An Algebraic-Geometric Derivation Using Linked Right Triangles

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Abstract

This paper presents an independent geometric derivation involving a couple of right triangles and algebraic substitution based on the Pythagorean theorem. The derivation develops a relationship from determining an unknown side length by connecting two right-triangle structures through a shared variable. By applying the Pythagorean theorem to multiple geometric configurations and performing algebraic elimination, a simplified expression for the unknown variable is obtained. The work emphasizes geometric visualization, structural reasoning, and step-by-step analytical derivation rather than the introduction of a new theorem. An example configuration is included to verify the consistency of the demonstration of the resulting relation. This exposition demonstrates how classical geometric principles may be reconstructed through independent mathematical reasoning and provides an educational illustration of algebraic-geometric problem solving using linked right-triangle systems.

Keywords: Pythagorean theorem; geometric derivation; right triangles; algebra substitution; variable elimination; Euclidean geometry; mathematical exposition.

Introduction

The Pythagorean theorem is one of the most fundamental results in Euclidean geometry and has played a central role in mathematics for centuries. The theorem establishes a relationship between the side lengths of a right triangle and serves as a foundation for many areas of geometry, algebra, trigonometry, physics, engineering, and mathematical analysis. In modern mathematics, the theorem is also closely related to coordinate geometry, distance formulas, vector spaces, and geometric modeling.

Historically, the Pythagorean theorem is commonly associated with Pythagoras, although geometric relations involving right triangles were studied by several ancient civilizations before the classical Greek period. Over time, mathematicians developed numerous proofs, geometric interpretations, and algebraic derivations related to the theorem.

https://mathworld.wolfram.com/PythagoreanTheorem.html

This paper presents an independent geometric derivation involving linked right triangles and algebraic substitution. Rather than introducing a new theorem, the objective of this work is to demonstrate how classical geometric principles may be reconstructed through visual reasoning and variable elimination technique. By connecting multiple right-triangle configurations through shared variables, a derived expression for an unknown side length is obtained using repeated applications of the Pythagorean theorem.

The derivation also illustrates the relationship between geometric visualization and algebraic manipulation. In particular, the method demonstrates how substitution and expansion techniques may be combined with geometric constructions to produce simplified expressions for unknown quantities. An example configuration is included to verify the consistency of the resulting relation and to provide a clearer interpretation of the derivation process.

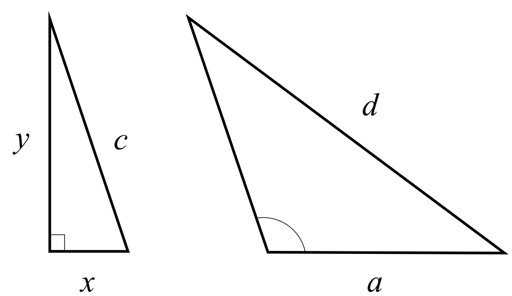
Geometric Setup

Consider a geometric configuration involving an initial right triangle and an adjacent non-right triangle connected within the same system. The first triangle is a right triangle with side lengths *x, y,* and hypotenuse *c,* where *x*, and *y* represent the perpendicular legs of the triangle. By the Pythagorean theorem, the relationship between these sides is given by:

$$c^2 = y^2 + x^2$$

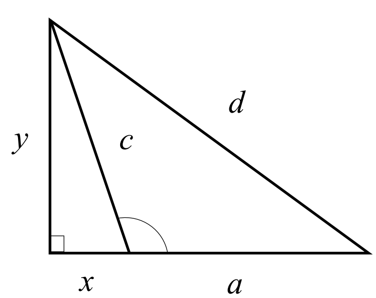
Rearranging the equation for y^2 gives:

$$y^2 = c^2 - x^2$$



Figures 1: initial right triangle and an adjacent non-right triangle that will be connected

An additional non-right triangular extension is then introduced along the horizontal direction. By combining these connected geometric components, a larger right-triangle configuration is formed, allowing the application of the Pythagorean theorem to multiple related structures. Let the base of this second triangle be represented by *(a + x),* where *a is* a fixed segment and *x* is the shared variable from the first triangle. Let the hypotenuse of the second triangle be denoted by *d,* while the vertical side remains *y.*



Figures 2: a larger right-triangle configuration

Applying the Pythagorean theorem to a larger right-triangle produces the relation:

$$d^2 = y^2 + (a + x)^2$$

The geometric construction therefore links both triangles through shared variable x and y. This configuration allows algebraic substitution and variable elimination techniques to be applied in the derivation of an explicit expression for the unknown side length.

Theorem

Consider a geometric configuration consisting of an initial right triangle with side lengths x, y, and hypotenuse c, together with an adjacent geometric extension that forms a second right triangle with hypotenuse d and horizontal base $(a + x)$, *where* $a > 0$. If both right-triangle configurations share the vertical side y, then the unknown side length x satisfies the relation.

$$x = \frac{d^2 - c^2 - a^2}{2a}$$

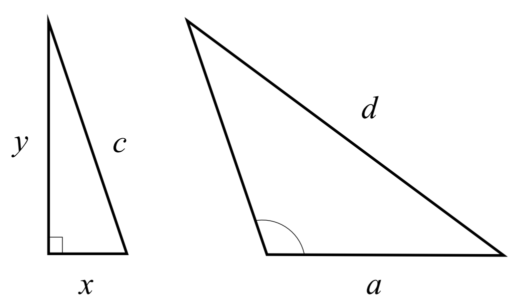
Proof

From the first right triangle, the Pythagorean theorem gives:

$$c^2 = y^2 + x^2$$

Rearranging for y^2 yields

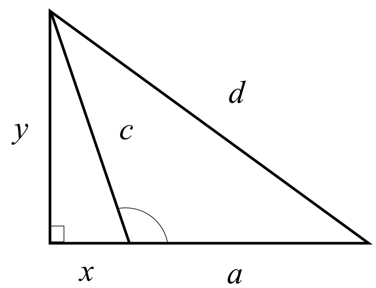
$$y^2 = c^2 - x^2$$



Figures 1: initial right triangle and an adjacent non-right triangle that will be connected

Next, consider the larger right triangle formed by the connected geometric configuration. Applying the Pythagorean theorem to the larger triangle gives:

$$d^2 = y^2 + (a + x)^2$$



Figures 2: a larger right-triangle configuration

Substituting the expression for y^2 into the second equation produces:

$$d^2 = (c^2 - x^2) + (a + x)^2$$

Expanding the squared term gives:

$$d^2 = c^2 - x^2 + a^2 + 2ax + x^2$$

Simplifying yields

$$d^2 = c^2 + a^2 + 2ax$$

Rearranging terms gives:

$$2ax = d^2 - c^2 - a^2$$

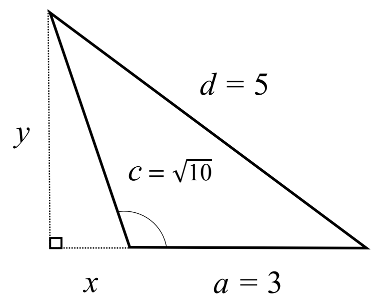
Finally, dividing both sides by 2*a,* where $a > 0$, gives the required expression:

$$x = \frac{d^2 - c^2 - a^2}{2a}$$

Thus, the unknown side length x is determined through algebraic substitution and repeated application of the Pythagorean theorem.

Example Verification

To verify the correctness of the derived relation, consider the geometric configuration shown in Figure 1. Let: a = 3, c = $\sqrt{10}$, d = 5.



Substituting these values into the derived formula:

$$x = \frac{d^2 - c^2 - a^2}{2a}$$

gives:

$$x = \frac{25 - 10 - 9}{2(3)}$$

$$x = \frac{6}{6}$$

$$\text{So, } x = 1$$

Next, substituting the value of x into:

$$y = \sqrt{c^2 - x^2}$$

yields:

$$y = \sqrt{10 - 1}$$

$$y = \sqrt{9}$$

$$y = 3$$

Therefore, the derived relation produces the values: $x = 1$, $y = 3$.

To further verify the result, these values are substituted into original Pythagorean relation of the first right triangle:

$$c^2 = x^2 + y^2$$

$$\left(\sqrt{10}\right)^2 = 1^2 + 3^2$$

$$10 = 1 + 9$$

$$10 = 10$$

Since both sides are equal, the Pythagorean theorem is satisfied. Consequently, the values obtained from the derived expression are consistent with the original geometric configuration. This verification confirms the correctness of the derivation for the chosen example.

Discussion

The derivation presented in this paper demonstrates how a geometric relationship may be obtained through the combination of multiple right-triangle structures and algebraic substitution. By connecting an initial right triangle with adjacent geometric extension and applying the Pythagorean theorem repeatedly, an explicit expression for the unknown side length x is derived. The result illustrates the interaction between geometric visualization and algebraic manipulations.

An important feature of the derivation is the use of shared variables between related triangles. Rather than treating each triangle independently, the method establishes relationships between these quantities, allowing algebraic elimination techniques to be applied. This approach provides a clear example of how geometric information may be translated into algebraic form and subsequently simplified to obtain useful relation.

Although the resulting expression is derived from classical geometric principles and does not constitute a new theorem, the derivation serves as an educational example of mathematical reconstruction. The process highlights the value of independent reasoning, geometric intuition, and systematic problem solving. In particular, it demonstrates how known mathematical tools may be combined in a structured manner to obtain nontrivial relationships among geometric quantities.

The method may also provide instructional value for students studying geometry and algebra, as it emphasizes visualization, equation formation, substitution, and verification. Such techniques are fundamental throughout mathematics and form the basis of more advanced mathematical analysis.

Finally, the derivation reinforces the central role of Pythagorean theorem in geometric reasoning. Even within relatively simple configurations, the theorem can be used repeatedly to establish relationships between unknown quantities and construct alternative pathways towards obtaining geometric results.

Conclusion

This paper presented an independent geometric derivation based on the Pythagorean theorem, algebraic substitution, and the interaction of connected triangle configurations. By relating multiple geometric structures through shared variables, an explicit expression for the unknown side length x was obtained and subsequently verified through a numerical example. The verification demonstrated that the derived relation is consistent with the original geometric construction and satisfies the Pythagorean theorem.

The derivation illustrates how classical geometric principles may be reconstructed through systematic reasoning and algebraic manipulation. In particular, the work highlights the effectiveness of variable elimination and geometric visualization in the development of mathematical relationships. Although the resulting expression is derived from established geometric concepts, the method provides an alternative perspective on the theorem within connected triangle systems.

More broadly, this exposition emphasizes the educational value of independent mathematical exploration. The process of constructing diagrams, forming equations, deriving relations, and verifying results reflects fundamental practices in mathematical problem solving. It is hoped that the presentation may serve as a useful example of how geometric intuition and algebraic techniques can be combined to investigate and understand mathematical structures.

Author's Note. The geometric derivation presented in this paper was independently conceived by the author in January 2025. The formal exposition and preparation of this manuscript began on 18th May 2026.

References

At minimum, include references such as:

i. *Euclid's Elements*, translated by Thomas L. Heath, Dover Publications.
ii. Geometry, Springer.
iii. Wolfram MathWorld: https://mathworld.wolfram.com/PythagoreanTheorem.html
iv. Encyclopaedia Britannica: https://www.britannica.com/science/Pythagorean-theorem