COLLATZ CONJECTURE: A SIMPLE PROOF FOR A SIMPLE PROBLEM

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1. Abstract

A simple proof of the Collatz conjecture is proposed. The approach relies on the fundamental properties of $\mathbb{N}$ and on their fruitful interplay with the Collatz transformation.

2. Introduction

The Collatz conjecture is defined as follows: When following the trajectory defined by the transformation $T(x)$ (see Eq. 1), all integers eventually end up in an infinite cycle of 1 and 2.

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 1 \pmod 2 \\ \frac{3x+1}{2} & \text{if } x \equiv 0 \pmod 2 \end{cases} \tag{1}$$

For instance, starting at $x = 5$, the trajectory $T^i(x)$ is $5 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1 \to ...$ It comes handy to describe the trajectory x, $T(x)$, $T^2(x)$, $T^3(x)$, ..., $T^i(x)$ as x_0, x_1, x_2, x_3, ..., x_i.

The conjecture is a well-known, yet unsolved, mathematical problem and was introduced by Lothar Collatz in 1937. Despite its simplicity, it has been proven remarkably challenging to be solved by conventional mathematical techniques. Paul Erdos coined the phrase that "mathematics is not yet ready for such problems" [1]. As state-of-the-art, Terence Tao recently showed that almost all Collatz trajectories attain almost bounded values [2]. Moreover, the conjecture has been verified numerically for all numbers up to at least 10^{18}. Also, a probabilistic, heuristic argument was put forward by Lagarias, concluding that the growth between two consecutive odd integers along a trajectory is on average $3/4$, which is less than 1, so that a net decrease should be observed. However, this did not account for the assumed perfect even/odd mixing of the transformation, so the argument was not enough to settle the conjecture [1].

3. Results

3.1. **Observations.** Exploring the trajectories (i.e. the value of the transformations x_i vs. the generation n) is rather confusing: Often, they shoot up to large values, before eventually falling to lower values and finally reaching the loop of 1 and 2. While for or certain start values such as $x_0 = 2^j$, the pattern is easily explained by a monotonous, exponential decrease, the $+1$ term in the odd transformation causes most trajectories to follow complex patterns (see Fig. 1).

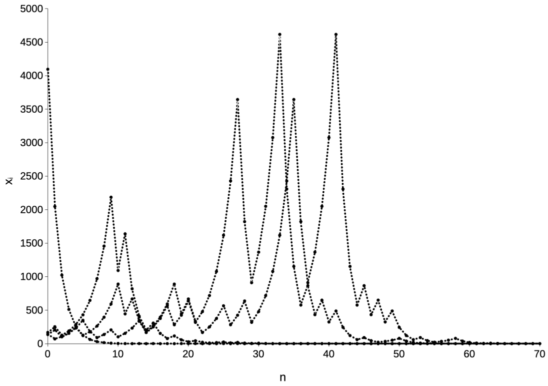

FIGURE 1. Three example trajectories (x_i vs. n). The start values x_0 are 137, 142, 169 and $2^{12} = 4096$, respectively. The stopping time (i.e. n, at which x_i first hits 1) for these examples is 58, 66, 33 and 12, respectively.

To be able to bring order into the transformation, it is much more helpful to look at graphs that do not focus on the trajectories (i.e. the intergeneration behavior), but on the collection of intrageneration values (see Fig. 2a). While the overall collection still seems rather erratic, some trends can already becoming apparent. At the latest, the trends become clear, when plotting only select generations, for instance the zeroth and first generation (G_0 and G_1, see Fig. 2b). Obviously, the density of even and odd integers is identical in G_0. This is the definition of $\mathbb{N}$. In consequence, every second x_0 value is folded downwards and every second upwards. This entails that the density of even and odd values is also identical on the two G_1 branches, at exactly half the density of G_0.

Now, as the key G_0 properties are conserved in G_1, they are also conserved in any G_n with any n (see two example graphs of G_2 and G_3 branches in Figs. 2c and 2d, respectively). In each generation, the point density of the branches is halved. The phenomenon of self-replicating patterns resembles the behavior of fractals, where also structures are replicated over and over again, when increasing the calculation depth.

For each generation, the same pattern occurs: The parent branch is split into two daughter branches with half of the point density of the parent branch. The slopes of the branches can be calculated with the help of a Pascal triangle, as they are multiplicative. Thus, the slope a_i of any branch is given by

$$a_i = 2^{-j} \left(\frac{3}{2} \right)^k \tag{2}$$

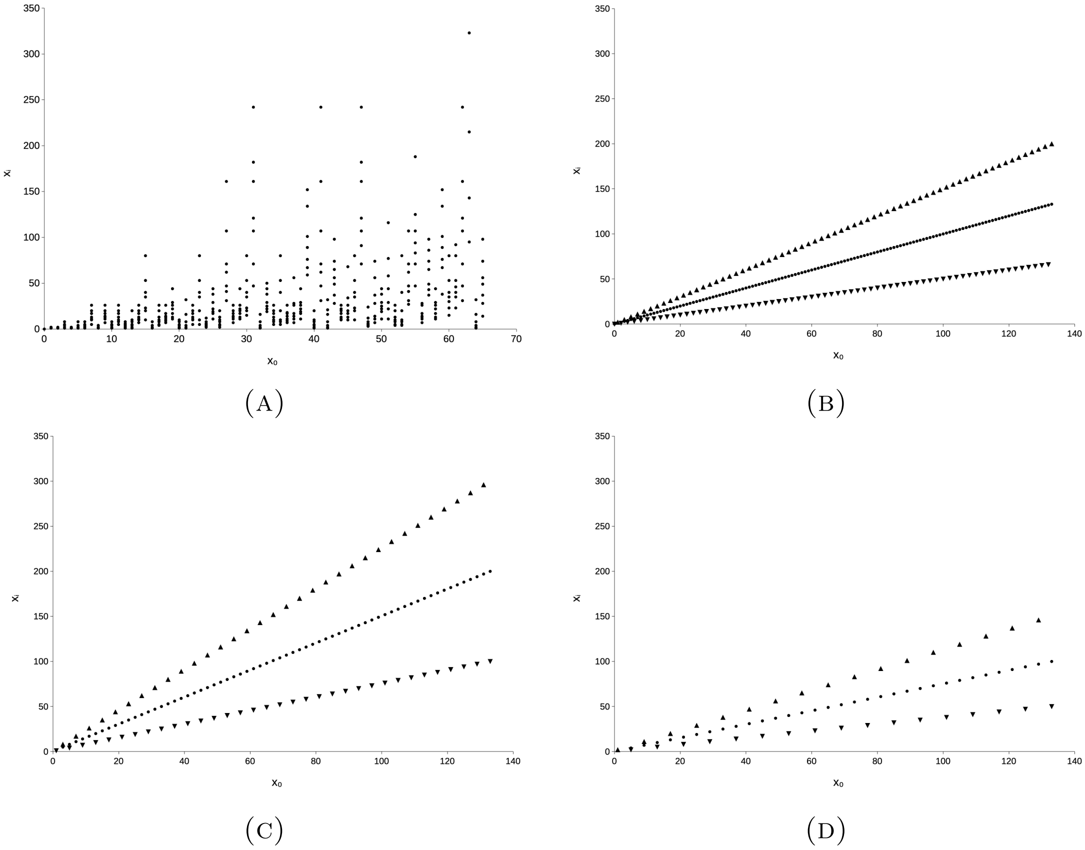

FIGURE 2. (A) Transformations $x_i(x_0)$ for $x_0 \in \{1,\ 65\}$ and i $\in \{1,\ 7\}$. (B) $x_o(x_0)$, denoted as dots, and the two associated $x_1(x_0)$ branches, denoted as upwards and downwards triangles, respectively. (C) The upper branch of $x_1(x_0)$, denoted as dots, and the two associated $x_2(x_0)$ branches, denoted as upwards and downwards triangles, respectively. (D) The lower branch of $x_2(x_0)$, denoted as dots, and the two associated $x_3(x_0)$ branches, denoted as upwards and downwards triangles, respectively.

The intercept b_i is calculated recursively and depends on the parity of the subset of the parent branch

$$b_i = \begin{cases} \frac{3}{2}b_{i-1} + \frac{1}{2} & \text{for odd subset of parent branch} \\ \frac{b_{i-1}}{2} & \text{for even subset of parent branch} \end{cases} \tag{3}$$

In that, $n = j + k$ ist the number of generations traversed, j is the number of even transformations and k the number of odd transformations.

3.2. **Evolution of Intertwined Sets.** As observed in section 2.1, the set of $\mathbb{N}$ is transformed into two sets of equal size: one set (folded up line) stemming from odd numbers and one set (folded down line) stemming from the even numbers.

It is convenient to define the point density (as a measure of the cardinality of sets of infinite size) of $\mathbb{N}$ as 1, simply meaning $\mathbb{N}$ contains an element at each integer value. Then, the point density of each of the two sets in the first generation is 2^{-1}. In the nth generation, the point density of the 2^n available sets is 2^{-n}. And each set in each generation consists of an equal amount of odd and even numbers.

For instance, the zeroth generation is comprised of a single set (read: "set 1 of generation 0"):

$$S(0,1) = \mathbb{N} \tag{4}$$

The set is comprised of two strictly intertwined subsets, containing odd and even numbers, respectively:

$$S(0,1) = S(0,1)_{odd} \cup S(0,1)_{even} = \mathbb{N}_{odd} \cup \mathbb{N}_{even} \tag{5}$$

The point density of the two subsets is identical:

$$r(S(0,1)_{odd}) = r(S(0,1)_{even}) = 2^{-1} \tag{6}$$

After transforming S(0,1), the first generation is comprised of two sets:

$$S(1,1) = \frac{3S(0,1)_{odd} + 1}{2} \tag{7}$$

$$S(1,2) = \frac{S(0,1)_{even}}{2} \tag{8}$$

These sets are both again comprised of two strictly intertwined subsets, containing odd and even numbers, respectively:

$$S(1,1) = S(1,1)_{odd} \cup S(1,1)_{even} \tag{9}$$

$$S(1,2) = S(1,2)_{odd} \cup S(1,2)_{even} \tag{10}$$

The point density of the four subsets is identical:

$$r(S(1,1)_{odd}) = r(S(1,1)_{even}) = r(S(1,2)_{odd}) = r(S(1,2)_{even}) = 2^{-2} \tag{11}$$

This behavior continues analogously in each generation. In the n^{th} generation, there are 2^n sets $S(n,i)$. Each set consists of two strictly intertwined subsets $S(n,i)_{odd}$ and $S(n,i)_{even}$. Thus, the number of subsets containing odd integers and the number of subsets containing even integers is identical at 2^n in each generation, totalling in 2^{n+1} subsets. The point density in each of the 2^{n+1} subsets is identical at

$$r(S(n,i)_{odd}) = r(S(n,i)_{even}) = 2^{-(n+1)} \tag{12}$$

For the purpose of verifying the Collatz conjecture, the following properties will be important

(1) The zeroth generation (G_0) of the Collatz transformation corresponds to $\mathbb{N}$ and can also be denoted as a single set of zeroth generation, $S(0,1)$. $S(0,1)$ has the property to consist of two strictly intertwined subsets, $S(0,1)_{odd}$ and $S(0,1)_{even}$, of equal point density
(2) Applying the Collatz transformation to the first generation (G_1) results in two sets $S(1,1)$ and $S(1,2)$. The key property of $S(0,1)$, i.e. consisting of two strictly intertwined subsets of odd and even numbers at equal point density, is self-replicated in the daughter sets $S(1,1)$ and $S(1,2)$. In fact, the property is self-replicated further in all sets $S(n,i)$ of all following generations
(3) The number of increasing or decreasing steps a given set of initial numbers undergoes – and the order of the transformations – is not governed by chance but by the binomial distibution
(4) The trajectory of any x_0 is coupled to the trajectories of its neighbors. Due to decreasing point density (2^{-n}), the neighbors get more distant (by 2^n) with each generation transversed
(5) Alternatingly, odd numbers in $S(n,i)$ are transformed to an odd number in $S(n+1,i')$ and an even number in $S(n+1,i')$. Likewise, alternatingly, even numbers in $S(n,i)$ are transformed to an odd number in $S(n+1,i'')$ and an even number in $S(n+1,i'')$. Thus, the transformation alternatingly changes the parity of the elements of $S(n,i)$. More specifically, purely odd subsets $S(n,i)_{odd}$ are transformed into a new set $S(n+1,i')$ that is composed of strictly intertwined odd and even subsets $S(n+1,i')_{odd}$ and $S(n+1,i')_{even}$. Likewise, purely even subsets in $S(n,i)$ are transformed into a new set $S(n+1,i'')$ that is composed of strictly intertwined odd and even subsets $S(n+1,i'')_{odd}$ and $S(n+1,i'')_{even}$

3.3. **Proof.**

(1) A large upper bound u_{even} (for example 100, to start with) is selected and $S(0,1)$ is expressed as the interval U = $[1, u_{even}]$
(2) Any odd number $x_{odd} \in$ U is selected, along with its neighbor $x_{odd}+1$, which is an even number x_{even}
(3) The growth factor of the x_{odd} and x_{even} pair, when moving from the G_0 set $S(0,1)$ to the two G_1 sets $S(1,1)$ and $S(1,2)$, is calculated:

$$f_g = \left(\frac{\frac{3x_{\text{odd}}+1}{2}}{x_{\text{odd}}}\right)^{1/2} \left(\frac{\frac{x_{\text{even}}}{2}}{x_{\text{even}}}\right)^{1/2} = \left(\frac{3x_{\text{odd}}+1}{2x_{\text{odd}}}\right)^{1/2} \left(\frac{1}{2}\right)^{1/2} \tag{13}$$

(4) For the lower extreme case (i.e. $x_{odd} = 1$, $x_{even} = 2$), the growth factor is simplified as

$$f_g = 2^{1/2}\left(\frac{1}{2}\right)^{1/2} = 1 \tag{14}$$

This finding accurately represents the fact that the pair 1 and 2 makes up the only loop, as 1 gets transformed into 2 and 2 into 1, thus this pair does not show any growth or recession.

When selecting a higher x_{odd} value, the growth factor gradually decreases. For the upper extreme case (i.e. $x_{odd} = 99$, $x_{even} = 100$), the plus 1 term can be neglected and growth factor approaches a constant value, which is smaller than 1:

$$f_g \approx \left(\frac{3}{2}\right)^{1/2}\left(\frac{1}{2}\right)^{1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2} \approx 0.87 \tag{15}$$

(5) Because of the intertwined character of the G_0 subsets $S(0,1)_{odd}$ and $S(0,1)_{even}$ and their equal point density, any other x_{odd}/x_{even} neighbor pair can be chosen from set $S(0,1)$ and the mean growth when moving the pair to the two G_1 sets $S(1,1)$ and $S(1,2)$ is also within $f_g \in (\sqrt{3}/2, 1]$. In fact, this procedure can be repeated until all elements of set $S(0,1)$ have been moved to the sets $S(1,1)$ and $S(1,2)$. Thus, the growth of all pairs in G_0, when transformed to G_1, is within $f_g \in (\sqrt{3}/2, 1]$

(6) Because of the intertwined character of the G_1 subsets $S(1,1)_{odd}$, $S(1,1)_{even}$, $S(1,2)_{odd}$ $S(1,2)_{even}$ and their equal point density, again an x_{odd}/x_{even} neighbor pair can be chosen from one of the sets $S(1,1)$ or $S(1,2)$ and the mean growth when moving the pair to one of the four G_2 sets $S(2,1)$, $S(2,2)$, $S(2,3)$ and $S(2,4)$ is again within $f_g \in (\sqrt{3}/2, 1]$. As before, this procedure can be repeated until all pairs of the G_1 sets $S(1,1)$ and $S(1,2)$ have been moved to the four G_2 sets $S(2,1)$, $S(2,2)$, $S(2,3)$ and $S(2,4)$. Thus, the growth of all pairs in G_1, when transformed to G_2, is within $f_g \in (\sqrt{3}/2, 1]$

(7) This behavior is repeated in each generation, such that the growth of all pairs – i.e. of every odd and every even number – in G_n, when transformed to G_{n+1}, is within $f_g \in (\sqrt{3}/2, 1]$. This statement is true for very large u_{even}. In fact, there is no limit to u_{even}, such that it can be chosen arbitrarily high, ultimately resulting in $\lim_{u_{even}\to\infty} U = \mathbb{N}$

(8) All G_n consist of 2^n sets $S(n,i)$. These sets consist of strictly intertwined odd/even subsets. As such, the recession observed when going from G_0 to G_1, i.e. $f_g \in (\sqrt{3}/2, 1]$, analogously happens on all G_n to G_{n+1} transformations

(9) The combination of the linear nature of G_0 and the linear nature of the Collatz transformation leads to linear $S(n,i)$ sets. All elements of a given $S(n,i)$ are on a line $x_i = a_i x_0 + b_i$. Just as there are no exceptions (outliers) in $\mathbb{N}$ – i.e. the elements are all on the trivial line $x_{i=0} = x_0$ – there are no exceptions (outliers) in $S(n,i)$. Thus, there is no number in $S(n,i)$ that can escape the alternate parity change, occurring when going to the $(n+1)^{th}$ generation and thus no number than can escape the recession that is associated with each number pair

4. Summary and Outlook

The Collatz conjecture was shown to be true by taking advantage of the fact that all transformation subsets $S(n,i)$ consist of strictly intertwinded odd/even subsets, which undergo alternate parity change when going to the next generation. All odd/even pairs in these sets shrink. For large numbers, the recession occurs with $f_g \approx \sqrt{3}/2$. For small numbers, the recession is less pronounced and at the extreme case vanishes, with $f_g = 1$. This occurs once a pair is composed of 1 and 2.

In a next step, it would be interesting to apply the concept on other, generalized problems with slope scaling factors other than $3/2$ and $1/2$, and with intercept additive terms other than 1. For instance, by changing the odd-associated factor from 3 to 5, equation 13 can be used to demonstrate that the growth factor is within $f_g \in [\sqrt{5}/2, \sqrt{6}/2)$. As such, the whole interval is larger than 1, which means that the pair will always grow and the trajectories will diverge.

References

[1] J.C. Lagarias, *"The Ultimate Challenge: The 3x+1 Problem"*, American Mathematical Society, Providence RI, 2010.

[2] T. Tao *"Almost all orbits of the Collatz map attain almost bounded values"*, Forum of Mathematics, Pi, 2022, 10 (12), doi:10.1017/fmp.2022.8.

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