2).$$

The following theorem was proved in [FW26c, Thm. 6.11].

Theorem 3.5. *Let A be an almost extendable weak Hessian field on U_1. Consider two points $u_-, u_+ \in U_2$ and a connecting path u. Assume that both asymptotic operators $A^{u_\mp}$ are isomorphisms as maps $H_1 \to H_0$. Then the operators*

$$\begin{aligned} \mathbb{D}^u &= \partial_s + A^u \colon W^{1,2}(\mathbb{R}, H_0) \cap L^2(\mathbb{R}, H_1) \to L^2(\mathbb{R}, H_0) \\ \mathbb{D}^u_2 &= \partial_s + A^u_2 \colon W^{1,2}(\mathbb{R}, H_1) \cap L^2(\mathbb{R}, H_2) \to L^2(\mathbb{R}, H_1) \end{aligned}$$

are both Fredholm operators of the same Fredholm index.

4 Hamiltonian Hessian field $\boldsymbol{A}$

Convention 4.1. To simplify the presentation in Section 4 we consider only spaces of loops $\mathcal{L}^\times_+ \mathfrak{Z} = C^\infty(\mathbb{S}^1, \mathfrak{Z}) \setminus \{0\}$ in an open subset $0 \in \mathfrak{Z} \subset \mathbb{C}$, as opposed to the space $\mathcal{L}^\times_- \mathfrak{Z}$ of twisted $(-)$ loops; cf. § 4.3. The use the notation

$$\Upsilon = (z, \eta) \in \mathcal{L}\mathfrak{Z} \times \mathcal{L}\mathbb{C} = T^*\mathcal{L}\mathfrak{Z}, \quad \Xi = (z, \xi) \in \mathcal{L}\mathfrak{Z} \times \mathcal{L}\mathbb{C} = T\mathcal{L}\mathfrak{Z}.$$

We canonically identify Euclidean $\mathbb{R}^2$ with $\mathbb{C} \simeq \mathbb{C}^* \simeq (\mathbb{R}^2)^*$.

Definition 4.2 (Analytic setup)**.** Let $0 \in \mathfrak{Z} \subset \mathbb{C}$ be an open subset of $\mathbb{C}$. Consider the Hilbert space triples defined by

$$\begin{aligned} (H_0, H_1, H_2) &:= \big(L^2(\mathbb{S}^1, \mathbb{C}^2), W^{1,2}(\mathbb{S}^1, \mathbb{C}^2), W^{2,2}(\mathbb{S}^1, \mathbb{C}^2)\big) \\ (h_0, h_1, h_2) &:= \big(L^2(\mathbb{S}^1, \mathbb{C}), W^{1,2}(\mathbb{S}^1, \mathbb{C}), W^{2,2}(\mathbb{S}^1, \mathbb{C})\big) \end{aligned}$$

the Hilbert space $h_r := W^{r,2}(\mathbb{S}^1, \mathbb{C})$, $r \in (\frac{1}{2}, 1)$, the open subsets

$$u_1 := \{z \in h_1 \mid z \not\equiv 0 \wedge \forall \tau \in \mathbb{S}^1 \colon z_\tau \in \mathfrak{Z}\}, \qquad u_2 := u_1 \cap W^{2,2}(\mathbb{S}^1, \mathbb{C}),$$

as well as $U_1 := u_1 \times h_1 \subset H_1$ and $U_2 := u_2 \times h_2 \subset H_2$.

4.1 Kepler case – $\boldsymbol{A}$ extends

We first consider the case $\mathcal{M} = 0$ (no magnetic term) which corresponds to the Kepler case.

Definition 4.3. The ***$\mathcal{H}$*-perturbed non-local symplectic action**

$$\mathcal{A} = \mathcal{A}_{\mathcal{H}}^{\Lambda} = i_{\mathcal{V}}\Lambda - \mathcal{H} \colon T^*\mathcal{L}^{\times}\mathfrak{Z} = \mathcal{L}^{\times}\mathfrak{Z} \times \mathcal{L}\mathbb{C} \to \mathbb{R}$$

takes at a point $\Upsilon = (z, \eta)$ and with $\mathcal{H}$ given by (2.9) the value

$$\begin{aligned}
\mathcal{A}(z,\eta) :&= \Lambda_{\Upsilon}\Upsilon' - \mathcal{H}(\Upsilon) \\
&= \langle \eta, z'\rangle - \tfrac{\|\eta\|^2}{8\|z\|^2} + \tfrac{1}{\|z\|^2} \\
&= -\langle \eta', z\rangle - \tfrac{\|\eta\|^2}{8\|z\|^2} + \tfrac{1}{\|z\|^2}.
\end{aligned}$$

Here the final equality is integration by parts. The functional $\mathcal{A}$ continuously extends to the Sobolev $W^{1,2}$-completion U_1 by the same formula.

The ***L^2*-gradient of *$\mathcal{A}$*** at $\Upsilon = (z, \eta) \in U_1$ is, by (2.12), of the form

$$\operatorname{grad} \mathcal{A}|_{(z,\eta)} = \begin{pmatrix} -\eta' + \frac{\|\eta\|^2 z}{4\|z\|^4} - \frac{2z}{\|z\|^4} \\ z' - \frac{\eta}{4\|z\|^2} \end{pmatrix} = J_0\Upsilon' - \operatorname{grad} \mathcal{H}|_{\Upsilon} \tag{4.18}$$

where

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, \quad \mathrm{Id} = \mathrm{Id}_{W^{1,2}(\mathbb{S}^1,\mathbb{R})}.$$

Linearizing the gradient at a pair $\Upsilon = (z, \eta) \in U_1$ in direction of a smooth vector field $\hat\Upsilon = (\hat z, \hat\eta) \in H_1$ along the loop Υ defines the **Hessian operator**

$$A^{\Upsilon}\hat\Upsilon := \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \operatorname{grad} \mathcal{A}(z_\varepsilon, \eta_\varepsilon) = J_0\hat\Upsilon' - \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \operatorname{grad} \mathcal{H}|_{\Upsilon_\varepsilon} \tag{4.19}$$

where $\varepsilon \to z_\varepsilon \not\equiv 0$ and $\varepsilon \to \eta_\varepsilon$ are smooth paths in loop space U_1, respectively in H_1, notation $\Upsilon_\varepsilon = (z_\varepsilon, \eta_\varepsilon)$, such that

$$\Upsilon_0 = (z, \eta), \qquad \tfrac{d}{d\varepsilon}\big|_0 \Upsilon_\varepsilon = \hat\Upsilon = (\hat z, \hat\eta). \tag{4.20}$$

The Hessian operator of the non-local Hamiltonian $\mathcal{H}$ in (2.9) is given by

$$\begin{aligned}
&\tfrac{d}{d\varepsilon}\big|_0 \operatorname{grad} \mathcal{H}|_{\Upsilon_\varepsilon} \\
&= \tfrac{d}{d\varepsilon}\big|_0 \begin{pmatrix} \frac{\|\eta_\varepsilon\|^2 z_\varepsilon}{4\|z_\varepsilon\|^4} - \frac{2z_\varepsilon}{\|z_\varepsilon\|^4} \\ -\frac{\eta_\varepsilon}{4\|z_\varepsilon\|^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\langle\eta,\hat\eta\rangle z}{2\|z\|^4} + \frac{\|\eta\|^2 \hat z}{4\|z\|^4} - \frac{\|\eta\|^2\langle z,\hat z\rangle z}{\|z\|^6} \overbrace{- \frac{2\hat z}{\|z\|^4} + \frac{8\langle z,\hat z\rangle z}{\|z\|^6}}^{=U^z} \\ -\frac{\hat\eta}{4\|z\|^2} + \frac{\langle z,\hat z\rangle \eta}{2\|z\|^4} \end{pmatrix} \\
&= \tfrac{1}{4\|z\|^6} \begin{pmatrix} 2\|z\|^2 z\,\eta^*\hat\eta + \|z\|^2\|\eta\|^2\hat z - 4\|\eta\|^2 z\,z^*\hat z - 8\|z\|^2\hat z + 32 z\,z^*\hat z \\ -\|z\|^4\hat\eta + 2\|z\|^2\eta\,z^*\hat z \end{pmatrix} \\
&= \tfrac{1}{4\|z\|^6} \begin{pmatrix} \|z\|^2\|\eta\|^2\mathrm{Id} - 4\|\eta\|^2 z\,z^* - 8\|z\|^2\mathrm{Id} + 32 z\,z^* & 2\|z\|^2 z\,\eta^* \\ 2\|z\|^2\eta\,z^* & -\|z\|^4\mathrm{Id} \end{pmatrix} \begin{pmatrix} \hat z \\ \hat\eta \end{pmatrix} \\
&=: \operatorname{Hess} \mathcal{H}^{(z,\eta)} \begin{pmatrix} \hat z \\ \hat\eta \end{pmatrix}.
\end{aligned} \tag{4.21}$$

Here $z^*\colon \hat z \mapsto \langle z, \hat z\rangle =: z^*\hat z$ is the duality pairing. The Hessian operator takes the form of a matrix $\operatorname{Hess}\mathcal{H}^\Upsilon$ called the **Hessian matrix** of the Hamiltonian $\mathcal{H}$. The matrix is symmetric since $(\eta z^*)^* = z^{**}\eta^* = z\eta^*$. Hence $\operatorname{Hess}\mathcal{H}^\Upsilon$ is H_0-symmetric. Along $U_1 \subset W^{1,2}\times W^{1,2}$ the multiplication operator is well defined as a bounded linear map

$$\operatorname{Hess}\mathcal{H}^\Upsilon\colon H_1 \xrightarrow{\text{bd}} H_1, \qquad \forall \Upsilon = (z,\eta)\in U_1 \tag{4.22}$$

since the product of two $W^{1,2}$ maps with domain $\mathbb{S}^1$ lies in $W^{1,2}$, in symbols, $W^{1,2}\cdot W^{1,2}\subset W^{1,2}$; see (C.52).

Lemma 4.4 (compact Hessian matrix)**.** *As a linear map between the spaces*

$$\forall \Upsilon\in U_1\colon \qquad \operatorname{Hess}\mathcal{H}^\Upsilon\colon \quad H_1 \xrightarrow{\text{cp}} H_0, \quad H_2 \xrightarrow{\text{cp}} H_1,$$

the Hessian matrix of $\mathcal{H}$ is a compact linear operator. It is continuous as a map

$$[\Upsilon\mapsto \operatorname{Hess}\mathcal{H}^\Upsilon] \in C^0(U_1,\mathcal{L}(H_1,H_0))\cap C^0(\underline{U_1},\mathcal{L}(H_2,H_1)).$$

Proof. The composition of a bounded and a compact operator is compact. Consider (4.22). On the target side there is the compact embedding $H_1\hookrightarrow H_0$ and on the domain side there is the compact embedding $H_2\hookrightarrow H_1$.

The two continuity assertions are a consequence of the compact embeddings together with continuity as a map

$$[(z,\eta)\mapsto \operatorname{Hess}\mathcal{H}^{(z,\eta)}]\in C^0(U_1,\mathcal{L}(H_1)).$$

This continuity follows from the continuity of the multiplication map $W^{1,2}\times W^{1,2}\to W^{1,2}$; see (C.52). This proves Lemma 4.4. □

4.1.1 Weak Hessian field extends

Lemma 4.5 (weak Hessian field)**.** *The Kepler Hessian operators*

$$A^\Upsilon = J_0\partial_\tau - \operatorname{Hess}\mathcal{H}^\Upsilon\colon H_1\to H_0 \tag{4.23}$$

one for each $\Upsilon\in U_1$, determine a weak Hessian field A on U_1.

Proof. There are three steps.

STEP 1 (SPACES). It holds that

$$\begin{aligned} A \in& C^0(U_1,\mathcal{L}(H_1,H_0)\cap\mathcal{L}(H_2,H_1))\\ &\subset C^0(U_1,\mathcal{L}(H_1,H_0))\cap C^0(U_2,\mathcal{L}(H_2,H_1)). \end{aligned} \tag{4.24}$$

The summand $J_0\partial_\tau$ is a constant map in $\mathcal{L}(H_1,H_0)\cap\mathcal{L}(H_2,H_1)$. Furthermore, by Lemma 4.4 the summand $\operatorname{Hess}\mathcal{H}$ lies in $C^0(U_1,\mathcal{L}(H_1))$.

STEP 2. The axiom (`Symmetry`) holds.

Both $J_0\partial_\tau$ and $\operatorname{Hess}\mathcal{H}^\Upsilon$ are H_0-symmetric at any point $\Upsilon\in U_1$.

STEP 3. The axiom (`Fredholm`) holds.

That $J_0\partial_\tau$ is Fredholm of index zero as a map $H_1 \to H_0$ and as a map $H_2 \to H_1$ is well known; see e.g. [FW26c, Thm. C.1]. In either case, by Lemma 4.4, the Hessian is a compact perturbation, but such preserve Fredholm property and index. This concludes the proof of Lemma 4.5. □

Observe that (4.24) shows that the non-magnetic weak Hessian field (4.23) is extendable, see Definition 3.2, which we state as

Proposition 4.6 (extends)**.** *The Kepler Hessian field A extends.*

4.2 With magnetic term $\mathcal{M}$

We use H_k, U_k, u_k as in Definition 4.2. Consider the action functional $\mathcal{A} = i_{\mathcal{V}}\Lambda + \pi^*\mathcal{M} - \mathcal{H}$ in (2.11). By (2.12), the L^2-gradient at $\Upsilon = (z,\eta) \in U_1$ is

$$\begin{aligned} \operatorname{grad}\mathcal{A}|_{(z,\eta)} &= \begin{pmatrix} -\eta' + \operatorname{grad}\mathcal{M}|_z + \frac{\|\eta\|^2 z}{4\|z\|^4} - \frac{2z}{\|z\|^4} \\ z' - \frac{\eta}{4\|z\|^2} \end{pmatrix} \\ &= J_0\Upsilon' + \begin{pmatrix} \operatorname{grad}\mathcal{M}|_z \\ 0 \end{pmatrix} - \operatorname{grad}\mathcal{H}|_\Upsilon \end{aligned}$$

where $\operatorname{grad}\mathcal{M}|_z$ is the sum of the last four terms in (2.8). Linearizing the gradient at a pair $\Upsilon = (z,\eta) \in U_1$ in direction of a vector field $\hat{\Upsilon} = (\hat{z},\hat{\eta}) \in H_1$ along the loop Υ yields, analogously to (4.19) and (4.21), **Hessian operators**

$$\begin{aligned} A^\Upsilon\hat{\Upsilon} = A^{(z,\eta)}(\hat{z},\hat{\eta}) &:= \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \operatorname{grad}\mathcal{A}(z_\varepsilon,\eta_\varepsilon) \\ &= J_0\hat{\Upsilon}' + \begin{pmatrix} M^z & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{z} \\ \hat{\eta} \end{pmatrix} - \operatorname{Hess}\mathcal{H}^{(z,\eta)}\begin{pmatrix} \hat{z} \\ \hat{\eta} \end{pmatrix} \end{aligned} \tag{4.25}$$

one operator at every element $\Upsilon = (z,\eta) \in U_1$. Lemma B.3 provides the Hessian operators M^z, one for each $z \in u_1$, of the magnetic function $\mathcal{M}$. The Hessian matrix of $\mathcal{H}$ is given by (4.21).

The task at hand is to show that $\Upsilon \mapsto A^\Upsilon$ defines a *weak Hessian field* along U_1 (Proposition 4.8) which *almost extends* (Theorem 4.11); cf. § 3.

Remark 4.7 (Magnetic Hessian M)**.** Of the three summands of the Hessian (4.25) it remains to analyze the second one, namely the map

$$U_1 = u_1 \times h_1 \supset u_1 \ni z \mapsto \begin{pmatrix} M^z & 0 \\ 0 & 0 \end{pmatrix} : H_1 \to H_0, \quad H_2 \to H_1.$$

Since $H_k = h_k \times h_k$, the magnetic Hessian is actually a linear map $M^z\colon h_1 \to h_0$, $h_2 \to h_1$ which is composed of 18 summands

$$\underbrace{0, C_{42}, C_{43}, F_{44}}_{T_4}, \underbrace{T_{51}, T_{52}, T_{53}, T_{54}}_{T_5}, \underbrace{T_{61}, T_{62}, T_{63}, T_{64}, T_{65}}_{T_6}, \underbrace{T_{71}, T_{72}, T_{73}, T_{74}, T_{75}}_{T_7}.$$

These linear operators are defined and analyzed one-by-one in Appendix C.3.2.

4.2.1 Weak Hessian field

Proposition 4.8 (weak Hessian field)**.** *The Hessian operators A^Υ in (4.25), one operator for every $\Upsilon \in U_1$, form a weak Hessian field A on U_1 (Definition 3.1).*

The proof of the proposition requires two lemmas which we state and prove thereafter (Lemma 4.9 and Lemma 4.10).

Proof. There are three steps.
STEP 1 (SPACES). It holds that $A \in C^0(U_1, \mathcal{L}(H_1, H_0)) \cap C^0(U_2, \mathcal{L}(H_2, H_1))$.
By the non-magnetic case (4.24) it suffices to show that the magnetic Hessian operators form a map $z \mapsto M^z \in C^0(u_1, \mathcal{L}(h_1, h_0)) \cap C^0(u_2, \mathcal{L}(h_2, h_1))$. This is true by the explicit formulas in § C.3.2.
STEP 2. The axiom (`Symmetry`) holds.
The Hessian operator A^Υ is the derivative of the L^2-gradient of the functional $\mathcal{A}$ in (2.11). Hence A^Υ is H_0-symmetric by Corollary A.2.
STEP 3. The axiom (`Fredholm`) holds.
On level one and also on level two there is 1 Fredholm operator of index zero $F^{1,\Upsilon}\colon H_1 \to H_0$, $\forall \Upsilon \in U_1$, and $F_2^{1,\Upsilon}\colon H_2 \to H_1$, $\forall \Upsilon \in U_2$ (Lemma 4.9) plus 17 compact perturbations $\operatorname{diag}(M^z_{ij}, 0)$ from the magnetic field (see § C.3.2 where to T_{65} and T_{75} we apply, in addition, Lemma 4.10) plus the compact perturbation $\operatorname{Hess}\mathcal{H}$ (Lemma 4.4). Since the Fredholm property as well as the index is stable under compact perturbation the Fredholm axiom follows. This proves Proposition 4.8. □

Fredholm perturbation

The following lemma enters step 3 of the proof of Proposition 4.8.

Lemma 4.9. *Let $(z, \eta) \in U_1$ and $f^z_{44} := -4b_{t_z}|_z j_0$, cf. (C.67). For $r \in [0,1]$ set*

$$F^{r,z} := J_0\partial_\tau + \operatorname{diag}(r \underbrace{f^z_{44}\partial_\tau}_{F^z_{44}}, 0) = \underbrace{\begin{pmatrix} r f^z_{44} & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}}_{=: \boldsymbol{b}^{r,z}} \partial_\tau \colon \quad H_1 \xrightarrow{F^{r,z}} H_0, \quad H_2 \xrightarrow{F_2^{r,z}} H_1.$$

Then both operators $F^{r,z}$ and $F_2^{r,z}$ are Fredholm of index zero for each r.

Proof. Pick $z \in u_1$. For $r = 0$ both operators $F^{0,z} = J_0\partial_\tau$ and $F_2^{0,z} = J_0\partial_\tau$ do not depend on z and are well known to be Fredholm of index zero; see proof of Proposition 4.6. Hence, because the semi-Fredholm index is invariant under homotopy, see e.g. [Mül07, §18 Cor. 3], it suffices to show that $F^{r,z}$ and $F_2^{r,z}$ are semi-Fredholm for every $r \in [0,1]$. To this end we derive a semi-Fredholm estimate for the level one operator $F^{r,z}$ and the level two operator $F_2^{r,z}$.
Step 1 (semi-Fredholm level 1). Given $z \in u_1$, there is $c_z > 0$ such that

$$\|\hat\Upsilon\|_{H_1} \le c_z \left(\|F^{r,z}\hat\Upsilon\|_{H_0} + \|\hat\Upsilon\|_{H_0} \right)$$

for all $\hat\Upsilon = (\hat z, \hat\eta) \in H_1$ and $r \in [0,1]$.

Proof. Fix $z \in u_1$. While $H_0 := L^2(\mathbb{S}^1, \mathbb{C}^2)$ by Definition 4.2, let us abbreviate

$$L^2 := L^2(\mathbb{S}^1, \mathbb{C}), \qquad H_0 = L^2 \times L^2.$$

For $r \in [0,1]$ the operator $\boldsymbol{b}^{r,z}$ is invertible with inverse

$$(\boldsymbol{b}^{r,z})^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & r f_{44}^z \end{pmatrix} : L^2 \times L^2 \to L^2 \times L^2$$
$$(\hat{z}, \hat{\eta}) \mapsto (\hat{\eta}, -\hat{z} + r f_{44}^z \hat{\eta}).$$

To estimate the operator norm of $(\boldsymbol{b}^{r,z})^{-1}$, let $\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in L^2 \times L^2$ and estimate

$$\begin{aligned}
\|(\boldsymbol{b}^{r,z})^{-1}\hat{\Upsilon}\|^2_{L^2\times L^2} &= \|\hat{\eta}\|^2_{L^2} + \|-\hat{z} + r f^z_{44}\hat{\eta}\|^2_{L^2} \\
&\le \|\hat{\eta}\|^2_{L^2} + 2\|\hat{z}\|^2_{L^2} + 2r^2\|f^z_{44}\hat{\eta}\|^2_{L^2} \\
&\le \|\hat{\eta}\|^2_{L^2} + 2\|\hat{z}\|^2_{L^2} + 2(4\beta^z_{\mathbb{S}^1})^2\|\hat{\eta}\|^2_{L^2} \quad \text{, by (C.67)} \\
&\le \max\{2, 1 + 2(4\beta^z_{\mathbb{S}^1})^2\}\|\hat{\Upsilon}\|^2_{L^2\times L^2}.
\end{aligned}$$

We used that $r \le 1$. Hence the operator norm of the inverse is bounded by

$$\|(\boldsymbol{b}^{r,z})^{-1}\|_{\mathcal{L}(L^2\times L^2)} \le \mu_z, \qquad \mu_z := \sqrt{\max\{2, 1 + 2(4\beta^z_{\mathbb{S}^1})^2\}}.$$

To get the semi-Fredholm estimate let $\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in W^{1,2} \times W^{1,2} = H_1$, then

$$\begin{aligned}
\|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} &= \|\hat{\Upsilon}\|^2_{L^2\times L^2} + \|\hat{\Upsilon}'\|^2_{L^2\times L^2} \\
&= \|\hat{\Upsilon}\|^2_{L^2\times L^2} + \|(\boldsymbol{b}^{r,z})^{-1}\boldsymbol{b}^{r,z}\hat{\Upsilon}'\|^2_{L^2\times L^2} \\
&= \|\hat{\Upsilon}\|^2_{L^2\times L^2} + \|(\boldsymbol{b}^{r,z})^{-1}F^{r,z}\hat{\Upsilon}\|^2_{L^2\times L^2} \\
&\le \|\hat{\Upsilon}\|^2_{L^2\times L^2} + \|(\boldsymbol{b}^{r,z})^{-1}\|^2_{\mathcal{L}(L^2\times L^2)}\|F^{r,z}\hat{\Upsilon}\|^2_{L^2\times L^2} \\
&\le \max\{1, \mu_z^2\}\left(\|\hat{\Upsilon}\|^2_{L^2\times L^2} + \|F^{r,z}\hat{\Upsilon}\|^2_{L^2\times L^2}\right) \\
&\le \max\{1, \mu_z^2\}\left(\|\hat{\Upsilon}\|_{L^2\times L^2} + \|F^{r,z}\hat{\Upsilon}\|_{L^2\times L^2}\right)^2.
\end{aligned}$$

This proves Step 1 for $c_z = \max\{1, \mu_z\}$. □

Step 2 (semi-Fredholm level 2). Given $z \in u_1$, there is $d_z > 0$ such that

$$\|\hat{\Upsilon}\|_{H_2} \le d_z \left(\|F_2^{r,z}\hat{\Upsilon}\|_{H_1} + \|\hat{\Upsilon}\|_{H_1}\right)$$

for all $\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in H_2$ and $r \in [0,1]$.

Proof. Fix $z \in u_1$. While $H_1 := W^{1,2}(\mathbb{S}^1, \mathbb{C}^2)$ by Definition 4.2, we abbreviate

$$W^{1,2} := W^{1,2}(\mathbb{S}^1, \mathbb{C}), \qquad H_1 = W^{1,2} \times W^{1,2}.$$

For $r \in [0,1]$ the operator $\boldsymbol{b}^{r,z}$ is invertible with inverse

$$(\boldsymbol{b}^{r,z})^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & r f^z_{44} \end{pmatrix} : W^{1,2} \times W^{1,2} \to W^{1,2} \times W^{1,2}$$
$$(\hat{z}, \hat{\eta}) \mapsto (\hat{\eta}, -\hat{z} + r f^z_{44} \hat{\eta}) .$$

To estimate the operator norm of $(\boldsymbol{b}^{r,z})^{-1}$, let $\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in W^{1,2} \times W^{1,2}$, then

$$\begin{aligned} \|(\boldsymbol{b}^{r,z})^{-1} \hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} &= \|\hat{\eta}\|^2_{W^{1,2}} + \|-\hat{z} + r f^z_{44}\hat{\eta}\|^2_{W^{1,2}} \\ &\le \|\hat{\eta}\|^2_{W^{1,2}} + 2\|\hat{z}\|^2_{W^{1,2}} + 2r^2 \|f^z_{44}\hat{\eta}\|^2_{W^{1,2}} \\ &\le \|\hat{\eta}\|^2_{W^{1,2}} + 2\|\hat{z}\|^2_{W^{1,2}} + 8\gamma^2_{\|z\|_{1,2}} \|\hat{\eta}\|^2_{W^{1,2}} \text{ , by (C.68)} \\ &\le \max\{2, 1 + 8\gamma^2_{\|z\|_{1,2}}\} \|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} . \end{aligned}$$

We used that $r \le 1$. Hence the operator norm of the inverse is bounded by

$$\|(\boldsymbol{b}^{r,z})^{-1}\|_{\mathcal{L}(W^{1,2}\times W^{1,2})} \le \kappa_z, \qquad \kappa_z := \sqrt{\max\{2, 1 + 8\gamma^2_{\|z\|_{1,2}}\}}.$$

To get the semi-Fredholm estimate let $\hat{\Upsilon} = (\hat{z}, \hat{\eta}) \in W^{1,2} \times W^{1,2} = H_1$, then

$$\begin{aligned} \|\hat{\Upsilon}\|^2_{W^{2,2}\times W^{2,2}} &= \|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} + \|\hat{\Upsilon}'\|^2_{W^{1,2}\times W^{1,2}} \\ &= \|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} + \|(\boldsymbol{b}^{r,z})^{-1} \boldsymbol{b}^{r,z} \hat{\Upsilon}'\|^2_{W^{1,2}\times W^{1,2}} \\ &= \|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} + \|(\boldsymbol{b}^{r,z})^{-1} F^{r,z} \hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} \\ &= \|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} + \|(\boldsymbol{b}^{r,z})^{-1}\|^2_{\mathcal{L}(W^{1,2}\times W^{1,2})} \|F^{r,z}\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} \\ &\le \max\{1, \kappa_z^2\} \Big(\|\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} + \|F^{r,z}\hat{\Upsilon}\|^2_{W^{1,2}\times W^{1,2}} \Big) \\ &\le \max\{1, \kappa_z^2\} \left(\|\hat{\Upsilon}\|_{W^{1,2}\times W^{1,2}} + \|F^{r,z}\hat{\Upsilon}\|_{W^{1,2}\times W^{1,2}} \right)^2 . \end{aligned}$$

This proves Step 2 for $d_z = \max\{1, \kappa_z\}$. □

Since the inclusion maps $H_1 \hookrightarrow H_0$ and $H_2 \hookrightarrow H_1$ are compact, Step 1 and Step 2 provide semi-Fredholm estimates, see e.g. [MS04, Le. A.1.1]. This completes the proof of Lemma 4.9. □

Compact perturbations

The following lemma enters step 3 of the proof of Proposition 4.8.

Lemma 4.10. *Pick $z \in u_1$. Both linear operators T^z_{65} and T^z_{75} in (C.70) and (C.72), respectively, are compact as operators $h_1 \to h_0$ and $h_2 \to h_1$.*

Proof. On level two compactness follows from the fact that $T^z_{65}, T^z_{75} \in \mathcal{L}(h_1)$, see estimates (C.71) and (C.73), and that the embedding $h_2 \hookrightarrow h_1$ is compact. Level one is harder. For that we show that the operators extend to bounded linear operators $h_r \to h_0$ for every $r \in (\frac{1}{2}, 1)$. Then the compactness follows from the compact inclusion $h_1 \hookrightarrow h_r$ whenever $r \in (\frac{1}{2}, 1)$. Fix $r \in (\frac{1}{2}, 1)$.

The operator T^z_{65}. We apply to (C.70) integration by parts to obtain

$$
\begin{aligned}
F^z_{65}\xi &:= \tfrac{2z_\tau}{\|z\|_2^2}\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, \xi'_\sigma\rangle_0 \, d\sigma\\
&= \tfrac{2z_\tau}{\|z\|_2^2}\left(\langle \dot{\boldsymbol{a}}_0|_{z_0},\xi_0\rangle_0 - \langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau},\xi_\tau\rangle_0 - \int_\tau^1 \langle (\dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma})',\xi_\sigma\rangle_0\, d\sigma\right)\\
&= \tfrac{2z_\tau}{\|z\|_2^2}\left(\langle \dot{\boldsymbol{a}}_0|_{z_0},\xi_0\rangle_0 - \langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau},\xi_\tau\rangle_0\right)\\
&\quad - \tfrac{2z_\tau}{\|z\|_2^2}\int_\tau^1 \left\langle \ddot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}\tfrac{|z_\sigma|^2}{\|z\|_2^2} + d\dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma} z'_\sigma, \xi_\sigma\right\rangle_0 d\sigma
\end{aligned}
$$

where we used $t_z(1)=1$ and 1-periodicity of $\dot{\boldsymbol{a}}$, z, and ξ. With the α-constants in (C.62) for $\xi \in W^{r,2}$ we estimate (cf. (C.69) for the last summand)

$$
\begin{aligned}
\|F^z_{65}\xi\|_{L^2} &\le \tfrac{2\dot\alpha^z_{\mathbb{S}^1}}{\|z\|_2}\|\xi\|_\infty + \tfrac{2\dot\alpha^z_{\mathbb{S}^1}}{\|z\|_2}\|\xi\|_\infty + \tfrac{2}{\|z\|_2}\tau\ddot\alpha^z_{\mathbb{S}^1}\tfrac{\|z\|_\infty^2}{\|z\|_2^2}\|\xi\|_\infty + \tfrac{2}{\|z\|_2}d\dot\alpha^z_{\mathbb{S}^1}\|z'\|_2\|\xi\|_2\\
&\le \mathit{const}(\|z\|_{1,2},r)\|\xi\|_{r,2}.
\end{aligned}
$$

This proves that $F^z_{65}\colon h_r \to h_0$ is bounded.

The operator T^z_{75}. We apply to (C.72) integration by parts to get

$$
\begin{aligned}
T^z_{75}\xi &:= -\tfrac{2z_\tau}{\|z\|_2^4}\int_0^1 \textstyle\int_0^s |z_\sigma|^2\, d\sigma\cdot\langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, \xi'_s\rangle_0\, ds\\
&= -\tfrac{2z_\tau}{\|z\|_2^4}\left(\|z\|_2^2\langle \dot{\boldsymbol{a}}_0|_{z_0},\xi_0\rangle_0 - 0\right) + \tfrac{2z_\tau}{\|z\|_2^4}\int_0^1 |z_s|^2\langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s},\xi_s\rangle\, ds\\
&\quad + \tfrac{2z_\tau}{\|z\|_2^4}\int_0^1 \textstyle\int_0^s |z_\sigma|^2\, d\sigma \left\langle \ddot{\boldsymbol{a}}_{t_z(s)}|_{z_s}\tfrac{|z_s|^2}{\|z\|_2^2} + d\dot{\boldsymbol{a}}_{t_z(s)}|_{z_s} z'_s, \xi_s\right\rangle_0 ds
\end{aligned}
$$

where we used $t_z(1)=1$ and 1-periodicity of $\dot{\boldsymbol{a}}$, z, and ξ. (We ignore that the first summand of T^z_{75} actually cancels the first summand of T^z_{65}.) With the α-constants in (C.62) for $\xi \in W^{r,2}$ we estimate (cf. (C.69) for the last summand)

$$
\begin{aligned}
\|F^z_{75}\xi\|_{L^2} &\le \tfrac{2\dot\alpha^z_{\mathbb{S}^1}}{\|z\|_2}\|\xi\|_\infty + \tfrac{2}{\|z\|_2^3}\|z\|_\infty^2\dot\alpha^z_{\mathbb{S}^1}\|\xi\|_\infty + \tfrac{2\ddot\alpha^z_{\mathbb{S}^1}}{\|z\|_2^3}\|z\|_\infty^2\|\xi\|_\infty + \tfrac{2d\dot\alpha^z_{\mathbb{S}^1}}{\|z\|_2}\|z'\|_2\|\xi\|_2\\
&\le \mathit{const}(\|z\|_{1,2},r)\|\xi\|_{r,2}.
\end{aligned}
$$

This proves that $F^z_{75}\colon h_r \to h_0$ is bounded. This proves Lemma 4.10. □

4.2.2 Almost extendable

Theorem 4.11. *The magnetic weak Hessian field A in (4.25) is almost extendable. The pair (F,C) defined for (z,η) in U_1, respectively U_2, by*

$$
\begin{aligned}
C^{(z,\eta)} &:= \operatorname{diag}(C^z_{42},0) + \operatorname{diag}(C^z_{43},0)\\
F^{(z,\eta)} &:= A^{(z,\eta)} - C^{(z,\eta)} = F^{1,z} + \textstyle\sum_1^{15}\operatorname{diag}(T_{ij},0) - \operatorname{Hess}\mathcal{H}^{(z,\eta)}
\end{aligned}
$$

is a decomposition (Definition 3.3).

The proof of the theorem requires two propositions which we state and prove thereafter (Proposition 4.12 and Proposition 4.13.

Proof. The proof has four steps 1, 2, (`C`), and (`F`). In Definition 4.2 we defined spaces $H_k := W^{k,2}(\mathbb{S}^1, \mathbb{C}^2) \supset U_k$ and $h_k := W^{k,2}(\mathbb{S}^1, \mathbb{C}) \supset u_k$.

Step 1. There is $r \in [0,1)$ such that $(z,\eta) \mapsto C^{(z,\eta)}$ lies in $C^0(U_1, \mathcal{L}(\underline{H_r}, H_0))$, where $H_r = W^{r,2}(\mathbb{S}^1, \mathbb{C}^2)$, and in $C^0(U_2, \mathcal{L}(\underline{H_1}, H_1))$; see (3.16).

Proof. The maps of the diagonal form $(z,\eta) \mapsto \operatorname{diag}(C^z_{42}, 0)$ have the required continuity property, since the diagonal block maps lie in the spaces

$$z \mapsto C^z_{42}, \quad z \mapsto C^z_{43} \quad \in C^0(u_1, \mathcal{L}(\underline{h_r}, h_0)) \cap C^0(u_2, \mathcal{L}(\underline{h_1}, h_1)) \tag{4.26}$$

by (C.65) and (C.66), respectively. □

Step 2. We need to show that the map $F\colon (z,\eta) \mapsto F^{(z,\eta)}$ is element of the space $C^0(U_1, \mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1))$; see (3.15).

Proof. The diagonal block $z \mapsto f^z_{44}\partial_\tau$ of $z \mapsto F^{1,z}$, see Lemma 4.9. is a map

$$f^z_{44}\partial_\tau \colon \quad h_1 \xrightarrow{\partial_\tau} h_0 \overset{\text{(C.67)}}{\longrightarrow} h_0, \quad h_2 \xrightarrow{\partial_\tau} h_1 \overset{\text{(C.68)}}{\longrightarrow} h_1,$$

which depends continuously on $z \in u_1$, see line two in (C.68). Since the other three blocks of $F^{1,z}$ are constant, this shows that $z \mapsto F^{1,z}$ has the required continuity property. The Hessian operator $\operatorname{Hess}\mathcal{H}$ has the required continuity property by Lemma 4.4. The fifteen operators of the diagonal form

$$\begin{aligned} z \mapsto \operatorname{diag}(T_{ij}, 0)\colon &\qquad H_1 \to H_0 &\qquad H_2 \to H_1 \\ z \mapsto T^z_{ij}\colon &\qquad h_1 \to h_0 &\qquad h_2 \to h_1 \end{aligned}$$

have the required continuity property since so do the diagonal blocks T_{ij} as shown for each of them in § C.3.2. This proves Step 2. □

Step (`C`). The map $(z,\eta) \mapsto C^{(z,\eta)} = \operatorname{diag}(C^z_{42}, 0) + \operatorname{diag}(C^z_{43}, 0)$ satisfies the scale Lipschitz estimate (3.17).

Proof. Due to the specific diagonal form it suffices to consider the upper diagonal operators (4.26). Proposition 4.12 and Proposition 4.13 prove Step (`C`). □

Step (`F`). $\forall \Upsilon = (z,\eta) \in U_1\colon\ F^{1,z}_2 + \sum_1^{15} \operatorname{diag}(T_{ij}, 0) - \operatorname{Hess}\mathcal{H}^\Upsilon \colon H_2 \to H_1$ is Fredholm of index zero.

Proof. Lemma 4.9 asserts that $F^{1,z}_2\colon H_2 \to H_1$ is Fredholm of index zero. Lemma 4.4 asserts that $\operatorname{Hess}\mathcal{H}^\Upsilon\colon H_2 \to H_1$ is compact. The fifteen operators of diagonal form $\operatorname{diag}(T_{ij}, 0)\colon H_2 \to H_1$ are compact since so are the diagonal blocks $T_{ij}\colon h_2 \to h_1$ as shown in § C.3.2. Since Fredholm property and index are stable under compact perturbation this proves Step (`F`). □

This concludes the proof of Theorem 4.11. □

4.2.3 Scale Lipschitz estimates

The following proposition is part of Step (C) of the proof of Theorem 4.11.

Proposition 4.12 (C_{43} is scale Lipschitz)**.** *For $z \mapsto C_{43}^z$ in (4.26) the scale Lipschitz estimate (C) holds true, namely (3.17) with lower case spaces h_k and u_1.*

Proof. Let $z \in u_1$ and $\xi \in h_1$. We consider the map in (C.66), namely

$$C^z \xi := C_{43}^z \xi := \left(db_{t_z(\tau)}|_{z_\tau} \xi_\tau\right) j_0 z_\tau'.$$

Since $z \colon \mathbb{S}^1 \to \mathfrak{Z}$ is continuous its image is compact. So the image of z admits an open neighborhood $\mathfrak{V}_z$ in $\mathfrak{Z} \subset \mathbb{R}^2$ of compact closure $\bar{\mathfrak{V}}_z$. By compactness $\bar{\mathfrak{V}}_z$ is contained in a ball of some radius $\rho = \rho(z)$, in symbols

$$\bar{\mathfrak{V}}_z \subset \mathfrak{B}_\rho := \{\nu \in \mathbb{C} \colon |\nu| < \rho\}. \tag{4.27}$$

We define by

$$\mathrm{v}_z := \{v \in W^{1,2}(\mathbb{S}^1, \mathfrak{V}_z) \colon \|v\|_2 > \tfrac{1}{2}\|z\|_2 > 0 \text{ and } \|v\|_{1,2} < 2\|z\|_{1,2}\} \tag{4.28}$$

an h_1-open neighborhood of z in u_1 uniformly bounded away from 0. Hence

$$v \in \mathrm{v}_z \quad \Rightarrow \quad \|v\|_2 \le \|v\|_\infty \le \rho.$$

Pick $v, w \in \mathrm{v}_z \cap h_2$. By definition of the $W^{1,2}$ norm, calculating the derivative ∂_τ, the triangle inequality and since $t_v'(\tau) = |v_\tau|^2 / \|v\|_2^2$ by (C.54), we get

$$\begin{aligned}
&\|(C^v - C^w)\xi\|_{1,2}\\
&\le \|\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau'\|_2\\
&\quad + \|\partial_\tau \left(\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau'\right) - \partial_\tau \left(\left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau'\right)\|_2\\
&\le \|\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau'\|_2\\
&\quad + \|\left(d\dot{b}_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v_\tau' - \left(d\dot{b}_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) \tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 w_\tau'\|_2\\
&\quad + \|\left(d^2 b_{t_v(\tau)}|_{v_\tau} \xi_\tau v_\tau'\right) j_0 v_\tau' - \left(d^2 b_{t_w(\tau)}|_{w_\tau} \xi_\tau w_\tau'\right) j_0 w_\tau'\|_2\\
&\quad + \|\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau'\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau'\right) j_0 w_\tau'\|_2\\
&\quad + \|\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau'' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau''\|_2.
\end{aligned}$$

We need to estimate the L^2-norms of five differences, notation **D1-D5**.

Difference D1. Add twice zero to write D1 as a sum $D_{11}\xi + D_{12}\xi + D_{13}\xi$

$$\begin{aligned}
&\left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau'\\
&= \left(db_{t_v(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau'\\
&\quad + \left(db_{t_w(\tau)}|_{v_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 v_\tau'\\
&\quad + \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 v_\tau' - \left(db_{t_w(\tau)}|_{w_\tau} \xi_\tau\right) j_0 w_\tau'
\end{aligned}$$

pointwise at $\tau \in \mathbb{S}^1$.

$\boldsymbol{D_{11}}$**.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the operator norm is finite

$$d\dot{\beta}_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{V}}_z} \|d\dot{b}\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})} < \infty. \tag{4.29}$$

By Taylor's theorem, then using (C.58) on $|t_v(\tau) - t_w(\tau)|$, we estimate

$$\begin{aligned} \|db_{t_v(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{v_\tau}\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})} &\le d\dot{\beta}_{\bar{\mathfrak{V}}_z}\, |t_v(\tau) - t_w(\tau)| \\ &\le 2d\dot{\beta}_{\bar{\mathfrak{V}}_z} \tfrac{\|v\|_2+\|w\|_2}{\|v\|_2^2} \|v-w\|_2 \\ &\le 2d\dot{\beta}_{\bar{\mathfrak{V}}_z} \tfrac{2^2 2\rho}{\|z\|_2^2} \|v-w\|_2 \\ &= \tfrac{16\rho(d\dot{\beta}_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2} \|v-w\|_2. \end{aligned} \tag{4.30}$$

The last step is by (4.27) and (4.28). Use this operator norm estimate to obtain

$$\begin{aligned} \|D_{11}\xi\|_2 &= \|\big((db_{t_v(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{v_\tau})\xi_\tau\big)\, j_0 v'_\tau\|_2 \\ &\le \tfrac{16\rho(d\dot{\beta}_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2} \|v-w\|_2 \|\xi\|_\infty \|v'\|_2 \\ &\le \tfrac{16\rho(d\dot{\beta}_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2}\ |v|_{h_1} \cdot |v-w|_{h_0}\ \|\xi\|_{1,2}. \end{aligned}$$

$\boldsymbol{D_{12}}$**.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the operator norm is finite

$$d^2\beta_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{V}}_z} \|d^2 b\|_{\mathcal{L}(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R})} < \infty. \tag{4.31}$$

By Taylor's theorem we estimate

$$\begin{aligned} \|D_{12}\xi\|_2 &= \|\big((db_{t_w(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{w_\tau})\xi_\tau\big)\, j_0 v'_\tau\|_2 \\ &\le d^2\beta_{\bar{\mathfrak{V}}_z} \|v-w\|_\infty \|\xi\|_\infty \|v'\|_2 \\ &\le d^2\beta_{\bar{\mathfrak{V}}_z}\ |v|_{h_1} \cdot |v-w|_{h_1}\ \|\xi\|_{1,2}. \end{aligned}$$

$\boldsymbol{D_{13}}$**.** With the constant $d\beta^w_{\mathbb{S}^1}$ defined by (C.63) we estimate

$$\begin{aligned} \|D_{13}\xi\|_2 &= \|(db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0\,(v'_\tau - w'_\tau)\|_2 \\ &\le d\beta^w_{\mathbb{S}^1} \|\xi\|_\infty \|v'-w'\|_2 \\ &\le d\beta^w_{\mathbb{S}^1}\ |v-w|_{h_1}\ \|\xi\|_{1,2} \end{aligned}$$

where the constant $d\beta^w_{\mathbb{S}^1}$ is defined by (C.63).

Difference D2 Add four zeroes to write D2 as a sum $D_{21}\xi + \dots + D_{25}\xi$

$$\begin{aligned} &(d\dot{b}_{t_v(\tau)}|_{v_\tau}\xi_\tau) \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 w'_\tau \\ &= \Big(d\dot{b}_{t_v(\tau)}|_{v_\tau} - d\dot{b}_{t_w(\tau)}|_{v_\tau}\Big)\, \xi_\tau \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau \\ &\quad + (d\dot{b}_{t_w(\tau)}|_{v_\tau}\xi_\tau) \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau \\ &\quad + (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau \\ &\quad + (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 v'_\tau \\ &\quad + (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 v'_\tau - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau) \tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 w'_\tau. \end{aligned}$$

$\boldsymbol{D_{21}}$**.** By compactness of $\bar{\mathfrak{Y}}_z$ the maximum of the operator norm is finite

$$d\ddot{\beta}_{\bar{\mathfrak{Y}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{Y}}_z} \|d\ddot{b}\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})} < \infty. \tag{4.32}$$

As in (4.30) we obtain an estimate, uniform in $\tau \in \mathbb{S}^1$, for the operator norm

$$\|d\dot{b}_{t_v(\tau)}|_{v_\tau} - d\dot{b}_{t_w(\tau)}|_{v_\tau}\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})} \le \tfrac{16\rho(d\ddot{\beta}_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2}\|v-w\|_2. \tag{4.33}$$

Use this operator norm estimate, as well as (4.27) and (4.28), to estimate

$$\begin{aligned}
\|D_{21}\xi\|_2 &= \|(d\dot{b}_{t_v}|_v - d\dot{b}_{t_w}|_v)\xi \tfrac{|v|^2}{\|v\|_2^2}\, j_0 v'\|_2\\
&\le \tfrac{16\rho(d\ddot{\beta}_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2}\|v-w\|_2\|\xi\|_\infty \tfrac{\|v\|_\infty^2}{\|v\|_2^2}\|v\|_{1,2}\\
&\le \tfrac{16\rho(d\ddot{\beta}_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2}\tfrac{2^2\rho^2}{\|z\|_2^2} 2\|z\|_{1,2}\|v-w\|_2\|\xi\|_{1,2}\\
&= c_{21}^z\ |v-w|_{h_0}\ \|\xi\|_{1,2} \quad ,\ c_{21}^z := \tfrac{128\rho^3\|z\|_{1,2}}{\|z\|_2^4} d\ddot{\beta}_{\bar{\mathfrak{Y}}_z}.
\end{aligned}$$

$\boldsymbol{D_{22}}$**.** By compactness of $\bar{\mathfrak{Y}}_z$ the maximum of the operator norm is finite

$$d^2\dot{\beta}_{\bar{\mathfrak{Y}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{Y}}_z} \|d^2\dot{b}\|_{\mathcal{L}(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R})} < \infty. \tag{4.34}$$

Similarly as for D_{21} we estimate

$$\begin{aligned}
\|D_{22}\xi\|_2 &= \|(d\dot{b}_{t_w(\tau)}|_{v_\tau} - d\dot{b}_{t_w(\tau)}|_{w_\tau})\xi_\tau \tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v_\tau'\|_2\\
&\le d^2\dot{\beta}_{\bar{\mathfrak{Y}}_z}\|v-w\|_\infty\|\xi\|_\infty \tfrac{\|v\|_\infty^2}{\|v\|_2^2}\|v\|_{1,2}\\
&\le d^2\dot{\beta}_{\bar{\mathfrak{Y}}_z}\|v-w\|_{1,2}\|\xi\|_{1,2}\tfrac{2^2\rho^2}{\|z\|_2^2} 2\|z\|_{1,2}\\
&\le c_{22}^z\ |v-w|_{h_1}\ \|\xi\|_{1,2} \quad ,\ c_{22}^z := \tfrac{8\rho^2\|z\|_{1,2}}{\|z\|_2^2} d^2\dot{\beta}_{\bar{\mathfrak{Y}}_z}.
\end{aligned}$$

$\boldsymbol{D_{23}}$**.** With $d\dot{\beta}^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}
\|D_{23}\xi\|_2 &= \|(d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|v_\tau|^2}{\|v\|_2^2}\, j_0 v_\tau' - (d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|v\|_2^2}\, j_0 v_\tau'\|_2\\
&= \|(d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|v_\tau|^2 - \langle v_\tau,w_\tau\rangle_0 + \langle v_\tau,w_\tau\rangle_0 - |w_\tau|^2}{\|v\|_2^2}\, j_0 v_\tau'\|_2\\
&= \|(d\dot{b}_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{\langle v_\tau, v_\tau - w_\tau\rangle + \langle v_\tau - w_\tau, w_\tau\rangle}{\|v\|_2^2}\, j_0 v_\tau'\|_2\\
&\le d\dot{\beta}^w_{\mathbb{S}^1}\|\xi\|_\infty\|v'\|_\infty \tfrac{\|v\|_\infty\|v-w\|_2 + \|w\|_\infty\|v-w\|_2}{\|v\|_2^2}\\
&\le \tfrac{2^2 2\|z\|_{1,2}}{\|z\|_2^2} d\dot{\beta}^w_{\mathbb{S}^1}\ |v|_{h_2}\cdot\ |v-w|_{h_0}\ \|\xi\|_{1,2}.
\end{aligned}$$

In the last step we used (4.28) in combination with $\|\cdot\|_\infty \le \|\cdot\|_{1,2}$.

$\boldsymbol{D_{24}}$**.** With $d\dot\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}
\|D_{24}\xi\|_2 &= \|(d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|v\|_2^2}\, j_0 v'_\tau - (d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|w\|_2^2}\, j_0 v'_\tau\|_2 \\
&= \|(d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)|w_\tau|^2\tfrac{\|w\|_2^2-\langle v,w\rangle+\langle v,w\rangle-\|v\|_2^2}{\|v\|_2^2\|w\|_2^2} j_0 v'_\tau\|_2 \\
&= \|(d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\,|w_\tau|^2\tfrac{\langle w-v,w\rangle+\langle v,w-v\rangle}{\|v\|_2^2\|w\|_2^2} j_0 v'_\tau\|_2 \\
&\le d\dot\beta^w_{\mathbb{S}^1}\|\xi\|_\infty \tfrac{\rho^2 2^2 2^2}{\|z\|_2^2\|z\|_2^2}\|v-w\|_2\,(\|w\|_2+\|v\|_2)\,\|v'\|_2 \\
&\le c^z_{24}\;|v-w|_{h_0}\;\|\xi\|_{1,2} \quad,\; c^z_{24}:=\tfrac{64\rho^3\|z\|_{1,2}}{\|z\|_2^4}d\dot\beta^w_{\mathbb{S}^1}.
\end{aligned}$$

The last step is by (4.27) and (4.28).

$\boldsymbol{D_{25}}$**.** With $d\dot\beta^w_{\mathbb{S}^1}$ from (C.63), as well as (4.27) and (4.28), we estimate

$$\begin{aligned}
\|D_{25}\xi\|_2 &= \|(d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|w\|^2}\, j_0 v'_\tau - (d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|w\|^2}\, j_0 w'_\tau\|_2 \\
&= \|(d\dot b_{t_w(\tau)}|_{w_\tau}\xi_\tau)\tfrac{|w_\tau|^2}{\|w\|^2}\, j_0\,(v'_\tau-w'_\tau)\|_2 \\
&\le d\dot\beta^w_{\mathbb{S}^1}\|\xi\|_\infty\tfrac{2^2\rho^2}{\|z\|_2^2}\|v-w\|_{1,2} \\
&\le c^z_{25}\;|v-w|_{h_1}\;\|\xi\|_{1,2} \quad,\; c^z_{25}:=\tfrac{4\rho^2}{\|z\|_2^2}d\dot\beta_{\bar{\mathfrak{V}}_z}.
\end{aligned}$$

Difference D3 Add three zeroes to write D3 as a sum $D_{31}\xi+\dots+D_{34}\xi$

$$\begin{aligned}
&(d^2b_{t_v(\tau)}|_{v_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau - (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau w'_\tau)\, j_0 w'_\tau \\
&= (d^2b_{t_v(\tau)}|_{v_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau - (d^2b_{t_w(\tau)}|_{v_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau \\
&\quad + (d^2b_{t_w(\tau)}|_{v_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau - (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau \\
&\quad + (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau v'_\tau)\, j_0 v'_\tau - (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau w'_\tau)\, j_0 v'_\tau \\
&\quad + (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau w'_\tau)\, j_0 v'_\tau - (d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau w'_\tau)\, j_0 w'_\tau .
\end{aligned}$$

$\boldsymbol{D_{31}}$**.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the operator norm is finite

$$d^2\ddot\beta_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{V}}_z}\|d^2\ddot b\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})}<\infty.$$

As in (4.30) we obtain an estimate, uniform in $\tau\in\mathbb{S}^1$, for the operator norm

$$\|d^2\dot b_{t_v(\tau)}|_{v_\tau} - d^2\dot b_{t_w(\tau)}|_{v_\tau}\|_{\mathcal{L}(\mathbb{R}^2,\mathbb{R})}\le\tfrac{16\rho(d^2\ddot\beta_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2}\|v-w\|_2.$$

Use this operator norm estimate to obtain

$$\begin{aligned}
\|D_{31}\xi\|_2 &= \|\big((d^2b_{t_v(\tau)}|_{v_\tau} - d^2b_{t_w(\tau)}|_{v_\tau})(\xi_\tau v'_\tau)\big)\, j_0 v'_\tau\|_2 \\
&\le \tfrac{16\rho(d^2\ddot\beta_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2}\|v-w\|_2\|\xi\|_\infty\|v'\|_2\|v'\|_\infty \\
&\le c^z_{31}\;|v|_{h_2}\cdot|v-w|_{h_0}\;\|\xi\|_{1,2} \quad,\; c^z_{31}:=\tfrac{32\rho\|z\|_{1,2}}{\|z\|_2^2}d^2\ddot\beta_{\bar{\mathfrak{V}}_z}.
\end{aligned}$$

***D*₃₂.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the operator norm is finite

$$d^3\beta_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{V}}_z} \|d^3 b\|_{\mathcal{L}(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R})} < \infty.$$

By Taylor's theorem we estimate

$$\begin{aligned}
\|D_{32}\xi\|_2 &= \|\left((d^2b_{t_w(\tau)}|_{v_\tau} - d^2b_{t_w(\tau)}|_{w_\tau})(\xi_\tau v'_\tau)\right) j_0 v'_\tau\|_2\\
&\le d^3\beta_{\bar{\mathfrak{V}}_z}\|v-w\|_\infty\|\xi\|_\infty\|v'\|_\infty\|v'\|_2\\
&\le 2\|z\|_{1,2} d^2\beta_{\bar{\mathfrak{V}}_z}\, |v|_{h_2}\cdot|v-w|_{h_1}\, \|\xi\|_{1,2}.
\end{aligned}$$

***D*₃₃.** With the constant $d^2\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}
\|D_{33}\xi\|_2 &= \|(d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau\,(v'_\tau - w'_\tau))\, j_0 v'_\tau\|_2\\
&\le d^2\beta^w_{\mathbb{S}^1}\|\xi\|_\infty\|v'-w'\|_\infty\|v'\|_2\\
&\le 2\|z\|_{1,2} d^2\beta^w_{\mathbb{S}^1}\, |v-w|_{h_2}\, \|\xi\|_{1,2}.
\end{aligned}$$

***D*₃₄.** With the constant $d^2\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}
\|D_{34}\xi\|_2 &= \|(d^2b_{t_w(\tau)}|_{w_\tau}\xi_\tau w'_\tau)\, j_0\,(v'_\tau - w'_\tau)\|_2\\
&\le d^2\beta^w_{\mathbb{S}^1}\|\xi\|_\infty\|w'\|_2\|v'-w'\|_\infty\\
&\le 2\|z\|_{1,2} d^2\beta^w_{\mathbb{S}^1}\, |v-w|_{h_2}\, \|\xi\|_{1,2}.
\end{aligned}$$

Difference D4 Add twice zero to write D4 as a sum $D_{41}\xi + D_{42}\xi + D_{43}\xi$

$$\begin{aligned}
&(db_{t_v(\tau)}|_{v_\tau}\xi'_\tau)\, j_0 v'_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi'_\tau)\, j_0 w'_\tau\\
&= (db_{t_v(\tau)}|_{v_\tau}\xi'_\tau)\, j_0 v'_\tau - (db_{t_w(\tau)}|_{v_\tau}\xi'_\tau)\, j_0 v'_\tau\\
&\quad + (db_{t_w(\tau)}|_{v_\tau}\xi'_\tau)\, j_0 v'_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi'_\tau)\, j_0 v'_\tau\\
&\quad + (db_{t_w(\tau)}|_{w_\tau}\xi'_\tau)\, j_0 v'_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi'_\tau)\, j_0 w'_\tau
\end{aligned}$$

***D*₄₁.** We use (4.30) to estimate

$$\begin{aligned}
\|D_{41}\xi\|_2 &= \|\left((db_{t_v(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{v_\tau})\xi'_\tau\right) j_0 v'_\tau\|_2\\
&\le \tfrac{16\rho(d\dot\beta_{\bar{\mathfrak{V}}_z})}{\|z\|_2^2}\|v-w\|_2\|\xi\|_\infty\|v'\|_2\\
&\le c^z_{41}\, |v-w|_{h_0}\, \|\xi\|_{1,2} \quad , \; c^z_{41} := \tfrac{32\rho\|z\|_{1,2}}{\|z\|_2^2} d\dot\beta_{\bar{\mathfrak{V}}_z}.
\end{aligned}$$

***D*₄₂.** By Taylor's theorem with constant $d^2\beta_{\bar{\mathfrak{V}}_z}$ in (4.31) we estimate

$$\begin{aligned}
\|D_{42}\xi\|_2 &= \|\left((db_{t_w(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{w_\tau})\xi'_\tau\right) j_0 v'_\tau\|_2\\
&\le d^2\beta_{\bar{\mathfrak{V}}_z}\|v-w\|_\infty\|\xi'\|_2\|v'\|_\infty\\
&\le d^2\beta_{\bar{\mathfrak{V}}_z}\, |v|_{h_2}\cdot|v-w|_{h_1}\, \|\xi\|_{1,2}.
\end{aligned}$$

D_{43}. With the constant $d\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}\|D_{43}\xi\|_2 &= \|(db_{t_w(\tau)}|_{w_\tau}\xi'_\tau)\, j_0\, (v'_\tau - w'_\tau)\|_2\\ &\le d\beta^w_{\mathbb{S}^1}\|\xi'\|_2\|v'-w'\|_\infty\\ &\le d\beta^w_{\mathbb{S}^1}\ |v-w|_{h_2}\ \|\xi\|_{1,2}.\end{aligned}$$

Difference D5 Add twice zero to write D5 as a sum $D_{51}\xi + D_{52}\xi + D_{53}\xi$

$$\begin{aligned}&(db_{t_v(\tau)}|_{v_\tau}\xi_\tau)\, j_0 v''_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0 w''_\tau\\ &= (db_{t_v(\tau)}|_{v_\tau}\xi_\tau)\, j_0 v''_\tau - (db_{t_w(\tau)}|_{v_\tau}\xi_\tau)\, j_0 v''_\tau\\ &\quad + (db_{t_w(\tau)}|_{v_\tau}\xi_\tau)\, j_0 v''_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0 v''_\tau\\ &\quad + (db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0 v''_\tau - (db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0 w''_\tau\end{aligned}$$

D_{51}. We use (4.30) to estimate

$$\begin{aligned}\|D_{51}\xi\|_2 &= \|\big((db_{t_v(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{v_\tau})\xi_\tau\big)\, j_0 v''_\tau\|_2\\ &\le \tfrac{16\rho(d\dot\beta_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2}\|v-w\|_2\|\xi\|_\infty\|v''\|_2\\ &\le c^z_{51}\ |v|_{h_2}\cdot|v-w|_{h_0}\ \|\xi\|_{1,2} \quad ,\ c^z_{51} := \tfrac{16\rho}{\|z\|_2^2} d\dot\beta_{\bar{\mathfrak{Y}}_z}.\end{aligned}$$

D_{52}. By Taylor's theorem with constant $d^2\beta_{\bar{\mathfrak{Y}}_z}$ in (4.31) we estimate

$$\begin{aligned}\|D_{52}\xi\|_2 &= \|\big((db_{t_w(\tau)}|_{v_\tau} - db_{t_w(\tau)}|_{w_\tau})\xi_\tau\big)\, j_0 v''_\tau\|_2\\ &\le d^2\beta_{\bar{\mathfrak{Y}}_z}\|v-w\|_\infty\|\xi\|_\infty\|v''\|_2\\ &\le d^2\beta_{\bar{\mathfrak{Y}}_z}\ |v|_{h_2}\cdot|v-w|_{h_1}\ \|\xi\|_{1,2}.\end{aligned}$$

D_{53}. With the constant $d\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate

$$\begin{aligned}\|D_{53}\xi\|_2 &= \|(db_{t_w(\tau)}|_{w_\tau}\xi_\tau)\, j_0\, (v''_\tau - w''_\tau)\|_2\\ &\le d\beta^w_{\mathbb{S}^1}\|\xi\|_\infty\|v''-w''\|_2\\ &\le d\beta^w_{\mathbb{S}^1}\ |v-w|_{h_2}\ \|\xi\|_{1,2}.\end{aligned}$$

Summarizing we have shown that there exists a constant $\kappa = \kappa(z)$ such that

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + |v|_{h_2}\cdot|v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. Interchanging the roles of v and w we get

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + |w|_{h_2}\cdot|v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. These two inequalities imply that

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + \min\{|v|_{h_2}, |w|_{h_2}\}\cdot|v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. This proves Proposition 4.12. □

The following proposition is part of Step (C) of the proof of Theorem 4.11.

Proposition 4.13 (C_{42} is scale Lipschitz)**.** *For $z \mapsto C_{42}^z$ in (4.26) the scale Lipschitz estimate (C) holds true, namely (3.17) with lower case spaces h_k and u_1.*

Proof. Let $z \in u_1$ and $\xi \in h_1$. We consider the map in (C.65), namely

$$C^z\xi := C_{42}^z\xi := \dot{b}_{t_z(\tau)}|_{z_\tau} (dt_z\xi)_\tau \, j_0 z'_\tau.$$

As in Proposition 4.12 let the ball $\bar{\mathfrak{V}}_z$ be defined by (4.27) and v_z by (4.28).

Pick $v, w \in \mathrm{v}_z \cap h_2$. By definition of the $W^{1,2}$ norm, calculating the derivative ∂_τ, the triangle inequality and since $t'_v(\tau) = \left|v_\tau\right|^2 / \|v\|_2^2$ by (C.54), we get

$$\begin{aligned}
&\|(C^v - C^w)\xi\|_{1,2} \\
&\le \|\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)_\tau \, j_0 v'_\tau - \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 w'_\tau\|_2 \\
&\quad + \|\partial_\tau \left(\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)_\tau \, j_0 v'_\tau\right) - \partial_\tau \left(\dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 w'_\tau\right)\|_2 \\
&\le \|\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)_\tau \, j_0 v'_\tau - \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 w'_\tau\|_2 \\
&\quad + \|\ddot{b}_{t_v(\tau)}|_{v_\tau} \tfrac{|v_\tau|^2}{\|v\|_2^2} (dt_v\xi)_\tau \, j_0 v'_\tau - \ddot{b}_{t_w(\tau)}|_{w_\tau} \tfrac{|w_\tau|^2}{\|w\|_2^2} (dt_w\xi)_\tau \, j_0 w'_\tau\|_2 \\
&\quad + \|\left(d\dot{b}_{t_v(\tau)}|_{v_\tau} v'_\tau\right) (dt_v\xi)_\tau \, j_0 v'_\tau - \left(d\dot{b}_{t_w(\tau)}|_{w_\tau} w'_\tau\right) (dt_w\xi)_\tau \, j_0 w'_\tau\|_2 \\
&\quad + \|\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)'_\tau \, j_0 v'_\tau - \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)'_\tau \, j_0 w'_\tau\|_2 \\
&\quad + \|\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)_\tau \, j_0 v''_\tau - \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 w''_\tau\|_2.
\end{aligned}$$

We need to estimate the L^2-norms of five differences, notation **D1-D5**.

Difference D1. Add three times zero to write D1 as a sum $D_{11}\xi + \dots + D_{14}\xi$

$$\begin{aligned}
&\dot{b}_{t_v(\tau)}|_{v_\tau} (dt_v\xi)_\tau \, j_0 v'_\tau - \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 w'_\tau \\
&= \left(\dot{b}_{t_v(\tau)}|_{v_\tau} - \dot{b}_{t_w(\tau)}|_{v_\tau}\right) (dt_v\xi)_\tau \, j_0 v'_\tau \\
&\quad + \left(\dot{b}_{t_w(\tau)}|_{v_\tau} - \dot{b}_{t_w(\tau)}|_{w_\tau}\right) (dt_v\xi)_\tau \, j_0 v'_\tau \\
&\quad + \dot{b}_{t_w(\tau)}|_{w_\tau} \left((dt_v\xi)_\tau - (dt_w\xi)_\tau\right) j_0 v'_\tau \\
&\quad + \dot{b}_{t_w(\tau)}|_{w_\tau} (dt_w\xi)_\tau \, j_0 \left(v'_\tau - w'_\tau\right)
\end{aligned}$$

pointwise at $\tau \in \mathbb{S}^1$.

***D*$_{11}$.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the absolute value is finite

$$\ddot{\beta}_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1 \times \bar{\mathfrak{V}}_z} |\ddot{b}| < \infty.$$

As in (4.30) we obtain an estimate, uniform in $\tau \in \mathbb{S}^1$, for the difference

$$\left|\dot{b}_{t_v(\tau)}|_{v_\tau} - \dot{b}_{t_w(\tau)}|_{v_\tau}\right| \le \tfrac{16\rho\ddot{\beta}_{\bar{\mathfrak{V}}_z}}{\|z\|_2^2} \|v - w\|_2. \tag{4.35}$$

Use this difference estimate, then (C.56) followed by (4.28), to obtain

$$\begin{aligned}
\|D_{11}\xi\|_2 &= \|\left(\dot b_{t_v(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{v_\tau}\right)(dt_v\xi)_\tau\, j_0 v'_\tau\|_2 \\
&\le \tfrac{16\rho\ddot\beta_{\mathfrak{Y}_z}}{\|z\|_2^2}\|v-w\|_2\|dt_v\xi\|_\infty\|v'\|_2 \\
&\le \tfrac{16\rho\ddot\beta_{\mathfrak{Y}_z}}{\|z\|_2^2}\|v-w\|_2\tfrac{4}{\|v\|_2}\|\xi\|_2\|v\|_{1,2} \\
&\le \tfrac{128\rho\ddot\beta_{\mathfrak{Y}_z}}{\|z\|_2^3}\ |v|_{h_1}\cdot|v-w|_{h_0}\ \|\xi\|_{1,2}.
\end{aligned} \tag{4.36}$$

***D*₁₂.** By Taylor's theorem and the constant $d\dot\beta_{\mathfrak{Y}_z}$ in (4.29) we estimate

$$\begin{aligned}
\|D_{12}\xi\|_2 &= \|\left(\dot b_{t_w(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{w_\tau}\right)(dt_v\xi)_\tau\, j_0 v'_\tau\|_2 \\
&\le d\dot\beta_{\mathfrak{Y}_z}\|v-w\|_\infty\|dt_v\xi\|_\infty\|v'\|_2 \\
&\le \tfrac{8d\dot\beta_{\mathfrak{Y}_z}}{\|z\|_2}\ |v|_{h_1}\cdot|v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned} \tag{4.37}$$

***D*₁₃.** With constants $\dot\beta^w_{\mathbb{S}^1}$ in (C.63) and $C_\rho^{\|z\|_2}$ in (C.61) we estimate

$$\begin{aligned}
\|D_{13}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}\left((dt_v\xi)_\tau - (dt_w\xi)_\tau\right) j_0 v'_\tau\|_2 \\
&\le \dot\beta^w_{\mathbb{S}^1}\|dt_v\xi - dt_w\xi\|_\infty\|v'\|_2 \\
&\le \dot\beta^w_{\mathbb{S}^1} 2C_\rho^{\|z\|_2}\ |v|_{h_1}\cdot|v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned} \tag{4.38}$$

***D*₁₄.** With $\dot\beta^w_{\mathbb{S}^1}$ in (C.63) and (C.56), followed by (4.28), we estimate

$$\begin{aligned}
\|D_{14}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0\left(v'_\tau - w'_\tau\right)\|_2 \\
&\le \dot\beta^w_{\mathbb{S}^1}\tfrac{4}{\|v\|_2}\|\xi\|_2\|v'-w'\|_2 \\
&\le \tfrac{8d\dot\beta_{\mathfrak{Y}_z}}{\|z\|_2}\ |v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned} \tag{4.39}$$

Difference D2 Add five zeroes to write D2 as a sum $D_{21}\xi + \dots + D_{26}\xi$

$$\begin{aligned}
&\ddot b_{t_v(\tau)}|_{v_\tau}\tfrac{|v_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau - \ddot b_{t_w(\tau)}|_{w_\tau}\tfrac{|w_\tau|^2}{\|w\|_2^2}(dt_w\xi)_\tau\, j_0 w'_\tau \\
&= \left(\ddot b_{t_v(\tau)}|_{v_\tau} - \ddot b_{t_w(\tau)}|_{v_\tau}\right)\tfrac{|v_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau \\
&\quad + \left(\ddot b_{t_w(\tau)}|_{v_\tau} - \ddot b_{t_w(\tau)}|_{w_\tau}\right)\tfrac{|v_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau \\
&\quad + \ddot b_{t_w(\tau)}|_{w_\tau}\tfrac{|v_\tau|^2-|w_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau \\
&\quad + \ddot b_{t_w(\tau)}|_{w_\tau}|w_\tau|^2\tfrac{\|w\|_2^2-\|v\|_2^2}{\|v\|_2^2\|w\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau \\
&\quad + \ddot b_{t_w(\tau)}|_{w_\tau}\tfrac{|w_\tau|^2}{\|w\|_2^2}\left((dt_v\xi)_\tau - (dt_w\xi)_\tau\right) j_0 v'_\tau \\
&\quad + \ddot b_{t_w(\tau)}|_{w_\tau}\tfrac{|w_\tau|^2}{\|w\|_2^2}(dt_w\xi)_\tau\, j_0\left(v'_\tau - w'_\tau\right).
\end{aligned}$$

$\boldsymbol{D_{21}}$**.** By compactness of $\bar{\mathfrak{V}}_z$ the maximum of the absolute value is finite

$$\ddot{\beta}_{\bar{\mathfrak{V}}_z} := \max_{\mathbb{S}^1\times\bar{\mathfrak{V}}_z} |\ddot{b}| < \infty.$$

As in (4.30) we obtain an estimate, uniform in $\tau \in \mathbb{S}^1$, for the operator norm

$$|\ddot{b}_{t_v(\tau)}|_{v_\tau} - \ddot{b}_{t_w(\tau)}|_{v_\tau}| \le \tfrac{16\rho\ddot{\beta}_{\bar{\mathfrak{V}}_z}}{\|z\|_2^2}\|v-w\|_2.$$

Use this operator norm estimate, as well as (4.27) and (4.28), to estimate

$$\begin{aligned}
\|D_{21}\xi\|_2 &= \|\left(\ddot{b}_{t_v(\tau)}|_{v_\tau} - \ddot{b}_{t_w(\tau)}|_{v_\tau}\right) \tfrac{|v_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le \tfrac{16\rho\ddot{\beta}_{\bar{\mathfrak{V}}_z}}{\|z\|_2^2}\|v-w\|_2 \tfrac{2^2\rho^2}{\|z\|_2^2}\tfrac{4}{\|z\|_2}\|\xi\|_2\|v'\|_2\\
&\le \tfrac{16\cdot 32\rho^3\ddot{\beta}_{\bar{\mathfrak{V}}_z}\|z\|_{1,2}}{\|z\|_2^5}\ |v-w|_{h_0}\ \|\xi\|_{1,2}.
\end{aligned}$$

$\boldsymbol{D_{22}}$**.** By Taylor's theorem and $d\ddot{\beta}_{\bar{\mathfrak{V}}_z}$ in (4.32) we estimate similarly as above

$$\begin{aligned}
\|D_{22}\xi\|_2 &= \|\left(\ddot{b}_{t_w(\tau)}|_{v_\tau} - \ddot{b}_{t_w(\tau)}|_{w_\tau}\right) \tfrac{|v_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le d\ddot{\beta}_{\bar{\mathfrak{V}}_z}\|v-w\|_\infty \tfrac{2^2\rho^2}{\|z\|_2^2}\tfrac{4}{\|z\|_2}\|\xi\|_2\|v'\|_2\\
&\le \tfrac{32\rho^2(d\ddot{\beta}_{\bar{\mathfrak{V}}_z})\|z\|_{1,2}}{\|z\|_2^3}\ |v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned}$$

$\boldsymbol{D_{23}}$**.** With $\ddot{\beta}^w_{\mathbb{S}^1}$ in (C.63) and (C.56), followed by (4.28), we estimate

$$\begin{aligned}
\|D_{23}\xi\|_2 &= \|\ddot{b}_{t_w(\tau)}|_{w_\tau} \tfrac{|v_\tau|^2 - \langle v_\tau,w_\tau\rangle_0 + \langle v_\tau,w_\tau\rangle_0 - |w_\tau|^2}{\|v\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le \ddot{\beta}^w_{\mathbb{S}^1}\tfrac{\|v-w\|_\infty(\|v\|_\infty+\|w\|_\infty)}{\|v\|_2^2}\|dt_v\xi\|_\infty\|v'\|_2\\
&\le \tfrac{128\rho\ddot{\beta}^w_{\mathbb{S}^1}\|z\|_{1,2}}{\|z\|_2^3}\ |v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned}$$

$\boldsymbol{D_{24}}$**.** With $\ddot{\beta}^w_{\mathbb{S}^1}$ in (C.63) and (C.56), followed by (4.28), we estimate

$$\begin{aligned}
\|D_{24}\xi\|_2 &= \|\ddot{b}_{t_w(\tau)}|_{w_\tau}|w_\tau|^2 \tfrac{\|w\|_2^2 - \langle v,w\rangle + \langle v,w\rangle - \|v\|_2^2}{\|v\|_2^2\|w\|_2^2}(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le \ddot{\beta}^w_{\mathbb{S}^1}\|w\|_\infty^2\tfrac{\|v-w\|_2(\|v\|_\infty+\|w\|_\infty)}{\|v\|_2^2\|w\|_2^2}\|dt_v\xi\|_\infty\|v'\|_2\\
&\le \tfrac{512\rho^3\ddot{\beta}^w_{\mathbb{S}^1}\|z\|_{1,2}}{\|z\|_2^5}\ |v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned}$$

$\boldsymbol{D_{25}}$**.** With constants $\ddot{\beta}^w_{\mathbb{S}^1}$ in (C.63) and $C_\rho^{\|z\|_2}$ in (C.61) we estimate

$$\begin{aligned}
\|D_{25}\xi\|_2 &= \|\ddot{b}_{t_w(\tau)}|_{w_\tau}\tfrac{|w_\tau|^2}{\|w\|_2^2}\left((dt_v\xi)_\tau - (dt_w\xi)_\tau\right) j_0 v'_\tau\|_2\\
&\le \ddot{\beta}^w_{\mathbb{S}^1}\tfrac{4\rho^2}{\|z\|_2^2}\|dt_v\xi - dt_w\xi\|_\infty\|v'\|_2\\
&\le \ddot{\beta}^w_{\mathbb{S}^1}\tfrac{4\rho^2}{\|z\|_2^2}2C_\rho^{\|z\|_2}\ |v|_{h_1}\cdot|v-w|_{h_1}\ \|\xi\|_{1,2}.
\end{aligned}$$

***D*₂₆.** With the constant $\ddot\beta^w_{\mathbb{S}^1}$ in (C.63) and (4.27), (4.28), (C.56) we estimate

$$
\begin{aligned}
\|D_{26}\xi\|_2 &= \|\ddot b_{t_w(\tau)}|_{w_\tau} \tfrac{|w_\tau|^2}{\|w\|_2^2} (dt_w\xi)_\tau\, j_0\,(v'_\tau - w'_\tau)\|_2\\
&\le \ddot\beta^w_{\mathbb{S}^1} \tfrac{2^2\rho^2}{\|z\|_2^2} \tfrac{4\cdot 2}{\|z\|_2} \|\xi\|_2 \|v'-w'\|_2\\
&\le \tfrac{32\rho^2 \ddot\beta_{\bar{\mathfrak{Y}}_z}}{\|z\|_2^3}\; |v-w|_{h_1}\; \|\xi\|_{1,2}.
\end{aligned}
$$

Difference D3 Add four zeroes to write D3 as a sum $D_{31}\xi + \dots + D_{35}\xi$

$$
\begin{aligned}
&\left(d\dot b_{t_v(\tau)}|_{v_\tau} v'_\tau\right)(dt_v\xi)_\tau\, j_0 v'_\tau - \left(d\dot b_{t_w(\tau)}|_{w_\tau} w'_\tau\right)(dt_w\xi)_\tau\, j_0 w'_\tau\\
&= \left((d\dot b_{t_v(\tau)}|_{v_\tau} - d\dot b_{t_w(\tau)}|_{v_\tau}) v'_\tau\right)(dt_v\xi)_\tau\, j_0 v'_\tau\\
&\quad + \left((d\dot b_{t_w(\tau)}|_{v_\tau} - d\dot b_{t_w(\tau)}|_{w_\tau}) v'_\tau\right)(dt_v\xi)_\tau\, j_0 v'_\tau\\
&\quad + \left(d\dot b_{t_w(\tau)}|_{w_\tau} (v'_\tau - w'_\tau)\right)(dt_v\xi)_\tau\, j_0 v'_\tau\\
&\quad + \left(d\dot b_{t_w(\tau)}|_{w_\tau} w'_\tau\right)((dt_v\xi)_\tau - (dt_w\xi)_\tau)\, j_0 v'_\tau\\
&\quad + \left(d\dot b_{t_w(\tau)}|_{w_\tau} w'_\tau\right)(dt_w\xi)_\tau\, j_0\,(v'_\tau - w'_\tau)\,.
\end{aligned}
$$

***D*₃₁.** By estimate (4.33) with constant $d\ddot\beta_{\bar{\mathfrak{Y}}_z}$ and (C.56) we obtain

$$
\begin{aligned}
\|D_{31}\xi\|_2 &= \|\left((d\dot b_{t_v(\tau)}|_{v_\tau} - d\dot b_{t_w(\tau)}|_{v_\tau}) v'_\tau\right)(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le \tfrac{16\rho(d\ddot\beta_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2} \|v-w\|_2\; \|v'\|_\infty \|dt_v\xi\|_\infty \|v'\|_2\\
&\le \tfrac{16\rho(d\ddot\beta_{\bar{\mathfrak{Y}}_z})}{\|z\|_2^2} \tfrac{8\cdot 2\|z\|_{1,2}}{\|z\|_2}\;\; |v|_{h_2}\cdot |v-w|_{h_0}\; \|\xi\|_{1,2}.
\end{aligned}
$$

***D*₃₂.** By Taylor's theorem and $d^2\dot\beta_{\bar{\mathfrak{Y}}_z}$ in (4.34) we estimate as above

$$
\begin{aligned}
\|D_{32}\xi\|_2 &= \|\left((d\dot b_{t_w(\tau)}|_{v_\tau} - d\dot b_{t_w(\tau)}|_{w_\tau}) v'_\tau\right)(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le d^2\dot\beta_{\bar{\mathfrak{Y}}_z} \|v-w\|_\infty\; \|v'\|_\infty \|dt_v\xi\|_\infty \|v'\|_2\\
&\le d^2\dot\beta_{\bar{\mathfrak{Y}}_z} \tfrac{8\cdot 2\|z\|_{1,2}}{\|z\|_2}\; |v|_{h_2}\cdot |v-w|_{h_1}\; \|\xi\|_{1,2}.
\end{aligned}
$$

***D*₃₃.** With $d\dot\beta^w_{\mathbb{S}^1}$ from (C.63) we estimate as above

$$
\begin{aligned}
\|D_{33}\xi\|_2 &= \|\left(d\dot b_{t_w(\tau)}|_{w_\tau} (v'_\tau - w'_\tau)\right)(dt_v\xi)_\tau\, j_0 v'_\tau\|_2\\
&\le d\dot\beta^w_{\mathbb{S}^1} \|v'-w'\|_\infty \|dt_v\xi\|_\infty \|v'\|_2\\
&\le d^2\dot\beta_{\bar{\mathfrak{Y}}_z} \tfrac{8\cdot 2\|z\|_{1,2}}{\|z\|_2}\; |v-w|_{h_2}\; \|\xi\|_{1,2}.
\end{aligned}
$$

***D_{34}*.** With constants $d\dot\beta^w_{\mathbb{S}^1}$ in (C.63) and $C_\rho^{\|z\|_2}$ in (C.61) we estimate

$$\begin{aligned}\|D_{34}\xi\|_2 &= \|\left(d\dot b_{t_w(\tau)}|_{w_\tau} w'_\tau\right)((dt_v\xi)_\tau - (dt_w\xi)_\tau)\, j_0 v'_\tau\|_2\\ &\le d\dot\beta^w_{\mathbb{S}^1}\|w'\|_2\|dt_v\xi - dt_w\xi\|_\infty\|v'\|_\infty\\ &\le d\dot\beta^w_{\mathbb{S}^1} 2\|z\|_{1,2} 2C_\rho^{\|z\|_2}\ |v|_{h_2}\cdot|v-w|_{h_1}\ \|\xi\|_{1,2}.\end{aligned}$$

***D_{35}*.** With $d\dot\beta^w_{\mathbb{S}^1}$ from (C.63) and (4.28) for $\|w\|_{1,2}$ we estimate as above

$$\begin{aligned}\|D_{35}\xi\|_2 &= \|\left(d\dot b_{t_w(\tau)}|_{w_\tau} w'_\tau\right)(dt_w\xi)_\tau\, j_0\,(v'_\tau - w'_\tau)\|_2\\ &\le d\dot\beta^w_{\mathbb{S}^1}\|w'\|_2\|dt_w\xi\|_\infty\|v'-w'\|_\infty\\ &\le d\dot\beta^w_{\mathbb{S}^1} 2\|z\|_{1,2}\tfrac{8}{\|z\|_2}\|\xi\|_2\|v-w\|_{2,2}\\ &\le \tfrac{16(d^2\dot\beta_{\bar{\mathfrak{Y}}_z})\|z\|_{1,2}}{\|z\|_2}\ |v-w|_{h_2}\ \|\xi\|_{1,2}.\end{aligned}$$

Difference D4 Add three zeroes to write D4 as a sum $D_{41}\xi + \ldots\xi + D_{44}\xi$

$$\begin{aligned}&\dot b_{t_v(\tau)}|_{v_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)'_\tau\, j_0 w'_\tau\\ &= \dot b_{t_v(\tau)}|_{v_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau - \dot b_{t_w(\tau)}|_{v_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau\\ &\quad \dot b_{t_w(\tau)}|_{v_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau\\ &\quad \dot b_{t_w(\tau)}|_{w_\tau}(dt_v\xi)'_\tau\, j_0 v'_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)'_\tau\, j_0 v'_\tau\\ &\quad \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)'_\tau\, j_0 v'_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)'_\tau\, j_0 w'_\tau.\end{aligned}$$

***D_{41}*.** We use (4.28), (4.35) and (C.57) to estimate

$$\begin{aligned}\|D_{41}\xi\|_2 &= \|\left(\dot b_{t_v(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{v_\tau}\right)(dt_v\xi)'_\tau\, j_0 v'_\tau\|_2\\ &\le \tfrac{16\rho\ddot\beta_{\bar{\mathfrak{Y}}_z}}{\|z\|_2^2}\|v-w\|_2\|(dt_v\xi)'\|_\infty\|v'\|_2\\ &\le \tfrac{16\rho\ddot\beta_{\bar{\mathfrak{Y}}_z}2\|z\|_{1,2}}{\|z\|_2^2}\left(\tfrac{2\|z\|_{1,2}}{\|z\|_2^2}+\tfrac{2\|z\|_{1,2}^2}{\|z\|_2^3}\right)\ |v-w|_{h_0}\ \|\xi\|_{1,2}.\end{aligned}$$

***D_{42}*.** By Taylor's theorem and $d\dot\beta_{\bar{\mathfrak{Y}}_z}$ in (4.29) and by (C.57) we estimate

$$\begin{aligned}\|D_{42}\xi\|_2 &= \|\left(\dot b_{t_w(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{w_\tau}\right)(dt_v\xi)'_\tau\, j_0 v'_\tau\|_2\\ &\le d\dot\beta_{\bar{\mathfrak{Y}}_z}\|v-w\|_\infty\|(dt_v\xi)'\|_\infty\|v'\|_2\\ &\le d\dot\beta_{\bar{\mathfrak{Y}}_z}2\|z\|_{1,2}\left(\tfrac{2\|z\|_{1,2}}{\|z\|_2^2}+\tfrac{2\|z\|_{1,2}^2}{\|z\|_2^3}\right)\ |v-w|_{h_1}\ \|\xi\|_{1,2}.\end{aligned}$$

***D_{43}*.** With constants $\dot\beta^w_{\mathbb{S}^1}$ in (C.63) and $C_\rho^{\|z\|_2}$ in (C.60) we estimate

$$\begin{aligned}\|D_{43}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}((dt_v\xi)'_\tau - (dt_w\xi)'_\tau)\, j_0 v'_\tau\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}\|(dt_v\xi)' - (dt_w\xi)'\|_\infty\|v'\|_2 \qquad (4.40)\\ &\le \dot\beta^w_{\mathbb{S}^1}2\|z\|_{1,2}2C_\rho^{\|z\|_2}\ |v-w|_{h_1}\ \|\xi\|_{1,2}.\end{aligned}$$

$\boldsymbol{D_{44}}$. With $\dot\beta^w_{\mathbb{S}^1}$ in (C.63) and by (C.57) we estimate

$$\begin{aligned}\|D_{44}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)'_\tau\, j_0\,(v'_\tau - w'_\tau)\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}\|(dt_v\xi)'\|_\infty\|v'-w'\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}\left(\tfrac{2\|z\|_{1,2}}{\|z\|_2^2}+\tfrac{2\|z\|_{1,2}^2}{\|z\|_2^3}\right)\ {\color{blue}|v-w|_{h_1}}\ \|\xi\|_{1,2}.\end{aligned}$$

Difference D5 Add three zeroes to write D5 as a sum $D_{51}\xi+\dots+D_{54}\xi$

$$\begin{aligned}&\dot b_{t_v(\tau)}|_{v_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0 w''_\tau\\ &= \dot b_{t_v(\tau)}|_{v_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau - \dot b_{t_w(\tau)}|_{v_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau\\ &\quad + \dot b_{t_w(\tau)}|_{v_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau\\ &\quad + \dot b_{t_w(\tau)}|_{w_\tau}(dt_v\xi)_\tau\, j_0 v''_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0 v''_\tau\\ &\quad + \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0 v''_\tau - \dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0 w''_\tau.\end{aligned}$$

$\boldsymbol{D_{51}}$. This is estimate (4.36) with v' replaced by v'', so

$$\begin{aligned}\|D_{51}\xi\|_2 &= \|\left(\dot b_{t_v(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{v_\tau}\right)(dt_v\xi)_\tau\, j_0 v''_\tau\|_2\\ &\le \tfrac{16\rho\ddot\beta_{\mathfrak{V}_z}}{\|z\|_2^2}\|v-w\|_2\|dt_v\xi\|_\infty\|v''\|_2\\ &\le \tfrac{128\rho\ddot\beta_{\mathfrak{V}_z}}{\|z\|_2^3}\ \ {\color{red}|v|_{h_2}}\cdot{\color{blue}|v-w|_{h_0}}\ \|\xi\|_{1,2}.\end{aligned}$$

$\boldsymbol{D_{52}}$. This is estimate (4.37) with v' replaced by v'', so

$$\begin{aligned}\|D_{52}\xi\|_2 &= \|\left(\dot b_{t_w(\tau)}|_{v_\tau} - \dot b_{t_w(\tau)}|_{w_\tau}\right)(dt_v\xi)_\tau\, j_0 v''_\tau\|_2\\ &\le d\dot\beta_{\mathfrak{V}_z}\|v-w\|_\infty\|dt_v\xi\|_\infty\|v''\|_2\\ &\le \tfrac{8d\dot\beta_{\mathfrak{V}_z}}{\|z\|_2}\ {\color{red}|v|_{h_2}\cdot|v-w|_{h_1}}\ \|\xi\|_{1,2}.\end{aligned}$$

$\boldsymbol{D_{53}}$. This is estimate (4.38) with v' replaced by v'', so

$$\begin{aligned}\|D_{53}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}\left((dt_v\xi)_\tau - (dt_w\xi)_\tau\right) j_0 v''_\tau\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}\|dt_v\xi - dt_w\xi\|_\infty\|v''\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}2C_\rho^{\|z\|_2}\ {\color{red}|v|_{h_2}\cdot|v-w|_{h_1}}\ \|\xi\|_{1,2}.\end{aligned}\tag{4.41}$$

$\boldsymbol{D_{54}}$. This is estimate (4.39) with $v'-w'$ replaced by $v''-w''$, so

$$\begin{aligned}\|D_{54}\xi\|_2 &= \|\dot b_{t_w(\tau)}|_{w_\tau}(dt_w\xi)_\tau\, j_0\,(v''_\tau - w''_\tau)\|_2\\ &\le \dot\beta^w_{\mathbb{S}^1}\tfrac{4}{\|v\|_2}\|\xi\|_2\|v''-w''\|_2\\ &\le \tfrac{8d\dot\beta_{\mathfrak{V}_z}}{\|z\|_2}\ {\color{red}|v-w|_{h_2}}\ \|\xi\|_{1,2}.\end{aligned}$$

Summarizing we have shown that there exists a constant $\kappa = \kappa(z)$ such that

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + |v|_{h_2} \cdot |v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. Interchanging the roles of v and w we get

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + |w|_{h_2} \cdot |v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. These two inequalities imply that

$$\|C^v - C^w\|_{\mathcal{L}(h_1)} \le \kappa\Big(|v-w|_{h_2} + \min\{|v|_{h_2}, |w|_{h_2}\} \cdot |v-w|_{h_1}\Big).$$

whenever $v, w \in \mathrm{v}_z$. This proves Proposition 4.13. □

4.3 Twisted loops

In the case of twisted loops $z(\tau+1) = -z(\tau)\ \forall \tau \in \mathbb{S}^1$ the Hilbert space triple has to be adjusted as follows

$$\begin{aligned} H_0^- &:= L^2([0,1],\mathbb{C}^2) \\ H_1^- &:= \{\Upsilon \in W^{1,2}([0,1],\mathbb{C}^2) \mid \Upsilon(1) = -\Upsilon(0)\} \\ H_2^- &:= \{\Upsilon \in W^{2,2}([0,1],\mathbb{C}^2) \mid \Upsilon(1) = -\Upsilon(0) \wedge \Upsilon'(1) = -\Upsilon'(0)\} \end{aligned}$$

and h_0^-, h_1^-, h_2^- are defined equally, up to replacing $\mathbb{C}^2$ by $\mathbb{C}$. Apart from this, the case of twisted loops proceeds completely analogous to the periodic case.

4.4 Proof of Theorem A and Theorem B

- Theorem B is Theorem 4.11.
- Theorem A follows from Theorem B and Theorem 3.5.

A Symmetry of the L^2-Hessian

Theorem A.1. *Let (H_0, H_1) be a Hilbert space pair[3] and $U_1 \subset H_1$ open. Suppose $f\colon U_1 \to \mathbb{R}$ is (i) a C^1 function such that (ii) there exists a C^1 map* $\operatorname{grad} f\colon U_1 \to H_0$ *and such that (iii) there is the identity*

$$df|_x\xi = \langle \operatorname{grad} f|_x, \xi\rangle_0$$

whenever $x \in U_1$ and $\xi \in H_1$. Then the following is true. The second derivative of f exists at every point $x \in U_1$, it is given by the formula

$$d^2 f|_x(\xi,\eta) = \langle d(\operatorname{grad} f)_x \xi, \eta\rangle_0 =: B_x(\xi,\eta),$$

and it varies continuously in x, that is $f \in C^2(U_1,\mathbb{R})$.

[3] H_0 and H_1 are both infinite dimensional Hilbert spaces, $H_1 \subset H_0$ is a dense subset, and inclusion $\iota\colon H_1 \to H_0$ is a compact linear map. H_0 and H_1 are separable by [FW24, Cor. A.5].

Proof. The proof has three steps. Without loss of generality we suppose that $|\cdot|_0 \le |\cdot|_1$, otherwise choose equivalent norms.

Step 1. $\forall x \in U_1$ the map $B_x \colon H_1 \times H_1 \to \mathbb{R}$ is a bounded bilinear form.

Proof. By definition of B followed by Cauchy-Schwarz and then pulling out the operator norm we estimate

$$\begin{aligned} |B_x(\xi,\eta)| = |\langle d(\operatorname{grad} f)_x \xi, \eta\rangle_0| &\le |d(\operatorname{grad} f)_x\xi|_0 |\eta|_0 \\ &\le \|d(\operatorname{grad} f)_x\|_{\mathcal{L}(H_1,H_0)} |\xi|_1 |\eta|_1. \end{aligned}$$

By assumption (ii) the right hand side is bounded. □

Step 2. The map[4] $B\colon U_1 \to \mathcal{L}(H_1,H_1;\mathbb{R})$, $x \mapsto B_x$, is continuous.

Proof. For $x, y \in U_1$ and $\xi, \eta \in H_1$ we estimate

$$\begin{aligned} |(B_x - B_y)(\xi,\eta)| &= \left|\langle d(\operatorname{grad} f)_x\xi - d(\operatorname{grad} f)_y \xi, \eta\rangle_0\right| \\ &\le |(d(\operatorname{grad} f)_x - d(\operatorname{grad} f)_y)\,\xi|_0 |\eta|_0 \\ &\le \|d(\operatorname{grad} f)_x - d(\operatorname{grad} f)_y\|_{\mathcal{L}(H_1,H_0)} |\xi|_1|\eta|_1. \end{aligned}$$

By assumption (ii) the map $x \mapsto d(\operatorname{grad} f)_x$ is continuous. □

Step 3. At any $x \in U_1$ the map B_x is the second derivative $d^2 f|_x$ of f at x.

Proof. The assertion of Step 3 is equivalent to

$$\sup_{|\eta|_1=1} \tfrac{1}{|\xi|_1} \left| df|_x\eta - df|_{x+\xi}\eta - B_x(\xi,\eta)\right| \longrightarrow 0 \quad , \text{ as } |\xi|_1 \to 0.$$

We use the definitions of $\operatorname{grad} f|_x$ and B_x to write

$$\begin{aligned} &\sup_{|\eta|_1=1} \tfrac{1}{|\xi|_1} \left|\langle \operatorname{grad} f|_x, \eta\rangle_0 - \langle \operatorname{grad} f|_{x+\xi},\eta\rangle_0 - \langle d(\operatorname{grad} f)_x\xi,\eta\rangle_0\right| \\ &= \sup_{|\eta|_1=1} \left|\left\langle \tfrac{\operatorname{grad} f|_x - \operatorname{grad} f|_{x+\xi} - d(\operatorname{grad} f)_x\xi}{|\xi|_1}, \eta\right\rangle_0\right| \\ &\le \sup_{|\eta|_1=1} \tfrac{|\operatorname{grad} f|_x - \operatorname{grad} f|_{x+\xi} - d(\operatorname{grad} f)_x\xi|_0}{|\xi|_1} |\eta|_0 \\ &\le \tfrac{|\operatorname{grad} f|_x - \operatorname{grad} f|_{x+\xi} - d(\operatorname{grad} f)_x\xi|_0}{|\xi|_1} \end{aligned}$$

whenever $x + \xi \in U_1$. The first inequality is by Cauchy Schwarz. The second inequality uses $|\cdot|_0 \le |\cdot|_1$. By definition of the derivative $d(\operatorname{grad} f)_x$ the right hand side converges to zero, as $|\xi|_1 \to 0$. Hence the left hand side converges to zero, as $|\xi|_1 \to 0$. This proves Step 3. □

By Step 3 the second derivative of f exists at every $x \in U_1$ and by Step 2 it varies continuously in x. This proves Theorem A.1. □

[4] By $\mathcal{L}(H_1, H_1;\mathbb{R})$ we denote the Banach space of bounded bilinear maps $H_1 \times H_1 \to \mathbb{R}$.

The following corollary is used to prove Proposition 4.8 (Step 2).

Corollary A.2 (H_0-symmetry)**.** *The derivative of the H_0-gradient is H_0-symmetric, namely* $\langle d(\operatorname{grad} f)_x\xi,\eta\rangle_0=\langle d(\operatorname{grad} f)_x\eta,\xi\rangle_0$ *for all* $\xi,\eta\in H_1$.

Proof. By Theorem A.1 we have $f\in C^2$. The second derivative of a C^2 function is symmetric by the infinite dimensional version of the classical Theorem of Schwarz; see e.g. [AP93, Thm. 3.4]. □

B Lagrangian Hessian field B

In B.1 we calculate the Hessian of the regularized Lagrangian functional $\mathcal{B}=\mathcal{K}-\mathcal{U}+\mathcal{M}$ defined by (2.7). The contributions from the magnetic functional $\mathcal{M}$ play a crucial role in the main Section 4 of this article. In Section B.2 we show that in the Kepler case ($\mathcal{M}=0$) the Lagrangian Hessian field almost extends.

B.1 Calculation of Hessian operators

Abbreviate $X_\varepsilon:=\operatorname{grad}\mathcal{B}(z_\varepsilon)$. To get rid of the fractions in formula (2.8) for $\operatorname{grad}\mathcal{B}$ we proceed as follows. Since $\|z_\varepsilon\|^4\neq 0$ and since by the product rule

$$\|z\|^4\underbrace{\tfrac{d}{d\varepsilon}\Big|_0 X_\varepsilon}_{B^z\xi}=\tfrac{d}{d\varepsilon}\Big|_0\left(\|z_\varepsilon\|^4X_\varepsilon\right)-\underbrace{\left(\tfrac{d}{d\varepsilon}\Big|_0\|z_\varepsilon\|^4\right)X_0}_{4\|z\|^2\langle z,\xi\rangle\operatorname{grad}\mathcal{B}(z)},$$

the Hessian operator B^z is equal to the difference

$$B^z\xi=\underline{\tfrac{1}{\|z\|^4}\tfrac{d}{d\varepsilon}\Big|_0\left(\|z_\varepsilon\|^4X_\varepsilon\right)}-\tfrac{4\langle z,\xi\rangle}{\|z\|^2}\operatorname{grad}\mathcal{B}(z),\qquad X_\varepsilon=\operatorname{grad}\mathcal{B}(z_\varepsilon),\tag{B.42}$$

where summand two is a sum of **seven terms $m1+\dots+m7$**, namely

$$\begin{aligned}
&-\tfrac{4\langle z,\xi\rangle}{\|z\|^2}(\operatorname{grad}\mathcal{B}|_z)_\tau\\
&\stackrel{(2.8)}{=}\underbrace{-16\langle z,\xi\rangle\tfrac{\|z'\|^2}{\|z\|^2}z_\tau}_{=:m_1\ (\tilde T_{11}+m_1=0)}+\underbrace{16\langle z,\xi\rangle z''_\tau}_{=:m_2\ (+\tilde T_{21})}+\underbrace{8\tfrac{\langle z,\xi\rangle}{\|z\|^6}z_\tau}_{=:m_3\ (\text{add to }\tilde T_{31})}\\
&\quad+\underbrace{\tfrac{8\langle z,\xi\rangle z_\tau}{\|z\|^6}\int_0^1\int_0^s|z_\sigma|^2\,d\sigma\cdot\langle\dot{\boldsymbol a}_{t_z(s)}|_{z_s},z'_s\rangle_0\,ds}_{=:m_7\ (\text{add to }\tilde T_{71})}+\underbrace{\tfrac{4\langle z,\xi\rangle|z_\tau|^2}{\|z\|^4}\dot{\boldsymbol a}_{t_z(\tau)}|_{z_\tau}}_{=:m_5\ (\text{add to }\tilde T_{56})}\\
&\quad-\underbrace{\tfrac{8\langle z,\xi\rangle z_\tau}{\|z\|^4}\int_\tau^1\langle\dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma},z'_\sigma\rangle_0\,d\sigma}_{=:m_6\ (\text{add to }\tilde T_{62})}+\underbrace{\tfrac{4\langle z,\xi\rangle}{\|z\|^2}\,b_{t_z(\tau)}|_{z_\tau}\,j_0z'_\tau}_{=:m_4\ (\text{add to }\tilde T_{41})}.
\end{aligned}\tag{B.43}$$

So the task at hand is to calculate summand one in (B.42), namely

$$\tfrac{d}{d\varepsilon}\Big|_0\tfrac{\|z_\varepsilon\|^4X_\varepsilon}{\|z\|^4}=\tfrac{1}{\|z\|^4}\tfrac{d}{d\varepsilon}\Big|_0\left(\|z_\varepsilon\|^4\operatorname{grad}\mathcal{K}|_{z_\varepsilon}-\|z_\varepsilon\|^4\operatorname{grad}\mathcal{U}|_{z_\varepsilon}+\|z_\varepsilon\|^4\operatorname{grad}\mathcal{M}|_{z_\varepsilon}\right).$$

To this end we differentiate and get the following sum of **seven terms T1–T7**

$$
\begin{aligned}
&\underline{\tfrac{1}{\|z\|^4}\ \tfrac{d}{d\varepsilon}\Big|_0 \left(\|z_\varepsilon\|^4 X_\varepsilon\right)}\\
&\stackrel{(2.8)}{=} \tfrac{1}{\|z\|^4}\ \tfrac{d}{d\varepsilon}\Big|_0 \Bigg(4\|z_\varepsilon\|^4\|z_\varepsilon'\|^2 z_\varepsilon - 4\|z_\varepsilon\|^6 z_\varepsilon'' - 2z_\varepsilon - \|z_\varepsilon\|^4\left(\operatorname{rot}\boldsymbol{a}_{t_{z_\varepsilon}}|_{z_\varepsilon}\right) j_0 z_\varepsilon'\\
&\quad - |z_\varepsilon|^2\, \|z_\varepsilon\|^2 \dot{\boldsymbol{a}}_{t_{z_\varepsilon}}|_{z_\varepsilon} + 2z_\varepsilon\|z_\varepsilon\|^2 \int_{\sigma=\tau}^1 \langle \dot{\boldsymbol{a}}_{t_{z_\varepsilon}(\sigma)}|_{z_{\varepsilon,\sigma}}, z_{\varepsilon,\sigma}'\rangle_0\, d\sigma\\
&\quad - 2z_\varepsilon \int_0^1 \textstyle\int_0^s |z_{\varepsilon,\sigma}|^2\, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_{z_\varepsilon}(s)}|_{z_{\varepsilon,s}}, z_{\varepsilon,s}'\rangle_0\, ds\Bigg).
\end{aligned}
$$

B.1.1 Terms T1–T7

Remark B.1 (Notation T, C, F)**.** In the following we encounter many summands, general notation T_{ij}, occasionally $\tilde{T}_{rs}$, also C_{42}, C_{43}, and F_{44}. The tilde indicates that some correction term m_r will eventually be added. So we can denote the sum $\tilde{T}_{rs}+m_r$ by T_{rs}. The naming of the F_{44} and the two C summands refers to the $A=F+C$ decomposition (3.14), while all the many T_{ij}'s will be compact perturbations of a Fredholm operator F. For the C summands one must establish the scale Lipschitz estimate (3.17). So one wishes to minimize the number of C summands. A summand enforces to be a C summand whenever the base point z is required to be in level two (i.e. $z\in u_2$ in the Lagrangian or $z\in U_2$ in the Hamiltonian case § 4, respectively).

Term T1. (Kinetic energy) Differentiate $\|z_\varepsilon\|^4$ as $(\|z_\varepsilon\|^2)^2$ to get

$$
\begin{aligned}
\tfrac{d}{d\varepsilon}\Big|_0 \tfrac{4\|z_\varepsilon\|^4\|z_\varepsilon'\|^2 z_\varepsilon}{\|z\|^4} &= \tfrac{1}{\|z\|^4}\left(16\|z\|^2\|z'\|^2\langle z,\xi\rangle z + 8\|z\|^4\langle z',\xi'\rangle z + 4\|z\|^4\|z'\|^2\xi\right)\\
&= \underbrace{16\tfrac{\|z'\|^2}{\|z\|^2}\langle z,\xi\rangle z}_{\tilde{T}^z_{11}\xi\ (+m_1=T^z_{11}\xi)} + \underbrace{8\langle z',\xi'\rangle z}_{T^z_{12}\xi} + \underbrace{\|z'\|^2\xi}_{T^z_{13}\xi}.
\end{aligned}
$$

Term T2. (Kinetic energy) Differentiate $\|z_\varepsilon\|^6$ as $(\|z_\varepsilon\|^2)^3$ to get

$$
-\tfrac{d}{d\varepsilon}\Big|_0 \tfrac{4\|z_\varepsilon\|^6 z_\varepsilon''}{\|z\|^4} = -\tfrac{24\|z\|^4\langle z,\xi\rangle z'' + 4\|z\|^6\xi''}{\|z\|^4} = \underbrace{-24\langle z,\xi\rangle z''}_{\tilde{C}^z_{21}\xi\ (+m_2=C^z_{21}\xi)}\ \underbrace{-4\|z\|^2\,\xi''}_{F^z_{22}\xi}.
$$

Term T3. (Potential) $-\tfrac{1}{\|z\|^4}\ \tfrac{d}{d\varepsilon}\Big|_0 2z_\varepsilon = -\tfrac{2\xi}{\|z\|^4} = \tilde{T}^z_{31}\ (+m_3=T^z_{31})$.

Remark B.2 (Magnetic T4-T7)**.** Consider the function and its differential

$$
\begin{aligned}
b_{t_z}|_z &:= \operatorname{rot}\boldsymbol{a}_{t_z}|_z = (\partial_1 a^2 - \partial_2 a^1)_{t_z}|_z\\
db_{t_z}|_z\xi &= (\partial_{11}a^2\xi^1 + \partial_{21}a^2\xi^2 - \partial_{12}a^1\xi^1 - \partial_{22}a^1\xi^1)_{t_z}|_z
\end{aligned}
\tag{B.44}
$$

in order to economize the computation of term T4. In the following we enumerate each summand appearing in terms T4-T7 in an obvious way and baptize it

C_{ij} or F_{ij} depending on to which side in the decomposition $F^z + C^z$ in (3.14) we put it. The underbracing of summands indicates to which of the four summands in (B.43) it will be added to obtain the magnetic Hessian operator M^z.

In the following formulas the term $(dt_z\xi)$ is given by (C.55).

Term T4. $-\frac{1}{\|z\|^4}\ \frac{d}{d\varepsilon}\big|_0 \|z_\varepsilon\|^4\, b_{t_{z_\varepsilon}}|_{z_\varepsilon}\, j_0 z'_\varepsilon =$

$$\underbrace{-\tfrac{4\langle z,\xi\rangle\, b_{t_z}|_z}{\|z\|^2}\, j_0 z'}_{\tilde T^z_{41}\xi\ (+m_4=T^z_{41}\xi)} \underbrace{-\dot b_{t_z}|_z\,(dt_z\xi)\, j_0 z'}_{C^z_{42}\xi} \underbrace{-(db_{t_z}|_z\xi)\, j_0 z'}_{C^z_{43}\xi} \underbrace{-4b_{t_z}|_z\, j_0\, \partial_\tau\xi}_{F^z_{44}\,\xi}.$$

Term T5. $-\frac{1}{\|z\|^4}\ \frac{d}{d\varepsilon}\big|_0 \left|z_\varepsilon\right|^2 \|z_\varepsilon\|^2 \dot{\boldsymbol a}_{t_{z_\varepsilon}}|_{z_\varepsilon} =$

$$\underbrace{-\tfrac{2\langle z_\tau,\xi_\tau\rangle_0}{\|z\|^2}\, \dot{\boldsymbol a}_{t_z}|_z}_{T^z_{51}\xi} \underbrace{-\tfrac{2|z_\tau|^2\langle z,\xi\rangle}{\|z\|^4}\, \dot{\boldsymbol a}_{t_z}|_z}_{\tilde T^z_{52}\xi\ (+m_5=T^z_{52}\xi)} \underbrace{-\tfrac{|z_\tau|^2}{\|z\|^2}\, \ddot{\boldsymbol a}_{t_z}|_z\,(dt_z\xi)}_{T^z_{53}\xi} \underbrace{-\tfrac{|z_\tau|^2}{\|z\|^2}\begin{pmatrix} d\dot a^1_{t_z}|_z\xi\\ d\dot a^2_{t_z}|_z\xi\end{pmatrix}}_{T^z_{54}\xi}.$$

Term T6. $\frac{1}{\|z\|^4}\ \frac{d}{d\varepsilon}\big|_0 2z_\varepsilon\|z_\varepsilon\|^2 \int_\tau^1 \langle \dot{\boldsymbol a}_{t_{z_\varepsilon}(\sigma)}|_{z_{\varepsilon,\sigma}}, z'_{\varepsilon,\sigma}\rangle_0\, d\sigma =$

$$\begin{aligned}
&\tfrac{2\xi}{\|z\|^2}\int_\tau^1 \langle \dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma && : T^z_{61}\xi\\
&+\tfrac{4z\langle z,\xi\rangle}{\|z\|^4}\int_\tau^1 \langle \dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma && : \tilde T^z_{62}\xi\ (+m_6 = T^z_{62})\\
&+\tfrac{2z}{\|z\|^2}\int_\tau^1 \langle \ddot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}(dt_z\xi)_\sigma, z'_\sigma\rangle_0\, d\sigma && : T^z_{63}\xi\\
&+\tfrac{2z}{\|z\|^2}\int_\tau^1 \left\langle \begin{pmatrix} d\dot a^1_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\\ d\dot a^2_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\end{pmatrix}, z'_\sigma\right\rangle_0 d\sigma && : T^z_{64}\xi\\
&+\tfrac{2z}{\|z\|^2}\int_\tau^1 \langle \dot{\boldsymbol a}_{t_z(\sigma)}|_{z_\sigma}, \xi'_\sigma\rangle_0\, d\sigma && : T_{65}\xi.
\end{aligned}$$

Term T7. $-\frac{1}{\|z\|^4}\ \frac{d}{d\varepsilon}\big|_0 2z_\varepsilon \int_0^1\int_0^s \left|z_{\varepsilon,\sigma}\right|^2 d\sigma\cdot \langle \dot{\boldsymbol a}_{t_{z_\varepsilon}(s)}|_{z_{\varepsilon,s}}, z'_{\varepsilon,s}\rangle_0\, ds =$

$$\begin{aligned}
&-\xi\tfrac{2}{\|z\|^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma\cdot \langle \dot{\boldsymbol a}_{t_z(s)}|_{z_s}, z'_s\rangle_0\, ds && : \tilde T^z_{71}\xi\ (+m_7 = T^z_{71})\\
&-z\tfrac{4}{\|z\|^4}\int_0^1 \int_0^s \langle z_\sigma,\xi_\sigma\rangle\, d\sigma\cdot \langle \dot{\boldsymbol a}_{t_z(s)}|_{z_s}, z'_s\rangle_0\, ds && : T^z_{72}\xi\\
&-z\tfrac{2}{\|z\|^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma\cdot \langle \ddot{\boldsymbol a}_{t_z(s)}|_{z_s}(dt_z\xi)_s, z'_s\rangle_0\, ds && : T^z_{73}\xi\\
&-z\tfrac{2}{\|z\|^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma\cdot \left\langle \begin{pmatrix} d\dot a^1_{t_z(s)}|_{z_s}\xi_s\\ d\dot a^2_{t_z(s)}|_{z_s}\xi_s\end{pmatrix}, z'_s\right\rangle_0 ds && : T^z_{74}\xi\\
&-z\tfrac{2}{\|z\|^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma\cdot \langle \dot{\boldsymbol a}_{t_z(s)}|_{z_s}, \xi'_s\rangle_0\, ds && : T^z_{75}\xi.
\end{aligned}$$

This finishes Remark B.2.

Lemma B.3. *The Hessian operators of $\mathcal{B} = \mathcal{K} + \mathcal{M} - \mathcal{U}$ are given by*

$$
\begin{aligned}
B^z\xi &= (T1 + T2 + T3 + T4 + T5 + T6 + T7) - \tfrac{4\langle z,\xi\rangle}{\|z\|^2} \operatorname{grad}\mathcal{B}|_z \\
K^z\xi &= (T1 + T2) - \tfrac{4\langle z,\xi\rangle}{\|z\|^2} \operatorname{grad}\mathcal{K}|_z \\
&= (T1 + T2) \underbrace{-16\tfrac{\|z'\|^2}{\|z\|^2}\langle z,\xi\rangle z}_{=m_1=-\tilde{T}_{11}} + \underbrace{16\langle z,\xi\rangle z''}_{m_2} \\
&= \underbrace{0}_{T_{11}} + \underbrace{8\langle z',\xi'\rangle z}_{T_{12}} + \underbrace{4\|z'\|^2\xi}_{T_{13}} \underbrace{-8\langle z,\xi\rangle z''}_{T_{21}=\tilde{T}_{21}+m_2} \underbrace{-4\|z\|^2\xi''}_{=T_{22}} \\
M^z\xi &= T4 + T5 + T6 + T7 + m_4 + m_5 + m_6 + m_7 \\
-U^z\xi &= T3 + m_3 = \tilde{T}^z_{31}\xi + m_3 \\
&= -\tfrac{2\xi}{\|z\|^4} + \tfrac{8\langle z,\xi\rangle z}{\|z\|^6} = T^z_{31}\xi
\end{aligned}
$$

at $z \in \mathcal{L}^\times\mathfrak{Z}$ and where $\operatorname{grad}\mathcal{B}|_z$ *is given by (2.8).*

Proof. Hessian (B.42), calculations thereafter, and gradient formula (2.8). □

B.2 Kepler case

B.2.1 Analytic setup

The Kepler case refers to $\mathcal{M} = 0$. We restrict ourselves to the case of periodic loops. The case of twisted periodic loops works analogously.

Definition B.4 (analytic setup in Lagrangian Section B)**.** In the Lagrangian setup target space is the configuration space instead of the phase space, therefore the target space $\mathbb{C}^2$ has to be replaced by $\mathbb{C}$. On the other hand, the Lagrangian case requires an additional derivative. Hence the Hilbert space triple in the Lagrangian case is defined by

$$(\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2) = (L^2(\mathbb{S}^1, \mathbb{C}), W^{2,2}(\mathbb{S}^1, \mathbb{C}), W^{4,2}(\mathbb{S}^1, \mathbb{C})) \tag{B.45}$$

and open subsets are defined by

$$\mathfrak{u}_1 := \{z \in \mathfrak{h}_1 \mid z \not\equiv 0 \wedge \forall \tau \in \mathbb{S}^1 \colon z(\tau) \in \mathfrak{Z}\}, \qquad \mathfrak{u}_2 := \mathfrak{u}_1 \cap \mathfrak{h}_2.$$

We denote spaces by lower case letters for distinction from Section 4 (uppercase).

Remark B.5 (twisted loops)**.** In the case of twisted loops $z(\tau + 1) = -z(\tau)$ $\forall \tau \in \mathbb{S}^1$ the Hilbert space triple has to be adjusted as follows

$$
\begin{aligned}
\mathfrak{h}_0^- &:= L^2([0,1], \mathbb{C}) \\
\mathfrak{h}_1^- &:= \{z \in W^{1,2}([0,1], \mathbb{C}) \mid z(1) = -z(0)\} \\
\mathfrak{h}_2^- &:= \{z \in W^{2,2}([0,1], \mathbb{C}) \mid z(1) = -z(0) \wedge z'(1) = -z'(0)\}
\end{aligned}
$$

Apart from this, the case of twisted loops proceeds completely analogous as the case of periodic loops.

Remark B.6 (open problem)**.** Since in the Lagrangian case the construction of the Hilbert space triple involves more derivatives, almost extendability is more involved. This is already seen in the Kepler case where in the Hamiltonian setup the Hessian is extendable whereas in the Lagrangian setup the Hessian is only almost extendable as we show in this section.

Almost extendability in the general Lagrangian setup (with magnetic term) is probably still true, but we did not check it due to the huge number of summands appearing.

On periodic loop space $\mathcal{L}_+^\times\mathfrak{Z}$, defined in (2.3), the regularized Lagrangian action functional $\mathcal{B}$ is given by formula (2.7). The functional $\mathcal{B}$ continuously extends to the Sobolev $W^{2,2}$-completion $\mathfrak{u}_1$ by the same formula.
Let $z \in \mathfrak{u}_1$. As $d\mathcal{B}|_z\cdot = \langle \operatorname{grad}\mathcal{B}|_z, \cdot\rangle \colon \mathfrak{h}_1 \to \mathbb{R}$ characterizes the L^2-gradient, the Hessian bi-linear form is characterized by $d^2\mathcal{B}|_z(\cdot,\cdot) = \langle d\operatorname{grad}\mathcal{B}|_z\cdot,\cdot\rangle$. In words, the derivative of the L^2-gradient is the linear operator $B^z := d\operatorname{grad}\mathcal{B}|_z \colon \mathfrak{h}_1 \to \mathfrak{h}_0$ representing the Hessian bi-linear form with respect to the L^2-inner product. This is formalized in Appendix A. See Theorem A.1 for the next definition.

Definition B.7. The **Hessian operator of $\mathcal{B}$ at $z \in \mathfrak{u}_1$** is the linearization[5]

$$B^z := d\operatorname{grad}\mathcal{B}|_z \colon \mathfrak{h}_1 \to \mathfrak{h}_0 \tag{B.46}$$

of the L^2-gradient (2.8) of the function $\mathcal{B} = \mathcal{K} - \mathcal{U} + \mathcal{M}$ in (2.7). By linearity

$$B^z = K^z - U^z + M^z$$

where the sum consists of the Hessian operators of $\mathcal{K}$, $\mathcal{U}$, and $\mathcal{M}$; cf. (2.7).

From now on we consider the Kepler case ($\mathcal{M} = 0$), in particular $M = 0$.

B.2.2 Weak Hessian field almost extends

Theorem B.8 (Kepler case almost extends)**.** *Let $B_0 = K - U \colon \mathfrak{u}_1 \ni z \mapsto B^z$ be defined by (B.46) with $\mathcal{M} = 0$. Then the following holds.*

(i) *B_0 defines a weak Hessian field along the set $\mathfrak{u}_1$.*

(ii) *The weak Hessian field B_0 almost extends in the sense of Definition 3.3.*

We prove the theorem in the form of two separate results. Part (i) is Lemma B.9 below, part (ii) is Theorem B.10.

Lemma B.9. *The non-magnetic Hessian operators $B_0^z \colon \mathfrak{h}_1 \to \mathfrak{h}_0$ given by*

$$B_0^z\xi = -4\|z\|^2\xi'' - 4\langle z'', z\rangle\xi - 8\langle z'',\xi\rangle z - 8\langle z,\xi\rangle z'' - \tfrac{2\xi}{\|z\|^4} + \tfrac{8\langle z,\xi\rangle z}{\|z\|^6}, \tag{B.47}$$

one for each $z \in \mathfrak{u}_1$, determine a weak Hessian field B_0 on $\mathfrak{u}_1$.

[5] Given $z \in \mathfrak{u}_1$ and $\xi \in \mathfrak{h}_1$, pick a smooth path of loops $\varepsilon \mapsto z_\varepsilon \in \mathfrak{u}_1$ with $z_0 = z$ and $\frac{d}{d\varepsilon}\big|_0 z_\varepsilon = \xi$, then the linearization is of the form $d\operatorname{grad}\mathcal{B}|_z\xi = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\operatorname{grad}\mathcal{B}(z_\varepsilon)$.

Proof. The formula for B_0^z is due to Lemma B.3 where, in addition, we applied to T_{12} and T_{13} integration by parts (boundary terms vanishes by periodicity). By Theorem C.1 in the form $W^{2,2}\times L^2\to L^2$ and $W^{4,2}\times W^{2,2}\to W^{2,2}$ the map $z\mapsto B_0^z$ is element of the spaces $C^0(\mathfrak{u}_1,\mathcal{L}(\mathfrak{h}_1,\mathfrak{h}_0))$ and $C^0(\mathfrak{u}_2,\mathcal{L}(\mathfrak{h}_2,\mathfrak{h}_1))$. We verify the (`Symmetry`) axiom (3.13): Pick $\xi,\eta\in\mathfrak{h}_0$, then

$$\begin{aligned}\langle B_0^z\xi,\eta\rangle &= \langle -4\|z\|^2\xi''-4\langle z'',z\rangle\xi-8\langle z'',\xi\rangle z-8\langle z,\xi\rangle z''-\tfrac{2\xi}{\|z\|^4}+\tfrac{8\langle z,\xi\rangle z}{\|z\|^6},\eta\rangle\\ &= -4\|z\|^2\langle\xi,\eta''\rangle-4\langle z'',z\rangle\langle\xi,\eta\rangle-8\langle z'',\xi\rangle\langle z,\eta\rangle-8\langle z,\xi\rangle\langle z'',\eta\rangle\\ &\quad -\tfrac{2\langle\xi,\eta\rangle}{\|z\|^4}+\tfrac{8\langle z,\xi\rangle\langle z,\eta\rangle}{\|z\|^6}\\ &= \langle\xi,B_0^z\eta\rangle.\end{aligned}$$

Concerning equality two, to summand one we applied twice integration by parts. Inequality three holds by symmetry of inner products and by interchanging the two summands with factor 8.
It remains to check the (`Fredholm`) axiom. The following operator as a map

$$-\partial_\tau\partial_\tau\colon\quad W^{2,2}\to L^2,\quad W^{4,2}\to W^{2,2},\qquad \forall z\in\mathfrak{u}_1$$

is positive semi-definite, symmetric, and $\ker\partial_\tau\partial_\tau=\mathbb{C}$ are the constant maps only (by periodicity affine maps are excluded). By symmetry, kernel and cokernel coincide, so $-\partial_\tau\partial_\tau$ is Fredholm of index zero. The factor $4\|z\|^2$ is non-zero since $z\in\mathfrak{u}_1$ is not the zero map, hence the **leading term** $-4\|z\|^2\partial_\tau\partial_\tau$ in the formula for B_0^z is Fredholm of index zero as well.
In the formula for B_0^z remain five summands. We separate $-8\langle z,\xi\rangle z''_\tau$ for later recycling in Theorem B.10 Step (`F`). The sum of the other four, denoted by

$$T^z\colon\xi\mapsto\left[\tau\mapsto -4\langle z'',z\rangle\xi_\tau-8\langle z'',\xi\rangle z_\tau-\tfrac{2\xi_\tau}{\|z\|^4}\tfrac{\|z\|^2}{\|z\|^2}+\tfrac{8\langle z,\xi\rangle z_\tau}{\|z\|^6}\right],\tag{B.48}$$

is bounded as a map $\mathfrak{h}_0\to\mathfrak{h}_0$ and even as a map $\mathfrak{h}_1\to\mathfrak{h}_1$ if $z\in\mathfrak{u}_1$. Indeed

$$\begin{aligned}\|T^z\xi\|_2 &\le 8\|z\|_{2,2}^2\left(\tfrac12+1+\tfrac{1}{4\|z\|_2^6}+\tfrac{1}{\|z\|_2^6}\right)\|\xi\|_2\\ \|(T^z\xi)''\|_2 &= \left\|-4\langle z'',z\rangle\xi''-8\langle z'',\xi\rangle z''-\tfrac{2\xi''}{\|z\|^4}\tfrac{\|z\|^2}{\|z\|^2}+\tfrac{8\langle z,\xi\rangle z''}{\|z\|^6}\right\|_2\\ &\le 8\|z\|_{2,2}^2\left(\tfrac12+1+\tfrac{1}{4\|z\|_2^6}+\tfrac{1}{\|z\|_2^6}\right)\|\xi\|_{2,2}.\end{aligned}$$

So, by compactness of the embeddings $\mathfrak{h}_1\hookrightarrow\mathfrak{h}_0$ and $\mathfrak{h}_2\hookrightarrow\mathfrak{h}_1$, both operators T^z, as $\mathfrak{h}_1\to\mathfrak{h}_0$ and as $\mathfrak{h}_2\to\mathfrak{h}_1$, are compact. But Fredholm property and index are stable under compact perturbation. So both operators

$$F^z\colon\xi\mapsto -4\|z\|^2\xi''+T^z\xi\colon\quad \mathfrak{h}_1\to\mathfrak{h}_0,\quad \mathfrak{h}_2\to\mathfrak{h}_1,\qquad\forall z\in\mathfrak{u}_1\tag{B.49}$$

are Fredholm of index zero where T^z is defined by (B.48). It remains to exhibit the yet missing summand as a compact perturbation of F^z. Indeed we estimate

$$\begin{aligned}\|8\langle z,\xi\rangle z''\|_2 &\le 8\|z\|_2\|\xi\|_2\|z''\|_2 \le 8\|z\|_{2,2}^2\|\xi\|_2\\ \|(8\langle z,\xi\rangle z'')''\|_2 &\le 8\|z\|_2\|\xi\|_2\|z''''\|_2\le 8\|z\|_{4,2}^2\|\xi\|_2.\end{aligned}$$

This shows that $\xi \mapsto 8\langle z,\xi\rangle z''$ is bounded as a map $\mathfrak{h}_0 \to \mathfrak{h}_0$ if $z \in \mathfrak{u}_1$ and as a map $\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_0 \to \mathfrak{h}_1$ if $z \in \mathfrak{u}_2$. Now the compactness and perturbation argument previous to (B.49) proves the (`Fredholm`) axiom and Lemma B.9. □

Theorem B.10. *The non-magnetic weak Hessian field B is almost extendable. Moreover, the pair (F,C) defined for $z \in \mathfrak{u}_1$ by*

$$\begin{aligned} F^z\xi &:= -4\|z\|^2\xi'' - 4\langle z'',z\rangle\xi - 8\langle z'',\xi\rangle z - \tfrac{2\xi}{\|z\|^4} + \tfrac{8\langle z,\xi\rangle z}{\|z\|^6} \\ C^z\xi &:= -8\langle z,\xi\rangle z'' \end{aligned} \tag{B.50}$$

is a decomposition (3.14).

Proof of Theorem B.10. The proof has four steps 1, 2, (`C`), and (`F`).

Step 1. By (C.52) the map $u \mapsto C^u$ is element of the spaces $C^0(\mathfrak{u}_1, \mathcal{L}(\underline{\mathfrak{h}_r}, \mathfrak{h}_0))$ and $C^0(\mathfrak{u}_2, \mathcal{L}(\underline{\mathfrak{h}_1}, \mathfrak{h}_1))$ where $\mathfrak{h}_r = W^{2r,2}(\mathbb{S}^1, \mathbb{R}^2)$ for some $r \in (\frac{1}{2}, 1)$.

Proof. Given $z \in \mathfrak{u}_1$, the estimate

$$\|C^z\xi\|_2 = \|8\langle z,\xi\rangle z''\|_2 \le 8\|z\|_2\|\xi\|_2\|z''\|_2 \le 8\|z\|_{2,2}^2\|\xi\|_2$$

shows that C is continuous as a map from $\mathfrak{u}_1$ even to $\mathcal{L}(\mathfrak{h}_0, \mathfrak{h}_0)$, so to $\mathcal{L}(\mathfrak{h}_r, \mathfrak{h}_0)$ whenever $r \in [0,1)$ due to the embedding $\mathfrak{h}_r \hookrightarrow \mathfrak{h}_0$. Given $z \in \mathfrak{u}_2$, then

$$\|(C^z\xi)''\|_2 = \|8\langle z,\xi\rangle z''''\|_2 \le 8\|z\|_2\|\xi\|_2\|z''''\|_2 \le 8\|z\|_{4,2}^2\|\xi\|_2$$

shows that C is continuous as a map from $\mathfrak{u}_2$ even to $\mathcal{L}(\mathfrak{h}_0, \mathfrak{h}_1)$, hence to $\mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_1)$ due to the embedding $\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_0$. □

Step 2. We need to show that the map $F\colon z \mapsto F^z$ is element of the space $C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_0) \cap \mathcal{L}(\mathfrak{h}_2, \mathfrak{h}_1))$; see (3.15).

Proof. We focus on $C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_2, \mathfrak{h}_1))$. Given $z \in \mathfrak{u}_1$, then the estimate

$$\begin{aligned} \|(F^z\xi)''\|_2 &= \left\| -4\|z\|_2^2\xi'''' - 4\langle z'',z\rangle\xi'' - 8\langle z'',\xi\rangle z'' - \tfrac{2\xi''}{\|z\|^4}\tfrac{\|z\|^2}{\|z\|^2} + \tfrac{8\langle z,\xi\rangle z''}{\|z\|^6} \right\|_2 \\ &\le 8\|z\|_{2,2}^2 \left(\tfrac{1}{2} + \tfrac{1}{2} + 1 + \tfrac{1}{4\|z\|_2^6} + \tfrac{1}{\|z\|_2^6}\right)\|\xi\|_{4,2} \end{aligned}$$

and an analogous estimate for the simpler case $\|F^z\xi\|_2$ show that F is continuous as a map from $\mathfrak{u}_1$ to $\mathcal{L}(\mathfrak{h}_2, \mathfrak{h}_1)$ and also to $\mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_0)$. □

Step (`C`). The map $z \mapsto C^z$ satisfies the scale Lipschitz estimate (3.17).

Proof. Fix a loop $z \in \mathfrak{u}_1 = W^{2,2}(\mathbb{S}^1, \mathfrak{Z}) \subset W^{2,2}(\mathbb{S}^1, \mathbb{C}) = \mathfrak{h}_1$. By continuity of z its image is compact. So there exists a bounded open neighborhood $\mathfrak{U}$ of $\operatorname{im} z$ in $\mathfrak{Z}$. Let κ_0 be a bound of $\mathfrak{U}$. An open neighborhood of z in $\mathfrak{u}_1$ is defined by

$$\mathrm{v}_z := W^{2,2}(\mathbb{S}^1, \mathfrak{U}) \supset W^{2,2}(\mathbb{S}^1, \mathfrak{Z}) = \mathfrak{u}_1 \ni z.$$

The L^∞ norm of the elements of v_z is bounded by κ_0. Let $\xi \in \mathfrak{h}_1$ and $v, w \in \mathrm{v}_z$. In the following we use the norm $\|f\|_2^2 + \|f''\|_2^2$ which is equivalent to the $W^{2,2}$ norm $\|f\|_2^2 + \|f'\|_2^2 + \|f''\|_2^2$. Adding twice zero we estimate

$$\begin{aligned}
\tfrac{1}{8^2}\|(C^v - C^w)\xi\|_{2,2}^2 &= \|\langle v,\xi\rangle v'' - \langle w,\xi\rangle w''\|_2^2 + \|\langle v,\xi\rangle v'''' - \langle w,\xi\rangle w''''\|_2^2 \\
&\le 2\|\langle v \underline{- w},\xi\rangle v''\|_2^2 + 2\|\langle w,\xi\rangle\,(\underline{v''} - w'')\|_2^2 \\
&\quad + 2\|\langle v \underline{-w},\xi\rangle v''''\|_2^2 + 2\|\langle w,\xi\rangle\,(\underline{v''''} - w'''')\|_2^2 \\
&\le 2\langle v-w,\xi\rangle^2\|v''\|_2^2 + 2\langle w,\xi\rangle^2\|v''-w''\|_2^2 \\
&\quad + 2\langle v-w,\xi\rangle^2\|v''''\|_2^2 + 2\langle w,\xi\rangle^2\|v''''-w''''\|_2^2 \\
&\le 2\|v-w\|_2^2\|\xi\|_2^2\|v\|_{4,2}^2 + 2\|w\|_2^2\|\xi\|_2^2\|v-w\|_{4,2}^2 \\
&\le 2|v-w|_{\mathfrak{h}_1}^2|\xi|_{\mathfrak{h}_1}^2|v|_{\mathfrak{h}_2}^2 + 2\|w\|_\infty^2|\xi|_{\mathfrak{h}_1}^2|v-w|_{\mathfrak{h}_2}^2 \\
&\le 2|v-w|_{\mathfrak{h}_1}^2|\xi|_{\mathfrak{h}_1}^2|v|_{\mathfrak{h}_2}^2 + 2\kappa_0^2|\xi|_{\mathfrak{h}_1}^2|v-w|_{\mathfrak{h}_2}^2.
\end{aligned}$$

To get inequality three we apply Cauchy-Schwarz and in addition we incorporate the first two summands of four into the estimate of the last two summands. With $\kappa := 8\sqrt{2}\max\{1,\kappa_0\}$ this implies that the operator norm is bounded by

$$\|C^v - C^w\|_{\mathcal{L}(\mathfrak{h}_1)} \le \kappa\Big(|v-w|_{\mathfrak{h}_2} + |v|_{\mathfrak{h}_2}|v-w|_{\mathfrak{h}_1}\Big).$$

Interchanging the roles of v and w we get

$$\|C^v - C^w\|_{\mathcal{L}(\mathfrak{h}_1)} \le \kappa\Big(|v-w|_{\mathfrak{h}_2} + |w|_{\mathfrak{h}_2}|v-w|_{\mathfrak{h}_1}\Big).$$

The above two estimates imply the scale Lipschitz estimate

$$\|C^v - C^w\|_{\mathcal{L}(\mathfrak{h}_1)} \le \kappa\Big(|v-w|_{\mathfrak{h}_2} + \min\{|v|_{\mathfrak{h}_2}, |w|_{\mathfrak{h}_2}\}\cdot|v-w|_{\mathfrak{h}_1}\Big) \tag{B.51}$$

which is precisely (3.17) in axiom (`C`). This proves Step (`C`). □

Step (`F`). $\forall z \in \mathfrak{u}_1\colon\ F_2^z := F^z|_{\mathfrak{h}_2}\colon \mathfrak{h}_2 \to \mathfrak{h}_1$ is Fredholm of index zero.
The proof of Step (`F`) was given in (B.49). This proves Theorem B.10. □

C Estimates for Hessian operator summands

In Appendix C we analyze the summands of the Hessian operator B^z in Lemma B.3. As a preparation we collect in C.1 a number of estimates.

In the following we suppose that z and ξ are smooth, so the calculations and estimates are justified and we can extract in each case the information in which Sobolev spaces z and ξ must at least be. By $\langle\cdot,\cdot\rangle$ we denote the L^2-inner product, by $\|\cdot\|_\rho$ the $L^\rho =: H_\rho$ norm, and by $\|\cdot\|_{k,2}$ the Sobolev $W^{k,2} =: H_k$ norm. We set $z_\tau := z(\tau)$. For Banach spaces X and Y let $\mathcal{L}(X,Y)$, and $\boldsymbol{\mathcal{C}(X,Y)}$ be the space of bounded, respectively **compact**, linear operators.

C.1 Utilities

Factor j_0. Since j_0 is rotation by $\frac{\pi}{2}$ it is an isometry, so it disappears in norms.

C.1.1 Fractional Sobolev spaces

Theorem C.1 (Multiplication map, [BH21, Thm. 7.4])**.** *Assume $\rho_1, \rho_2, \rho \geq 0$ are real and satisfy $\rho_1 \geq \rho$ and $\rho_2 \geq \rho$ and $\rho_1 + \rho_2 > \frac{1}{2} + \rho$. Then the following is true. If $v \in W^{\rho_1,2}(\mathbb{S}^1)$ and $w \in W^{\rho_2,2}(\mathbb{S}^1)$, then $vw \in W^{\rho,2}(\mathbb{S}^1)$ and pointwise multiplication of functions is a continuous bi-linear map*

$$W^{\rho_1,2}(\mathbb{S}^1) \times W^{\rho_2,2}(\mathbb{S}^1) \to W^{\rho,2}(\mathbb{S}^1).$$

Most important for us is the theorem in the form

$$W^{r,2} \times L^2 \xrightarrow{r>\frac{1}{2}} L^2, \qquad W^{1,2} \times W^{1,2} \to W^{1,2}. \tag{C.52}$$

The **Sobolev inequality** 2 on $\mathbb{S}^1$ asserts,[6] respectively implies, that

$$\|z\|_2 \leq \|z\|_\infty \overset{2}{\leq} \|z\|_{1,2}, \qquad \|z'\|_\infty \leq \|z'\|_{1,2} \leq \|z\|_{2,2}.$$

All norms are on the domain $\mathbb{S}^1$, unless indicated differently. In fact, not only H_1 embedds into C^0, but even the larger spaces H_r whenever r is strictly larger then the borderline value $\frac{1}{2}$. This improvement is crucial for us, see e.g. term C_{43} further below.

Proposition C.2. *For any $\alpha \in (0, 1)$ the inclusion map $H_{\frac{1}{2}+\alpha}(\mathbb{R}) \subset C^{0,\alpha}(\mathbb{R})$ into the space of **α-Hölder continuous** maps is continuous. Here*

$$u \in C^{0,\alpha}(\mathbb{R}) \quad \Leftrightarrow \quad u \text{ bounded and } \exists C \colon |u(x + y) - u(x)| \leq C|y|^\alpha \ \forall x, y.$$

Proof. See e.g. [Tay96, Ch. 4 Prop. 1.5]. □

By Proposition C.2, given $r \in (\frac{1}{2}, 1)$, there exists a constant c_r such that $\|\cdot\|_{C^0} \leq \|\cdot\|_{C^{0,r-1/2}} \leq c_\alpha \|\cdot\|_{H_{1/2+\alpha}}$ on $\mathbb{S}^1$. Writing $\|\cdot\|_\infty = \|\cdot\|_{C^0}$ and $r = \frac{1}{2} + \alpha$, then for $r \in (\frac{1}{2}, 1]$ it holds that

$$\|\cdot\|_2 \leq \|\cdot\|_\infty \leq c_\alpha \|\cdot\|_{r,2}, \qquad \|\cdot\|_{r,2} \leq \|\cdot\|_{1,2}. \tag{C.53}$$

C.1.2 Factor t_z

Recall from (2.6) the Barutello-Ortega-Verzini reparametrization

$$t_z(\tau) := \frac{\int_0^\tau |z(s)|^2 \, ds}{\|z\|^2}, \qquad t'_z(\tau) = \frac{|z(\tau)|^2}{\|z\|^2}, \qquad t_z \in C^1. \tag{C.54}$$

[6] for a proof with constant 1 see e.g. [FW25, §A.6 Sobolev embedding]

which takes a loop z to a map $t_z := t(z)\colon \mathbb{S}^1 \to \mathbb{S}^1$, $\tau \mapsto t_z(\tau)$. There are two obvious identities

$$\begin{aligned} \tfrac{d}{d\tau} t_z(\tau) &= \frac{\left|z(\tau)\right|^2}{\|z\|^2} \\ (dt|_z\xi)_\tau &= \frac{2}{\|z\|^2} \int_0^\tau \langle z_\sigma, \xi_\sigma\rangle \, d\sigma - \frac{2\,\langle z, \xi\rangle}{\|z\|^4} \int_0^\tau \left|z_\sigma\right|^2 d\sigma. \end{aligned} \tag{C.55}$$

We estimate

$$\|t_z'\|_2 \overset{(C.54)}{=} \left\| \tfrac{|z(\cdot)|^2}{\|z\|_2^2} \right\|_2 \le \frac{\|z\|_\infty \|z\|_2}{\|z\|_2^2} \le \frac{\|z\|_{1,2}}{\|z\|_2}$$

and

$$\|t_z'\|_\infty : = \sup_{\tau\in\mathbb{S}^1} \left|\tfrac{d}{d\tau} t_z(\tau)\right| \overset{(C.54)}{=} \sup_{\tau\in\mathbb{S}^1} \frac{\left|z(\tau)\right|^2}{\|z\|_2^2} \le \frac{\|z\|_\infty^2}{\|z\|_2^2} \le \frac{\|z\|_{1,2}^2}{\|z\|_2^2}.$$

By (C.55) we have and by the triangle and Cauchy-Schwarz inequalities we get

$$\begin{aligned} \|(dt|_z\xi)\|_2 &\le \|(dt|_z\xi)\|_{\infty} \\ &= \sup_{\tau\in\mathbb{S}^1} \left| \tfrac{2}{\|z\|_2^2} \int_0^\tau \langle z_\sigma, \xi_\sigma\rangle_0 \, d\sigma - \tfrac{2\langle z,\xi\rangle}{\|z\|_2^4} \int_0^\tau \left|z_\sigma\right|^2 d\sigma \right| \\ &\le \tfrac{2}{\|z\|_2^2} \sup_{\tau\in\mathbb{S}^1} \left|\langle z, \xi\rangle_{L^2_{0,\tau}}\right| + \tfrac{2\|z\|_2\|\xi\|_2}{\|z\|_2^4} \sup_{\tau\in\mathbb{S}^1} \left|\langle z, z\rangle_{L^2_{0,\tau}}\right| \\ &\le \tfrac{2}{\|z\|_2^2} \sup_{\tau\in\mathbb{S}^1} \left(\|z\|_{L^2_{0,\tau}} \|\xi\|_{L^2_{0,\tau}} \right) + \tfrac{2\|\xi\|_2}{\|z\|_2^3} \sup_{\tau\in\mathbb{S}^1} \|z\|_{L^2_{0,\tau}}^2 \\ &\le \tfrac{2}{\|z\|_2} \|\xi\|_2 + \tfrac{2}{\|z\|_2} \|\xi\|_2 \\ &\le \tfrac{4}{\|z\|_2} \|\xi\|_{2}. \end{aligned} \tag{C.56}$$

Differentiate (C.55) to get equality one here

$$\begin{aligned} \|(dt|_z\xi)'\|_2 &= \left\| \tfrac{2}{\|z\|_2^2} \langle z_\cdot, \xi_\cdot\rangle_0 - \tfrac{2\langle z,\xi\rangle}{\|z\|_2^4} \left|z_\cdot\right|^2 \right\|_2 \\ &\le \tfrac{2}{\|z\|_2^2} \left\|\langle z_\cdot, \xi_\cdot\rangle_0\right\|_2 + \tfrac{2\|z\|_\infty\|\xi\|_2}{\|z\|_2^4} \|\langle z_\cdot, z_\cdot\rangle_0\|_2 \\ &\le \tfrac{2}{\|z\|_2^2} \|z\|_\infty \|\xi\|_2 + \tfrac{2\|z\|_{1,2}\|\xi\|_2}{\|z\|_2^4} \|z\|_\infty \|z\|_2 \\ &\le \left(\tfrac{2\|z\|_{1,2}}{\|z\|_2^2} + \tfrac{2\|z\|_{1,2}^2}{\|z\|_2^3} \right) \|\xi\|_2 \\ &: = d'\mathfrak{t}^{z_{1,2}} \|\xi\|_{2} \end{aligned}$$

and to get equality one here

$$\begin{aligned} \|(dt|_z\xi)'\|_{\infty} &= \sup_{\tau\in\mathbb{S}^1} \left| \tfrac{2}{\|z\|_2^2} \langle z_\tau, \xi_\tau\rangle - \tfrac{2\langle z,\xi\rangle}{\|z\|_2^4} \left|z_\tau\right|^2 \right| \\ &\le \tfrac{2}{\|z\|_2^2} \|z\|_\infty \|\xi\|_\infty + \tfrac{2\|\xi\|_2}{\|z\|_2^3} \|z\|_\infty^2 \\ &\le \tfrac{2\|z\|_{1,2}}{\|z\|_2^2} \|\xi\|_{1,2} + \tfrac{2\|z\|_{1,2}^2}{\|z\|_2^3} \|\xi\|_2 \\ &\le d'\mathfrak{t}^{z_{1,2}} \|\xi\|_{1,2}. \end{aligned} \tag{C.57}$$

By definition (2.6) of t_z and adding zero we estimate

$$\begin{aligned} |t_v(\tau) - t_w(\tau)| &= \left| \tfrac{\int_0^\tau |v_s|^2\, ds}{\|v\|_2^2} - \tfrac{\int_0^\tau |w_s|^2\, ds}{\|v\|_2^2} + \tfrac{\int_0^\tau |w_s|^2\, ds}{\|v\|_2^2} - \tfrac{\int_0^\tau |w_s|^2\, ds}{\|w\|_2^2} \right| \\ &\le \tfrac{1}{\|v\|_2^2} \left| \int_0^\tau |v_s|^2 - \langle v_s, w_s\rangle_0 + \langle v_s, w_s\rangle_0 - |w_s|^2\, ds \right| \\ &\quad + \int_0^\tau |w_s|^2\, ds \left| \tfrac{\|w\|_2^2 - \langle v,w\rangle + \langle v,w\rangle - \|v\|_2^2}{\|v\|_2^2 \|w\|_2^2} \right| \\ &= \tfrac{1}{\|v\|_2^2} \left| \int_0^\tau \langle v_s, v_s - w_s\rangle_0 + \langle v_s - w_s, w_s\rangle_0\, ds \right| \\ &\quad + \int_0^\tau |w_s|^2\, ds \left| \tfrac{\langle v, v-w\rangle + \langle v-w, w\rangle}{\|v\|_2^2 \|w\|_2^2} \right| \\ &\overset{3}{\le} 2 \tfrac{\|v\|_2 \|v-w\|_2 + \|v-w\|_2 \|w\|_2}{\|v\|_2^2} = 2\|v-w\|_2 \tfrac{\|v\|_2 + \|w\|_2}{\|v\|_2^2}. \end{aligned} \tag{C.58}$$

In step 3 , on summand one, we take the absolute value inside the integral, then enlarge $\int_0^\tau$ to $\int_0^1$, then we apply Cauchy-Schwarz. On summand two we enlarge $\int_0^\tau$ to $\int_0^1$ in which case $\|w\|_2^2$ cancels.

We need in (4.40) the following estimate. Differentiate (C.55) to get step 1. Step 2 is by the triangle inequality and produces a sum of two absolute values. In summand one add zero and in summand two add two times zero, then apply the triangle inequality to get step 3 where in addition we insert three zeroes

$$\begin{aligned} &\tfrac{1}{2} |(dt_v \xi)'_\tau - (dt_w \xi)'_\tau| \\ &\overset{1}{=} \left| \tfrac{1}{\|v\|_2^2} \langle v_\tau, \xi_\tau\rangle_0 - \tfrac{\langle v,\xi\rangle}{\|v\|_2^4} |v_\tau|^2 - \tfrac{1}{\|w\|_2^2} \langle w_\tau, \xi_\tau\rangle_0 + \tfrac{\langle w,\xi\rangle}{\|w\|_2^4} |w_\tau|^2 \right| \\ &\overset{2}{\le} \left| \tfrac{1}{\|v\|_2^2} \langle v_\tau, \xi_\tau\rangle_0 - \tfrac{1}{\|w\|_2^2} \langle w_\tau, \xi_\tau\rangle_0 \right| + \left| \tfrac{\langle v,\xi\rangle}{\|v\|_2^4} |v_\tau|^2 - \tfrac{\langle w,\xi\rangle}{\|w\|_2^4} |w_\tau|^2 \right| \\ &\overset{3}{\le} \left| \tfrac{\|w\|_2^2 - \langle w,v\rangle + \langle w,v\rangle - \|v\|_2^2}{\|v\|_2^2 \|w\|_2^2} \right| |\langle v_\tau, \xi_\tau\rangle_0| + |\langle v_\tau - w_\tau, \xi_\tau\rangle_0| \tfrac{1}{\|w\|_2^2} \\ &\quad + |\langle v-w, \xi\rangle| \tfrac{|v_\tau|^2}{\|v\|_2^4} \\ &\quad + \left| \tfrac{\|w\|_2^4 - \|w\|_2^2\|v\|_2^2 + \|w\|_2^2\|v\|_2^2 - \|v\|_2^4}{\|v\|_2^4 \|w\|_2^4} \right| |v_\tau|^2 |\langle w,\xi\rangle| \\ &\quad + \tfrac{|\langle w,\xi\rangle|}{\|w\|_2^4} \left| |v_\tau|^2 - \langle w_\tau, v_\tau\rangle_0 + \langle w_\tau, v_\tau\rangle_0 - |w_\tau|^2 \right| \\ &\overset{4}{\le} \|v-w\|_2 \tfrac{\|v\|_2 + \|w\|_2}{\|v\|_2^2 \|w\|_2^2} \cdot |v_\tau| \cdot |\xi_\tau| + |v_\tau - w_\tau| \cdot \tfrac{|\xi_\tau|}{\|w\|_2^2} \\ &\quad + \|v-w\|_2 \|\xi\|_2 \tfrac{|v_\tau|^2}{\|v\|_2^4} \\ &\quad + \|v-w\|_2 \left(\|v\|_2 + \|w\|_2\right) \left(\|v\|_2^2 + \|w\|_2^2\right) |v_\tau|^2 \|w\|_2 \|\xi\|_2 \\ &\quad + \tfrac{\|\xi\|_2}{\|w\|_2^3} |v_\tau - w_\tau| \left(|v_\tau| + |w_\tau|\right) \end{aligned} \tag{C.59}$$

pointwise for $\tau \in \mathbb{S}^1$. In step 4 we used the Cauchy-Schwarz inequality for the Euclidean inner product $\langle\cdot,\cdot\rangle_0$ on $\mathbb{R}^2$ and for the $L^2(\mathbb{S}^1, \mathbb{R}^2)$ inner product $\langle\cdot,\cdot\rangle$.

Take the supremum over τ and simplify with $\|\cdot\|_2 \le \|\cdot\|_\infty$ we get the estimate

$$
\begin{aligned}
&\tfrac{1}{2}\|(dt_v\xi)' - (dt_w\xi)'\|_\infty \\
&\le \|v-w\|_\infty \|\xi\|_\infty \cdot \\
&\quad \left(\tfrac{(\|v\|_\infty+\|w\|_\infty)^2}{\|v\|_2^2\|w\|_2^2} + \tfrac{1}{\|w\|_2^2} + \tfrac{\|v\|_\infty}{\|v\|_2^2} + (\|v\|_\infty+\|w\|_\infty)^6 + \tfrac{\|v\|_\infty+\|w\|_\infty}{\|w\|_2^3} \right) \\
&\le C_\rho^{\|z\|_2} \|v-w\|_\infty \|\xi\|_\infty \quad , \; C_\rho^{\|z\|_2} := \left(64\rho^6 + \tfrac{4+\rho^4}{\|z\|_2^2} + \tfrac{16\rho}{\|z\|_2^3} + \tfrac{64\rho^2}{\|z\|_2^4} \right).
\end{aligned}
\tag{C.60}
$$

where the last inequality uses hypothesis (4.28) met in the applications in § 4.2.2, namely $\|v\|_2, \|w\|_2 \ge \frac{1}{2}\|z\|_2$ and $\|v\|_\infty, \|w\|_\infty \le \rho$.

We need in (4.41) an estimate for the following L^∞ norm. Step 2 is by (C.55)

$$
\begin{aligned}
&\tfrac{1}{2}\|dt_v\xi - dt_w\xi\|_\infty \\
&= \tfrac{1}{2} \sup_{\tau\in\mathbb{S}^1} |(dt_v\xi)_\tau - (dt_w\xi)_\tau| \\
&\overset{2}{=} \sup_{\tau\in\mathbb{S}^1} \left| \int_0^\tau \left(\tfrac{\langle v_\sigma, \xi_\sigma\rangle_0}{\|v\|_2^2} - \tfrac{\langle w_\sigma, \xi_\sigma\rangle_0}{\|w\|_2^2} - \tfrac{\langle v, \xi\rangle}{\|v\|_2^4} |v_\sigma|^2 + \tfrac{\langle w, \xi\rangle}{\|w\|_2^4} |w_\sigma|^2 \right) d\sigma \right| \\
&\overset{3}{\le} \int_0^1 \left| \tfrac{\langle v_\sigma, \xi_\sigma\rangle_0}{\|v\|_2^2} - \tfrac{\langle w_\sigma, \xi_\sigma\rangle_0}{\|w\|_2^2} - \tfrac{\langle v, \xi\rangle}{\|v\|_2^4} |v_\sigma|^2 + \tfrac{\langle w, \xi\rangle}{\|w\|_2^4} |w_\sigma|^2 \right| d\sigma \\
&\overset{4}{\le} \sup_{\tau\in\mathbb{S}^1} \left| \tfrac{\langle v_\tau, \xi_\tau\rangle_0}{\|v\|_2^2} - \tfrac{\langle w_\tau, \xi_\tau\rangle_0}{\|w\|_2^2} - \tfrac{\langle v, \xi\rangle}{\|v\|_2^4} |v_\tau|^2 + \tfrac{\langle w, \xi\rangle}{\|w\|_2^4} |w_\tau|^2 \right| \\
&\overset{5}{=} \tfrac{1}{2}\|(dt_v\xi)' - (dt_w\xi)'\|_\infty \\
&\overset{6}{\le} C_\rho^{\|z\|_2} \|v-w\|_\infty \|\xi\|_\infty.
\end{aligned}
\tag{C.61}
$$

Step 3 takes the absolute value inside the integral, but then $\int_0^\tau |\cdot| \le \int_0^1 |\cdot|$ and the supremum over τ becomes void. Step 4 estimates the integral by the supremum over the integrand times the length of the integration interval $[0,1]$. Step 5 is by equality 1 in (C.59). Step 6 is (C.60).

C.1.3 Factor $\boldsymbol{a}$

Recall Remark 2.4 (iii) that $\boldsymbol{a} = (a^1, a^2)\colon \mathbb{S}^1 \times \mathfrak{Z} \to \mathbb{R}^2$ is the vector potential of the twisted-periodic 1-form ϑ. Restricted to the compact set $\mathbb{S}^1 \times \operatorname{im} z$, see factor $\boldsymbol{b}$, the following are finite constants (which depend on $\|z\|_\infty \le \|z\|_{1,2}$)

$$
\begin{aligned}
\alpha_{\mathbb{S}^1}^z &:= \left\| \boldsymbol{a}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty & \dot\alpha_{\mathbb{S}^1}^z &:= \left\| \dot{\boldsymbol{a}}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty \\
d\alpha_{\mathbb{S}^1}^z &:= \left\| d\boldsymbol{a}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty & \ddot\alpha_{\mathbb{S}^1}^z &:= \left\| \ddot{\boldsymbol{a}}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty \\
d^2\alpha_{\mathbb{S}^1}^z &:= \left\| d^2\boldsymbol{a}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty & d\dot\alpha_{\mathbb{S}^1}^z &:= \left\| d\dot{\boldsymbol{a}}|_{\mathbb{S}^1\times \operatorname{im} z} \right\|_\infty .
\end{aligned}
\tag{C.62}
$$

C.1.4 Factor $b = \operatorname{rot} \boldsymbol{a}$

Remark C.3 (Estimates for $b = \operatorname{rot} \boldsymbol{a}$)**.** Recall that b defined in (B.44) abbreviates the rotational. We further abbreviate

$$\begin{gathered}\beta^z_{\mathbb{S}^1} := \left\|b|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty, \quad \dot\beta^z_{\mathbb{S}^1} := \left\|\dot b|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty, \quad d\beta^z_{\mathbb{S}^1} := \left\|db|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty, \\ d\dot\beta^z_{\mathbb{S}^1} := \left\|d\dot b|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty, \quad d^2\beta^z_{\mathbb{S}^1} := \left\|d^2 b|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty.\end{gathered} \tag{C.63}$$

These values are finite: Firstly b is a smooth map $b\colon \mathbb{S}^1\times\mathfrak{Z}\to\mathbb{R}$. Secondly, since z is in U_2 (could be even U_1), it is in particular a continuous map $\mathbb{S}^1\to\mathfrak{Z}$ and therefore its image as a subset $\operatorname{im}(z)\subset\mathfrak{Z}$ is compact. Therefore the sup norm of the restriction of b to $\mathbb{S}^1\times\operatorname{im}(z)$ is finite, similarly for derivatives of b. Hence the β-constants, analogously the β-constants above, depend on $\|z\|_\infty\le\|z\|_{1,2}$.

With the chain rule we differentiate to obtain the estimate

$$\begin{aligned}\|(db_{t_z}|_z)'\|_2 &= \|d\dot b_{t_z}|_z\, t_z' + d^2 b_{t_z}|_z z'\|_2 \\ &\le d\dot\beta^z_{\mathbb{S}^1}\|t_z'\|_2 + d^2\beta^z_{\mathbb{S}^1}\|z\|_{1,2} \\ &\le \left(d\dot\beta^z_{\mathbb{S}^1}\tfrac{1}{\|z\|_2} + d^2\beta^z_{\mathbb{S}^1}\right)\|z\|_{1,2} =: (d\beta)_2^{\prime,z_{1,2}}.\end{aligned}$$

and the estimate

$$\begin{aligned}\|(db_{t_z}|_z)'\|_\infty &= \|d\dot b_{t_z}|_z\, t_z' + d^2 b_{t_z}|_z z'\|_\infty \\ &\le d\dot\beta^z_{\mathbb{S}^1}\|t_z'\|_\infty + d^2\beta^z_{\mathbb{S}^1}\|z'\|_\infty \\ &\le d\dot\beta^z_{\mathbb{S}^1}\tfrac{\|z\|^2_{1,2}}{\|z\|_2^2} + d^2\beta^z_{\mathbb{S}^1}\|z\|_{2,2} =: (d\beta)_\infty^{\prime,z_{2,2}}.\end{aligned}$$

This concludes Remark C.3.

C.2 Lagrangian scale $(\mathfrak{h}_0,\mathfrak{h}_1,\mathfrak{h}_2) = (L^2, W^{2,2}, W^{4,2})$

For the notation $\mathfrak{h}_k$ and $\mathfrak{u}_k$ see Definition B.4.

C.2.1 Kinetic terms T1 and T2

Term T1 – $T_{11} = 0 = (\tilde T_{11} + m_1)$, T_{12}, T_{13}

T_{11}. This term is zero.
T_{12}. We set $T^z\xi := 8\langle z',\xi'\rangle z_\tau$ and estimate

$$\begin{aligned}\|T^z\xi\|_2 &\le 8\|z'\|_2\|\xi'\|_2\|z\|_2 \le 8\|z\|_2\|z\|_{1,2}\|\xi\|_{1,2} \\ \|(T^z\xi)'\|_2 &= \|8\langle z',\xi'\rangle z'\|_2 \le 8\|z'\|_2\|\xi'\|_2\|z'\|_2 8\|z\|^2_{1,2}\|\xi\|_{1,2} \\ \|(T^z\xi)''\|_2 &= \|8\langle z',\xi'\rangle z''\|_2 \le 8\|z'\|_2\|\xi'\|_2\|z''\|_2 \le 8\|z\|^2_{2,2}\|\xi\|_{1,2}.\end{aligned}$$

This shows that

$$[z\mapsto T^z]\in C^0(\mathfrak{u}_1\subset U_1, \mathcal{L}(\mathfrak{h}_1\hookrightarrow W^{1,2},\mathfrak{h}_0))\cap C^0(\mathfrak{u}_1,\mathcal{L}(\mathfrak{h}_2\hookrightarrow\mathfrak{h}_1\hookrightarrow W^{1,2},\mathfrak{h}_1)).$$

Hence on both levels $T = T_{12}$ takes values in the *compact* linear operators

$$T_{12} \in C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_1, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_2, \mathfrak{h}_1)).$$

***T*₁₃.** We set $T^z\xi := 4\|z'\|_2^2 \xi_\tau$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &= 4\|z'\|_2^2\|\xi\|_2 \le 4\|z\|_{1,2}^2\|\xi\|_2 \\
\|(T^z\xi)'\|_2 &= 4\|z'\|_2^2\|\xi'\|_2 \le 4\|z\|_{1,2}^2\|\xi\|_{1,2} \\
\|(T^z\xi)''\|_2 &= 4\|z'\|_2^2\|\xi''\|_2 \le 4\|z\|_{1,2}^2\|\xi\|_{2,2}.
\end{aligned}$$

This shows that

$$[z \mapsto T^z] \in C^0(\mathfrak{u}_1 \subset U_1, \mathcal{L}(\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_0, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1 \subset U_1, \mathcal{L}(\mathfrak{h}_2 \hookrightarrow \mathfrak{h}_1, \mathfrak{h}_1)).$$

Hence on both levels $T = T_{13}$ takes values in the *compact* linear operators

$$T \in C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_1, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_2, \mathfrak{h}_1)).$$

Term T2 – $C_{21} = (\tilde{T}_{21} + m_2)$, F_{22}

C_{21}. We set $C^z\xi := -8\langle z, \xi\rangle z''_\tau$ and estimate

$$\begin{aligned}
\|C^z\xi\|_2 &\le 8\|z\|_2\|z''\|_2\|\xi\|_2 \le 8\|z\|_2\|z\|_{2,2}\|\xi\|_2 \\
\|(C^z\xi)'\|_2 &\le 8\|z\|_2\|z'''\|_2\|\xi\|_2 \le 8\|z\|_2\|z\|_{3.2}\|\xi\|_2 \\
\|(C^z\xi)''\|_2 &\le 8\|z\|_2\|z''''\|_2\|\xi\|_2 \le 8\|z\|_2\|z\|_{4,2}\|\xi\|_2.
\end{aligned}$$

This shows that, given any $r \in (\frac{1}{2}, 1)$, we have

$$[z \mapsto C^z] \in C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_r \hookrightarrow \mathfrak{h}_0, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_2, \mathcal{L}(\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_0, \mathfrak{h}_1)).$$

Hence on both levels $C = C_{21}$ takes values in the *compact* linear operators

$$C_{21} \in C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_r, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_2, \mathcal{C}(\mathfrak{h}_1)).$$

Remark C.4. Since on level two C_{21} does not extend to $\mathfrak{u}_1$ this operator necessarily goes into the C-part of the decomposition (B.50). This is in general very undesirable since for the C-part one must show the scale Lipschitz estimate (3.17). For C_{21}, due to its extremely simple formula, this estimate is still relatively short, see Step (C) in the proof of Theorem B.10.

The following operator F_{22} extends on level two to $\mathfrak{u}_1$, it goes in the F-part of the decomposition (B.50). In fact, it is Fredholm of index zero. The previous operators T_{12} and T_{13}, as well as T_{31} below, are compact perturbations of F_{22}.

F_{22}. We set $F^z\xi := -4\|z\|^2\xi''_\tau$ and estimate

$$\begin{aligned}
\|F^z\xi\|_2 &\le 4\|z\|_2^2\|\xi''\|_2 \le 4\|z\|_2^2\|\xi\|_{2,2} \\
\|(F^z\xi)'\|_2 &\le 4\|z\|_2^2\|\xi'''\|_2 \le 4\|z\|_2^2\|\xi\|_{3,2} \\
\|(F^z\xi)''\|_2 &\le 4\|z\|_2^2\|\xi''''\|_2 \le 4\|z\|_2^2\|\xi\|_{4,2}.
\end{aligned}$$

This shows that

$$[z \mapsto F^z] \in C^0(\mathfrak{u}_0 \subset \mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_0 \subset \mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_2, \mathfrak{h}_1)).$$

Hence on both levels $z \mapsto F^z = F^z_{22}$ takes values in the bounded linear operators and on level two it extends to $\mathfrak{u}_1$

$$F_{12} \in C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_2, \mathfrak{h}_1)).$$

C.2.2 Potential term T3

$\boldsymbol{T_{31} = \tilde{T}_{31} + m_3}$. We set $T^z\xi := -\frac{2\xi_\tau}{\|z\|_2^4} + \frac{8\langle z,\xi\rangle z_\tau}{\|z\|_2^6}$ and estimate

$$\begin{aligned} \|T^z\xi\|_2 &\le \tfrac{10}{\|z\|_2^4}\|\xi\|_2 \\ \|(T^z\xi)'\|_2 &\le \tfrac{2}{\|z\|_2^4}\left(1 + 4\tfrac{\|z\|_{1,2}}{\|z\|_2}\right)\|\xi\|_{1,2} \\ \|(T^z\xi)''\|_2 &\le \tfrac{2}{\|z\|_2^4}\left(1 + 4\tfrac{\|z\|_{2,2}}{\|z\|_2}\right)\|\xi\|_{2,2}. \end{aligned} \tag{C.64}$$

This shows that

$$[z \mapsto T^z] \in C^0(\mathfrak{u}_0 \subset \mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_0, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1, \mathcal{L}(\mathfrak{h}_2 \hookrightarrow \mathfrak{h}_1, \mathfrak{h}_1)).$$

Hence on both levels $T = T_{13}$ takes values in the *compact* linear operators

$$T \in C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_1, \mathfrak{h}_0)) \cap C^0(\mathfrak{u}_1, \mathcal{C}(\mathfrak{h}_2, \mathfrak{h}_1)).$$

C.3 Hamiltonian scale $(h_0, h_1, h_2) = (L^2, W^{1,2}, W^{2,2})$

In this section we analyze the potential contribution $z \mapsto U^z$ and, as described in Remark B.2 and most importantly, the magnetic contribution $z \mapsto M^z$ to the Hessian field $z \mapsto B^z$; see Lemma B.3.

For the notation $h_k \supset u_k$ see Definition 4.2.

C.3.1 Potential term T3

$\boldsymbol{T_{31} = \tilde{T}_{31} + m_3}$. We set $T^z\xi := -\frac{2\xi_\tau}{\|z\|_2^4} + \frac{8\langle z,\xi\rangle z_\tau}{\|z\|_2^6}$.
The first two estimates in (C.64) show that

$$[z \mapsto T^z] \in C^0(u_0 \subset u_1, \mathcal{L}(h_1 \hookrightarrow h_0, h_0)) \cap C^0(u_1, \mathcal{L}(h_2 \hookrightarrow h_1, h_1)).$$

Hence on both levels $T = T_{13}$ takes values in the *compact* linear operators

$$T_{31} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

C.3.2 Magnetic terms T4-T7

Concerning notation $h_k \supset u_k$ see Definition 4.2.

To carry out the following $17 + 1$ estimates we use the utility estimates prepared in § C.1, in particular for the vector potential $\boldsymbol{a}$, the function $b = \operatorname{rot} \boldsymbol{a}$, and the Barutello-Ortega-Verzini reparametrization t_z.

Term T4 – $\boldsymbol{T_{41}} = \mathbf{0} = \tilde{T}_{41} + m_4$, $\boldsymbol{C_{42}}$, $\boldsymbol{C_{43}}$, $\boldsymbol{F_{44}}$

$\boldsymbol{T_{41} + m_4}$**.** This term is zero.

$\boldsymbol{C_{42}}$**.** We define $C^z\xi := \dot{b}_{t_z(\tau)}|_{z_\tau}(dt_z\xi)_\tau \, j_0 z'_\tau$ and estimate

$$\begin{aligned}
\|C^z\xi\|_2 &\le \|\dot{b}|_{\mathbb{S}^1\times \operatorname{im} z}\|_\infty \|dt_z\xi\|_\infty \|z'\|_2 \le \dot{\beta}^z_{\mathbb{S}^1} \tfrac{4\|z\|_{1,2}}{\|z\|_2} \|\xi\|_2 \\
\|(C^z\xi)'\|_2 &\le \|(\dot{b}_{t_z}|_z)'\|_2 \|dt_z\xi\|_\infty \|z'\|_\infty + \|\dot{b}|_{\mathbb{S}^1\times \operatorname{im} z}\|_\infty \|(dt_z\xi)'\|_2 \|z'\|_\infty \\
&\quad + \|\dot{b}|_{\mathbb{S}^1\times \operatorname{im} z}\|_\infty \|dt_z\xi\|_\infty \|z''\|_2 \\
&\le \left((d\beta)_2^{\prime, z_{1,2}} \tfrac{4}{\|z\|_2} + \dot{\beta}^z_{\mathbb{S}^1}\, d'\mathfrak{t}^{z_{1,2}}_\infty + \dot{\beta}^z_{\mathbb{S}^1} \tfrac{4}{\|z\|_2} \right) \|z\|_{2,2} \|\xi\|_2 .
\end{aligned}$$

This shows that, given any $r \in (\frac{1}{2}, 1)$, we have

$$[z \mapsto C^z] \in C^0(u_1, \mathcal{L}(h_r \hookrightarrow h_0, h_0)) \cap C^0(u_2, \mathcal{L}(h_1 \hookrightarrow h_0, h_1)).$$

Hence on both levels $C = C_{42}$ takes values in the *compact* linear operators

$$\begin{aligned}
C_{42} &\in C^0(u_1, \mathcal{C}(h_r, h_0)) \cap C^0(u_2, \mathcal{C}(h_1)) \\
&\subset C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_2, \mathcal{C}(h_2, h_1)).
\end{aligned} \tag{C.65}$$

Remark C.5. Since on level two C_{42} does not extend to u_1 this operator necessarily goes into the C-part of the decomposition (3.14). This is in general very undesirable, since for the C-part one must show the scale Lipschitz estimate (3.17). For C_{42} this cumbersome task is carried out in Proposition 4.13 and for the following operator C_{43} in Proposition 4.12.

As a rule of thumb, all operators which extend on level two to u_1 should be put into the F-part of the decomposition (3.14). One of them should be Fredholm of index zero and others compact perturbations. In the case at hand f_{44} below contributes to the Fredholm operator in Lemma 4.9. Luckily all the many summands in T5 T6 T7 further below play the role of compact perturbations. So these do not require any further work to prove almost extendable.

$\boldsymbol{C_{43}}$**.** We define $C^z\xi := (db_{t_z(\tau)}|_{z_\tau}\xi_\tau)\, j_0 z'_\tau$, pick $r \in (\frac{1}{2}, 1)$, and estimate

$$\begin{aligned}
\|C^z\xi\|_2 &\le \|db_{t_z}|_z\|_\infty \|\xi\|_\infty \|z'\|_2 \overset{\text{(C.53)}}{\le} d\beta^z_{\mathbb{S}^1} \|z\|_{1,2} \|\xi\|_{r,2} \\
\|(C^z\xi)'\|_2 &\le \|(db_{t_z}|_z)'\|_2 \|\xi\|_\infty \|z'\|_\infty + \|db_{t_z}|_z\|_\infty \|\xi'\|_2 \|z'\|_\infty \\
&\quad + \|db_{t_z}|_z\|_\infty \|\xi\|_\infty \|z''\|_2 \\
&\le \left((d\beta)_2^{\prime, z_{1,2}} + 2 d\beta^z_{\mathbb{S}^1} \right) \|z\|_{2,2} \|\xi\|_{1,2} .
\end{aligned}$$

This shows that, given any $r \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned}
C_{43} &\in C^0(u_1, \mathcal{L}(h_r, h_0)) \cap C^0(u_2, \mathcal{L}(h_1)) \\
&\subset C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_2, \mathcal{C}(h_2, h_1)).
\end{aligned} \tag{C.66}$$

$F_{44} = f_{44}\partial_\tau$. Set $F^z_{44} = f^z_{44}\partial_\tau$ and $f^z_{44}\xi := -4b_{t_z}|_z\, j_0\xi$. With the estimates for t_z in § C.1.2 and the β-constants in (C.63) we get the estimates

$$\begin{aligned}
\|f^z_{44}\xi\|_2 &\le \|4b_{t_z}|_z\, j_0\xi\|_2 \le 4\beta^z_{\mathbb{S}^1}\|\xi\|_2\\
\|(f^z_{44}\xi)'\|_2 &\le 4\|(b_{t_z}|_z)'\|_2\|\xi\|_\infty + 4\|b_{t_z}|_z\|_\infty\|\xi'\|_2\\
&\le 4\left(\|\dot b_{t_z}|_z\, t'_z\|_2 + \|db_{t_z}|_z z'\|_2 + \beta^z_{\mathbb{S}^1}\right)\|\xi\|_{1,2}\\
&\le \underbrace{4\left(\dot\beta^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2} + d\beta^z_{\mathbb{S}^1}\|z\|_{1,2} + \beta^z_{\mathbb{S}^1}\right)}_{=:\gamma_{\|z\|_{1,2}}}\|\xi\|_{1,2}.
\end{aligned} \tag{C.67}$$

This shows the following estimate

$$\begin{aligned}
\|f^z_{44}\xi\|_{1,2} \le \|f^z_{44}\xi\|_2 + \|(f^z_{44}\xi)'\|_2 &\le 2\gamma_{\|z\|_{1,2}}\|\xi\|_{1,2}\\
[z\mapsto f^z_{44}] &\in C^0(u_1,\mathcal{L}(h_1)),
\end{aligned} \tag{C.68}$$

while the continuity assertion follows by writing out the formula for ${f^z_{44}}'$ together with smoothness of $b_{t_z}|_z$ and $t_z(\tau)$ in both variables. Since $F^z_{44} = f^z_{44}\partial_\tau = -4b_{t_z}|_z\, j_0\partial_\tau$ the estimates above immediately tell that

$$\begin{aligned}
\|F^z_{44}\xi\|_2 &= \|f^z_{44}\xi'\|_2 \le 4\beta^z_{\mathbb{S}^1}\|\xi\|_{1,2}\\
\|(F^z_{44}\xi)'\|_2 &= \|(f^z_{44}\xi')'\|_2 \le \ \gamma_{\|z\|_{1,2}}\|\xi\|_{2,2}.
\end{aligned}$$

This shows that on both levels $z\mapsto F^z = F^z_{44}$ takes values in the bounded linear operators and on level two it extends to u_1

$$F_{44}\in C^0(u_1,\mathcal{L}(h_1,h_0))\cap C^0(u_1,\mathcal{L}(h_2,h_1)).$$

Term T5 – T_{51}, $T_{52} = \tilde T_{52} + m_5$, T_{53}, T_{54}

T_{51}. We define $T^z\xi := -\frac{2\langle z_\tau,\xi_\tau\rangle_0}{\|z\|_2^2}\,\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|z\|_2\|\xi\|_2}{\|z\|_2^2}\,\dot\alpha^z_{\mathbb{S}^1} \le 2\dot\alpha^z_{\mathbb{S}^1}\tfrac{1}{\|z\|_2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &= \Big\|\tfrac{2\langle z'_\tau,\xi_\tau\rangle_0}{\|z\|_2^2}\,\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} + \tfrac{2\langle z_\tau,\xi'_\tau\rangle_0}{\|z\|_2^2}\,\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\\
&\quad + \tfrac{2\langle z_\tau,\xi_\tau\rangle_0}{\|z\|_2^2}\,\ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\, t'_z(\tau) + \tfrac{2\langle z_\tau,\xi_\tau\rangle_0}{\|z\|_2^2}\, d\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} z'_\tau\Big\|_2\\
&\le \tfrac{2}{\|z\|_2^2}\Big(\dot\alpha^z_{\mathbb{S}^1}\|\xi\|_\infty\|z'\|_2 + \dot\alpha^z_{\mathbb{S}^1}\|z\|_\infty\|\xi'\|_2\\
&\quad + \ddot\alpha^z_{\mathbb{S}^1}\|z\|_\infty\|\xi\|_2\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2} + d\dot\alpha^z_{\mathbb{S}^1}\|z\|_\infty\|\xi\|_\infty\|z'\|_2\Big)\\
&\le \tfrac{2}{\|z\|_2^2}\left(2\dot\alpha^z_{\mathbb{S}^1} + \ddot\alpha^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2} + d\dot\alpha^z_{\mathbb{S}^1}\|z\|_{1,2}\right)\|z\|_{1,2}\|\xi\|_{1,2}.
\end{aligned}$$

This shows that

$$[z\mapsto T^z]\in C^0(u_1,\mathcal{L}(h_1\hookrightarrow h_0,h_0))\cap C^0(u_1,\mathcal{L}(h_2\hookrightarrow h_1,h_1)).$$

Hence on both levels $T = T_{51}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{51} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

$\boldsymbol{T_{52} + m_5}$**.** We define $T^z\xi := \frac{2|z_\tau|^2 \langle z,\xi\rangle}{\|z\|_2^4}\, \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|z\|_\infty \|z\|_2^2 \|\xi\|_2}{\|z\|_2^4}\, \dot{\alpha}^z_{\mathbb{S}^1} \le 2\dot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2} \|\xi\|_2\\
\|(T^z\xi)'\|_2 &= \Big\| \tfrac{4\langle z_\tau, z_\tau'\rangle \langle z,\xi\rangle}{\|z\|_2^4}\, \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\\
&\qquad + \tfrac{2|z_\tau|^2\langle z,\xi\rangle}{\|z\|_2^4}\, \ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} t_z'(\tau) + \tfrac{2|z_\tau|^2\langle z,\xi\rangle}{\|z\|_2^4}\, d\dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} z_\tau' \Big\|_2\\
&\le 4\dot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty \|z'\|_2 \|z\|_2 \|\xi\|_2}{\|z\|_2^4} + 2\ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty \|z\|_2 \|z\|_2 \|\xi\|_2}{\|z\|_2^4} \tfrac{\|z\|_{1,2}^2}{\|z\|_2^2}\\
&\qquad + 2d\dot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty \|z\|_\infty \|z\|_2 \|\xi\|_2}{\|z\|_2^4} \|z'\|_2\\
&\le \tfrac{\|z\|_{1,2}^2}{\|z\|_2^3} \left(4\dot{\alpha}^z_{\mathbb{S}^1} + 2\ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2} + 2d\dot{\alpha}^z_{\mathbb{S}^1} \|z\|_{1,2}\right) \|\xi\|_2.
\end{aligned}$$

This shows that on both levels $T = T_{52}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{52} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

$\boldsymbol{T_{53}}$**.** We define $T^z\xi := -\frac{|z_\tau|^2}{\|z\|_2^2}\, \ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\, (dt_z\xi)_\tau$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{\|z\|_\infty \|z\|_2}{\|z\|_2^2}\, \|\ddot{\boldsymbol{a}}_{t_z}|_z\|_\infty \|dt_z\xi\|_\infty \le 4\ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2} \|\xi\|_2\\
\|(T^z\xi)'\|_2 &= \Big\| \tfrac{2\langle z_\tau, z_\tau'\rangle}{\|z\|_2^2}\, \ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\, (dt_z\xi)_\tau + \tfrac{|z_\tau|^2}{\|z\|_2^2}\, \ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} t_z'(\tau)\, (dt_z\xi)_\tau\\
&\qquad + \tfrac{|z_\tau|^2}{\|z\|_2^2}\, d\ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau} z_\tau'\, (dt_z\xi)_\tau + \tfrac{|z_\tau|^2}{\|z\|_2^2}\, \ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}\, (dt_z\xi)_\tau' \Big\|_2\\
&\le 2\ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty \|z'\|_2}{\|z\|_2^2} \tfrac{4\|\xi\|_2}{\|z\|_2} + \ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty^2}{\|z\|_2^2} \tfrac{\|z\|_{1,2}}{\|z\|_2} \tfrac{4\|\xi\|_2}{\|z\|_2}\\
&\qquad + d\ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty^2}{\|z\|_2^2} \|z'\|_2 \tfrac{4\|\xi\|_2}{\|z\|_2} + \ddot{\alpha}^z_{\mathbb{S}^1} \tfrac{\|z\|_\infty^2}{\|z\|_2^2} 2\tfrac{\|z\|_{1,2}}{\|z\|_2^2} \left(1 + \tfrac{\|z\|_{1,2}}{\|z\|_2}\right) \|\xi\|_2\\
&\le const(\|z\|_{1,2})\, \|\xi\|_2.
\end{aligned}$$

This shows that on both levels $T = T_{53}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{53} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

$\boldsymbol{T_{54}}$**.** We define $T^z\xi := -\frac{\langle z_\tau, z_\tau\rangle}{\|z\|_2^2}\begin{pmatrix} d\dot{a}^1_{t_z(\tau)}|_{z_\tau}\xi_\tau \\ d\dot{a}^2_{t_z(\tau)}|_{z_\tau}\xi_\tau\end{pmatrix}$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{\|z\|_\infty^2}{\|z\|_2^2}\, d\dot{\alpha}^z_{\mathbb{S}^1}\|\xi\|_2 \le d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2}\|\xi\|_2 \\
\|(T^z\xi)'\|_2 &= \Bigg\| \tfrac{2\langle z_\tau, z'_\tau\rangle}{\|z\|_2^2}\begin{pmatrix} d\dot{a}^1_{t_z(\tau)}|_{z_\tau}\xi_\tau \\ d\dot{a}^2_{t_z(\tau)}|_{z_\tau}\xi_\tau\end{pmatrix} \\
&\quad + \tfrac{|z_\tau|^2}{\|z\|_2^2}\begin{pmatrix} d\ddot{a}^1_{t_z(\tau)}|_{z_\tau}t'_z(\tau)\xi_\tau + d^2\dot{a}^1_{t_z(\tau)}|_{z_\tau}z'_\tau\xi_\tau + d\dot{a}^1_{t_z(\tau)}|_{z_\tau}\xi'_\tau \\ d\ddot{a}^2_{t_z(\tau)}|_{z_\tau}t'_z(\tau)\xi_\tau + d^2\dot{a}^2_{t_z(\tau)}|_{z_\tau}z'_\tau\xi_\tau + d\dot{a}^2_{t_z(\tau)}|_{z_\tau}\xi'_\tau\end{pmatrix}\Bigg\|_2 \\
&\le \mathit{const}(\|z\|_{1,2})\,\|\xi\|_{1,2}.
\end{aligned}$$

This shows that on both levels $T = T_{54}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{54} \in C^0(u_1, \mathcal{C}(h_1,h_0)) \cap C^0(u_1, \mathcal{C}(h_2,h_1)).$$

Term T6 – $\boldsymbol{T_{61}}$, $\boldsymbol{T_{62} = \tilde{T}_{62} + m_6}$, $\boldsymbol{T_{63}}$, $\boldsymbol{T_{64}}$, $\boldsymbol{T_{65}}$

$\boldsymbol{T_{61}}$**.** We define $T^z\xi := \frac{2\xi_\tau}{\|z\|_2^2}\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|\xi\|_2}{\|z\|_2^2}\sup_{\tau\in\mathbb{S}^1}\left|\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma\right| \\
&\le \tfrac{2\|\xi\|_2}{\|z\|_2^2}\sup_{\tau\in\mathbb{S}^1}\left(\|\dot{\boldsymbol{a}}_{t_z}|_z\|_{L^2_{[\tau,1]}}\|z'\|_{L^2_{[\tau,1]}}\right) \\
&\le 2\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_2 \\
\|(T^z\xi)'\|_2 &\le \tfrac{2\|\xi'\|_2}{\|z\|_2^2}\sup_{\tau\in\mathbb{S}^1}\left|\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma\right| + \tfrac{2\|\xi\|_\infty}{\|z\|_2^2}\left\|\langle \dot{\boldsymbol{a}}_{t_z(\cdot)}|_{z_\cdot}, z'_\cdot\rangle_0\right\|_2 \\
&\le 2\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_{1,2} + 2\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z'\|_2}{\|z\|_2^2}\|\xi\|_{1,2}.
\end{aligned} \tag{C.69}$$

This shows that on both levels $T = T_{61}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{61} \in C^0(u_1, \mathcal{C}(h_1,h_0)) \cap C^0(u_1, \mathcal{C}(h_2,h_1)).$$

$\boldsymbol{T_{62}}$**.** We define $T^z\xi := -\frac{4z_\tau\langle z,\xi\rangle}{\|z\|^4}\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le 4\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_2 \qquad \text{(see } T_{61}\text{)} \\
\|(T^z\xi)'\|_2 &\le \tfrac{4\|z'\|_2\|z\|_2\|\xi\|_2}{\|z\|_2^4}\sup_{\tau\in\mathbb{S}^1}\left|\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, z'_\sigma\rangle_0\, d\sigma\right| \qquad \text{(see } T_{61}\text{)} \\
&\quad + \tfrac{4\|z\|_\infty\|z\|_2\|\xi\|_2}{\|z\|_2^4}\left\|\tau\mapsto \langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}, z'_\tau\rangle_0\right\|_2 \\
&\le 8\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^3}\|\xi\|_2.
\end{aligned}$$

This shows that on both levels $T = T_{62}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{62} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

***T*₆₃.** We define $T^z\xi := \frac{2z_\tau}{\|z\|_2^2}\int_\tau^1 \langle \ddot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}\, (dt_z\xi)_\sigma, z'_\sigma\rangle_0 \, d\sigma$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|z\|_2}{\|z\|_2^2}\left(\|\ddot{\boldsymbol{a}}_{t_z}|_z\|_{L^\infty_{[\tau,1]}}\|dt_z\xi\|_{L^2_{[\tau,1]}}\|z'\|_{L^2_{[\tau,1]}}\right) && \text{(see } T_{61})\\
&\le \tfrac{2}{\|z\|_2}\ddot{\alpha}^z_{\mathbb{S}^1}\tfrac{4}{\|z\|_2}\|\xi\|_2\|z'\|_2\\
&\le 8\ddot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &\le \tfrac{2\|z'\|_2}{\|z\|_2^2}\left(\|\ddot{\boldsymbol{a}}_{t_z}|_z\|_{L^\infty_{[\tau,1]}}\|dt_z\xi\|_{L^2_{[\tau,1]}}\|z'\|_{L^2_{[\tau,1]}}\right) && \text{(see } \|T^z\xi\|_2)\\
&\quad + \tfrac{2\|z\|_\infty}{\|z\|_2^2}\|\tau\mapsto\langle\underbrace{\ddot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}}_{L^\infty}\underbrace{(dt_z\xi)_\tau}_{L^\infty},\underbrace{z'_\tau}_{L^2}\rangle_0\|_2 && \text{(see } T_{61})\\
&\le 16\ddot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^3}\|\xi\|_2.
\end{aligned}$$

This shows that on both levels $T = T_{63}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{63} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

***T*₆₄.** Define $T^z\xi := \frac{2z_\tau}{\|z\|_2^2}\int_\tau^1 \langle (d\dot{a}^1_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma, d\dot{a}^2_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma), z'_\sigma\rangle_0\, d\sigma$ and estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|z\|_2}{\|z\|_2^2}\sup_{\tau\in\mathbb{S}^1}\left|\int_\tau^1\left\langle\begin{pmatrix} d\dot{a}^1_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\\ d\dot{a}^2_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\end{pmatrix}, z'_\sigma\right\rangle_0 d\sigma\right|\\
&= \tfrac{2}{\|z\|_2}\sup_{\tau\in\mathbb{S}^1}\left|\left\langle\begin{pmatrix} d\dot{a}^1_{t_z}|_z\xi\\ d\dot{a}^2_{t_z}|_z\xi\end{pmatrix}, z'\right\rangle_{L^2_{[\tau,1]}}\right|\\
&\le \tfrac{2}{\|z\|_2}\left\|\begin{pmatrix} d\dot{a}^1_{t_z}|_z\xi\\ d\dot{a}^2_{t_z}|_z\xi\end{pmatrix}\right\|_2\|z'\|_2\\
&\le \tfrac{2}{\|z\|_2}\sqrt{2}\left\|d\dot{\boldsymbol{a}}|_{\mathbb{S}^1\times \operatorname{im} z}\right\|_\infty\|\xi\|_2\|z'\|_2\\
&\le 2\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &\le \tfrac{2\|z'\|_2}{\|z\|_2^2}\sup_{\tau\in\mathbb{S}^1}\left|\int_\tau^1\left\langle\begin{pmatrix} d\dot{a}^1_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\\ d\dot{a}^2_{t_z(\sigma)}|_{z_\sigma}\xi_\sigma\end{pmatrix}, z'_\sigma\right\rangle_0 d\sigma\right| \quad \text{(see } \|T^z\xi\|_2)\\
&\quad + \tfrac{2\|z\|_\infty}{\|z\|_2^2}\left\|\tau\mapsto\left\langle\begin{pmatrix} d\dot{a}^1_{t_z(\tau)}|_{z_\tau}\xi_\tau\\ d\dot{a}^2_{t_z(\tau)}|_{z_\tau}\xi_\tau\end{pmatrix}, z'_\tau\right\rangle_0\right\|_2\\
&\le 2\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2}\|\xi\|_2 + 2\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\underbrace{\left\|\begin{pmatrix} d\dot{a}^1_{t_z}|_z\xi\\ d\dot{a}^2_{t_z}|_z\xi\end{pmatrix}\right\|_\infty}_{\le\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\|\xi\|_\infty}\|z'\|_2\\
&\le 4\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2}\|\xi\|_{1,2}.
\end{aligned}$$

This shows that on both levels $T = T_{64}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{64} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

T_{65}. We define

$$F^z\xi := \tfrac{2z_\tau}{\|z\|_2^2} \int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, \xi'_\sigma\rangle_0 \, d\sigma \tag{C.70}$$

and we estimate

$$\begin{aligned}
\|F^z\xi\|_2 &\le 2\dot\alpha^z_{\mathbb{S}^1} \tfrac{1}{\|z\|_2}\|\xi'\|_2 \le 2\dot\alpha^z_{\mathbb{S}^1} \tfrac{1}{\|z\|_2}\|\xi\|_{1,2} \qquad (\text{see } C_{61})\\
\|(F^z\xi)'\|_2 &= \tfrac{2}{\|z\|_2^2} \left\| \tau \mapsto z'_\tau \textstyle\int_\tau^1 \langle \dot{\boldsymbol{a}}_{t_z(\sigma)}|_{z_\sigma}, \xi'_\sigma\rangle_0 \, d\sigma - z_\tau \langle \dot{\boldsymbol{a}}_{t_z(\tau)}|_{z_\tau}, \xi'_\tau\rangle_0 \right\|_2 \\
&\le 4\dot\alpha^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_{1,2}. \qquad (\text{see } C_{61})
\end{aligned} \tag{C.71}$$

As usual on level two $z \mapsto T_{65}^z$ extends to u_1 and induces a compact operator

$$h_2 \overset{\text{cp.}}{\hookrightarrow} h_1 \xrightarrow{\text{bd}} h_1.$$

On level one we only get boundedness $h_1 \to h_0$, to prove compactness is harder; see Lemma 4.10.

Term T7 – $T_{71} = \tilde{T}_{71} + m_7$, T_{72}, T_{73}, T_{74}, T_{75}

T_{71}. For $T^z\xi := \left(\frac{8\langle z,\xi\rangle z_\tau}{\|z\|_2^6} - \frac{2\xi_\tau}{\|z\|_2^4}\right) \int_0^1 \int_0^s |z_\sigma|^2 \, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z'_s\rangle_0 \, ds$ we estimate

$$\begin{aligned}
\|T^z\xi\|_2 &\le \left(\tfrac{8\|\xi\|_2}{\|z\|_2^4} + \tfrac{2\|\xi\|_2}{\|z\|_2^4}\right) \left| \int_0^1 \underbrace{\textstyle\int_0^s |z_\sigma|^2 \, d\sigma}_{=:f(s)\le f(1)} \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z'_s\rangle_0 \, ds \right| \\
&= \tfrac{10\|\xi\|_2}{\|z\|_2^4} |\langle f\dot{\boldsymbol{a}}_{t_z}|_z, z'\rangle| \le \tfrac{10\|\xi\|_2}{\|z\|_2^4} \|(\dot{\boldsymbol{a}}_{t_z}|_z)\|_\infty \|\underbrace{f}_{\le f(1)}\|_2 \|z'\|_2 \\
&\le \tfrac{10\|\xi\|_2}{\|z\|_2^4} \dot\alpha^z_{\mathbb{S}^1} \underbrace{f(1)}_{=\|z\|_2^2} \underbrace{\|1\|_2}_{=1} \|z\|_{1,2} \\
&\le 10\dot\alpha^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2} \|\xi\|_2 \\
\|(T^z\xi)'\|_2 &\le 8\dot\alpha^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2} \left(\tfrac{\|z\|_{1,2}}{\|z\|_2} + 1\right) \|\xi\|_{1,2}. \qquad (\text{cf. } \|T^z\xi\|_2)
\end{aligned}$$

This shows that on both levels $T = T_{71}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{71} \in C^0(u_1, \mathcal{C}(h_1, h_0)) \cap C^0(u_1, \mathcal{C}(h_2, h_1)).$$

***T*₇₂.** For $T^z\xi := -z_\tau \frac{4}{\|z\|_2^4}\int_0^1 \int_0^s \langle z_\sigma, \xi_\sigma\rangle\, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z_s'\rangle_0\, ds$ we estimate

$$
\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{4\|z\|_2}{\|z\|_2^4}\int_0^1 \underbrace{\int_0^s \langle z_\sigma, \xi_\sigma\rangle\, d\sigma}_{=:g(s)} \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z_s'\rangle_0\, ds \qquad \text{(cf. } T_{71}\text{)}\\
&\le \tfrac{4}{\|z\|_2^3}\|\dot{\boldsymbol{a}}|_{\mathbb{S}^1\times \operatorname{im} z}\|_\infty \|g\|_2\|z'\|_2 \qquad , \|g\|_2 \le \|z\|_2\|\xi\|_2\\
&\le 4\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &= \left\| {\color{red}z'} \tfrac{4}{\|z\|_2^4}\int_0^1 \int_0^s \langle z_\sigma, \xi_\sigma\rangle\, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, z_s'\rangle_0\, ds\right\|_2\\
&\le 4\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^3}\|\xi\|_2. \qquad \text{(cf. } \|T^z\xi\|_2\text{)}
\end{aligned}
$$

This shows that on both levels $T = T_{72}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{72} \in C^0(u_1, \mathcal{C}(h_1,h_0)) \cap C^0(u_1, \mathcal{C}(h_2,h_1)).$$

***T*₇₃.** For $T^z\xi := -z_\tau \frac{2}{\|z\|_2^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma \cdot \langle \ddot{\boldsymbol{a}}_{t_z(s)}|_{z_s}\, (dt_z\xi)_s, z_s'\rangle_0\, ds$ we estimate

$$
\begin{aligned}
\|T^z\xi\|_2 &\le \tfrac{2\|z\|_2}{\|z\|_2^4}\|\underbrace{f}_{\le f(1)}\|_2\, \|(\ddot{\boldsymbol{a}}_{t_z}|_z)\|_\infty \|dt_z\xi\|_\infty\|z'\|_2 \qquad \text{(see } T_{71}\text{)}\\
&\le 8\ddot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2^2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &= \left\| {\color{red}z'} \tfrac{2}{\|z\|_2^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma \cdot \langle \ddot{\boldsymbol{a}}_{t_z(s)}|_{z_s} (dt_z\xi)_s, z_s'\rangle_0\, ds\right\|_2\\
&\le 8\ddot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^3}\|\xi\|_2. \qquad \text{(cf. } \|T^z\xi\|_2\text{)}
\end{aligned}
$$

This shows that on both levels $T = T_{73}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{73} \in C^0(u_1, \mathcal{C}(h_1,h_0)) \cap C^0(u_1, \mathcal{C}(h_2,h_1)).$$

***T*₇₄.** For $T^z\xi := -z_\tau \frac{2}{\|z\|^4}\int_0^1 \int_0^s |z_\sigma|^2\, d\sigma \cdot \left\langle \begin{pmatrix} d\dot{a}^1_{t_z(s)}|_{z_s}\xi_s \\ d\dot{a}^2_{t_z(s)}|_{z_s}\xi_s \end{pmatrix}, z_s'\right\rangle_0 ds$ we estimate

$$
\begin{aligned}
\|T^z\xi\|_2 &\le 2\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}}{\|z\|_2}\|\xi\|_2\\
\|(T^z\xi)'\|_2 &\le\le 2\sqrt{2}d\dot{\alpha}^z_{\mathbb{S}^1}\tfrac{\|z\|_{1,2}^2}{\|z\|_2^2}\|\xi\|_2. \qquad \text{(cf. } \|T^z\xi\|_2\text{)}
\end{aligned}
$$

This shows that on both levels $T = T_{74}$ takes values in the *compact* linear operators and on level two it extends to u_1

$$T_{74} \in C^0(u_1, \mathcal{C}(h_1,h_0)) \cap C^0(u_1, \mathcal{C}(h_2,h_1)).$$

***T_{75}*.** We define

$$T^z\xi := -z_\tau \tfrac{2}{\|z\|_2^4} \int_0^1 \textstyle\int_0^s |z_\sigma|^2\, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, \xi'_s\rangle_0\, ds \tag{C.72}$$

and estimate

$$\begin{aligned} \|T^z\xi\|_2 &\le 2\dot\alpha^z_{\mathbb{S}^1} \tfrac{1}{\|z\|_2}\|\xi'\|_2 \le 2\dot\alpha^z_{\mathbb{S}^1} \tfrac{1}{\|z\|_2} \|\xi\|_{1,2} \qquad (\text{see } T_{71}) \\ \|(T^z\xi)'\|_2 &\le \tfrac{2}{\|z\|_2^4}\|z'_\tau\|_2 \left| \int_0^1 \textstyle\int_0^s |z_\sigma|^2\, d\sigma \cdot \langle \dot{\boldsymbol{a}}_{t_z(s)}|_{z_s}, \xi'_s\rangle_0\, ds \right| \\ &\le 2\dot\alpha^z_{\mathbb{S}^1} \tfrac{\|z\|_{1,2}}{\|z\|_2^2} \|\xi\|_{1,2}. \qquad (\text{see } T_{71}) \end{aligned} \tag{C.73}$$

As usual on level two $z \mapsto T^z_{75}$ extends to u_1 and induces a compact operator

$$h_2 \overset{\text{cp.}}{\hookrightarrow} h_1 \overset{\text{bd}}{\longrightarrow} h_1.$$

On level one we only get boundedness $h_1 \to h_0$, to prove compactness is harder; see Lemma 4.10.

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