Limitation in Ramanujan positivity and Zeta Zeros of Riemann

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Abstract

Ramanujan's divisor-sum identity gives one of the most analytical positivity arguments in the theory of the Riemann zeta-function: in Ingham's work it yields the non-vanishing of $\zeta(s)$on the line $\Re s = 1$. This paper revisits that mechanism and examines what is required to move it toward the critical strip. We first give a self-contained proof of the Ramanujan–Ingham zero-free line. We then prove that the direct critical-strip analogue fails for a precise Euler-factor reason: the positive Ramanujan square acquires an obstructing pole, while removing that pole destroys positivity already at prime level. This obstruction leads naturally to the Nyman–Beurling Hilbert-space formulation. Using Mellin transforms, we express the relevant closure problem through centered Ramanujan fractional-part functions and derive the exact finite-dimensional Gram system for optimal approximation. We prove fixed-window density of the associated boundary functions and separate the remaining problem into compact approximation and tail control. The main conclusion is a rigorous reduction: within this Ramanujan–Beurling framework, the remaining obstruction to the Riemann Hypothesis is an explicit uniform growing-window approximation estimate with controlled coefficient mass.

Keywords: Ramanujan identity; Riemann zeta-function; Nyman–Beurling criterion; Hilbert-space approximation; divisor sums; Riemann Hypothesis

1. Introduction

During studying the Ramanujan's paper titled *Some formulæ in the analytic theory of numbers* which contains a divisor-sum identity that later became a very useful tool in analytic number theory [1]. The formula was included in the collected edition of Ramanujan's papers [2], and Murty's exposition emphasizes that Ramanujan's work on the zeta-function was not confined to special values but also contained identities relevant to non-vanishing questions and the distribution of primes [3]. Among these identities, the product formula for generalized divisor sums is especially important: Ingham used it to prove the non-vanishing of $\zeta(s)$on the line $\Re s = 1$[4].

The present paper begins from that Ramanujan–Ingham mechanism. The purpose is not to claim a proof of the Riemann Hypothesis. Rather, the goal is to identify exactly what remains missing if one tries to extend Ramanujan's positivity method from the boundary line $\Re s = 1$into the open half-strip $1/2 < \Re s < 1$. Standard background on $\zeta(s)$, its continuation, functional equation, critical strip, and classical estimates may be found in Titchmarsh's monograph [5].

The first part of this paper gives a complete proof of the zero-free line using Ramanujan's identity. The proof is included because it displays the essential structure: a zeta quotient is represented by a Dirichlet series with nonnegative coefficients, and a hypothetical zero forces a contradiction through Landau's principle for positive Dirichlet series [6].

The second part explains why the same argument does not directly prove more. A hypothetical zero $\rho = \beta + i\gamma$, $1/2 < \beta < 1$, can be inserted into Ramanujan's identity in such a way that the left-hand side remains a square. However, the analytic side contains an additional zeta factor that produces a real pole at $s = 3 - 2\beta$, which lies to the right of $s = 1$. Dividing away this pole gives the desired zero detector, but the resulting Euler product has negative first prime coefficients for infinitely many primes. Thus the direct coefficient-positive extension fails for a local arithmetic reason.

The appropriate replacement is a Hilbert-space closure problem. Nyman introduced a functional-analytic formulation of the Riemann Hypothesis [7], and Beurling sharpened it into a closure criterion [8]. Balazard and Saias gave a convenient Mellin-transform form of this criterion [9], while Báez-Duarte proved the important strengthening that the dilation parameters may be restricted to positive integers [10]. Subsequent work studied lower bounds, Hilbert-space reformulations, and optimal Dirichlet-polynomial choices in this setting [11–14]. Related real-variable reformulations appear in the work of Bercovici–Foias and Báez-Duarte [15,16].

The main contribution of this paper is the organization of these ingredients into a connected uniform reduction. We derive the exact finite-dimensional Gram system for optimal approximation, prove fixed-window density on the critical boundary, and isolate the remaining global estimate. The final obstruction is a uniform growing-window approximation problem: one must approximate $1/\zeta(1/2 + it)$by finite Ramanujan-centered Dirichlet polynomials on expanding intervals while keeping the coefficients sufficiently controlled to make the weighted tails vanish. This is the precise point at which a Ramanujan-based proof along this route would require a genuinely new quantitative estimate.

2. Ramanujan's divisor-sum identity and the zero-free line

Regarding Ramanujan's divisor for $a \in \mathbb{C}$, let

$$\sigma_a(n) = \sum_{d|n} d^a . \qquad (1)$$

Ramanujan's identity states that, in the half-plane of absolute convergence,

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}. \qquad (2)$$

Both sides then continue meromorphically. We recall the Euler-factor verification, since it is short and clarifies the arithmetic structure. Let $p$be prime and set $x = p^{-s}$. Since

$$\sigma_a(p^k) = 1 + p^a + \cdots + p^{ka},$$

a direct summation gives

$$\sum_{k=0}^{\infty} \sigma_a\,(p^k)\sigma_b(p^k)x^k = \frac{1 - p^{a+b}x^2}{(1-x)(1-p^a x)(1-p^b x)(1-p^{a+b}x)}. \qquad (3)$$

Multiplication over all primes gives (2).

The positivity appears when $a = it$and $b = -it$, with $t \in \mathbb{R}$. Then

$$\sigma_{-i}\ (n) = \sigma_{\overline{it}}(n),$$

and (2) becomes in the form of

$$F_t(s) := \sum_{n=1}^{\infty} \frac{|\ \sigma_{it}(n)\ |^2}{n^s} = \frac{\zeta(s)^2\zeta(s+it)\zeta(s-it)}{\zeta(2s)}. \qquad (4)$$

The coefficients on the left are nonnegative, and the first coefficient is 1.

We shall use Landau's theorem in the following fundamental form.

Lemma 1. Let

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, a_n \geq 0,$$

have finite abscissa of convergence σ_c. Then $D(s)$cannot be holomorphic at the real point $s = \sigma_c$.

Proof. Suppose that $D$is holomorphic in a disc centered at σ_c. Choose $\sigma_0 > \sigma_c$so close to $\sigma_c$that the Taylor expansion at $\sigma_0$reaches a real point $\sigma_1 < \sigma_c$. For $k \geq 0$,

$$(-1)^k D^{(k)}(\sigma_0) = \sum_{n=1}^{\infty} a_n (\log\ n)^k n^{-\sigma_0} \geq 0.$$

The Taylor expansion at σ_0, evaluated at σ_1, gives

$$D(\sigma_1) = \sum_{k=0}^{\infty} \frac{D^{(k)}(\sigma_0)}{k!} (\sigma_1 - \sigma_0)^k.$$

Using the preceding derivative identity and monotone convergence,

$$D(\sigma_1) = \sum_{n=1}^{\infty} a_n n^{-\sigma_0} \sum_{k=0}^{\infty} \frac{(\sigma_0 - \sigma_1)\log\ n)^k}{k!} = \sum_{n=1}^{\infty} a_n n^{-\sigma_1}.$$

Thus the Dirichlet series converges at σ_1, contradicting $\sigma_1 < \sigma_c$.

Theorem 2. For every real t,

$$\zeta(1 + it) \neq 0. \qquad (5)$$

Proof of Theorem 2. If $t = 0$, then $\zeta(s)$has a pole at $s = 1$, so there is no zero. Assume $t \neq 0$, and suppose that $\zeta(1 + it) = 0$. By conjugation symmetry, $\zeta(1 - it) = 0$.

In (4), the factor $\zeta(s)^2$has a double pole at $s = 1$, while $\zeta(s + it)\zeta(s - it)$has a double zero there. Hence the quotient on the right of (4) is holomorphic at $s = 1$.

For real $\sigma \in (1/2,1]$, the denominator $\zeta(2\sigma)$is nonzero. Therefore the quotient in (4) is regular throughout this interval. By Lemma 1, the abscissa of convergence of the positive Dirichlet series $F_t$cannot lie in $(1/2,1]$. Consequently the series converges for every real $\sigma > 1/2$, and for such σ,

$$F_t(\sigma) \geq | \sigma_{it}(1) |^2 = 1. \qquad (6)$$

On the other hand, the analytic expression in (4) tends to 0as $\sigma \downarrow 1/2$, because $\zeta(2\sigma) \to +\infty$while the numerator remains finite. This contradicts (6). Thus $\zeta(1+it) \neq 0$.

Therefore Ramanujan's identity gives the zero-free boundary line

$$\zeta(s) \neq 0 (\Re s = 1). \qquad (7)$$

The success of the argument depends on two features occurring simultaneously: the quotient detects a hypothetical zero, and the same quotient is represented by a Dirichlet series with nonnegative coefficients.

3. The critical-strip obstruction

We now examine the same strategy in the open right half of the critical strip. Let

$$\rho = \beta + i\gamma, \frac{1}{2} < \beta < 1. \qquad (8)$$

The zero-detecting substitution in Ramanujan's identity is

$$a = 1 - \rho, b = 1 - \bar{\rho}.$$

Then $b = \bar{a}$, and the left-hand side of (2) is again a positive square:

$$D_\rho(s) = \sum_{n=1}^{\infty} \frac{| \sigma_{1-\rho}(n) |^2}{n^s}. \qquad (9)$$

Using (2), one obtains

$$D_\rho(s) = \frac{\zeta(s)\zeta(s-1+\rho)\zeta(s-1+\bar{\rho})\zeta(s+2\beta-2)}{\zeta(2s+2\beta-2)}. \qquad (10)$$

If $\zeta(\rho) = 0$, then at $s = 1$the two middle factors vanish. The pole of $\zeta(s)$at $s = 1$is simple, so the analytic side of (10) vanishes at $s = 1$. This resembles the mechanism of Section 2, but an obstruction appears before one can use positivity.

The extra factor $\zeta(s + 2\beta - 2)$has a pole when

$$s + 2\beta - 2 = 1,$$

namely at

$$s = 3 - 2\beta. \qquad (11)$$

Because $\beta < 1$, this pole lies strictly to the right of $s = 1$. The positive Dirichlet series is therefore blocked before it reaches the point where the contradiction would occur.

The natural repair is to remove the obstructing factor and consider

$$R_\rho(s) = \frac{\zeta(s)\zeta(s - 1 + \rho)\zeta(s - 1 + \bar{\rho})}{\zeta(2s + 2\beta - 2)}. \qquad (12)$$

Under the zero assumption, $R_\rho(s)$would vanish at $s = 1$. If $R_\rho$had nonnegative Dirichlet coefficients and positive first coefficient, the contradiction would follow. The next proposition shows that this hope fails at the prime level.

Proposition 3. The Euler product corresponding to $R_\rho(s)$ cannot have all Dirichlet coefficients nonnegative.

Proof. Put

$$\alpha = 1 - \beta, 0 < \alpha < \frac{1}{2}.$$

At a prime p, write

$$x = p^{-s}, u = p^{1-\rho}.$$

The local factor of (12) is

$$\frac{1 - u\bar{u}\, x^2}{(1-x)(1-ux)(1-\bar{u}x)}. \qquad (13)$$

The coefficient of x is

$$1 + u + \bar{u} = 1 + 2p^{\alpha}\cos(\gamma \log p). \qquad (14)$$

If $\gamma = 0$, then ρ is real. The zeta-function has no real zero in $(0, 1)$, since $\zeta(s) < 0$ on this interval. Thus a nontrivial zero in the open strip has $\gamma \neq 0$.

Choose an arc $J \subset \mathbb{R}/2\pi\mathbb{Z}$ on which $\cos\theta < -1/2$. The condition $\gamma \log p \in J$ selects infinitely many intervals of the form $[e^{A}, e^{B}]$ with $B - A$ bounded below by a positive constant. The prime number theorem gives primes in all sufficiently large such intervals. Along these primes, (14) is negative for all sufficiently large p. Hence the Dirichlet coefficients of R_ρ cannot all be nonnegative. A view of the obstruction proved in Proposition 3 is provided in **Figure1** . After the obstructing pole is removed from the Ramanujan quotient, the first nontrivial local coefficient at a prime p becomes a constant term plus an oscillatory term whose amplitude grows like $p^{1-\beta}$. In the plotted example, $\beta = 0.75$, so the amplitude grows like $p^{0.25}$, while the phase is governed by $\gamma \log p$, here with $\gamma = 17$. The repeated crossings below the horizontal axis show that the coefficient is not merely oscillatory; it becomes genuinely negative along prime subsequences.

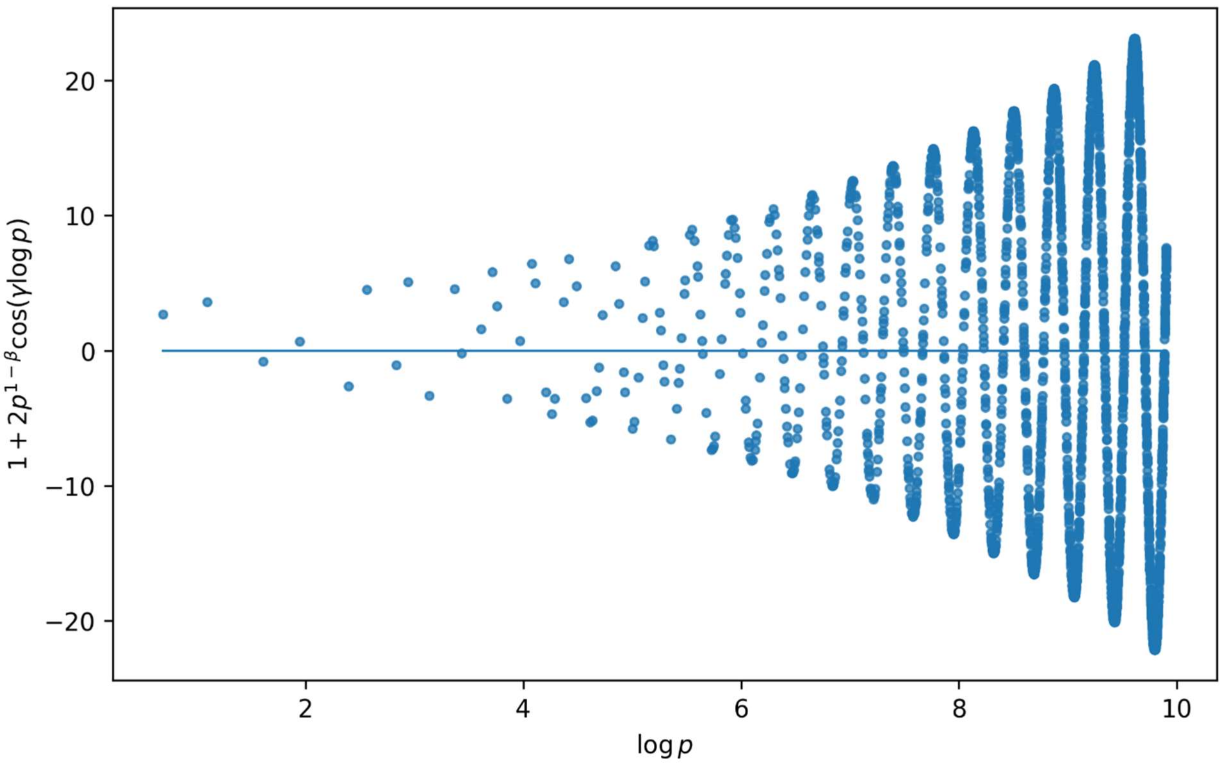

Figure 1. Prime-level obstruction for the pole-removed Ramanujan quotient

Remark 1. Proposition 3 is the local reason the direct Ramanujan–Ingham extension fails. The original square keeps positivity but carries the pole at $s = 3 - 2\beta$. Removing the pole restores the desired analytic vanishing at $s = 1$, but destroys positivity at the first prime coefficient. A successful extension must therefore replace coefficient positivity by a different mechanism.

4. The Ramanujan–Beurling Hilbert-space formulation

For the start, let

$$\mathcal{H} = L^2(0,1).$$

For $n \geq 2$, define the centered fractional-part function

$$b_n(x) = \left\{\frac{1}{nx}\right\} - \frac{1}{n}\left\{\frac{1}{x}\right\}, 0 < x < 1. \tag{15}$$

The centering cancels the pole at $s = 1$in the Mellin transform. For $0 < \theta \leq 1$and $\Re s > 1$,

$$\int_0^1 \left\{\frac{\theta}{x}\right\} x^{s-1}\, dx = \frac{\theta}{s-1} - \frac{\theta^s}{s}\zeta(s). \qquad (16)$$

Indeed, after the substitution $u = \theta/x$, the integral is reduced to a sum over intervals on which $\lfloor u \rfloor$is constant. Taking $\theta = 1/n$and subtracting n^{-1}times the case $\theta = 1$, we get

$$\mathcal{M}b_n(s) := \int_0^1 b_n(x) x^{s-1}\, dx = \frac{\zeta(s)}{s}\left(\frac{1}{n} - n^{-s}\right). \qquad (17)$$

The Mellin transform satisfies the Plancherel identity

$$\| f \|_{L^2(0,1)}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \mathcal{M}f\left(\frac{1}{2} + it\right) \right|^2 dt. \qquad (18)$$

This follows from $x = e^{-u}$and the Fourier Plancherel theorem applied to $e^{-u/2} f(e^{-u})$.

Let

$$\mathcal{B} = \overline{\operatorname{span}\{b_n : n \geq 2\}}^{L^2(0,1)}. \qquad (19)$$

The integer-dilation Nyman–Beurling criterion says that the Riemann Hypothesis is equivalent to

$$1 \in \mathcal{B}. \qquad (20)$$

We use this as a standard theorem, but we record the zero-obstruction direction because it is the mechanism behind the present reduction.

Proposition 4. If $\zeta(\rho) = 0$for some $\rho$with $1/2 < \Re \rho < 1$, then $1 \notin \mathcal{B}$.

Proof. For $f \in L^2(0,1)$, point evaluation of $\mathcal{M}f$at $\rho$is bounded:

$$| \mathcal{M}f(\rho) | \leq \| f \|_2 \left(\int_0^1 x^{2\Re\rho - 2}\, dx\right)^{1/2}. \qquad (21)$$

The integral is finite because $\Re\rho > 1/2$. By (17), each $\mathcal{M}b_n(s)$is divisible by $\zeta(s)$. Therefore every element of $\mathcal{B}$has Mellin transform equal to 0at $s = \rho$. But

$$\mathcal{M}1(\rho) = \frac{1}{\rho} \neq 0.$$

Thus $1 \notin \mathcal{B}$.

This Hilbert-space formulation avoids the failed prime-coefficient positivity of Section 3. Instead of demanding positivity coefficient by coefficient, it asks whether the constant function belongs to a closed subspace generated by Ramanujan-centered fractional parts.

5. Finite projections and the Gram system

Define the finite-dimensional spaces

$$\mathcal{B}_N = \text{span}\,\{b_2, \dots, b_N\}. \qquad (22)$$

Let

$$d_N = \text{dist}_{\,L^2(0,1)}(1, \mathcal{B}_N). \qquad (23)$$

Since the spaces $\mathcal{B}_N$are nested, the sequence $d_N$is nonincreasing.

For $2 \leq j, k \leq N$, define

$$G_N(j,k) = \langle b_k, b_j \rangle_{L^2(0,1)}, u_N(j) = \langle 1, b_j \rangle_{L^2(0,1)}. \qquad (24)$$

Lemma 5. The Gram matrix $G_N$is positive definite for every $N \geq 2$.

Proof. Suppose

$$\sum_{k=2}^{N} c_k\, b_k = 0$$

in $L^2(0,1)$. Taking Mellin transforms and using (17), we can obtain

$$\zeta(s)\sum_{k=2}^{N} c_k\left(\frac{1}{k}-k^{-s}\right)=0 \qquad (25)$$

for $\Re s>1$. Since $\zeta(s)$is not identically zero, the finite Dirichlet expression must vanish identically. Letting $\Re s\to+\infty$ gives:

$$\sum_{k=2}^{N}\frac{c_k}{k}=0.$$

Thus

$$\sum_{k=2}^{N} c_k\, k^{-s}=0$$

identically. The functions $k^{-s}=e^{-s\log k}$are linearly independent for distinct k, so every $c_k=0$.

Write the orthogonal projection of 1onto $\mathcal{B}_N$as

$$P_N 1=\sum_{k=2}^{N} a_{N,k}\, b_k. \qquad (26)$$

The normal equations are

$$G_N a_N=u_N. \qquad (27)$$

The exact squared distance is

$$d_N^2=1-u_N^* G_N^{-1} u_N. \qquad (28)$$

Thus the Hilbert-space problem has a completely explicit finite-dimensional form.

We now pass to the boundary representation. We put

$$s_t=\frac{1}{2}+it, d\nu(t)=\frac{dt}{2\pi\mid s_t\mid^2}. \qquad (29)$$

For coefficients $c_2, \dots, c_N$, define

$$A(s) = \sum_{n=2}^{N} c_n \left(\frac{1}{n} - n^{-s}\right). \qquad (30)$$

By (17) and Plancherel,

$$\| 1 - \sum_{n=2}^{N} c_n b_n \|_2^2 = \int_{-\infty}^{\infty} | 1 - \zeta(s_t) A(s_t) |^2 \, d\nu(t). \qquad (31)$$

Consequently,

$$d_N^2 = \inf_A \int_{-\infty}^{\infty} | 1 - \zeta(s_t) A(s_t) |^2 \, d\nu(t), \qquad (32)$$

where the infimum is over all polynomials of the form (30).

Proposition 6. The Riemann Hypothesis is equivalent to

$$d_N \longrightarrow 0. \qquad (33)$$

Proof. By the integer-dilation Nyman–Beurling theorem, RH is equivalent to $1 \in \mathcal{B}$. Since $\mathcal{B}$is the closure of $\bigcup_N \mathcal{B}_N$, this is equivalent to the finite distances tending to zero.

The system (27) is the exact finite least-squares system. The difficulty is no longer formal; it is the asymptotic estimation of the inverse Gram form in (28), or equivalently of the boundary approximation problem in (32).

6. Fixed-window density and compact–tail separation

The boundary formulation (32) involves the whole real line. We first prove that approximation has no local obstruction on bounded intervals.

Fix $T > 0$, and set $I_T = [-T, T]$. For $n \geq 2$, let

$$\psi_n(t) = \frac{1}{n} - n^{-1/2-it}. \qquad (34)$$

Lemma 7. The linear span of $\{\psi_n : n \geq 2\}$ is dense in $L^2(I_T)$.

Proof. Let $g \in L^2(I_T)$be orthogonal to every ψ_n. We define:

$$F(z) = \int_{-T}^{T} \bar{g}(t)\, e^{-itz}\, dt. \qquad (35)$$

The function F is entire and of exponential type at most T. Orthogonality gives

$$F(\log\, n) = n^{-1/2} F(0), n \geq 2. \qquad (36)$$

Hence

$$H(z) = F(z) - F(0)e^{-z/2}$$

vanishes at every $\log n,\ n \geq 2$. The function $H$is entire of finite exponential type. If it were not identically zero, Jensen's formula would imply that the number of its zeros in $|z| \leq R$is $O(R)$. But the zeros $\log\, n \leq R$already give $e^R + O(1)$zeros. Therefore $H \equiv 0$.

Thus $F(z) = F(0)e^{-z/2}$. On the real axis, F is the Fourier transform of a compactly supported L^2-function and hence belongs to $L^2(\mathbb{R})$. Since $e^{-x/2} \notin L^2(\mathbb{R})$, we must have $F(0) = 0$. Thus $F \equiv 0$, and the injectivity of the Fourier transform gives $g = 0$.

Multiplication by the zeta boundary value preserves density.

Proposition 8. For every $T > 0$,

$$\operatorname{span}\left\{\zeta\left(\frac{1}{2}+it\right)\psi_n(t) : n \geq 2\right\} = L^2(I_T). \qquad (37)$$

Proof. Suppose $f \in L^2(I_T)$is orthogonal to every function in (37). Then

$$f(t)\overline{\zeta\left(\frac{1}{2}+it\right)}$$

is orthogonal to all ψ_n. By Lemma 7, this product is zero almost everywhere. The zeros of $\zeta(1/2 + it)$are isolated and hence have measure zero in I_T. Therefore $f = 0$almost everywhere.

Thus, for each fixed T, one can approximate the constant function 1in $L^2(I_T)$by functions of the form $\zeta(s_t)A(s_t)$. The global problem is harder because Tmust tend to infinity.

For

$$A(s) = \sum_{n \le N} c_n \left(\frac{1}{n} - n^{-s}\right),$$

define the coefficient mass

$$L(A) = \sum_{n \le N} | c_n | n^{-1/2}. \qquad (38)$$

Lemma 9. For every $\varepsilon > 0$and $T \ge 2$,

$$\int_{|t|>T} | 1 - \zeta(s_t) A(s_t) |^2 \, d\nu(t) \ll_\varepsilon T^{-1} + L(A)^2 T^{-1/2+2\varepsilon}. \qquad (39)$$

Proof. The classical convexity bound gives

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon (1 + | t |)^{1/4+\varepsilon}. \qquad (40)$$

Also,

$$| A(1/2 + it) | \le L(A).$$

Therefore, for $| t | > T$,

$$| 1 - \zeta(s_t) A(s_t) |^2 \ll_\varepsilon 1 + L(A)^2 (1 + | t |)^{1/2+2\varepsilon}.$$

Since $d\nu(t) \asymp t^{-2} dt$for large $| t |$, integration over $| t | > T$gives (39).

Fixed-window density proves local approximation. Lemma 9 shows that global approximation follows if the approximants can be chosen with controlled coefficient mass. The remaining difficulty is precisely this uniformity.

7. Conclusion in main reduction theorem

We now state the final result for this frame. It is an exact reduction within the Ramanujan–Beurling framework.

Theorem 10. The following statements are equivalent.

(i) The Riemann Hypothesis holds.

(ii) There exist polynomials

$$A_N(s) = \sum_{n=2}^{N} c_{N,n} \left(\frac{1}{n} - n^{-s}\right) \qquad (41)$$

such that

$$\int_{-\infty}^{\infty} | 1 - \zeta \left(\frac{1}{2}+it\right) A_N \left(\frac{1}{2}+it\right) |^2 \frac{dt}{2\pi(\frac{1}{4}+t^2)} \longrightarrow 0. \qquad (42)$$

Proof of Theorem 10. By (32), the left-hand side of (42), minimized over all polynomials of the form (41), is exactly d_N^2. Therefore (42) is equivalent to $d_N \to 0$. Proposition 6 identifies this with the integer-dilation Nyman–Beurling criterion, which is equivalent to RH.

Theorem 10 is exact but still global. The next corollary separates the global estimate into a finite-window approximation and a tail condition.

Corollary 11. It is sufficient to construct $A_N$of the form (41) and numbers $T_N \to \infty$such that

$$\int_{-T_N}^{T_N} | 1 - \zeta \left(\frac{1}{2}+it\right) A_N \left(\frac{1}{2}+it\right) |^2 \, d\nu(t) \longrightarrow 0, \qquad (43)$$

and, for some $\varepsilon < 1/4$,

$$L(A_N)^2 T_N^{-1/2+2\varepsilon} \longrightarrow 0. \qquad (44)$$

Proof. Split the integral in (42) into $| t | \leq T_N$and $| t | > T_N$. The first part tends to zero by (43). The second tends to zero by Lemma 9 and (44).

The fixed-window density theorem proves that approximation on each fixed compact interval is possible. What remains is uniformity as the interval expands. We therefore isolate the final problem.

Problem 1. Construct Ramanujan-centered Dirichlet polynomials $A_N$of the form (41) and heights $T_N \to \infty$satisfying (43) and (44).

Problem 1 is the precise remaining obstruction in this approach. Ramanujan's divisor identity proves the zero-free line through positivity. The direct critical-strip extension fails because the positive square carries a pole and the pole-removed quotient loses prime-level positivity. The Hilbert-space formulation replaces coefficient positivity by closure, and the finite Gram system makes the approximation problem exact. The remaining issue is no longer a formal zero-location statement; it is the construction of uniformly effective boundary mollifiers with controlled coefficient mass. Thus the conclusion is deliberately sharp. Any Ramanujan–Beurling proof of the Riemann Hypothesis along this route must solve Problem 1.

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