A Geometric Explanation of Dark Matter Based on General Relativity

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Abstract

Within the framework of standard general relativity[1], under the assumptions of staticity, spherical symmetry and the strong energy condition, we prove that a geometric transition zone — the "reverse-bending zone" — must appear in the periphery of any finite self-gravitating system, where the t–r sectional curvature changes sign from negative to positive. This zone is bounded by the curvature zero r_0, the curvature peak r_{peak}, and the matter boundary R; in the interval (r_0, R) the sectional curvature smoothly transforms from matter-dominated spherical compression to vacuum saddle-shaped stretching. The reverse-bending zone is not a free vacuum but a forced geometry locked jointly by the interior baryonic potential well and the far-field boundary condition. Within this zone, the Misner–Sharp-type gravitational mass $M(r)$ continues to grow: it grows faster than linearly in the region $r_0 \to r_{\text{peak}}$, and although the growth slows down in the region $r_{\text{peak}} \to R$, it never ceases. The resulting geometric Weyl stretching together with the self-energy of the gravitational field provide an extra centripetal acceleration, which naturally manifests itself, in the weak-field approximation, as an approximately logarithmic potential and a flattening of the rotation curves. The theoretical sectional curvature formula precisely reproduces the Friedmann acceleration equation, the Newtonian radial tidal formula (and, by integration, the Newtonian gravitational acceleration), the TOV equation, and the Schwarzschild solution, among others — not approximately, not asymptotically, but exactly in their respective limits.It yields parameter-free, falsifiable predictions that can be directly tested with existing rotation-curve and photometric data. This result demonstrates that, without introducing new particles or modifying the field equations, forced geometry within general relativity alone can generate a "dark-matter-like" gravitational effect on galactic scales.

Keywords: general relativity, dark matter, sectional curvature, continuous metric, reverse-bending zone

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1 Introduction

The nature of dark matter is one of the deepest unsolved mysteries in fundamental physics. The standard ΛCDM paradigm attributes the phenomenon to unknown particle species. However, after decades of intense experimental searches, dark-matter particles have not been detected directly. Moreover, all the various theoretical explanations for dark matter have so far not been accepted by the scientific community as a perfect description.

In this paper we find an alternative and understandable path to the problem from within relativity. We prove that, without introducing any new physical entity or free parameter, the mathematical structure of general relativity already contains a geometric mechanism capable of producing the observational features of dark matter[6]. The key point is that we use standard relativistic methods to demonstrate that, when the spacetime curvature around a matter aggregation transitions to the far field, a saddle-shaped geometric transition zone necessarily forms in the spacetime outside the matter. This saddle-shaped spacetime geometry is precisely the geometric origin of the "dark-matter" effect.For standard references on general relativity, see [2, 3, 5].

2 Geometric Transition Theorem

2.1 Prerequisites

We consider a static, spherically symmetric, self-gravitating bound system described by Einstein's general relativity, satisfying the following physically realizable and mathematically self-consistent conditions:

1. **Finite, smooth distribution of matter.** The energy-momentum tensor $T_{\mu\nu}$ is confined within a sphere of coordinate radius $r = R$. At the boundary $r = R$, the energy density $\rho(r)$ and radial pressure $p(r)$ satisfy

$$\lim_{r\to R^-} \rho(r) = 0, \qquad \lim_{r\to R^-} p(r) = 0,$$

   and $\rho(r), p(r) \in C^1[0, R]$ (first-order derivatives continuous). The system possesses a finite positive ADM mass $M > 0$, consistent with the integrated interior mass:

$$M = \lim_{r\to\infty} \frac{c^2 r}{2G}\left(1 - e^{-2\lambda(r)}\right) = \int_0^\infty 4\pi r^2 \rho(r)\, dr.$$

   The equation of state $p = p(\rho)$ satisfies the causality condition $0 \le dp/d\rho \le c^2$. This theorem applies to self-gravitating bound systems composed of normal matter (stars, galaxies, galaxy clusters), and excludes systems that contain spacetime singularities, such as black holes or naked singularities. For systems like galaxies, whose evolutionary timescale is much longer than the dynamical timescale, the static and spherical symmetry assumption is a good approximation.

2. **Strong energy condition.** For all timelike vector fields u^μ, the energy-momentum tensor satisfies

$$T_{\mu\nu} u^\mu u^\nu \ge \frac{1}{2} T^\lambda_{\ \lambda} u^\sigma u_\sigma.$$

For a static spherically symmetric ideal fluid (the most general form of matter compatible with spherical symmetry), this condition is equivalent to

$$\rho c^2 + 3p \geq 0.$$

The strong energy condition guarantees the attractive nature of gravity. All known macroscopic normal matter (stars, gas, plasma) strictly satisfies this condition.

3. **Regularity of spacetime.** The spacetime metric tensor $g_{\mu\nu} \in C^2(\mathcal{M})$, i.e., the metric and its first and second derivatives are continuous throughout the entire manifold $\mathcal{M}$. This ensures that the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ is globally well-defined and all its components are bounded, ruling out the unphysical situation of a singular thin shell of curvature (which would correspond to infinite stress) at the boundary $r = R$.

4. **Exterior vacuum.** In the region $r > R$, $T_{\mu\nu} = 0$, and the spacetime satisfies the vacuum Einstein field equations with a cosmological constant $\Lambda \geq 0$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor and Λ has the dimension of m^{-2}.

5. **Asymptotic behaviour.** As $r \to \infty$, the spacetime tends asymptotically to Minkowski spacetime (if $\Lambda = 0$) or to de Sitter spacetime (if $\Lambda > 0$).

2.2 Statement of the Theorem

Theorem 2.1 (Geometric Transition Theorem)**.** *Under the above premises, there necessarily exists a* ***unique critical radius*** r_0 *inside the sphere, satisfying* $0 < r_0 < R$*, such that the sectional curvature of the* t*-*r *plane in an orthonormal frame,*

$$K_{tr}(r) \equiv R_{\hat{t}\hat{r}\hat{t}\hat{r}}(r),$$

vanishes at that point, i.e., $K_{tr}(r_0) = 0$*. Moreover:*

- *In the region* $0 < r < r_0$, $K_{tr}(r) < 0$*, corresponding to* ***spherical curvature*** *(radial compression, characteristic of a matter-dominated gravitational potential well);*

- *In the region* $r_0 < r < R$, $K_{tr}(r) > 0$*, corresponding to* ***saddle-shaped curvature*** *(radial stretching, an intrinsic feature of the Schwarzschild vacuum geometry).*

A neighbourhood $\mathcal{T} = (r_0 - \delta, r_0 + \delta)$ *containing this zero constitutes the* ***geometric transition zone*** *(reverse-bending zone). Its core geometric feature is that the spacetime curvature smoothly transforms from purely spherical bending induced by matter to the saddle-shaped bending inherent to the vacuum — precisely the mathematical expression of anti-elastic bending in elasticity theory.*

Physical clarification: Although mathematically the reverse-bending zone lies inside the matter boundary, physically it is already a geometrically dominated region; this is precisely the region that traditional post-Newtonian approximations treat as a free vacuum (detailed in Section 4.1).

3 Proof of the Theorem

3.1 Metric, tetrad and sectional curvature

Adopt the most general static spherically symmetric line element (in SI units, explicitly retaining c):

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $\Phi(r), \lambda(r) \in C^2[0,\infty)$ are real-valued functions depending only on the radial coordinate r.

Introduce an orthonormal tetrad adapted to the metric:

$$\mathbf{e}_{\hat{t}} = e^{-\Phi(r)}c^{-1}\partial_t, \quad \mathbf{e}_{\hat{r}} = e^{-\lambda(r)}\partial_r, \quad \mathbf{e}_{\hat{\theta}} = r^{-1}\partial_\theta, \quad \mathbf{e}_{\hat{\phi}} = (r\sin\theta)^{-1}\partial_\phi, \tag{2}$$

with the dual basis:

$$\omega^{\hat{t}} = e^{\Phi(r)}c\,dt, \quad \omega^{\hat{r}} = e^{\lambda(r)}dr, \quad \omega^{\hat{\theta}} = r\,d\theta, \quad \omega^{\hat{\phi}} = r\sin\theta\,d\phi. \tag{3}$$

In this frame, the metric reduces to the Minkowski metric: $g = -\omega^{\hat{t}}\otimes\omega^{\hat{t}} + \omega^{\hat{r}}\otimes\omega^{\hat{r}} + \omega^{\hat{\theta}}\otimes\omega^{\hat{\theta}} + \omega^{\hat{\phi}}\otimes\omega^{\hat{\phi}}$.

From the first Cartan structure equation[3, 4] $d\omega^{\hat{\mu}} = -\omega^{\hat{\mu}}{}_{\hat{\nu}}\wedge\omega^{\hat{\nu}}$, the non-vanishing connection 1-forms are:

$$\omega^{\hat{t}}{}_{\hat{r}} = \omega^{\hat{r}}{}_{\hat{t}} = e^{-\lambda}\Phi'\,\omega^{\hat{t}}, \quad \omega^{\hat{\theta}}{}_{\hat{r}} = -\omega^{\hat{r}}{}_{\hat{\theta}} = \frac{e^{-\lambda}}{r}\omega^{\hat{\theta}}, \quad \omega^{\hat{\phi}}{}_{\hat{r}} = -\omega^{\hat{r}}{}_{\hat{\phi}} = \frac{e^{-\lambda}}{r}\omega^{\hat{\phi}}, \quad \omega^{\hat{\phi}}{}_{\hat{\theta}} = -\omega^{\hat{\theta}}{}_{\hat{\phi}} = \frac{\cot\theta}{r}\omega^{\hat{\phi}}. \tag{4}$$

Here and below, a prime denotes differentiation with respect to the radial coordinate r.

From the second Cartan structure equation $\Omega^{\hat{\mu}}{}_{\hat{\nu}} = d\omega^{\hat{\mu}}{}_{\hat{\nu}} + \omega^{\hat{\mu}}{}_{\hat{\lambda}}\wedge\omega^{\hat{\lambda}}{}_{\hat{\nu}}$, the curvature 2-form of the t-r plane is:

$$\Omega^{\hat{t}}{}_{\hat{r}} = d\omega^{\hat{t}}{}_{\hat{r}} + \omega^{\hat{t}}{}_{\hat{\lambda}}\wedge\omega^{\hat{\lambda}}{}_{\hat{r}} = -e^{-2\lambda}\Big[\Phi'' + (\Phi')^2 - \Phi'\lambda'\Big]\,\omega^{\hat{t}}\wedge\omega^{\hat{r}}. \tag{5}$$

The Riemann curvature tensor and the curvature 2-form are related by $\Omega^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\,\omega^{\hat{\rho}}\wedge\omega^{\hat{\sigma}}$. Comparing with (5) and using the metric signature convention $g_{\hat{t}\hat{t}} = -1$, we obtain:

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = g_{\hat{t}\hat{t}}R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}. \tag{6}$$

Hence, the exact expression for the sectional curvature is:

$$\boxed{K_{tr}(r) \equiv -R_{\hat{t}\hat{r}\hat{t}\hat{r}}(r) = -e^{-2\lambda(r)}\Big[\Phi''(r) + \big(\Phi'(r)\big)^2 - \Phi'(r)\lambda'(r)\Big]}. \tag{7}$$

This expression is a purely differential-geometric result that depends solely on the metric functions and holds universally for all static spherically symmetric spacetimes. For the self-gravitating bound system studied in this paper, the matter source is described as an isotropic ideal fluid. In the orthonormal frame (2) adapted to the metric, its energy-momentum tensor takes the diagonal form

$$T_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(\rho c^2,\, p,\, p,\, p), \tag{8}$$

where $\rho(r)$ is the proper energy density and $p(r)$ is the isotropic pressure.

3.2 Exact algebraic identity based on the field equations

In order to rigorously analyse the sign change of $K_{tr}(r)$, we start directly from the Einstein field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ and derive their exact algebraic expression **without neglecting the cosmological constant** Λ.

For an ideal fluid (or effective fluid) matter source, the Einstein equations in the orthonormal frame yield:

$$\frac{2}{r}e^{-2\lambda}\lambda' + \frac{1}{r^2}(1 - e^{-2\lambda}) - \Lambda = \frac{8\pi G}{c^2}\rho, \tag{9}$$

$$\frac{2}{r}e^{-2\lambda}\Phi' - \frac{1}{r^2}(1 - e^{-2\lambda}) + \Lambda = \frac{8\pi G}{c^4}p, \tag{10}$$

$$e^{-2\lambda}\left[\Phi'' + (\Phi')^2 - \Phi'\lambda' + \frac{\Phi' - \lambda'}{r}\right] + \Lambda = \frac{8\pi G}{c^4}p. \tag{11}$$

Define the interior mass $M(r) = \int_0^r 4\pi r'^2 \rho(r')dr'$, which satisfies

$$e^{-2\lambda(r)} = 1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda}{3}r^2. \tag{12}$$

The geometric definition of the sectional curvature is $K_{tr}(r) \equiv R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -e^{-2\lambda}\big[\Phi'' + (\Phi')^2 - \Phi'\lambda'\big]$. Eliminating the second-derivative term using the angular equation (11), we obtain

$$K_{tr} = -\frac{8\pi G}{c^4}p + \Lambda + \frac{e^{-2\lambda}}{r}(\Phi' - \lambda'). \tag{13}$$

Solving (9) and (10) for λ' and Φ' respectively gives

$$\lambda' = e^{2\lambda}\left[\frac{4\pi G}{c^2}\rho r - \frac{GM}{c^2 r^2} + \frac{\Lambda}{3}r\right], \qquad \Phi' = e^{2\lambda}\left[\frac{GM}{c^2 r^2} + \frac{4\pi G}{c^4}pr - \frac{\Lambda}{3}r\right].$$

Subtracting the two equations yields

$$\Phi' - \lambda' = e^{2\lambda}\left[\frac{2GM}{c^2 r^2} + \frac{4\pi G}{c^4}pr - \frac{4\pi G}{c^2}\rho r - \frac{2\Lambda}{3}r\right].$$

Substituting this back into (13) and simplifying algebraically (the Λ terms cancel exactly) yields the **exact algebraic identity**:

$$\boxed{K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}}. \tag{14}$$

Physical interpretation:

- The first term $\dfrac{2GM(r)}{c^2 r^3}$ represents the non-local contribution from the interior mass; in the vacuum region $r > R$ it is the Weyl stretching (vacuum tide).
- The second term $-\dfrac{4\pi G}{c^2}\left(\rho + \dfrac{p}{c^2}\right)$ represents the Ricci compression due to the local **active gravitational mass density** (note the coefficient 1 in front of the pressure term, consistent with standard general relativity).

- The third term $\frac{\Lambda}{3}$ is a uniform background curvature provided by the cosmological constant. Inside a galaxy (small r) this term is dozens of orders of magnitude smaller than the matter terms and can be completely neglected; however, in the far field it ensures that the curvature tends to the de Sitter constant curvature, which is crucial for the smooth junction between interior and exterior.

Note: Equation (14) is an exact relativistic identity without any omission, valid throughout the entire space — uniformly applicable in the interior, at the matter boundary, and in the exterior vacuum. On galactic scales, pressure enters the curvature in the form of the equivalent mass density p/c^2. Since the typical gas velocity in galaxies is far below the speed of light, $p \sim \rho v^2$, one has

$$\frac{p}{c^2} \sim \rho \left(\frac{v}{c}\right)^2,$$

which is merely a second-order relativistic small quantity compared to the density ($\sim 10^{-6}\rho$). The solitary factor c^2 in its denominator is sufficient to demonstrate that its contribution is negligible relative to the density term. As for the cosmological-constant term $\Lambda/3$, its intrinsic curvature scale ($\sim 10^{-52}\,\mathrm{m}^{-2}$) is dozens of orders of magnitude smaller than the typical matter-induced curvature inside a galaxy ($\sim 10^{-43}\,\mathrm{m}^{-2}$), and is likewise entirely negligible.

3.3 Sign of the sectional curvature and existence of the zero

From the exact formula (14) we can rigorously determine the sign of $K_{tr}(r)$ at the centre and in the far field, and prove the existence of a zero.

Compression feature at the centre: As $r \to 0$, $M(r) \approx \frac{4}{3}\pi r^3 \rho(0)$, p is finite, and the $\Lambda/3$ term is negligible compared with the matter terms. Substituting into (14):

$$K_{tr}(0) \approx \frac{8\pi G}{3c^2}\rho(0) - \frac{4\pi G}{c^2}\left(\rho(0) + \frac{p(0)}{c^2}\right) = -\frac{4\pi G}{3c^2}\left(\rho(0) + \frac{3p(0)}{c^2}\right) < 0.$$

As long as the strong energy condition holds, the sectional curvature at the centre is strictly negative, corresponding to radial compression.

Positive curvature feature of the exterior vacuum: At the matter boundary $r = R$, the prerequisites give $\rho(R) = p(R) = 0$ and $M(R) = M_{\text{total}} > 0$. Substituting into (14) yields

$$K_{tr}(R) = \frac{2GM_{\text{total}}}{c^2R^3} + \frac{\Lambda}{3} > 0.$$

Thus, the sectional curvature is already strictly positive at the matter boundary. By Birkhoff's theorem[4] , the exterior vacuum region $r > R$ is uniquely described by the Schwarzschild–de Sitter solution, and $K_{tr}(r) > 0$ is precisely an intrinsic feature of that solution.

Existence and physical location of the zero: Because the matter distribution is C^2 smooth, $M(r)$, $\rho(r)$ and $p(r)$ are continuous functions; therefore $K_{tr}(r)$ is continuous in the whole space. It is negative at $r = 0$ ($K_{tr}(0) < 0$) and positive at $r = R$ ($K_{tr}(R) > 0$). By the Intermediate Value Theorem for continuous functions, there must exist at least one zero $r_0 \in (0, R)$ such that $K_{tr}(r_0) = 0$.

Setting (14) to zero, and using the fact that in the outer part of a galaxy the pressure is negligible ($p \approx 0$) and the $\Lambda/3$ term is far smaller than the matter term on this scale, the zero position is determined by the equation

$$\frac{2GM(r_0)}{c^2 r_0^3} = \frac{4\pi G}{c^2}\rho(r_0) \quad \Longrightarrow \quad \frac{M(r_0)}{\frac{4}{3}\pi r_0^3} = \frac{3}{2}\rho(r_0).$$

Defining the mean mass density inside r_0 as $\bar{\rho}(r_0) = M(r_0)/(\frac{4}{3}\pi r_0^3)$, the zero condition can be written concisely as

$$\boxed{\bar{\rho}(r_0) = 1.5\,\rho(r_0)}. \tag{15}$$

For a three-dimensional spherically symmetric exponential volume density distribution $\rho(r) = \rho_0 e^{-r/h}$[7] ,substituting the cumulative mass $M(r) = 4\pi\rho_0 h^3\big[2 - e^{-r/h}(r^2/h^2 + 2r/h + 2)\big]$ into $\bar{\rho}(r_0) = 1.5\rho(r_0)$, and letting $x_0 \equiv r_0/h$, elimination of ρ_0 yields the transcendental equation

$$4e^{x_0} = x_0^3 + 2x_0^2 + 4x_0 + 4. \tag{16}$$

Numerical solution gives $x_0 \approx 1.441$, hence

$$\boxed{r_0 \approx 1.44\,h}. \tag{17}$$

This result is derived solely from the zero condition of the sectional curvature in general relativity, without any free parameters.

This relation shows that the reverse-bending curvature zero appears exactly at the radius where "the interior mean density equals 1.5 times the local density". For common density profiles that are single-peaked, monotonically decreasing radially and without oscillatory tails (such as the exponential disc or power-law profiles of galaxies): near the centre $M(r) \approx \frac{4}{3}\pi r^3\rho(0)$, so $\bar{\rho} \approx \rho$, and the ratio $\bar{\rho}/\rho \approx 1 < 1.5$; as r increases, the local density ρ decays faster than the mean density $\bar{\rho}$, so the ratio $\bar{\rho}/\rho$ increases strictly monotonically, and at the matter boundary $r = R$ we have $\rho(R) = 0$ but $\bar{\rho}(R) > 0$, so the ratio diverges to $+\infty$. By the Intermediate Value Theorem, there exists a unique r_0 satisfying $\bar{\rho}(r_0) = 1.5\rho(r_0)$.

It is worth emphasising that the reverse-bending zero r_0 is the starting point of the reverse-bending zone, not the starting point of the Schwarzschild vacuum. In the Schwarzschild vacuum at finite radius the sectional curvature is strictly positive ($K_{tr} = 2GM/(c^2r^3) > 0$), whereas here $K_{tr}(r_0) = 0$. If one tried to match the Schwarzschild solution directly at r_0, the curvature would jump from zero to a positive value, violating the C^2 continuity requirement of the metric.

3.4 Peak of the sectional curvature in the reverse-bending zone

Theorem 2.1 and the exact expression (14) establish that $K_{tr}(r)$ in the reverse-bending zone changes from negative to positive at r_0, remains positive at the matter boundary $r = R$, and in the far field decays to the de Sitter constant curvature background $\Lambda/3$. This means that $K_{tr}(r)$ starts from zero, rises in some interval, reaches a positive maximum, then decreases and eventually tends to the far-field vacuum value.

Under the conditions of Theorem 2.1, let $K_{tr}(r) \in C^1[r_0, \infty)$ satisfy $K_{tr}(r_0) = 0$, $K_{tr}(r) > 0$ for all $r > r_0$, and $\lim_{r\to\infty} K_{tr}(r) = \Lambda/3$. Choose $R_1 > R$ such that for $r > R_1$, $|K_{tr}(r) - \Lambda/3| < \varepsilon$, and take ε sufficiently small so that $\Lambda/3 < K_{tr}(R)$. The continuous function K_{tr} on the closed interval $[r_0, R_1]$ must attain a maximum, and this maximum cannot be at r_0 (because $K_{tr}(r_0) = 0$) nor at infinity (because the limit is $\Lambda/3 < K_{tr}(R)$); therefore there exists at least one interior maximum point $r_{\text{peak}} \in (r_0, R_1)$ such that

$$K_{tr}(r_{\text{peak}}) = \max_{r \geq r_0} K_{tr}(r) > 0, \qquad K'_{tr}(r_{\text{peak}}) = 0.$$

To further confirm that K_{tr} indeed first rises to the right of r_0, one can directly examine the derivative at the zero. Using the approximation $K_{tr} \approx 2GM/(c^2r^3) - 4\pi G\rho/c^2$ in the outer part of the galaxy and the zero condition $\bar{\rho}(r_0) = 1.5\rho(r_0)$, we obtain

$$K'_{tr}(r_0) = -\frac{4\pi G}{c^2}\Big(\frac{\rho(r_0)}{r_0} + \rho'(r_0)\Big).$$

For a radially monotonically decreasing density distribution, $\rho'(r_0) < 0$, and when $r_0 > h$ we have $|\rho'| > \rho/r_0$; thus $K'_{tr}(r_0) > 0$, and the curvature immediately rises on the outer side of the zero. This guarantees the existence of a rising segment, which together with the far-field decay forces at least one interior maximum.

To determine the specific position of the peak, we use the approximation that the pressure in the outer part of the galaxy is negligible ($p \approx 0$) and the $\Lambda/3$ term is far smaller than the matter term on this scale, simplifying the exact expression (14) to

$$K_{tr}(r) \approx \frac{2GM(r)}{c^2r^3} - \frac{4\pi G}{c^2}\rho(r). \tag{18}$$

Differentiating with respect to r and setting the derivative to zero, using $M'(r) = 4\pi r^2\rho(r)$, gives the general equation satisfied by the peak position r_{peak}:

$$\boxed{3M(r_{\text{peak}}) = 4\pi r_{\text{peak}}^3\, \rho(r_{\text{peak}}) - 2\pi r_{\text{peak}}^4\, \rho'(r_{\text{peak}})}. \tag{19}$$

This equation contains no free parameters and applies universally to any static spherically symmetric density distribution.

Now take the volume density of the galactic visible matter as a three-dimensional spherically symmetric exponential distribution $\rho(r) = \rho_0 e^{-r/h}$ [7](h being the radial scale length), with cumulative mass

$$M(r) = 4\pi\rho_0 h^3\Big[2 - e^{-r/h}\Big(\frac{r^2}{h^2} + \frac{2r}{h} + 2\Big)\Big].$$

Letting $x \equiv r/h$ and substituting into Eq. (19), after eliminating ρ_0 and simplifying we obtain the transcendental equation for x:

$$\boxed{12e^x = x^4 + 2x^3 + 6x^2 + 12x + 12}. \tag{20}$$

This equation has only one non-trivial real root in the range $x > 0$ (the other root is $x = 0$, corresponding to the degenerate centre solution). Solving numerically with Newton's method yields

$$x_{\text{peak}} \approx 2.8413 \quad \Longrightarrow \quad \boxed{r_{\text{peak}} \approx 2.84\, h}. \tag{21}$$

Non-Schwarzschild matching feature at the peak. The above analysis reveals a crucial geometric fact about the reverse-bending zone: at the curvature peak r_{peak}, $K'_{tr}(r_{\text{peak}}) = 0$. If one attempted to match the interior solution directly to the

Schwarzschild vacuum at this point, the exterior curvature $K_{tr}^{\mathrm{Sch}} = 2GM/(c^2r^3)$ would have the derivative $K_{tr}'^{\mathrm{Sch}} = -6GM/(c^2r^4) < 0$ (for all finite r), which contradicts the peak condition $K'_{tr}(r_{\mathrm{peak}}) = 0$. From the basic requirement of continuity we have $0 < r_0 < r_{\mathrm{peak}} < R$, i.e. the peak point must lie inside the matter boundary (R). Therefore, under the premises of staticity, spherical symmetry and a C^2 metric, one cannot smoothly join the interior forced geometry to the exterior free Schwarzschild vacuum at r_{peak}. In summary, geometrically the reverse-bending zone is neither purely interior baryon-dominated nor exterior free vacuum; it must complete a smooth transition from growth to decay within a finite extended interval, with a decay rate that is overall slower than $1/r^3$ in order to achieve a smooth connection of the curvature and its derivative. This will be elaborated in Section 4.

3.5 Transition to the exterior Schwarzschild–de Sitter vacuum

Beyond the curvature peak r_{peak}, as the radius continues to increase, the local matter density $\rho(r)$ and pressure $p(r)$ decay exponentially (or faster), and the growth of the cumulative mass $M(r)$ saturates, gradually approaching the total mass of the system M_{total}. From the exact formula (14)

$$K_{tr}(r) = \frac{2GM(r)}{c^2r^3} - \frac{4\pi G}{c^2}\Big(\rho(r) + \frac{p(r)}{c^2}\Big) + \frac{\Lambda}{3}$$

we see that the second term (local Ricci compression) vanishes together with the matter density, the first term (Weyl stretching) decays as M_{total}/r^3, and the third term $\Lambda/3$ remains as a uniform background curvature. Hence $K_{tr}(r)$ decreases smoothly from the peak and approaches the pure vacuum value as one nears the matter boundary $r = R$.

According to the prerequisites of Theorem 2.1, at the boundary $r = R$ we have

$$\rho(R) = p(R) = 0, \qquad M(R) = M_{\mathrm{total}} > 0.$$

Substituting into (14) immediately gives

$$K_{tr}(R^-) = \frac{2GM_{\mathrm{total}}}{c^2R^3} + \frac{\Lambda}{3}.$$

In the exterior region $r > R$, the matter energy-momentum tensor $T_{\mu\nu} = 0$, and the spacetime satisfies the vacuum Einstein equations with the cosmological constant Λ. By Birkhoff's theorem generalised to include Λ, the static spherically symmetric vacuum solution is uniquely described by the Schwarzschild–de Sitter (Kottler) metric, with the exterior total mass parameter exactly equal to M_{total}, and the t-r sectional curvature is

$$K_{tr}(R^+) = \frac{2GM_{\mathrm{total}}}{c^2R^3} + \frac{\Lambda}{3}.$$

Therefore $K_{tr}(R^-) = K_{tr}(R^+)$, and the sectional curvature is **automatically continuous** at the matter boundary, without the need for any additional matching conditions or thin-shell constructions.

This fact has a fundamental physical significance: Eq. (14), as an exact algebraic identity of the Einstein field equations, already **threads through the entire spacetime**. The interior forced geometry and the exterior free vacuum are not two different solutions that need to be stitched together, but rather continuous manifestations of the same formula on the two sides of the natural matter boundary.

The physical meaning of the boundary R is the dividing line between the interior forced geometry and the exterior free vacuum. At the end of the reverse-bending zone, the exterior vacuum description takes over. Thus, starting from the negative curvature (baryonic compression) at the galactic centre, passing through the curvature zero r_0, the reverse-bending rising segment, the curvature peak r_{peak}, the transition zone, and finally reaching the exterior Schwarzschild–de Sitter vacuum, the evolution of the sectional curvature of the entire spacetime is completely described by the single unified exact formula (14). This closed loop constitutes the rigorous general-relativistic foundation for the geometric explanation of dark matter.

3.6 Geometric transition zone and its physical significance

The above proof using the standard spherically symmetric model establishes that, from the centre to infinity, $K_{tr}(r)$ inevitably undergoes the process "negative (spherical compression) → crosses zero → positive (saddle-shaped stretching)". The critical radius r_0 is the exact position of the curvature sign reversal; the interval from there outward up to the matter boundary R, (r_0, R), constitutes the **geometric transition zone** — that is, the **reverse-bending zone**. In this region, $K_{tr}(r) > 0$, and spacetime exhibits saddle-shaped curvature, but this stretching is not the free Schwarzschild stretching of the exterior vacuum at $r > R$; rather, it is a **forced geometry** locked jointly by the interior matter distribution and the far-field asymptotically flat condition.

Hierarchy one: existence proof. The spherically symmetric geometry acts as a "detection tool". It uses a known, rigorous geometric structure to prove that, for a given matter distribution and vacuum boundary conditions, a curvature zero necessarily exists inside. This step requires no new physics and is entirely an internal deduction of standard general relativity.

Hierarchy two: properties of the reverse-bending zone. With the zero proven to exist, the reverse-bending zone is established as a geometric entity. However, whether the metric form and gravitational effects in this zone still obey the logic of the standard vacuum solution is a separate question. The Schwarzschild metric proves that r_0 necessarily exists and $r_0 < R$, but that does not mean the Schwarzschild metric correctly describes the reverse-bending zone. Although the reverse-bending zone is not a vacuum ($\rho \neq 0$), the local matter density is already extremely low, its Ricci compression contribution is negligible, and the geometric effect is completely dominated by the non-local Weyl stretching produced by the interior mass. (A detailed argument is given in Section 4.)

On the universality of the reverse-bending zone: The five conditions on which the proof of Theorem 2.1 depends form a set of sufficient conditions for the existence of the reverse-bending zone, not necessary conditions. The spherically symmetric results show that the reverse-bending mechanism does not rely on any special microscopic assumptions, but stems from the combined action of a finite matter distribution, the exterior vacuum boundary, and geometric regularity. This suggests that the mechanism can be extended to more general self-gravitating systems, although its rigorous non-spherically symmetric form awaits future investigation.

The role of the spherical symmetry assumption: It is not a prerequisite for the existence of the reverse-bending zone, but rather a way to convert the general physical conditions into an exactly solvable mathematical model. Under spherical symmetry, the sectional curvature K_{tr} happens to correspond to an eigenvalue of the tidal force, and the position of its zero can be analytically determined as $\bar{\rho}(r_0) = 1.5\,\rho(r_0)$. For realistic non-spherically symmetric galaxies, the reverse-bending zone still exists, but its precise geometric form requires solving the corresponding non-linear boundary value problem, which will be a direction for future numerical work. Spherical symmetry in general relativity has never been an exact description of reality, but a standard methodological tool for uncovering universal physical mechanisms. All cornerstone results of general relativity — Birkhoff's theorem, the Schwarzschild solution, the Oppenheimer-Volkoff equation, Friedmann cosmology, etc. — are built on the assumption of spherical symmetry, yet this in no way diminishes their universality.

4 Forced vacuum geometry and the emergence of dark-matter effects

The geometric transition theorem of Section 3 rigorously proves that a geometric transition zone (reverse-bending zone) with curvature sign reversal exists in the periphery of the visible matter, with the zero r_0 located at $\bar{\rho}(r_0) = 1.5\,\rho(r_0)$. In this section we demonstrate that, once this inner zero-curvature boundary is crossed, the spacetime metric enters a forced geometric state constrained jointly by the interior integrated mass and the exterior boundary. In its weakest form this state causes the effective mass parameter in the tail to deviate from the bare baryonic extrapolation, and when further satisfying a self-similarity condition in the mature zone it can evolve into a scale-invariant logarithmic potential zone, giving rise to flat rotation curves and the associated scaling relations.

4.1 Forced geometry: the range of applicability of the post-Newtonian approximation and Birkhoff's theorem

In astrophysical research, when analysing gravitational effects in the outer parts of galaxies, the post-Newtonian approximation is commonly adopted, and it is assumed by default that spacetime reduces to the Schwarzschild vacuum described by Birkhoff's theorem.

According to the spherically symmetric derivation, in the reverse-bending region $0 < r_0 < r_{\text{peak}} < R$, the curvature zero appears exactly at the radius where "the interior mean density equals 1.5 times the local density", and the peak r_{peak} lies inside the matter boundary R. This result indicates that the low-density outer part of a galaxy belongs geometrically to a forced transition zone: its spacetime metric is neither dominated solely by the local tenuous matter, nor does it obey the free Schwarzschild vacuum form, but is constrained jointly by the interior baryonic potential well and the far-field boundary condition. In this region the local matter density can be extremely low, yet its spacetime curvature and the non-linear geometric self-energy contributions may not be negligible.

Birkhoff's theorem is strictly valid in spherically symmetric source-free regions where the matter density is exactly zero. In contrast, the dynamical information from the outer part of a galaxy comes from tracer objects embedded in that region; as far as the system attribution is concerned, this region still belongs to the self-gravitating matter envelope, rather than being an independent vacuum exterior completely detached from the central matter. Therefore, treating it entirely as a strictly source-free region in the sense of Birkhoff's theorem can at best be regarded as an approximate treatment. Especially in the reverse-bending zone, where the curvature has already undergone a reorganisation from negative to positive sign, the geometric structure is still controlled by the interior mass distribution and non-linear boundary conditions. Hence, Birkhoff's theorem applies to the strictly vacuum exterior region outside the reverse-bending zone, but cannot serve as a direct substitute for the geometric structure inside the reverse-bending transition layer.

4.2 Physical effects of the forced geometry in the reverse-bending zone: the emergence of dark-matter phenomena

4.2.1 Dominance of the Weyl tensor and the self-energy of the gravitational field

The most central non-linear essence of general relativity is: **the gravitational field itself carries energy, and the energy of the gravitational field also produces gravity**.

We have already shown above that the reverse-bending zone is geometrically constrained by three points — the zero r_0, the peak r_{peak}, and the matter boundary R — thereby forming a forced geometric transition layer. Even under the most conservative treatment — matching to the Schwarzschild vacuum immediately after the curvature reaches its positive peak r_{peak} — the first half of the reverse-bending enhancement zone is ineliminable. In this zone the local baryon density has already been continuously decreasing, yet the radial sectional curvature K_{tr} rises from zero to its maximum positive value $K_{\max}$, which means that the gravitational behaviour here is no longer controlled by the local Ricci source term alone, but must be understood as the joint result of the overall integrated mass, the boundary conditions, and geometric non-linearity. Therefore, even this first half of the reverse-bending alone is sufficient to produce additional gravitational effects that deviate from the Newtonian expectation.

According to the expression for the sectional curvature (14)

$$K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3},$$

when matched to the Schwarzschild-type tail after $r = r_{\text{peak}}$, the mass parameter of the latter must satisfy

$$K_{\max} = \frac{2GM_{\text{peak}}}{c^2 r_{\text{peak}}^3} \quad \Rightarrow \quad M_{\text{peak}} = \frac{c^2 r_{\text{peak}}^3}{2G} K_{\max}, \tag{22}$$

yielding the conclusion that even if the tail still has the form $K_{tr} \propto r^{-3}$, the "mass parameter" M_{peak} that it decays from is no longer determined solely by the bare baryonic mass inside r_{peak}, but is determined jointly by the curvature peak built up in the first half of the reverse-bending enhancement zone through the matching condition. In dynamical inversion, this manifests itself as an effective mass parameter that may deviate significantly from the extrapolated bare baryonic value. This conclusion relies on staticity, spherical symmetry and the weak-field approximation, but does not depend on whether there exists a long-range logarithmic tail beyond the peak, thus constituting a robust conclusion under minimal assumptions.

4.2.2 Definition and qualitative analysis of the total mass in the reverse-bending zone

The $M(r)$ defined in Eq. (12) of this paper is fully consistent with the standard *Misner–Sharp* quasi-local mass definition in static spherically symmetric spacetime. Because this term is mathematically exactly equivalent to the standard Misner–Sharp mass, it completely captures the contributions of all forms of gravitational sources within radius r, a conclusion widely accepted in the physics community. In view of the fact that Eq. (14) exhibits an exact algebraic coupling between $M(r)$ and the sectional curvature K_{tr}, in the reverse-bending zone the growth of $M(r)$ no longer follows the linear extrapolation of the bare baryonic matter, but is an inevitable consequence of the non-linear superposition of the baryon distribution and the geometric self-energy of the gravitational field. To facilitate distinction, we give it a new name in the reverse-bending zone, defining it as the "gravitational mass". The essence behind this name remains unchanged, and is meant to emphasise that, in the forced geometric transition zone, the depth of the gravitational potential well is supported jointly by matter and spacetime curvature. According to the exact algebraic identity (14) from Section 3.2:

$$K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}.$$

Solving for the non-local mass term $M(r)$, we define the gravitational mass as:

$$\boxed{M_{\mathrm{geom}}(r) \equiv M(r) \equiv \frac{c^2 r^3}{2G}\left[K_{tr}(r) + \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) - \frac{\Lambda}{3}\right]}. \tag{23}$$

In (23), $M_{\mathrm{geom}}(r)$ is the total mass including the baryonic contribution and the geometric self-energy contribution. This is a strict algebraic identity that does not depend on any approximation. The evolution of the gravitational mass can be read off directly. In the reverse-bending zone ($r_0 < r < R$), the local density and pressure are strictly positive, the sectional curvature is strictly positive, and the cosmological constant term can be neglected.

Rapid growth segment ($r_0 \to r_{\mathrm{peak}}$): K_{tr} rises from zero to the maximum $K_{\max}$, while $\rho + p/c^2$ decreases but still contributes positively to the growth. Superposing the two, the expression inside the brackets increases as a whole, and multiplying by the explicit factor r^3 from the prefactor $c^2 r^3/(2G)$ — this makes the gravitational mass grow rapidly. This is the most drastic phase of gravitational mass growth.

Decelerating growth segment ($r_{\mathrm{peak}} \to R$): After passing the peak, K_{tr} decreases from the maximum but remains positive; $\rho + p/c^2$ continues to decrease but still contributes positive growth. The change of the gravitational mass here depends on the competition between the rate of decline of K_{tr} and the growth rate of the r^3 factor in the prefactor $c^2 r^3/(2G)$.

Note: ($R < r$) is already outside the reverse-bending zone, the Schwarzschild decay segment governed by Birkhoff's theorem.

4.2.3 Quantitative proof of the persistent growth of the gravitational mass in the reverse-bending zone

1. Rapid growth segment ($r_0 \to r_{\mathrm{peak}}$) — the gravitational mass grows faster than linearly

In the first half of the reverse-bending zone $r_0 < r < r_{\rm peak}$, the gravitational mass $M(r)$ grows faster than linearly. This conclusion can be verified by exact calculation for the exponential density distribution. Introduce the dimensionless radial coordinate $x \equiv r/h$ and the normalised mass function

$$S(x) \equiv \frac{M(r)}{4\pi\rho_0 h^3} = 2 - e^{-x}(x^2 + 2x + 2), \tag{24}$$

where ρ_0 is the central density and h is the radial scale length. The curvature zero x_0 is determined by the condition $\bar{\rho}(r_0) = 1.5\,\rho(r_0)$ as $x_0 \approx 1.441$ (see (17)), and the curvature peak $x_{\rm peak}$ is given by the peak equation (20) with the numerical solution $x_{\rm peak} \approx 2.841$.

At these two positions the normalised masses are

$$S(x_0) \approx 0.353, \qquad S(x_{\rm peak}) \approx 1.080.$$

Now construct a linear-growth reference curve starting from the zero:

$$S_{\rm lin}(x) \equiv S(x_0)\,\frac{x}{x_0}.$$

This curve describes the case where the mass grows strictly as $M(r) \propto r$. At the peak, the mass extrapolated linearly is

$$S_{\rm lin}(x_{\rm peak}) = 0.353 \times \frac{2.841}{1.441} \approx 0.696.$$

The ratio of the actual mass to the linear extrapolation is

$$\frac{S(x_{\rm peak})}{S_{\rm lin}(x_{\rm peak})} \approx \frac{1.080}{0.696} \approx 1.55.$$

This ratio is significantly larger than 1, proving that in the interval $r_0 \to r_{\rm peak}$ the growth rate of the total mass $M(r)$ is strictly faster than linear. From a differential viewpoint, the mass growth in this interval satisfies

$$\frac{dM}{dr} > \frac{M}{r},$$

i.e. the local mass increment exceeds the linear growth required by a uniform distribution. The fundamental reason is that the curvature K_{tr} rapidly climbs from zero to its maximum $K_{\rm max}$, driven jointly with the gradually decaying but still positive local matter term $\rho + p/c^2$, and further amplified by the r^3 volume factor. The mass reaching 1.55 times the linear extrapolation implies that the circular orbital rotation speed $v_{\rm rot} \propto \sqrt{M/r}$ at the end of this interval is enhanced by a factor of about $\sqrt{1.55} \approx 1.245$, i.e., relative to the case of pure linear mass growth, the rotation speed increases by an additional $\sim 24.5\%$. This magnitude is sufficiently large to be resolved by current galaxy rotation-curve observations, providing a clear numerical basis for testing the theory.

2. Decelerating growth segment ($r_{\rm peak} \to R$) — the growth rate of the gravitational mass slows down but the mass still continues to increase

According to the three-stage analysis of mass evolution in Section 4.2.2, in the stage $r_{\rm peak} \to R$ the competition between the growth rates of the sectional curvature and the

r^3 factor is key. Here we quantify this rigorously. Differentiating the exact identity (14) directly and substituting $M'(r) = 4\pi r^2 \rho(r)$ gives the exact curvature evolution equation

$$K'_{tr} = -\frac{3}{r} K_{tr} - \frac{4\pi G}{c^2 r}\left(\rho + \frac{3p}{c^2}\right) - \frac{4\pi G}{c^2}\left(\rho' + \frac{p'}{c^2}\right) + \frac{\Lambda}{r}. \tag{25}$$

In the reverse-bending zone $r_0 < r < R$, we have $\rho > 0,\ p > 0,\ \rho' < 0,\ p' < 0$, and Λ/r is negligible on galactic scales. Analysing the signs of the various contributions on the right-hand side:

- The second term $-\frac{4\pi G}{c^2 r}(\rho + \frac{3p}{c^2})$ is a negative contribution (since $\rho > 0,\ p > 0$);
- The third term $-\frac{4\pi G}{c^2}\rho'$ is a **strictly positive** contribution (since $\rho' < 0$);
- The fourth term $-\frac{4\pi G}{c^2}\frac{p'}{c^2}$ is a **strictly positive** contribution (since $p' < 0$).

For the common exponentially decaying density profile $\rho \propto e^{-r/h}$, we have $\rho' = -\rho/h$. Neglecting the pressure terms, the positive contribution is $\frac{4\pi G}{c^2}\frac{\rho}{h}$ and the absolute value of the negative contribution is $\frac{4\pi G}{c^2}\frac{\rho}{r}$. When $r > h$ the positive contribution strictly exceeds the negative contribution. Inside the reverse-bending zone $r > r_0 \approx 1.44h > h$, this condition is automatically satisfied. For general single-peaked monotonically decaying density profiles, in the region $r > h$ we also have $|\rho'| > \rho/r$, and the conclusion that the positive contributions dominate still holds. Therefore, the overall net contribution of the last three terms on the right-hand side is strictly positive.

Dropping the net positive contribution, we obtain a strict inequality for the curvature decay:

$$\boxed{K'_{tr} > -\frac{3}{r} K_{tr}}. \tag{26}$$

Physical meaning: The persistent presence of matter provides support to the sectional curvature, making its decay rate strictly slower than the Schwarzschild vacuum law $K'_{tr} = -3K_{tr}/r$.

Derivation of a lower bound for the mass: Now connect this curvature inequality with the mass-curvature correspondence in the outer region. In the outer low-density region ($\rho, p/c^2 \to 0$), the definition of the gravitational mass (23) reduces to

$$M_{\text{geom}}(r) \propto r^3 K_{tr}(r).$$

Differentiating gives the mass change rate:

$$M'_{\text{geom}} \propto 3r^2 K_{tr} + r^3 K'_{tr} = r^2 K_{tr}\left(3 + \frac{rK'_{tr}}{K_{tr}}\right).$$

Substituting the rigorous conclusion (26) for the curvature evolution, since $K'_{tr} > -3K_{tr}/r$, the term in parentheses is strictly positive, and therefore

$$\boxed{M'_{\text{geom}} > 0}.$$

Throughout the second half of the reverse-bending zone, the curvature decay is necessarily slower than that of the Schwarzschild vacuum, and the gravitational mass continues to increase, its growth rate merely slowing down as the curvature decreases.

3. Summary

In this section we have proved and quantified, through a combination of analytic and numerical methods, the behaviour of the "reverse-bending zone" for a static spherically symmetric exponential density distribution: in the first half, from the curvature zero r_0 to the peak r_{peak}, the growth rate of the gravitational mass $M(r)$ is significantly faster than a simple linear extrapolation; after crossing the peak, although the growth rate begins to slow down, the effective mass still continues to increase all the way to the matter boundary R.

The reverse-bending zone is not a simple transition layer, but a non-linear gravitational enhancement zone locked jointly by the baryon distribution, the boundary conditions, and the geometric self-energy. Without introducing any new particle hypothesis, it naturally induces extra gravity that deviates from the Newtonian expectation. This growth of the gravitational mass can dynamically manifest itself as the flattening of rotation curves, and geometrically ensures the continuity of the metric at the matter boundary.

In summary, proving that the gravitational mass grows inside the reverse-bending zone proves that, relative to the expectation from the bare baryon distribution, additional gravitational effects necessarily exist. This provides a rigorous general-relativistic foundation for understanding the gravitational anomalies on galactic scales.

4.3 Logarithmic potential as an effective approximate description of the forced geometry

Because in the first half of the reverse-bending zone ($r_0 < r < r_{\text{peak}}$) the mass grows faster than linearly, while in the second half ($r_{\text{peak}} < r < R$) the growth continues but at a decelerating rate, and eventually outside the matter boundary it transitions to the constant-mass Schwarzschild tail, there necessarily exists a natural intermediate window $[r_1, r_2] \subset (r_{\text{peak}}, R)$ in the second half such that the total gravitational mass approximately satisfies $M(r) \propto r$. Within this window, the gravitational potential exhibits approximately logarithmic growth and the rotation curve is approximately flat.

To quantitatively characterise the evolution pattern of the mass growth, introduce the local power-law index

$$\gamma(r) \equiv \frac{d \ln M}{d \ln r}.$$

The physical meaning of $\gamma(r)$ is the instantaneous power-law dependence of the mass growth on radius: if $M(r) \propto r^{\gamma}$, then $d \ln M / d \ln r = \gamma$. From the quantification in Section 4.2.3, we have proved that in the whole reverse-bending zone:

- In $r_0 < r < r_{\text{peak}}$ (first half), $\gamma(r) > 1$, the mass grows super-linearly, and the rotation speed continues to rise;
- At $r = r_{\text{peak}}$, $\gamma(r)$ starts to decrease from a value greater than 1;
- In $r_{\text{peak}} < r < R$ (second half), $\gamma(r)$ remains positive but gradually decreases, the mass still grows, albeit with a decelerating rate.

Combining the complete boundary conditions — $\gamma > 1$ in the first half, $\gamma > 0$ and decreasing in the second half, and $\gamma \to 0$ outside the matter boundary (the constant-mass Schwarzschild tail) — one concludes that in the second half there must exist an intermediate window $[r_1, r_2] \subset (r_{\text{peak}}, R)$ where

$$\gamma(r) \approx 1.$$

Within this window, the gravitational mass approximately satisfies linear growth

$$M(r) \approx \mu r, \qquad \mu = \text{constant}.$$

From the circular orbital velocity formula, we obtain

$$v_{\text{rot}}(r) \approx \sqrt{\frac{G\mu}{r} \cdot r} = \text{constant}, \tag{27}$$

and the gravitational potential is approximately logarithmic:

$$\Phi(r) \sim G\mu \ln r. \tag{28}$$

Thus, in this paper the "logarithmic potential" should be understood as an **effective approximate description** of a small segment in the second half of the reverse-bending zone, not as a strict exact solution over the whole interval.

4.4 Geometric origin of the maximum of the rotation curve

It has been explained in the preceding sections that, inside the effective gravitational window corresponding to the forced vacuum geometry, the cumulative mass $M(r)$ continues to grow over a certain radial range and can induce an effective dynamics that approaches a flat rotation curve. In this section we discuss further the question: for a given mass distribution, where does the maximum of the circular orbital speed occur?

It must be emphasised that the derivation below is carried out within a **spherically symmetric three-dimensional exponential density proxy model**. This model can serve as an analytic approximation to the visible-matter distribution of disk galaxies and is used to uncover the geometric scale associated with the position of the speed maximum. It is not equivalent to a rigorous axisymmetric thin-disk model; the conclusions of this section should therefore be understood as **conclusions of an approximate model**, not as unconditional theorems for all real disk galaxies.

4.4.1 Universal criterion for the speed maximum

In the weak-field approximation the square of the circular orbital speed is

$$v^2(r) = \frac{GM(r)}{r}, \tag{29}$$

where $M(r)$ is the total gravitational mass inside radius r, strictly defined by Eqs. (12) and (14). The maximum of the speed satisfies

$$\frac{d}{dr}(v^2) = G\frac{d}{dr}\left(\frac{M(r)}{r}\right) = 0.$$

From the quotient rule,

$$\frac{M'(r)\, r - M(r)}{r^2} = 0,$$

i.e.

$$M'(r)\, r = M(r).$$

Substituting the continuity equation $M'(r) = 4\pi r^2 \rho(r)$ yields the **absolute geometric criterion for the position of the speed maximum**:

$$\boxed{4\pi r^3 \rho(r) = M(r)}. \tag{30}$$

This criterion depends only on the mass distribution itself and contains no free parameters.

4.4.2 Exact solution for an exponential density distribution

Take the visible matter of the galaxy to follow a three-dimensional spherically symmetric exponential distribution $\rho(r) = \rho_0 e^{-r/h}$[7] ,where h is the radial scale length. The cumulative mass is

$$M(r) = \int_0^r 4\pi s^2 \rho(s)\, ds = 4\pi\rho_0 h^3 \Big[2 - e^{-r/h}\Big(\frac{r^2}{h^2} + \frac{2r}{h} + 2\Big)\Big]. \tag{31}$$

Introduce the dimensionless coordinate $x \equiv r/h$ and substitute $\rho(r)$ and $M(r)$ into the extremum criterion (30):

$$4\pi(xh)^3\rho_0 e^{-x} = 4\pi\rho_0 h^3 \big[2 - e^{-x}(x^2 + 2x + 2)\big].$$

Cancelling the common factor $4\pi\rho_0 h^3$ gives

$$x^3 e^{-x} = 2 - e^{-x}(x^2 + 2x + 2).$$

Multiplying both sides by e^x and rearranging yields the **characteristic transcendental equation** that determines the position of the speed maximum:

$$\boxed{2e^x = x^3 + x^2 + 2x + 2}. \tag{32}$$

Solving Eq. (32) gives a unique real root

$$x_{v\,\max} \approx 3.386,$$

i.e.

$$\boxed{r_{v\,\max} \approx 3.39\,h}. \tag{33}$$

4.4.3 Ordering of the three characteristic points

If the zero-point position r_0 and the curvature-peak position $r_{\rm peak}$ defined earlier are obtained under the same approximation conditions, the same geometric framework and the same dimensionless scale length h, and satisfy

$$r_0 \approx 1.44h, \qquad r_{\rm peak} \approx 2.84h,$$

then, within the exponential proxy framework adopted in this paper, the three characteristic radii numerically satisfy the ordering relation

$$\boxed{r_0 \approx 1.44h \;<\; r_{\rm peak} \approx 2.84h \;<\; r_{v\,\max} \approx 3.39h}. \tag{34}$$

The corresponding physical stages can be summarised as follows:

- $r < r_0$: central rising segment, where the speed is mainly controlled by the interior cumulative mass;
- $r_0 < r < r_{\rm peak}$: rising segment in which geometric enhancement gradually appears;
- $r_{\rm peak} < r < r_{v\,\max}$: the speed continues to rise but the growth rate weakens;
- $r > r_{v\,\max}$: the speed enters a flat or slowly declining region, where the system is dominated by the outer asymptotic behaviour.

This ordering provides, within the adopted proxy model, a clear segmented picture of how the rotation curve rises from the centre, passes through a stage of geometric enhancement, and then approaches a flat outer region.

5 Theoretical Extensions

5.1 Exact Emergence of the Friedmann Acceleration Equation

The sectional curvature identity derived earlier (Eq. 14) offers a novel unified perspective on the dark energy problem. Taking the homogeneous and isotropic limit, this identity, which holds rigorously throughout all space, reduces precisely to the Friedmann acceleration equation that describes the accelerated expansion of the Universe.

Assume that the Universe is homogeneous and isotropic on large scales, so that $\rho(r) = \bar{\rho} = \text{const}$, $p(r) = p(\bar{\rho})$, and the enclosed mass naturally satisfies $M(r) = \frac{4}{3}\pi r^3 \bar{\rho}$. Substituting these conditions into the identity (14) and multiplying by c^2 gives the acceleration gradient:

$$\begin{aligned} c^2 K_{tr} &= \frac{2G}{r^3}\left(\frac{4}{3}\pi r^3 \bar{\rho}\right) - 4\pi G\left(\bar{\rho} + \frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \\ &= \frac{8\pi G}{3}\bar{\rho} - 4\pi G\bar{\rho} - \frac{4\pi G p}{c^2} + \frac{\Lambda c^2}{3} \\ &= -\frac{4\pi G}{3}\left(\bar{\rho} + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \end{aligned} \tag{35}$$

This is precisely the Friedmann acceleration equation of standard cosmology:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{36}$$

Thus we obtain the exact correspondence:

$$\boxed{c^2 K_{tr} = \frac{\ddot{a}}{a}}. \tag{37}$$

Deep physical implications of this relation:

- **Dual identity of a geometric quantity**: The sectional curvature K_{tr} serves, in local astrophysics, as the geometric carrier of the radial tidal acceleration gradient; in the cosmological limit, it becomes directly equal to the acceleration of the cosmic scale factor $\ddot{a}/a$.

- **Unified description of dark energy**: The cosmological constant Λ appears in the identity (14) as the uniform background curvature $\Lambda/3$. It is negligible inside stars, requires delicate consideration on galactic scales, and dominates the accelerated expansion on cosmological scales. It is not an ad hoc "dark energy" parameter, but an inherent integration constant of the Einstein field equations, whose effect manifests itself self-consistently in the sectional curvature.

- **From local tides to cosmic expansion**: One and the same algebraic expression describes, in a non-uniform spherically symmetric system, Newtonian tides, relativistic pressure corrections and Weyl stretching; in a homogeneous medium, it transforms automatically into the Friedmann equation. This eliminates the conceptual barrier between "local gravitational physics" and "cosmology", demonstrating that the so-called dark energy effect is nothing but the manifestation of spacetime curvature on different scales.

Theoretical significance: The identity (14) together with its homogeneous limit (35) constitutes a bridge that rigorously unifies the most fundamental local curvature structures of general relativity with the global evolution of the Universe on the largest scales. Not only does it require no ad hoc dark energy model, but it also integrates all contributions—pressure, density, mass and the cosmological constant—into a transparent and observable tidal-force framework, providing a solid theoretical foundation for re-examining the nature of the accelerated cosmic expansion.

5.2 Exact Emergence of the Classical Newtonian Radial Tidal Expression

1.Starting point: the relativistic tide

Multiplying the identity (14) by the square of the speed of light yields the exact relativistic radial tidal acceleration gradient:

$$c^2 K_{tr}(r) = \frac{2GM(r)}{r^3} - 4\pi G\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{38}$$

In galactic or lower-density astrophysical environments:

- The pressure term satisfies $p/c^2 \sim \rho(v/c)^2 \ll \rho$, and is thus a second-order relativistic small quantity;
- The cosmological-constant term $\Lambda c^2/3$ is dozens of orders of magnitude smaller than the matter terms.

Consequently, in the non-relativistic weak-field limit, equation (38) reduces to:

$$c^2 K_{tr}(r) \approx \frac{2GM(r)}{r^3} - 4\pi G\rho(r). \tag{39}$$

2.Expressing in terms of mean density

Introduce the mean density inside radius r:

$$\bar{\rho}(r) \equiv \frac{M(r)}{\frac{4}{3}\pi r^3}, \tag{40}$$

so that $M(r) = \frac{4}{3}\pi r^3 \,\bar{\rho}(r)$. Substituting into the first term of (39):

$$\frac{2G}{r^3} \cdot \frac{4}{3}\pi r^3 \,\bar{\rho}(r) = \frac{8\pi G}{3}\,\bar{\rho}(r). \tag{41}$$

Hence:

$$c^2 K_{tr}(r) \approx \frac{8\pi G}{3}\bar{\rho}(r) - 4\pi G\rho(r) = 4\pi G\left(\frac{2}{3}\,\bar{\rho}(r) - \rho(r)\right). \tag{42}$$

3.The classical Newtonian radial tide

In Newtonian gravity, the gravitational potential Φ of a spherically symmetric mass distribution satisfies Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. The gravitational acceleration is

$-\partial_r \Phi = GM(r)/r^2$; differentiating with respect to r gives the radial tidal acceleration gradient:

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{GM(r)}{r^2}\right) \\ &= \frac{G}{r^2}\frac{dM}{dr} - \frac{2GM}{r^3} \\ &= \frac{G}{r^2}\,4\pi r^2 \rho(r) - \frac{2GM(r)}{r^3} \\ &= 4\pi G\rho(r) - \frac{2GM(r)}{r^3}.\end{aligned} \tag{43}$$

Defining the radial tidal acceleration gradient as $-\partial_r^2 \Phi$ (positive value corresponds to stretching, negative to compression), we obtain:

$$-\frac{\partial^2 \Phi}{\partial r^2} = \frac{2GM(r)}{r^3} - 4\pi G\rho(r) = 4\pi G\left(\frac{2}{3}\,\bar{\rho}(r) - \rho(r)\right). \tag{44}$$

Comparing (42) with (44), we immediately arrive at:

$$\boxed{c^2 K_{tr}(r) \xrightarrow{\text{weak-field limit}} -\frac{\partial^2 \Phi}{\partial r^2}}. \tag{45}$$

4.Recovering Newtonian gravity from the tide

Using the mass continuity equation $M'(r) = 4\pi r^2 \rho(r)$, the weak-field limit (39) can be rewritten as

$$c^2 K_{\rm tr}(r) = \frac{2GM(r)}{r^3} - \frac{GM'(r)}{r^2} = -\frac{d}{dr}\left(\frac{GM(r)}{r^2}\right). \tag{46}$$

Define the magnitude of the spherically symmetric Newtonian gravitational acceleration $g(r) \equiv GM(r)/r^2$ (positive towards the center). Then

$$c^2 K_{\rm tr}(r) = -g'(r). \tag{47}$$

Integrating with the far-field boundary condition $g(\infty) = 0$ (guaranteed by finite total mass) immediately yields

$$\boxed{g(r) = -\int_r^\infty c^2 K_{\rm tr}(r')\,dr'}. \tag{48}$$

Therefore, under the condition of static spherical symmetry, the Newtonian gravitational acceleration is uniquely determined by the radial integral of the sectional curvature $K_{\rm tr}$.**Equation (47) establishes a direct dual relationship between the sectional curvature and the gravitational acceleration. All conclusions of this paper concerning the zero point, the peak, the velocity extremum, and the flattening of rotation curves in the anti-bending region can be retroactively verified for self-consistency through this equation, while the physical picture becomes more intuitive and easier to grasp**.

5.Physical meaning

- The sectional curvature of general relativity multiplied by c^2 corresponds, in the weak-field low-velocity limit, not approximately but **exactly** to the Newtonian radial tidal acceleration gradient.

- The structure $\frac{2}{3}\bar{\rho}(r) - \rho(r)$ in Eq. (42) is precisely the classical Newtonian criterion: inside a uniform density sphere one finds compressive tides ($\rho = \bar{\rho}$, negative result), while outside one finds stretching tides ($\rho = 0$, positive result).

- The relativistic identity (38) can be regarded as the **complete generalisation** of the Newtonian tidal formula: through the addition of the pressure term p/c^2 and the cosmological-constant term $\Lambda c^2/3$ it covers the entire range of physical scales, from compact objects to cosmic expansion.

This emergence process clearly demonstrates that general relativity does not overthrow Newtonian theory, but subsumes it as the low-velocity weak-field limit. The sectional curvature K_{tr} is the geometric bridge that connects these two languages.

5.3 Exact Emergence of the TOV Equation

The identity (14) contains the complete geometric information of the Einstein field equations for a static, spherically symmetric perfect fluid. Combined with the definition of the enclosed mass and the conservation of the stress–energy tensor, it permits a direct algebraic reconstruction of the Tolman–Oppenheimer–Volkoff (TOV) equation of hydrostatic equilibrium.

From the field equations one obtains the radial derivative of the metric potential Φ as

$$\Phi'(r) = e^{2\lambda(r)} \left[\frac{GM(r)}{c^2 r^2} + \frac{4\pi G}{c^4} p(r) r - \frac{\Lambda}{3} r \right], \tag{49}$$

while the (r, r) component of the field equations, together with the mass definition $M'(r) = 4\pi r^2 \rho$, fixes

$$e^{-2\lambda(r)} = 1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda}{3} r^2. \tag{50}$$

For a static perfect fluid the radial component of the conservation law $\nabla_\mu T^{\mu\nu} = 0$ reduces to

$$\frac{dp}{dr} = -(\rho c^2 + p) \frac{d\Phi}{dr}. \tag{51}$$

Substituting (49) and (50) into (51) yields immediately

$$\boxed{\frac{dp}{dr} = -(\rho c^2 + p) \frac{\dfrac{GM(r)}{c^2 r^2} + \dfrac{4\pi G}{c^4} p(r) r - \dfrac{\Lambda}{3} r}{1 - \dfrac{2GM(r)}{c^2 r} - \dfrac{\Lambda}{3} r^2}}. \tag{52}$$

For $\Lambda = 0$ this is the standard TOV equation of relativistic stellar structure:

$$\frac{dp}{dr} = -\left(\rho + p/c^2\right) \frac{GM(r) + 4\pi G r^3 p/c^2}{r^2 \left(1 - 2GM(r)/(c^2 r)\right)}. \tag{53}$$

The derivation makes no use of the second-order field equation beyond what is already encoded in the identity (14). Hence the sectional curvature formula (14) is not merely compatible with the TOV equation — it *implies* it. Together with the vacuum Schwarzschild–de Sitter limit and the homogeneous Friedmann limit, this demonstrates that Eq. (14) serves as a unified geometric origin of the fundamental equations governing static, spherical, and cosmological gravitational fields.

5.4 Summary

The exact emergence of the Schwarzschild solution and other classical results will not be elaborated one by one in this paper. Under the conditions of a static, spherically symmetric ideal fluid, Eq. (14) is the common geometric destination of all known exact solutions in relativity. This means that, within the classification system of exact solutions to the Einstein field equations, Eq. (14) should be recognised as the algebraically complete form of the radial sectional curvature for the static spherically symmetric case. Equation (14) expresses the curvature directly as an algebraic function of the matter distribution $\rho(r)$, the pressure $p(r)$ and the enclosed mass $M(r)$, so that **without the need to solve nonlinear differential equations** one can fully determine the sectional curvature, and it is straightforward to understand. It is not a new metric solution *per se*, but a **meta-theorem** that governs all static spherically symmetric solutions.

6 Complete Step-by-Step Derivation of the Spherically Symmetric Sectional Curvatures

6.1 Complete Derivation of the Sectional Curvatures

1. Notation and conventions

- Metric signature: $(-,+,+,+)$.
- The speed of light is explicitly retained as c.
- We adopt the static, spherically symmetric metric (coordinate basis (t,r,θ,ϕ)):

  $$ds^2 = -A(r)\,c^2dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

  where $A(r) = e^{2\Phi(r)}$ and

  $$B(r) = \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}.$$

  $m(r)$ is the Misner–Sharp cumulated mass, satisfying $m'(r) = 4\pi r^2\rho(r)$.
- The orthonormal frame is denoted by a hat $\hat{}$. In this frame $g_{\hat a\hat b} = \mathrm{diag}(-1,+1,+1,+1)$.
- Riemann tensor convention: $R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\sigma\nu} - \partial_\nu\Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\alpha\mu}\Gamma^\alpha{}_{\sigma\nu} - \Gamma^\rho{}_{\alpha\nu}\Gamma^\alpha{}_{\sigma\mu}$.
- The sectional curvatures are defined as:

  $$\begin{aligned} K_{tr} &\equiv -R_{\hat t\hat r\hat t\hat r}, \\ K_{t\theta} &\equiv -R_{\hat t\hat\theta\hat t\hat\theta}, \\ K_{r\theta} &\equiv -R_{\hat r\hat\theta\hat r\hat\theta}, \\ K_{\theta\phi} &\equiv -R_{\hat\theta\hat\phi\hat\theta\hat\phi}. \end{aligned}$$

2. Required Christoffel symbols (coordinate basis) Metric components (coordinate basis):

$$g_{tt} = -Ac^2, \quad g_{rr} = B, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta.$$

The non-zero Christoffel symbols directly relevant for the following Riemann component calculations are:

$$\begin{aligned}
\Gamma^t{}_{tr} &= \frac{A'}{2A} = \Phi'(r), \\
\Gamma^r{}_{tt} &= \frac{c^2 A'}{2B}, \\
\Gamma^r{}_{rr} &= \frac{B'}{2B}, \\
\Gamma^r{}_{\theta\theta} &= -\frac{r}{B}, \\
\Gamma^r{}_{\phi\phi} &= -\frac{r}{B}\sin^2\theta, \\
\Gamma^\theta{}_{r\theta} &= \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \\
\Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^\phi{}_{\theta\phi} &= \cot\theta.
\end{aligned}$$

Here $' \equiv d/dr$.

3. Calculation of the key Riemann components (coordinate basis → orthonormal frame)

$R_{\hat{t}\hat{r}\hat{t}\hat{r}}$ has already been rigorously proved in identity (14) of the main text using the Cartan structure equations. In the orthonormal frame:

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{2Gm(r)}{c^2 r^3} + \frac{4\pi G}{c^2}\Big(\rho(r) + \frac{p(r)}{c^2}\Big) - \frac{\Lambda}{3}.$$

Hence $K_{tr} = -R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{2Gm(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\big(\rho + p/c^2\big) + \frac{\Lambda}{3}$.

For $R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}$, a direct calculation in the coordinate basis gives

$$R_{t\theta t\theta} = \frac{Ac^2 r\Phi'}{B}.$$

Transforming to the orthonormal frame:

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \frac{R_{t\theta t\theta}}{|g_{tt}|\, g_{\theta\theta}} = \frac{Ac^2 r\Phi'/B}{Ac^2 \cdot r^2} = \frac{\Phi'}{Br}.$$

Using the Einstein equations (with Λ) to solve for Φ' (see Section 4), substituting and simplifying yields

$$\boxed{R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \frac{Gm(r)}{c^2 r^3} + \frac{4\pi G}{c^4}p(r) - \frac{\Lambda}{3}}.$$

$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$ is obtained by a similar calculation (or by inference from the Ricci constraints). The result in the orthonormal frame is

$$\boxed{R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{Gm(r)}{c^2 r^3} + \frac{4\pi G}{c^2}\rho(r) + \frac{\Lambda}{3}}.$$

$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ follows from the spherical symmetry:

$$\boxed{R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2Gm(r)}{c^2 r^3} + \frac{\Lambda}{3}}.$$

4. Solving for Φ' using the Einstein equations (with Λ) The Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, for a perfect fluid $T_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(\rho c^2, p, p, p)$, give the following Ricci components in the orthonormal frame:

$$R_{\hat{t}\hat{t}} = \frac{4\pi G}{c^2}\Big(\rho + \frac{3p}{c^2}\Big) - \Lambda,$$
$$R_{\hat{r}\hat{r}} = \frac{4\pi G}{c^2}\Big(\rho - \frac{p}{c^2}\Big) + \Lambda,$$
$$R_{\hat{\theta}\hat{\theta}} = \frac{4\pi G}{c^2}\Big(\rho - \frac{p}{c^2}\Big) + \Lambda.$$

The mass equation $m'(r) = 4\pi r^2 \rho(r)$, together with the expression derived from $G^r{}_r + \Lambda g^r{}_r$, yields

$$\Phi'(r) = \frac{Gm(r) + \frac{4\pi G}{c^2} r^3 p(r) - \frac{\Lambda c^2}{3} r^3}{c^2 r^2 \Big(1 - \frac{2Gm(r)}{c^2 r}\Big)}.$$

Substituting into $R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \Phi'/(Br)$ and using $B^{-1} = 1 - 2Gm/(c^2 r)$ confirms the result of Section 3.2.

5. Explicit expressions for the sectional curvatures K From the definition $K_{ab} \equiv -R_{\hat{a}\hat{b}\hat{a}\hat{b}}$, we obtain the four independent sectional curvatures:

$$K_{tr}(r) = \frac{2Gm(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\Big(\rho(r) + \frac{p(r)}{c^2}\Big) + \frac{\Lambda}{3}, \tag{54}$$
$$K_{t\theta}(r) = -\frac{Gm(r)}{c^2 r^3} - \frac{4\pi G}{c^4} p(r) + \frac{\Lambda}{3}, \tag{55}$$
$$K_{r\theta}(r) = \frac{Gm(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\rho(r) - \frac{\Lambda}{3}, \tag{56}$$
$$K_{\theta\phi}(r) = -\frac{2Gm(r)}{c^2 r^3} - \frac{\Lambda}{3}. \tag{57}$$

The four formulas above constitute the algebraically complete solution of the static, spherically symmetric Einstein field equations at the curvature level, encoding the full geometric information of spherically symmetric gravity in the most compact form.

6.2 Consistency checks

1. **Ricci trace relations**: For instance, $R_{\hat{t}\hat{t}} = -K_{tr} - 2K_{t\theta}$. Substituting the above expressions recovers $R_{\hat{t}\hat{t}} = (4\pi G/c^2)(\rho + 3p/c^2) - \Lambda$.

2. **Schwarzschild vacuum** ($\rho = p = \Lambda = 0,\ m = \mathrm{const}$):

$$K_{tr} = \frac{2GM}{c^2 r^3},\ K_{t\theta} = -\frac{GM}{c^2 r^3},\ K_{r\theta} = \frac{GM}{c^2 r^3},\ K_{\theta\phi} = -\frac{2GM}{c^2 r^3},$$

which matches the classical tidal structure $2 : -1 : -1$.

3. **de Sitter** ($m = \rho = p = 0,\ \Lambda \neq 0$):

$$K_{tr} = K_{t\theta} = \frac{\Lambda}{3}, \quad K_{r\theta} = K_{\theta\phi} = -\frac{\Lambda}{3},$$

consistent with $R_{\hat{t}\hat{t}} = -\Lambda,\ R_{\hat{i}\hat{i}} = +\Lambda$.

4. **Newtonian weak field** ($p \to 0,\ \Lambda \to 0$):

$$c^2 K_{tr} \to \frac{2Gm}{r^3} - 4\pi G\rho, \quad c^2 K_{t\theta} \to -\frac{Gm}{r^3},$$

corresponding to the Poisson/tidal formulas.

5. **Weyl–Ricci decomposition**: The Weyl part coefficients are $\frac{Gm}{c^2 r^3} \times \{+2, -1, +1, -2\}$ for $\{K_{tr}, K_{t\theta}, K_{r\theta}, K_{\theta\phi}\}$. In the Ricci part, ρ mainly enters $K_{r\theta}$, p enters $K_{t\theta}$, and K_{tr} contains the $\rho + p/c^2$ combination. Λ enters as $\pm\Lambda/3$.

6.3 Circular motion is determined solely by the sectional curvature K_{tr}

Define the geometric effective mass $M_{\text{geom}}(r) \equiv \frac{c^2 r^3}{2G} K_{tr}(r)$. Then

$$m(r) = M_{\text{geom}}(r) + \frac{2\pi r^3}{c^2}\left(\rho + \frac{p}{c^2}\right) - \frac{c^2 r^3}{6G}\Lambda.$$

In the outer regions of a galaxy (p, Λ negligible, Ricci term small), $M_{\text{geom}} \approx m(r)$, i.e., the geometry dominates.

Direct link between circular motion and K_{tr}: In the static, spherically symmetric weak-field approximation, the radial acceleration for circular motion is given by the geodesic equation:

$$\frac{v^2}{r} = \frac{Gm(r) + \dfrac{4\pi G}{c^2} r^3 p(r) - \dfrac{\Lambda c^2}{3} r^3}{r^2\left(1 - \dfrac{2Gm(r)}{c^2 r}\right)} e^{2\Phi(r)}.$$

In the limit of a weak field ($e^{2\Phi} \approx 1$), low pressure ($p \approx 0$), and negligible cosmological constant ($\Lambda \approx 0$), the above reduces to

$$\frac{v^2}{r} = \frac{Gm(r)}{r^2} + O\left(\frac{G^2 m^2}{c^2 r^3}\right), \tag{58}$$

which is precisely the Newtonian equilibrium condition for circular motion.

This acceleration is completely determined by $m(r)$. And K_{tr} is the *only* sectional curvature that can form a closed algebraic-differential system with the mass continuity equation $m'(r) = 4\pi r^2 \rho$: in the weak-field, low-pressure approximation, combining Eq. (54) with Eq. (58) allows $m(r)$ and $\rho(r)$ to be self-consistently solved from the observed $v(r)$ without introducing any additional matter hypothesis. The remaining sectional curvatures $K_{t\theta}, K_{r\theta}, K_{\theta\phi}$ contribute only to higher-order tidal deformations (such as the velocity ellipsoid or disk thickness) and do not affect the first-order dynamics of rotation curves. Therefore, for the problem of galactic rotation curves, it is sufficient to focus on K_{tr}, while the other curvature components can be regarded as a passive response.

7 Falsifiable predictions and outlook

7.1 Falsifiable predictions

This theory yields the following rigorous, falsifiable predictions that can be clearly distinguished from the particle dark-matter paradigm:

1. **Geometric locking of the starting position of the reverse-bending zone.** The zero condition of Theorem 2.1, $\bar{\rho}(r_0) = 1.5\rho(r_0)$, locks the starting radius of the reverse-bending zone r_0 into a definite ratio with the disc scale length h of the galaxy's visible matter. For a three-dimensional spherically symmetric exponential density distribution, the strict solution is $r_0 \approx 1.44\,h$ (corresponding to the volume density form); for the surface density form of a thin exponential disc, the corresponding value of r_0/h can be derived separately by integrating the surface density. This prediction contains no free parameters and can be tested directly by statistical comparison of large-sample galaxy photometric data (yielding h) with characteristic radii of rotation curves.

2. **Deterministic relationship between the effective dark-matter profile and the visible matter distribution.** The geometric effective mass distribution of the reverse-bending zone $0 < r_0 < r_{\text{peak}} < R$ is uniquely determined by the visible matter density profile via the Einstein field equations. The overall position and shape of the reverse-bending zone and the total mass growth relationship are essentially determined, and can be tested against observations. This is the fundamental difference between this theory and particle dark-matter models.

3. **Positive correlation between the strength of the reverse-bending zone and the compactness of the system.** The non-local term $2GM/(c^2r^3)$ in Eq. (14) endows the sectional curvature with sensitivity to the scale of the system. For galaxies of the same total mass but smaller scale length h, r_0 is smaller, the curvature peak $K_{\max}$ is larger, and the geometric enhancement of the reverse-bending zone is more pronounced. Hence the theory predicts that the ratio of dynamical mass to baryonic mass, $M_{\text{dyn}}/M_{\text{bar}}$, should be positively correlated with the compactness $1/r_0$ of the system. Compact systems such as dwarf spheroidal galaxies should exhibit systematically larger mass discrepancies, a trend that can be tested with existing survey data.

4. **Correspondence of characteristic points in the rotation-curve morphology.** The theory predicts that the rotation curve undergoes a curvature sign reversal near r_0, and the sectional curvature reaches its maximum at the curvature peak $r_{\text{peak}} \approx 2.84\,h$. In this interval the total mass grows super-linearly, so it is predicted that this interval corresponds to a rising segment of the velocity curve. These features can be directly compared with the characteristic points (such as inflection points, flat-starting points) of observed rotation curves, constituting a parameter-free morphological test.

5. **Universal ratio between the position of the maximum rotation speed and the disk scale length.** Within the spherically symmetric three-dimensional exponential density proxy model adopted in this paper, the theory predicts that the maximum of the rotation curve occurs around $r_{v\,\max} \approx 3.39\,h$. This ratio contains no free parameters and can be tested directly by statistical comparison of large-sample galaxy photometric data (yielding the disk scale length h) with the peak radius of the rotation curve. If the statistical results show that this ratio deviates significantly from 3.39 for exponential disk galaxies, the theory is falsified.

7.2 Conclusions and outlook

Working strictly within the framework of standard general relativity and employing canonical textbook relativistic computation, we have derived the sectional curvature formula (14). This formula reproduces exactly the Friedmann acceleration equation, exactly recovers the classical Newtonian radial tidal formula, exactly yields the TOV equation, and exactly reduces to the Schwarzschild solution. To question formula (14) is tantamount to doubting simultaneously relativity, the theory of cosmic expansion, and classical Newtonian mechanics.

Since formula (14) is rigorously correct, the curvature sign-reversal point and the curvature maximum point deduced from it — which delineate the *anti-bending region* arising from the nonlinear interplay between non-local Weyl stretching and the self-energy of the gravitational field — rest on a very solid foundation. Under the condition of forced geometry, the gravitational mass inside the anti-bending region continues to grow; in the first half it grows super-linearly and, within a local window, approximates a linear increase, thereby manifesting itself as an approximately flat rotation curve. The zero-point condition and the peak condition provide quantitative predictions with *no free parameters*, which can be statistically tested against large samples of galaxy surveys, offering a sharp contrast to the particle dark matter paradigm.

The present framework is currently built upon the assumption of static spherical symmetry. Future work can proceed along the following directions:

(1) Generalise the spherical-symmetry theorem to axisymmetric and generally static non-spherically symmetric configurations, in order to describe the structure of real spiral galaxies more precisely;

(2) Study the formation and evolution of the reverse-bending zone in dynamical spacetimes, and explore its possible role on cosmological scales;

(3) Use large-sample rotation-curve databases such as SPARC and weak-lensing survey data to carry out systematic tests of the quantitative predictions made in this paper.

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8 Appendix A: Complete Inverse Verification of Eq. (14) and Its Theoretical Significance

A.1 Purpose

In the main text, Eq. (14) gives the relationship between the t-r sectional curvature $K_{tr}(r)$ and the enclosed mass, local density (and, if necessary, pressure and cosmological constant) in the static, spherically symmetric case. If this relationship only holds in the forward direction and cannot be "inverted", the whole theoretical framework might be nothing more than a formal rewriting, rather than a truly self-consistent structure.

Therefore, the aims of this appendix are:

1. Start from an arbitrarily given sectional curvature profile $K_{tr}(r)$;
2. Recover *in reverse* the full set of dynamical quantities

$$\{g(r),\ v^2(r),\ M(r),\ \rho(r)\};$$

3. Substitute them back into the original curvature formula, Eq. (14);
4. Prove rigorously that the input $K_{tr}(r)$ is recovered exactly.

This constitutes a genuine "inverse verification": it proves that Eq. (14) is not a one-way derivation, but a reversible, mathematically closed mapping.

A.2 Starting point: Eq. (14) and its weak-field, spherically symmetric form

In the static, spherically symmetric case, Eq. (14) reads

$$K_{tr}(r) = \frac{2GM(r)}{c^2r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}. \tag{A.1}$$

In the weak-field, low-pressure limit appropriate for galactic scales,

$$\frac{p}{c^2} \ll \rho, \qquad \text{the } \Lambda \text{ term is negligible,}$$

so that Eq. (A.1) reduces to

$$K_{tr}(r) = \frac{2GM(r)}{c^2r^3} - \frac{4\pi G}{c^2}\rho(r). \tag{A.2}$$

Moreover, the spherically symmetric mass continuity equation holds exactly:

$$M'(r) = 4\pi r^2\rho(r). \tag{A.3}$$

A.3 The four-step inverse mapping

Step 1: Recover the gravitational acceleration $g(r)$ from $K_{tr}(r)$.

Define the magnitude of the spherically symmetric gravitational acceleration as

$$g(r) \equiv \frac{GM(r)}{r^2} > 0. \tag{A.4}$$

From Eq. (47) of the main text,

$$c^2K_{tr}(r) = -g'(r), \tag{A.5}$$

integration yields

$$g(r) = g(\infty) + \int_r^\infty c^2 K_{tr}(r')\, dr'. \tag{A.6}$$

If the total mass of the system is finite, the far-field boundary condition

$$g(\infty) = 0 \tag{A.7}$$

holds, and therefore

$$\boxed{g(r) = \int_r^\infty c^2 K_{tr}(r')\, dr'.} \tag{A.8}$$

Differentiating immediately verifies

$$g'(r) = -c^2 K_{tr}(r), \tag{A.9}$$

which is strictly consistent with (A.5).

Step 2: Recover the rotation curve $v^2(r)$ from $g(r)$.

For a stable circular orbit,

$$v^2(r) = r\, g(r). \tag{A.10}$$

This step introduces no new hypothesis; it is merely the basic dynamical relation in a spherically symmetric central potential. Hence, once $K_{tr}(r)$ is given, the rotation curve is uniquely determined.

Step 3: Recover the enclosed mass $M(r)$ from $g(r)$.

From the definition (A.4) one directly obtains

$$\boxed{M(r) = \frac{r^2 g(r)}{G}.} \tag{A.11}$$

Thus, given $K_{tr}(r)$, the enclosed mass function is uniquely fixed.

Step 4: Recover the matter density $\rho(r)$ from $M(r)$.

Using the spherically symmetric mass continuity equation (A.3):

$$\boxed{\rho(r) = \frac{M'(r)}{4\pi r^2}.} \tag{A.12}$$

Substituting (A.11) into the above,

$$M'(r) = \frac{d}{dr}\left(\frac{r^2 g(r)}{G}\right) = \frac{2rg(r) + r^2 g'(r)}{G}. \tag{A.13}$$

With $g'(r) = -c^2 K_{tr}(r)$ we obtain

$$M'(r) = \frac{2rg(r) - r^2 c^2 K_{tr}(r)}{G}. \tag{A.14}$$

Hence

$$\rho(r) = \frac{2rg(r) - r^2 c^2 K_{tr}(r)}{4\pi G r^2} = \frac{g(r)}{2\pi G r} - \frac{c^2 K_{tr}(r)}{4\pi G}. \tag{A.15}$$

Thus

$$\boxed{\rho(r) = \frac{g(r)}{2\pi G r} - \frac{c^2 K_{tr}(r)}{4\pi G}.} \tag{A.16}$$

A.4 Substituting $\{M(r), \rho(r)\}$ back into Eq. (14)
Now insert (A.11) and (A.16) into the weak-field curvature expression (A.2):

$$K_{tr}^{(\text{check})}(r) = \frac{2G}{c^2 r^3} \cdot \frac{r^2 g(r)}{G} - \frac{4\pi G}{c^2}\left(\frac{g(r)}{2\pi G r} - \frac{c^2 K_{tr}(r)}{4\pi G}\right)$$
$$= \frac{2g(r)}{c^2 r} - \frac{2g(r)}{c^2 r} + K_{tr}(r). \tag{A.17}$$

This yields the exact identity

$$\boxed{K_{tr}^{(\text{check})}(r) = K_{tr}(r).} \tag{A.18}$$

The inverse verification of Eq. (14) is thus complete.

A.5 Mathematical significance of the inverse verification

The importance of this result goes beyond the mere fact that "the algebra was not messed up":

A.5.1 It proves that Eq. (14) constitutes an invertible map. From any $K_{tr}(r)$ satisfying the boundary and smoothness conditions, one can uniquely recover

$$K_{tr}(r) \longleftrightarrow g(r) \longleftrightarrow M(r) \longleftrightarrow \rho(r).$$

Hence, the sectional curvature and the dynamical quantities are not loosely related; they stand in a one-to-one correspondence.

A.5.2 It eliminates any suspicion of circular reasoning. If the theory only proceeded as "first assume a mass distribution, then compute the curvature, and finally claim that the curvature explains the dynamics", it would risk a hidden circular definition. The inverse verification proves that, even starting entirely from the curvature function, one can independently reconstruct all dynamical quantities and exactly recover the original formula. This shows that the theory is closed and does not rely on a priori assumptions dressed up as retroactive substitution.

A.5.3 It turns curvature into an observable that can be inverted. Galactic rotation curves measure $v(r)$, from which one obtains $g(r)$ and $g'(r)$. Because

$$c^2 K_{tr}(r) = -g'(r),$$

one can directly infer the sectional curvature profile from observations, and then use Eq. (14) to check its consistency with the baryonic distribution. Thus K_{tr} is no longer an abstract geometric quantity, but a physical object that can be directly compared with galactic observations.

A.6 Conclusion of this appendix

Appendix A has demonstrated that, under the static, spherically symmetric, weak-field and low-pressure conditions, Eq. (14) is not only usable in the forward direction, but can also be completely inverted and closed. Given $K_{tr}(r)$, the four-step mapping recovers $\{g, v^2, M, \rho\}$, and substituting them back into Eq. (14) reproduces the input $K_{tr}(r)$ exactly. This constitutes the first pillar of the self-consistency foundation of the theoretical framework presented in the paper.

9 Appendix B: Exact Derivation, Physical Meaning of Eq. (47) and Its Rigorous Consequences for the Mass Growth Law

B.1 Purpose

The most crucial relation in the main text is

$$\boxed{c^2 K_{tr}(r) = -g'(r)} \tag{B.1}$$

namely Eq. (47).

The tasks of this appendix are threefold:

1. Prove rigorously that Eq. (47) follows from the field equations, and explain why it is not an accidental approximation but a direct bridge between geometry and dynamics;
2. Clarify its physical meaning: it establishes a point-by-point one-to-one correspondence between "spacetime curvature" and "gravitational gradient / tidal force";
3. Starting directly from Eq. (47) and without relying on any specific density model, prove that the mass growth is superlinear in the first half of the anti-bending region and sublinear in the second half.

B.2 Rigorous derivation of Eq. (47)

Begin with the exact relativistic identity (A.1):

$$K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}. \tag{B.2}$$

Multiplying by c^2 gives

$$c^2 K_{tr}(r) = \frac{2GM(r)}{r^3} - 4\pi G\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{B.3}$$

On galactic scales:

- $p/c^2 \ll \rho$, so the pressure term is negligible;
- $\Lambda c^2/3$ is many orders of magnitude smaller than the local matter term;

hence

$$c^2 K_{tr}(r) = \frac{2GM(r)}{r^3} - 4\pi G\rho(r). \tag{B.4}$$

Using

$$M'(r) = 4\pi r^2 \rho(r) \quad \Longrightarrow \quad 4\pi G\rho(r) = \frac{GM'(r)}{r^2}, \tag{B.5}$$

we obtain

$$c^2 K_{tr}(r) = \frac{2GM(r)}{r^3} - \frac{GM'(r)}{r^2}. \tag{B.6}$$

Define

$$g(r) \equiv \frac{GM(r)}{r^2}, \tag{B.7}$$

so that

$$g'(r) = \frac{GM'(r)}{r^2} - \frac{2GM(r)}{r^3}. \tag{B.8}$$

Therefore we arrive at Eq. (47):

$$\boxed{c^2 K_{tr}(r) = -g'(r).} \tag{B.9}$$

B.3 In what sense is Eq. (47) an "exact result"?

It is necessary to distinguish two levels of exactness:

B.3.1 At the relativistic level. Equation (B.2) is an exact consequence of the Einstein field equations; it contains no approximation.

B.3.2 At the level of application to galaxies. The steps from (B.2) to (B.9) use the galactic-scale approximations of low pressure, weak field, and negligible Λ. Hence, for the problem of galactic rotation curves, Eq. (47) is the rigorously valid dominant relation within its domain of applicability; it is neither an empirical fit nor an extra hypothesis.

In other words, the status of Eq. (47) is analogous to that of the Poisson equation in the Newtonian limit: it is not a "guess", but a direct consequence of the exact field equations in a specific physical limit.

B.4 Physical meaning of Eq. (47)

Equation (47) brings into one-to-one correspondence two objects that appear entirely different:

- On the left, $K_{tr}(r)$: a purely geometric quantity, describing the sectional curvature of the t-r plane;
- On the right, $-g'(r)$: a purely dynamical quantity, describing the rate at which the gravitational acceleration changes with radius, i.e., the tidal gradient.

Thus,

$$\boxed{\text{sectional curvature} \iff \text{gravitational gradient}}$$

This tells us that the dark-matter-like effect in galaxies does not reside in "how large the gravitational pull is", but in "how slowly the gravitational acceleration decreases with radius". The precise quantitative language for "decreases slowly" is $g'(r)$; its geometric formulation is $K_{tr}(r)$.

B.5 Mass growth law deduced directly from Eq. (47)

Define the local logarithmic slope of the enclosed mass,

$$\gamma(r) \equiv \frac{d \ln M}{d \ln r} = \frac{rM'(r)}{M(r)}. \tag{B.10}$$

If $\gamma > 1$, the mass grows faster than linearly (*superlinear growth*); if $\gamma = 1$, the growth is exactly linear; if $\gamma < 1$, the growth is *sublinear*.

Deriving $\gamma(r)$ from Eq. (47).

From

$$M(r) = \frac{r^2 g(r)}{G}, \tag{B.11}$$

differentiation gives

$$M'(r) = \frac{2rg(r) + r^2 g'(r)}{G}. \tag{B.12}$$

Therefore

$$\gamma(r) = \frac{rM'(r)}{M(r)} = \frac{r \cdot \frac{2rg+r^2g'}{G}}{\frac{r^2g}{G}} = 2 + \frac{rg'(r)}{g(r)}. \tag{B.13}$$

Using Eq. (47), $g' = -c^2 K_{tr}$, we obtain

$$\boxed{\gamma(r) = 2 - \frac{rc^2 K_{tr}(r)}{g(r)}.} \tag{B.14}$$

This is the most essential local mass-growth criterion.

B.6 Superlinear growth in the first half of the anti-bending region

Let r_0 be the zero of the anti-bending region, satisfying

$$K_{tr}(r_0) = 0. \tag{B.15}$$

Substituting into (B.14) yields

$$\gamma(r_0) = 2. \tag{B.16}$$

Hence, right at the zero point the mass growth is strictly superlinear.

Because $K_{tr}(r)$ and $g(r)$ are physically continuous, there exists a right neighbourhood of r_0 in which

$$\gamma(r) > 1. \tag{B.17}$$

This shows that **as soon as one enters the anti-bending region, the mass growth is necessarily superlinear at first.**

B.7 The superlinear–linear–sublinear transition point

For a circular orbit,

$$v^2(r) = rg(r). \tag{B.18}$$

Differentiating,

$$\frac{d(v^2)}{dr} = g(r) + rg'(r) = g(r) - rc^2 K_{tr}(r). \tag{B.19}$$

The velocity maximum $r_{v_{\max}}$ satisfies

$$\frac{d(v^2)}{dr} = 0 \quad \Longleftrightarrow \quad g(r_{v_{\max}}) = r_{v_{\max}} c^2 K_{tr}(r_{v_{\max}}). \tag{B.20}$$

Substituting this into (B.14) gives

$$\gamma(r_{v_{\max}}) = 2 - 1 = 1. \tag{B.21}$$

Consequently:

- For $r < r_{v_{\max}}$ still inside the anti-bending region, if

  $$rc^2 K_{tr} < g,$$

  then $\gamma > 1$, i.e., superlinear mass growth;

- At $r = r_{v_{\max}}$, $\gamma = 1$, i.e., exactly linear mass growth;

- For $r > r_{v_{\max}}$, if

  $$rc^2 K_{tr} > g,$$

  then $\gamma < 1$, i.e., sublinear mass growth.

We thus obtain the rigorous piecewise conclusion:

$$\boxed{\begin{aligned} r_0 < r < r_{v_{\max}} &\implies \gamma(r) > 1 \quad \text{(superlinear)}, \\ r = r_{v_{\max}} &\implies \gamma(r) = 1 \quad \text{(linear)}, \\ r > r_{v_{\max}} &\implies \gamma(r) < 1 \quad \text{(sublinear)}. \end{aligned}} \tag{B.22}$$

B.8 Why this conclusion is important

This result is crucial because it does *not* depend on any specific model for $\rho(r)$. That is,

- No exponential disk needs to be assumed;
- No NFW halo needs to be assumed;
- No empirical fitting of flat rotation curves is required.

As long as Eq. (47) holds and the system possesses an anti-bending region, the following are guaranteed:

1. In the neighbourhood of the zero point, mass growth is necessarily superlinear;
2. This growth persists until the velocity peak;
3. Beyond that point, the growth automatically transitions to sublinear.

This demonstrates that dark-matter-like dynamical behavior is not an accidental consequence of a particular density distribution, but a universal geometric outcome of the evolution of the sectional curvature.

B.9 Theoretical implications of Appendix B

B.9.1 *Implications for general relativity itself.* It shows that K_{tr} is not a dispensable curvature component, but the most direct and central geometric quantity in galactic dynamics. Among all curvature components, it is the one that controls the gravitational gradient, and therefore controls both the rotation curve and the growth of the effective mass.

B.9.2 *Implications for the dark matter problem.* It provides a mechanism that is entirely within standard GR: it does not modify the Einstein field equations; it does not introduce additional particles; it does not resort to empirical fitting; purely through the sign change and evolution of the sectional curvature, a dark-matter-like dynamical effect emerges. Hence, at least at the level of existence, the dark matter problem no longer necessarily demands new matter ingredients; geometry itself is sufficient to produce such an effect.

B.10 Conclusions of Appendix B and overall significance of the two appendices

Appendix B demonstrates that Eq. (47) is not an empirical formula, but a direct consequence of the Einstein field equations in the galactic weak-field, spherically symmetric limit. It establishes an exact dual relationship between the sectional curvature and the gravitational gradient, and further rigorously derives that within the anti-bending region, mass growth inevitably follows the universal evolution: "superlinear in the first half — linear at the velocity peak — sublinear in the second half." This conclusion is independent of any specific density model and constitutes the second core pillar of the theoretical framework of this paper.

Overall significance of the two appendices

Taken together, these two appendices form the strongest theoretical closure of the paper:

Appendix A proves the reversible invertibility of Eq. (14), showing that the passage from geometry to dynamics and back to geometry is rigorously closed.

Appendix B proves the origin, physical meaning, and direct constraints of Eq. (47) on the mass growth law, demonstrating that the anti-bending region is not an accidental feature of an image, but the decisive geometric mechanism governing the dynamical behavior.

Together they establish:

$$\boxed{K_{tr}(r) \iff g'(r) \iff M(r),\, v(r),\, \rho(r)}$$

Thus, for the first time, a reversible, falsifiable, and directly numerically testable bridge is established among "spacetime geometry — galactic dynamics — observed rotation curves."