Article

Resolving Zeno's Paradox Through the Intrinsic Progression of Time

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Abstract

This study presents a resolution to the 2500-year-old Zeno's paradox, restoring the relationship between Space and time to the fundamentals of classical understanding. Further, the calculation shows the instability of the relativistic space-time relationship in resolving the paradox.

Zeno's paradox has four main canonical variants, Dichotomy, Achilles and Tortoise, Arrow Paradox and Moving Rows. A comprehensive analysis is conducted of both classical and contemporary relativistic approaches proposed to resolve the paradox, including standard calculus, the summation of infinite series, models invoking quantised space, quantised time, and various adaptations within relativistic framework. Despite their theoretical sophistication, these approaches are mathematically argued to fall short of providing a satisfactory resolution.

Coherent resolution emerges only under the hypothesis that time progresses as an independent, intrinsic and continuous entity, rather than being tied to spatial or kinematic variables.

Classical mechanics tends to treat time as an independent parameter, but this feature has not been explicitly utilised in prior attempts to resolve Zeno's paradox at a fundamental level. The present work proves where previous attempts, including the relativistic framework, fail and re-examines the paradox by explicitly enforcing the independence and the continuity of time as a primary principle, to consistently resolve the paradox.

Keywords: Relativity; Space Time; Lorentz; Length Contraction; Time dilation; Zeno's paradox; Quantum; Einstein; Newton Laws; Physics, Philosophy;

1. Introduction

Zeno's paradox presents a significant challenge to the description of motion. It has four main canonical variants, The Dichotomy (The Racecourse Paradox), Achilles and the Tortoise, The Arrow Paradox and The Stadium (Moving Rows). All the four are resolved in this study through a systematic analysis of The Achilles and the tortoise which is the most challenging. This variant proclaims that a faster body can never overtake a slower one when the motion is decomposed into an infinite sequence of intervals. In its commonly cited form, a faster runner attempts to catch a slower runner given an initial lead. Each time the faster runner reaches the position previously occupied by the slower runner (stages,say), the latter has advanced further, leading to an apparently infinite regress that precludes overtaking.

Notwithstanding this reasoning, empirical observation contradicts the paradoxical conclusion: in physical reality, the faster body inevitably overtakes the slower one. This discrepancy between logical construction and observed outcome forms the Zeno's paradox.

Not only the classical kinematics could not resolve the paradox, but also the contemporary Relativistic and Quantum kinematics have not been successful in resolving the paradox.

It is demonstrated, in this study, that a coherent resolution emerges only under the hypothesis that Time progresses as an independent and intrinsic entity, rather than as a derivative of spatial or kinematic variables.

Within this framework, the paradox does not arise as a physical inconsistency, thereby providing a resolution consistent with observed motion. This perspective offers a novel interpretative framework for addressing the longstanding conceptual difficulties associated with Zeno's paradox.

Although classical mechanics treats time as an independent parameter, this feature has not been explicitly utilised in prior attempts to resolve Zeno's paradox at a fundamental level. Moreover, within special relativity, the intrinsic coupling of space and time into space-time further complicates such a resolution. In this context, the present work proves where previous attempts, including the relativistic framework fail and re-examines the paradox by explicitly enforcing the independence of time as a primary principle.

The other variants of the Zeno's paradox also are then successfully tested for the framework of independent and continuous time.

2. Method of Analysis

The present study employs a theoretical and comparative analytical method to examine Zeno's paradox. Previously proposed resolutions based on mathematical formulation, summation of convergent infinite series, quantised space, quantised time, and relativistic effects are tested for their internal consistency and physical plausibility. The Achilles variant is used to test the effectiveness of the above attempts. Subsequently, an alternative interpretation is developed by explicitly treating time as an independently evolving and continuous parameter. Then all the other major variations of the Zeno's paradox are tested for this alternative.

3. Results

Analyses of the various attempts proposed to resolve the Achilles and the tortoise variant of Zeno's paradox.

3.1 The ultimate lead

If the velocity of the faster body is Vh, velocity of slower body is Vt and the initial lead is X0 ;

Time taken by the faster body to cover the initial lead X0 (to reach the 1st position of the slower body)

$$= (X_0/V_h)$$

During which the slower body has progressed to reach the new position (2nd position) by a distance

$$= (X_0/V_h) * V_t$$

This sequence continues, and the distance traversed by the faster body after n positions

$$S_h = X_0 + (X_0/V_h) * Vt + ((X_0/V_h)\ V_t/V_h) * Vt + (((X_0/V_h) * V_t/V_h) * V_t/V_h) * Vt + \ldots\ldots\ldots\ldots$$

$$= X_0 + X_0(V_t/V_h)^1 + X_0\ (V_t/V_h)^2 + X_0(V_t/V_h)^3 + X_0(V_t/V_h)^4 + \ldots.. X_0(V_t/V_h)^{n-1} \qquad (01)$$

where n denotes the number of stages corresponding to the successive positions of the slower body reached by the faster body.

The distance traversed by the slower body by then is given by

$$S_t = X_0(V_t/V_h)^1 + X_0\ (V_t/V_h)^2 + X_0(V_t/V_h)^3 + X_0(V_t/V_h)^4 + \ldots.. X_0(V_t/V_h)^{n-1} + X_0(V_t/V_h)^n \qquad (02)$$

Given the initial lead X0, the residual separation (lead) between the two is

$$S_t + X_0 - S_h = X_0(V_t/V_h)^n \qquad (03)$$

For the faster body to overtake the slower body, the residual separation must vanish. This condition requires $n \to \infty$, implying that the faster body must traverse an infinite number of stages. This constitutes Zeno's paradox.

In practical treatments, it is often assumed that for sufficiently large n, the residual term X0 $(V_t/V_h)^n$ becomes negligibly small and approximated as zero, thereby implying that the faster body overtakes the slower body.

However, this approximation does not resolve Zeno's paradox at a fundamental level. An exact physical description requires that the limiting process be treated rigorously, without assuming the vanishing of the residual separation.

3.2 Previous Approaches to resolve the Zeno's Paradox

3.2.1 Summation of Infinite Series

Mathematically, the total distance traversed by the faster body can be expressed as the sum of a convergent infinite series, given by

$$S_h = X_0 (1 - K^n)/(1 - K) \quad (04)$$

Where $K = V_t/V_h$

n = number of slower body stages the faster body has passed in its run

Similarly, for the slower body, given the initial lead of X0

$$S_t + X_0 = X_0 (1 - K^{n+1})/(1 - K) \quad (05)$$

For the faster body to reach the slower body

$S_h = S_t + X_0 = X_0(1 - K^n)/(1 - K) = X_0(1 - K^{n+1})/(1 - K)$

$[1 - K^n] = [1 - K^{n+1}]$

$$\frac{[1 - K^n]}{[1 - K^{n+1}]} = 1 \quad (06)$$

This condition is satisfied only in the limit $n \rightarrow \infty$.

Consequently, the mathematical summation of a convergent series does not resolve Zeno's paradox.

3.2.2 Quantised Space

A proposed resolution to Zeno's paradox invokes the hypothesis of quantised space, wherein spatial extent is assumed to consist of discrete units ("space quanta") that define a minimum length scale. Under this assumption, once the separation between the two bodies becomes smaller than a single spatial quantum, the faster body must traverse at least one discrete unit, thereby necessarily overtaking the slower body.

However, no empirical evidence supports a fundamental lower bound on spatial divisibility in this form. Consequently, although the quantised space hypothesis offers a conceivable resolution, it cannot be regarded as physically established.

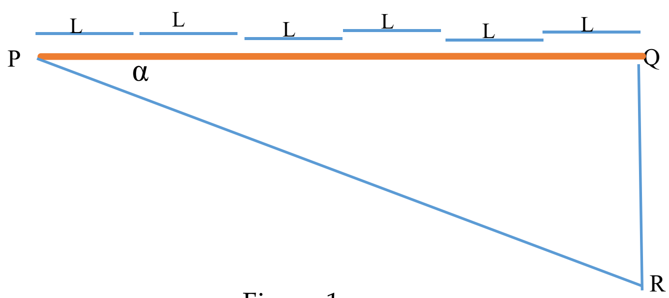

Figure 1

This limitation can be further illustrated with reference to Fig. 1, demonstrating that spatial length must be continuously divisible. Consider a line segment PQ partitioned into equal sublengths L, assumed to represent indivisible spatial quanta. If this assumption holds, any constructed segment, such as PR must consist of an integer multiple of L. However, for arbitrary angles α, such a constraint generally cannot be satisfied, implying that PS cannot be consistently defined.

Further, within the length of space quanta (if assumed to exist) the Newton Laws break, as the object has to jump the distance instantly. Which also scatters the notion on the maximum possible velocity.

3.2.3 Quantized Time

A similar resolution to Zeno's paradox can be formulated by invoking quantised time, which imposes a minimum temporal interval (time quanta) for motion. Under this assumption, the faster body advances in discrete steps corresponding to this fundamental time unit, irrespective of the instantaneous separation. Consequently, when the distance to the slower body becomes less than or equal to the distance traversed within a single time quanta, overtaking necessarily occurs within that interval.

A figure similar to Figure 1 can be constructed for time too, by lines representing the times taken to traverse the distances in Figure 1 at a given velocity. The same argument applies to lengths stands valid for time and it shows the impracticability of having time quanta.

Since no empirical evidence indicates a fundamental limit to the divisibility of time in physics, the introduction of quantised time cannot be regarded as a valid resolution of Zeno's paradox.

3.2.4 The relativistic approach

3.2.4.1 Length contraction for the faster body

In the framework of special relativity, spatial lengths are frame-dependent and undergo Lorentz contraction, characterised by the factor γ.

Where $$\gamma = \gamma_h = \frac{1}{\sqrt{(1-(\frac{v_h}{c})^2}} \qquad (07)$$

c is the velocity of light

As shown in [1], Quiroga contends that it is sufficient to consider the contraction factor γ associated solely with the faster body.

Thus, equation 4 becomes

$$S_h = \frac{X_0}{\gamma_h}(1 - K^n)/(1 - K) \qquad (08)$$

Hence, equation 6 becomes

$$\frac{[1 - K^n]}{[1 - K^{n+1}]} = \gamma_h \qquad (09)$$

This yields a finite value for n, suggesting that, within a relativistic framework that accounts for length contraction of the faster body alone, a resolution of Zeno's paradox can be obtained. However, such a treatment is incomplete, as length contraction must be consistently applied to both bodies; neglecting its effect on the slower body renders the argument physically inconsistent.

3.2.4.2 Length contraction for both the faster body and slower body

When length contraction of the slower body is also taken into account, Eq. (5) is correspondingly modified.

$$S_t + X_0 = \frac{X_0}{\gamma_t}(1 - K^{n+1})/(1 - K) \tag{10}$$

Where γ_t is the Lorentz length contraction factor for the slower body

This yields

$$\frac{1-k^n}{1-k^{n+1}} = \frac{\gamma_h}{\gamma_t} \tag{11}$$

which is inconsistent, as the left-hand side is strictly less than unity, whereas the right-hand side exceeds unity ($\gamma_h > \gamma_t$)

When length contraction is consistently applied to both the faster and slower bodies, the relativistic approach does not resolve Zeno's paradox.

3.2.4.3 Initial Lead Without Length Contraction

The slower body does not traverse the initial lead; accordingly, length contraction shall not be applied to the initial separation.

In the above analysis, the Lorentz contraction factor was applied to the initial lead of the slower body. We now reconsider the formulation by excluding length contraction from this initial separation.

Equation 2 is rewritten

$$St = X_0(V_t/V_h)^1 [1 + (V_t/V_h)^1 + (V_t/V_h)^2 + (V_t/V_h)^3 + (V_t/V_h)^4 + \ldots.. (V_t/V_h)^{n-1}]$$

$$= X_0 k [(1-k^n)/(1-k)]$$

For the motion of the slower body, length contraction is not applied to the initial lead.

$$\frac{S_t}{\gamma_t} = \frac{X_0}{\gamma_t}(1 - K^n)/(1 - K)$$

The slower body possesses an initial lead

$$\frac{S_t}{\gamma_t} + X_0 = X_0 + \frac{X_0}{\gamma_t}(1 - K^n)/(1 - K)$$

For the faster body to overtake the slower body,

$$\frac{S_t}{\gamma_t} + X_0 \leq \frac{S_h}{\gamma_h}$$

$$X_0 + \frac{X_0}{\gamma_t}(1 - K^n)/(1 - K) \leq \frac{X_0}{\gamma_h}(1 - K^n)/(1 - K)$$

$$X_0 \leq \frac{X_0}{\gamma_h}(1 - K^n)/(1 - K) - \frac{X_0}{\gamma_t}(1 - K^n)/(1 - K)$$

$$X_0 \leq X_0\left(\frac{1}{\gamma_h} - \frac{1}{\gamma_t}\right)(1 - K^n)/(1 - K)$$

$X_0 \leq 0$, as $\gamma_t < \gamma_h$, and $k < 1$

Therefore, for the faster body to reach or overtake the slower body, the initial separation X_0 must be zero or negative. This condition is physically inconsistent with the assumed setup.

Consequently, the relativistic approach does not resolve Zeno's paradox.

3.2.4.4 Length Contraction During Acceleration

If the faster body attains its maximum velocity, Vh , prior to the start of the race and subsequently moves with constant velocity, the effect of acceleration of the faster body does not contribute to the analysis. An analogous argument applies to the slower body.

Therefore, the relativistic framework does not resolve Zeno's paradox.

3.2.4.5 Time Dilation

In all the above calculations, we considered velocities and lengths. Lengths do not have a time component and only the length contraction is applicable. Velocities do have a time component. However, the length contraction factor and the time dilation factor in velocities cancel off and no time dilation alone is applicable.

V classical = L/T

V relativistic = $(L/\gamma) / (T/\gamma)$ = V classical (numeric value)

Therefore the time dilation has no effect on the above analysis and the conclusion remains, that, relativistic approach does not resolve Zeno's paradox.

3.3 Time as an Independent Parameter on the variants of Zeno's Paradox

Consider the hypothesis that time evolves independently of spatial processes, i.e., it progresses irrespective of events occurring in space and space itself. Under this assumption, the motion of the bodies is described as a continuous sequence of positional changes in space, while time advances concurrently as an independent parameter.

Velocity is then defined by comparing the spatial displacement with the corresponding independently elapsed time interval, rather than treating time as derived from spatial progression.

In this framework, the underlying mechanism differs fundamentally: time progresses independently of spatial processes and is not influenced by events occurring in space. Consequently, physical analysis must be formulated by comparing spatial displacements over a given interval of unattached time. If, within a specified time span, a body traverses a certain distance, its velocity is then defined by the ratio of that displacement to the corresponding time interval, which had no binding to the dimensions and directions of space.

3.3.1 Achilles and the Tortoise

For illustration, consider Zeno's paradox with a finite time interval of one second (This is a take from continuous time and not a discrete time period). To maintain a constant velocity V_h (m/s), the faster body must traverse a distance V_h (m) within this interval, irrespective of the instantaneous separation from the slower body. If, at any stage, the distance between the two bodies becomes less than or equal to Vh (m), the faster body must necessarily overtake the slower body within the next subsequent second. It cannot restrict its motion to merely reaching the previous position of the slower body within that time period without reducing its velocity. In the independent time framework, the one second considered in the illustration is not a time quanta. It is infinitely divisible and always bigger than zero as time is flowing independently and has no halting.

This allows the faster body to overtake the slower body resolving the Zeno's paradox.

3.3.2 The Dichotomy (The Racecourse Paradox)

Before an object can reach a destination, it must reach the halfway point.
Before halfway, it must reach quarter-way and the pattern continues requiring an infinite number of tasks to complete to start moving. Hence motion cannot begin, was the challenge of Zeno. This can similarly argued to say that no journey can finish.

Of the infinitely divisible space, this breaking into halves shall be continued indefinitely. However, while the division is being done, time progresses independently. We can keep dividing, but time flows. Since time advances independently, spatial displacement must continuously occur in accordance with the specified velocity. Thus, the continuous progression of time necessarily enforces the corresponding spatial displacements in accordance with Newtonian Laws, making the initiation, continuation, and completion of motion possible.

Reiterate, Zeno argued that an object has to move the first ever smallest possible distance to start motion, which it cannot because that first smallest possible distance too can be divided and so on. The fact that it has to move the nearest smallest spatial distance to start motion is correct. In the uniform space this assumed smallwst spatial distance is bordering zero, but never zero. Newtonian Laws enforces that if the object is already moving it would continue to move at the same velocity if no force is applied. Therefore, it keeps moving the distances that has to match the passing of time and its velocity. The size of the nearest smallest spatial distance is immaterial..

In this framework, the state of division proposed by Zeno has no direct relevance to the actual movement of objects. Motion proceeds in accordance with Newtonian laws.

3.3.3 The Stadium (Moving Rows / The Rows Paradox)

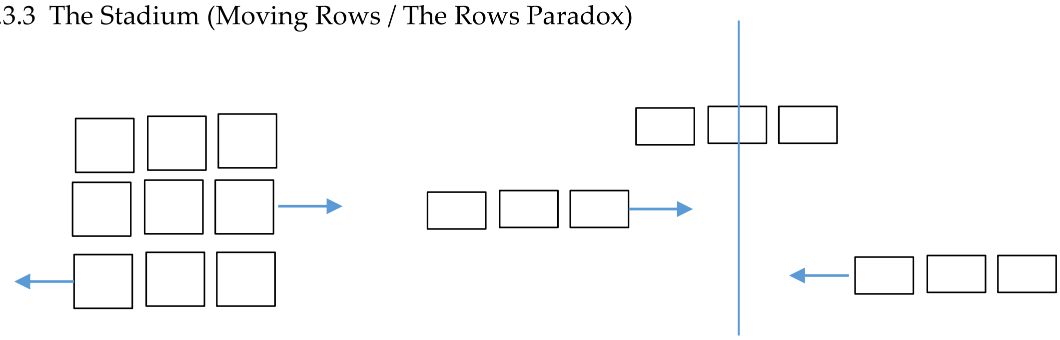

Fig 02 – Moving Rows

The challenge here is logically misleading and brings no paradox. It is discussed here only for the sake of completion.

Consider three rows of equal sized compartments (Fig 02), one stationary, one moving right and the other moving left at the equal speed of the one moving right.

The right moving row takes a certain time for a number of its compartments to cover a same number of compartments of the stationary row, and the same number of compartments of the left moving row covers the same set of compartments of the stationary row at the same time. Zeno argued that with respect to the stationary row, the left and right moving rows each has passed equal number of their compartments, therefore two have also covered the same number of compartments of each other. But taken separately, the two oppositely moving rows need half the time to cover the same number of compartments, making the paradox that half the time equals the full time.

In the hindsight, there seems a no paradox as the right and left moving bodies start at two different points, the left and the right ends of the set of the compartments of the stationary row. Before begin to pass each other, the two moving rows have to meet, for which they travel half the distance of full length of the stationary compartment set. That consumes a half time which had been hidden in the Zeno's demonstration. It then needs another half time to pass each other and reach the opposite ends of the stationary set of compartments. In total it is one full time interval, which is equal to the time that moving rows take to pass the stationary row. There is apparently no paradox. Even if the lengths of the compartments are infinitesimally small the argument stands and no paradox persists.

However, if the length of the compartments is zero, the moving rows starts crossing each other right away without having to travel to meet. Then the Paradox appears intact, half the time equals full time. Yet, now each of the three rows contains no lengths. None has therefore passed nothing. No movement has happened and there will be no paradox.

3.3.4 The Arrow Paradox

“A flying arrow occupies a space exactly equal to itself at any given instant. At that instant, it occupies only that amount of space and not moving. For all the instants the argument stands. Thus throughout all time the arrow is at rest.”

Moving arrow is at rest all the time, is the Zeno’s Arrow paradox.

Of the independent and continuous framework of time, time is independently and continuously progressing and cannot consider of instants of no progress. . All the instants of time have positive magnitudes, however much smaller it could be. The arrow moves the distance relevant to its velocity at that small instant. It therefore traverses a larger space than its size during that instant. Thus during all the instances, the Arrow is moving.

Therefore, within the framework in which time evolves as continuous and independent parameter, Zeno’s time paradoxes ceases to exist.

4. Discussion

Zeno’s paradox remained a paradox for so long because the attempted resolutions have not considered the independence and the continuity of time and its separation from the spatial activities in its real context. The classical frameworks though carried this feature inherently, surprisingly not used extensively in resolving the paradox. The relativistic framework, on the other hand, lacked this factual reality of independence of time, hence was not successful in resolving the paradox. The continuity and the independence of time from the space has consistently resolved the Zeno’s paradox. Interestingly, the resolution yields an irredeemable challenge to Space-Time relationship in relativity.

The essential distinction lies in the treatment of time, which is an independent parameter that evolves irrespective of spatial dynamics. Paradox persists only if time can be halted until the process of division is done. Instead time progresses continuously and independently where one can relate spatial activities within the framework of independent time, for demonstrating the motion.

Accordingly, the formulation of motion must consistently respect this independence of time.

The quantised space in addition to not physically standing sound, breaks the Newton Laws within the space quanta as objects have to jump the quanta instantly for motion to happen. This also breaks the notion of the maximum possible speed.

Time flows, spatial acts do not affect time (and vice versa), and is a useful tool to compare spatial activities. If an object moved a certain distance, and the time lapsed during that movement gives us a notion of the velocity. If, moving the same distance is made during a different lapse of time, the velocity is different.

Within a framework in which time evolves as a continuous and independent parameter, together with continuous space, Zeno’s paradoxes are consistently resolved and no longer arise as physical inconsistencies. The infinite divisibility of space does not prevent motion, because the uninterrupted progression of time necessarily permits the corresponding continuous spatial displacement in accordance with the laws of motion.

It is thereby concluded that a framework in which time progresses continuously and independently of space should be regarded as the physically consistent description of reality. Within this framework, motion and spatial displacement remain possible regardless of the infinite divisibility of space.

Abbreviations

S_h - Distance travelled by the faster body

S_t - Distance travelled by the slower body

X_0 - Initial lead

V_h - Velocity of the faster body

V_t - Velocity of the slower body

γ - Lorentz contraction factor / Lorentz time dilation factor

γ_h - Lorentz contraction factor for the faster body

γ_t - Lorentz contraction factor for the faster body

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