A NEW CLOSED-FORM EXPANSION OF THE RIEMANN ZETA FUNCTION VIA ELEMENTARY SYMMETRIC POLYNOMIALS ON CONSECUTIVE INTEGERS

AYOUB ZAROUAL

Abstract. We establish a closed-form expansion of the Riemann zeta function $\zeta(s)$ at any complex point $s = a + ib$ in terms of an auxiliary real-analytic function $L(a)$ built from the Bernoulli numbers and the rising factorial. The cornerstone of the derivation is a clean differentiation identity for the elementary symmetric polynomial function $K_n^p(a)$ on consecutive integers, which we prove by a generating-function argument. Specialising the expansion to the critical line $\mathrm{Re}(s) = 1/2$ recasts the Riemann hypothesis as the simultaneous vanishing of two real power series in the imaginary part b.

1. Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1,$$

admits an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole of residue 1 at $s = 1$, and is the central object of analytic number theory [6, 16, 9, 10]. The Riemann hypothesis (RH), formulated in [14], asserts that all non-trivial zeros of ζ lie on the critical line $\mathrm{Re}(s) = 1/2$; it is one of the seven Millennium Problems [2] and remains open despite extensive numerical and structural progress [5, 4, 3].

The classical *Euler–Maclaurin expansion* of $\zeta(s)$ [8, 12, 1] provides one of the standard tools for both the asymptotic study and the high-precision numerical evaluation of the zeta function. In its symbolic form,

$$\zeta(s) \;=\; \tfrac{1}{2} \;-\; \frac{1}{1-s} \;+\; \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\,(s)_{2j-1}, \tag{1}$$

where $(B_{2j})_{j\geq 1}$ are the Bernoulli numbers at even index and $(s)_m = s(s+1)\cdots(s+m-1)$ is the rising factorial.

The aim of this note is to recast (1) in a form whose analytic content is carried entirely by an auxiliary real-analytic function L on the real line, and whose combinatorial content is encapsulated by the elementary symmetric polynomial function $K_n^p(a)$ evaluated on consecutive integers. The resulting closed form, our Theorem 6.1, separates the real and imaginary parts of $\zeta(a+ib)$ as two real power series in b, with coefficients given by the even and odd derivatives of L at the real abscissa a. Specialising to $a = 1/2$ then expresses RH as the simultaneous vanishing of two real power series in the variable b.

The combinatorial heart of the construction is Theorem 3.1, an elementary differentiation identity for the rising factorial which states that its n-th derivative is $n!$ times the elementary symmetric polynomial of degree $p - n$ on the same arithmetic progression. This identity is of independent interest in the theory of finite differences and elementary symmetric polynomials [15, 11]; we prove it by a one-line generating-function argument.

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The starting point of our construction is a closed-form rearrangement of (1) which appeared as relation (10) in our earlier work [13]; we record it here as Theorem 5.1 for the sake of self-containedness.

Outline. Section 2 fixes the notation and introduces the elementary symmetric polynomial function K_n^p. Section 3 states and proves the differentiation lemma. Section 4 introduces the auxiliary function $L(a)$ and computes its derivatives in closed form. Section 5 records the underlying Euler–Maclaurin identity. Section 6 contains our main theorem and its proof. Section 7 reformulates the result entirely in terms of K_n^p and the Bernoulli numbers, and Section 8 draws out the implication for the Riemann hypothesis. Section 9 collects concluding remarks and points to several directions for further work.

2. Notation and the function K_n^p

For non-negative integers n, p with $n \geq p \geq 0$, and for $a \in \mathbb{R}$, define

$$K_n^p(a) := \sum_{a \leq j_1 < j_2 < \cdots < j_p \leq a+n-1} j_1 \, j_2 \, \cdots \, j_p, \tag{2}$$

that is, $K_n^p(a)$ is the sum of the products of all p-element non-repeating combinations chosen from the n consecutive integers $\{a, a+1, a+2, \ldots, a+n-1\}$. We use the standard conventions

$$K_n^0(a) = 1, \qquad K_n^p(a) = 0 \quad \text{whenever } p < 0 \text{ or } p > n.$$

Equivalently, $K_n^p(a)$ is the elementary symmetric polynomial of degree p evaluated on the n consecutive integers starting at a [15, 11]. In particular,

$$K_p^p(a) = a(a+1)(a+2)\cdots(a+p-1) = (a)_p, \tag{3}$$

the rising factorial (Pochhammer symbol).

3. The differentiation lemma

Lemma 3.1. *For all $p, n \in \mathbb{N}$ and $a \in \mathbb{R}$,*

$$\left[K_p^p(a)\right]^{(n)} = n! \, K_p^{p-n}(a), \tag{4}$$

where (n) denotes the n-th derivative with respect to a.

Proof. Introduce the bivariate generating polynomial

$$P(x,a) := \prod_{k=0}^{p-1} \big(x + a + k\big). \tag{5}$$

Expanding $P(x,a)$ in powers of x, the coefficient of x^{p-q} is, by the very definition of elementary symmetric polynomials,

$$P(x,a) = \sum_{q=0}^{p} K_p^q(a) \, x^{p-q}. \tag{6}$$

Setting $u := x + a$, we obtain $P(x,a) = \prod_{k=0}^{p-1}(u+k) = (u)_p$, so P is a function of u alone. Consequently

$$\frac{\partial}{\partial a} P(x,a) = \frac{\partial}{\partial x} P(x,a),$$

and, by induction, $\partial_a^n P = \partial_x^n P$.

Specialising to $x = 0$, observe that $K_p^p(a) = P(0,a)$ by (5). Therefore

$$\left[K_p^p(a)\right]^{(n)} = \partial_a^n P(x,a)\Big|_{x=0} = \partial_x^n P(x,a)\Big|_{x=0}.$$

By Taylor's theorem, $\partial_x^n P(x,a)\big|_{x=0}$ equals $n!$ times the coefficient of x^n in $P(x,a)$. Reading off this coefficient from (6), the term contributing the power x^n has $p - q = n$, i.e. $q = p - n$. Hence

$$\left[K_p^p(a)\right]^{(n)} = n! \, K_p^{p-n}(a).$$

The identity (4) is valid for every $n \geq 0$. When $n > p$, the right-hand side vanishes by the convention $K_p^q = 0$ for $q < 0$, in agreement with the fact that the n-th derivative of a polynomial of degree p is zero as soon as $n > p$. □

Corollary 3.2. *Specialising Theorem 3.1 to $p = 2j - 1$ for every integer $j \geq 1$, we obtain the even/odd identities*

$$\left[K_{2j-1}^{2j-1}(a)\right]^{(2n)} = (2n)!\, K_{2j-1}^{2j-1-2n}(a) = (2n)!\, K_{2j-1}^{2(j-n)-1}(a), \tag{7}$$

$$\left[K_{2j-1}^{2j-1}(a)\right]^{(2n+1)} = (2n+1)!\, K_{2j-1}^{2j-1-2n-1}(a) = (2n+1)!\, K_{2j-1}^{2(j-n-1)}(a). \tag{8}$$

The right-hand sides vanish unless the upper index is non-negative, i.e. for $j \geq n+1$ in both (7) *and* (8)*.*

4. The auxiliary function L

Let $(b_{2j})_{j\geq 1}$ be the canonical sequence of Bernoulli numbers B_{2j} at even index [7, Ch. 9]. Define

$$L(a) \;:=\; \tfrac{1}{2} \;+\; \sum_{j=1}^{\infty} \frac{b_{2j}}{(2j)!}\, K_{2j-1}^{2j-1}(a), \qquad a \in \mathbb{R}. \tag{9}$$

The series should be understood as the Euler–Maclaurin asymptotic expansion of ζ minus its principal pole [12, 8]; it is a bona fide analytic function on $\mathbb{C} \setminus \{1\}$, and the term-by-term operations below are valid in the corresponding sectors of convergence.

Proposition 4.1. *For every $n \in \mathbb{N}$ and $a \in \mathbb{R}$,*

$$\frac{L^{(2n)}(a)}{(2n)!} = \sum_{j=n+1}^{\infty} \frac{b_{2j}}{(2j)!}\, K_{2j-1}^{2(j-n)-1}(a), \tag{10}$$

$$\frac{L^{(2n+1)}(a)}{(2n+1)!} = \sum_{j=n+1}^{\infty} \frac{b_{2j}}{(2j)!}\, K_{2j-1}^{2(j-n-1)}(a). \tag{11}$$

Proof. Differentiating (9) term by term and applying Theorem 3.2,

$$L^{(2n)}(a) \;=\; \sum_{j=1}^{\infty} \frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2j-1}(a)\right]^{(2n)} \;=\; (2n)! \sum_{j=1}^{\infty} \frac{b_{2j}}{(2j)!}\, K_{2j-1}^{2(j-n)-1}(a).$$

For $j \leq n$ the upper index $2(j-n)-1 \leq -1$ is negative, hence those terms vanish, and the sum collapses to $j \geq n+1$. Dividing by $(2n)!$ proves (10). Identity (11) is identical, with $(2n+1)$ in place of $(2n)$ and the odd half of Theorem 3.2 in place of (7). □

5. The starting expansion

The starting point of the construction, which appeared as relation (10) in our previous work [13], is a closed-form rearrangement of the classical Euler–Maclaurin expansion of $\zeta(s)$; we record it here for the sake of self-containedness.

Lemma 5.1. *For every $s \in \mathbb{C} \setminus \{1\}$,*

$$\zeta(s) \;=\; -\frac{1}{1-s} \;+\; L(s), \tag{12}$$

where L is given by (9) *with $b_{2j} = B_{2j}$ (Bernoulli numbers).*

Proof. Apply the Euler–Maclaurin summation formula [8, 12] to the partial sums $\sum_{n=1}^{N} n^{-s}$ with $f(x) = x^{-s}$. Letting $N \to \infty$ (in the sense of asymptotic expansions, with the regularising convention $b_{2j} = B_{2j}$) and using $f^{(2k-1)}(1) = -(s)_{2k-1}$, one obtains

$$\zeta(s) \;=\; \tfrac{1}{2} \;-\; \frac{1}{1-s} \;+\; \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\,(s)_{2j-1},$$

which is exactly (12) once L is identified through (9) and using $K_{2j-1}^{2j-1}(s) = (s)_{2j-1}$ from (3). □

6. Main theorem

Theorem 6.1. *Let $s = a + ib$ with $a, b \in \mathbb{R}$ and $s \neq 1$. Then*

$$\begin{aligned} \zeta(s) \;=\; & -\frac{1-a}{(1-a)^2+b^2} \;+\; \sum_{n=0}^{\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(a) \\ & + \; i\left[\sum_{n=0}^{\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(a) \;-\; \frac{b}{(1-a)^2+b^2}\right]. \end{aligned} \tag{13}$$

Proof. By Theorem 5.1, we have the identity

$$\zeta(s) \;=\; -\frac{1}{1-s} \;+\; L(s). \tag{14}$$

Step 1 — Real and imaginary parts of $1/(1-s)$. With $s = a+ib$ we have $1-s = (1-a)-ib$, hence

$$\frac{1}{1-s} \;=\; \frac{1}{(1-a)-ib} \;=\; \frac{(1-a)+ib}{(1-a)^2+b^2} \;=\; \frac{1-a}{(1-a)^2+b^2} \;+\; i\,\frac{b}{(1-a)^2+b^2}. \tag{15}$$

Step 2 — Taylor expansion of L around a. L is real-analytic on $\mathbb{R}$ and analytic in a complex neighbourhood of a. Hence

$$L(s) \;=\; L(a+ib) \;=\; \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} L^{(n)}(a).$$

Splitting the sum into even and odd indices and using $i^{2n} = (-1)^n$, $i^{2n+1} = i\,(-1)^n$,

$$L(s) \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(a) \;+\; i\sum_{n=0}^{\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(a). \tag{16}$$

Step 3 — Substitution. Plugging (15) and (16) into (14),

$$\begin{aligned} \zeta(s) \;=\; & -\left[\frac{1-a}{(1-a)^2+b^2} + i\,\frac{b}{(1-a)^2+b^2}\right] \\ & + \sum_{n=0}^{\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(a) \;+\; i\sum_{n=0}^{\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(a). \end{aligned}$$

Regrouping real and imaginary parts yields exactly (13). □

7. Reformulation in terms of K_n^p

By Theorem 4.1, the derivatives $L^{(2n)}(a)$ and $L^{(2n+1)}(a)$ in Theorem 6.1 can be replaced by their explicit K-series. Doing so in (13) gives the announced *new formula*, expressed entirely through the elementary symmetric polynomial function K_n^p on consecutive integers and the Bernoulli numbers:

$$\begin{aligned} \zeta(s) \;=\; & -\frac{1-a}{(1-a)^2+b^2} \;+\; \sum_{n=0}^{\infty}(-1)^n\, b^{2n} \sum_{j=n+1}^{\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \\ & + \; i\left[\sum_{n=0}^{\infty}(-1)^n\, b^{2n+1} \sum_{j=n+1}^{\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \;-\; \frac{b}{(1-a)^2+b^2}\right]. \end{aligned} \tag{17}$$

8. Implication for the Riemann hypothesis

The Riemann hypothesis (RH) asserts that all non-trivial zeros of ζ lie on the critical line $\mathrm{Re}(s) = 1/2$ [14, 2]. Setting $a = 1/2$ in Theorem 6.1 and demanding $\zeta(\frac{1}{2} + ib) = 0$ separates into the real and imaginary equations

$$-\frac{1/2}{1/4+b^2} + \sum_{n=0}^{\infty}(-1)^n \frac{b^{2n}}{(2n)!} L^{(2n)}(\tfrac{1}{2}) = 0, \tag{18}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{b^{2n+1}}{(2n+1)!} L^{(2n+1)}(\tfrac{1}{2}) - \frac{b}{1/4+b^2} = 0. \tag{19}$$

The Riemann hypothesis is therefore equivalent to the statement that every non-trivial real number b satisfying both (18) and (19) corresponds to a zero of $\zeta(\frac{1}{2} + ib)$: the location of the non-trivial zeros is governed entirely by these two real series in the variable b, evaluated at $a = 1/2$.

9. Concluding remarks

(1) **Combinatorial content.** The sole non-trivial analytic ingredient of the derivation is Theorem 3.1: the n-th derivative of the rising factorial $(a)_p$ equals $n!$ times the elementary symmetric polynomial of degree $p-n$ on the same arithmetic progression. The proof is a one-line consequence of the generating-function identity $\partial_a P = \partial_x P$ for $P(x,a) = \prod_{k=0}^{p-1}(x+a+k)$.

(2) **Convergence.** The series defining $L(a)$ in (9) is the Euler–Maclaurin asymptotic expansion of ζ [12, 8]; it is divergent for any fixed a but is asymptotic in suitable sectors. All the derivative identities of Section 4 hold formally and pointwise after appropriate truncation. The Taylor expansion of L used in Step 2 of the proof of Theorem 6.1 is convergent in a complex neighbourhood of a once L is identified with its analytic continuation $\zeta(s)+1/(1-s)$.

(3) **Sequence freedom.** If the sequence (b_{2j}) is replaced by a different choice of weights, formula (13) continues to hold with ζ replaced by the corresponding modified Dirichlet-like function. The Bernoulli choice $b_{2j} = B_{2j}$ is the unique one for which the residual $\zeta(s) + 1/(1-s)$ admits the prescribed expansion through the rising factorial.

(4) **Independent interest.** Theorem 3.1 is, by itself, a clean elementary fact about elementary symmetric polynomials on arithmetic progressions, and may be useful in other contexts (combinatorial identities, finite-difference calculus, generating functions of Stirling type [15, 7]).

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Independent reseacher
Email address: ayoub.zaroual31@gmail.com