A Moment Framework for the Riemann Hypothesis

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Abstract

The Riemann Hypothesis remains one of the deepest open problems in mathematics because it connects the hidden geometry of complex zeros with the distribution of prime numbers. This paper develops a structured analytic framework that transfers the problem from its usual complex form into a real-variable setting where the desired zero structure can be studied through positivity. The central idea is to work with the completed zeta function after normalization and transformation, and to identify a precise positivity principle that would force the corresponding zeros into the correct location.

The framework reduces the problem to a compact moment condition for the central logarithmic coefficients of the normalized completed function. This condition is then connected with complete Bernstein functions, Stieltjes functions, Hankel positivity, finite-difference inequalities, and an equivalent logarithmic integral representation. Each step is formulated as a rigorous implication, so the remaining difficulty is isolated in one concrete positivity theorem rather than hidden inside formal manipulation.

The approach is designed to avoid the common weaknesses of many proposed arguments for the Riemann Hypothesis. It does not use the Euler product outside its valid region, does not infer the result from symmetry alone, does not replace the original zeta function with a modified object, and does not rely on numerical evidence as proof. Instead, it gives a connected chain from central derivative positivity to a Stieltjes logarithmic derivative, from there to a negative-real-axis zero structure, and finally back to the critical-line statement.

The paper does not claim a completed proof of the Riemann Hypothesis. Its contribution is a clean reduction that identifies a single remaining positivity problem in a form suitable for rigorous verification. This gives a clear and testable route for future work, with finite conditions that can be studied through moment theory, operator theory, and the analytic theory of special functions.

1 Introduction

Riemann's zeta function is initially defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad \Re(s) > 1. \tag{1.1}$$

Riemann's continuation and functional equation reveal that this function is governed by a symmetric entire object after completion. The Riemann Hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$. The problem goes back to Riemann's memoir [1] and remains one of the Clay Millennium Prize Problems [2]. The present paper does not use the Dirichlet series or Euler product outside their natural domain. It works instead with the completed function, where the analytic symmetry of the problem is exact.

The completed zeta function is

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.2}$$

It is entire and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{1.3}$$

The critical-line variable is

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right). \tag{1.4}$$

Then Ξ is an even real entire function,

$$\Xi(-z) = \Xi(z), \qquad \Xi(\overline{z}) = \overline{\Xi(z)}. \tag{1.5}$$

The Riemann Hypothesis becomes the real-zero statement

$$\text{RH} \iff Z(\Xi) \subset \mathbb{R}. \tag{1.6}$$

The present framework transforms this real-zero problem into a complete-Bernstein problem. After normalization,

$$G(z) = \frac{\Xi(z)}{\Xi(0)}, \tag{1.7}$$

we introduce

$$F(t) = G(i\sqrt{t}). \tag{1.8}$$

Because G is even, F is a single-valued entire function of t. Its zeros lie on the negative real axis exactly when the zeros of G are real. Thus

$$\text{RH} \iff Z(F) \subset (-\infty, 0). \tag{1.9}$$

The main idea is to look not at F alone, but at its logarithmic derivative. If

$$\Psi(t) = \log F(t), \tag{1.10}$$

then the desired structural property is that Ψ is a complete Bernstein function, or equivalently that Ψ' is a Stieltjes function. This paper develops that criterion in full detail and reduces the final unproved part to explicit positivity conditions on central logarithmic derivatives of ξ.

2 Classical analytic setting

The classical references for the analytic theory of $\zeta(s)$ include Titchmarsh and Edwards [3, 4]. The completed function ξ eliminates the pole of ζ at $s = 1$, incorporates the gamma factor, and turns the functional equation into the simple symmetry

$$\xi(s) = \xi(1 - s). \tag{2.1}$$

The zeros of ξ are precisely the nontrivial zeros of ζ, counted with multiplicity.

The transformation

$$s = \frac{1}{2} + iz \tag{2.2}$$

maps the critical line $\Re(s) = 1/2$ onto the real z-axis. Thus a zero $s = \rho$ of ξ corresponds to

$$z = -i\left(\rho - \frac{1}{2}\right). \tag{2.3}$$

Consequently

$$\Re(\rho) = \frac{1}{2} \quad \Longleftrightarrow \quad z \in \mathbb{R}. \tag{2.4}$$

The completed function admits a Fourier representation of the form

$$\Xi(z) = 2\int_0^\infty \Phi(u)\cos(zu)\,du, \tag{2.5}$$

where Φ is the Riemann kernel. One standard form is

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}. \tag{2.6}$$

This kernel is smooth and rapidly decreasing as $u \to +\infty$. The kernel representation is not used here as a formal proof of zero localization; rather, it provides an analytic source for the central coefficients and motivates the positivity structures introduced below.

Hardy's theorem that infinitely many zeros lie on the critical line [5], and the large-scale numerical verifications of Platt and Trudgian [6], give important evidence, but neither type of result proves RH. The present framework is instead aimed at a global structural criterion.

3 From real zeros to negative-axis zeros

The normalized function G is even and satisfies $G(0) = 1$. Hence there exists an entire function H such that

$$G(z) = H(z^2). \tag{3.1}$$

The choice

$$F(t) = G(i\sqrt{t}) \tag{3.2}$$

is therefore the same as

$$F(t) = H(-t). \tag{3.3}$$

Since

$$G(z) = F(-z^2), \tag{3.4}$$

a zero z_0 of G corresponds to a zero

$$t_0 = -z_0^2 \tag{3.5}$$

of F. Thus $z_0 \in \mathbb{R}$ if and only if $t_0 \in (-\infty, 0]$. Because $G(0) = 1$, zero is not a zero. We have the exact equivalence

$$Z(G) \subset \mathbb{R} \quad \Longleftrightarrow \quad Z(F) \subset (-\infty, 0). \tag{3.6}$$

Since $Z(G) = Z(\Xi)$, this gives

$$\mathrm{RH} \quad \Longleftrightarrow \quad Z(F) \subset (-\infty, 0). \tag{3.7}$$

This conversion is important because negative-real-zero entire functions are naturally connected to Stieltjes transforms, complete Bernstein functions, continued fractions, and moment theory. It changes the geometry of the problem from a vertical line in the s-plane to a one-dimensional support condition on the negative real axis in the t-plane.

4 Stieltjes and complete Bernstein functions

A Stieltjes function is a function S admitting a representation

$$S(t) = a + \int_0^\infty \frac{d\sigma(x)}{t + x}, \qquad a \geq 0, \quad \sigma \geq 0, \tag{4.1}$$

under the usual local integrability condition. Such functions are analytic on

$$\mathbb{C} \setminus (-\infty, 0], \tag{4.2}$$

positive on $(0, \infty)$, and map the upper half-plane into the lower half-plane. Complete Bernstein functions are closely related: a function Ψ is complete Bernstein if and only if Ψ' is Stieltjes, under standard normalization [7, 8].

The representation most suited to the present problem is

$$\Psi(t) = \int_0^R \log(1 + tx)\, d\mu(x), \qquad \mu \geq 0, \quad R < \infty. \tag{4.3}$$

Differentiating gives

$$\Psi'(t) = \int_0^R \frac{x}{1 + tx}\, d\mu(x). \tag{4.4}$$

This derivative is a Stieltjes-type transform after the change of variable $y = 1/x$, and its singularities lie only on the negative real axis.

The key observation is therefore the following: if

$$\Psi(t) = \log F(t) \tag{4.5}$$

admits the positive logarithmic representation above, then F cannot have zeros away from the negative real axis. This is the analytic backbone of the framework.

5 Main complete-Bernstein criterion

Theorem 5.1 (Complete-Bernstein criterion)**.** *Let*

$$G(z) = \frac{\Xi(z)}{\Xi(0)}, \qquad F(t) = G(i\sqrt{t}). \tag{5.1}$$

Suppose there exist $R > 0$ and a positive measure μ on $[0, R]$ such that

$$\log F(t) = \int_0^R \log(1 + tx)\, d\mu(x). \tag{5.2}$$

Then the Riemann Hypothesis is true.

Proof. Differentiation gives

$$\frac{F'(t)}{F(t)} = \int_0^R \frac{x}{1 + tx}\, d\mu(x). \tag{5.3}$$

The right-hand side is analytic away from the negative real axis. Therefore the poles of F'/F, which are exactly the zeros of F counted with multiplicity, can occur only on $(-\infty, 0)$. Thus

$$Z(F) \subset (-\infty, 0). \tag{5.4}$$

Since $F(t) = G(i\sqrt{t})$, all zeros of G are real. Since $G(z) = \Xi(z)/\Xi(0)$, all zeros of Ξ are real. This is RH. □

The theorem is deliberately phrased as a criterion rather than as a completed proof. The remaining problem is to prove the positive logarithmic representation for the transformed completed zeta function.

6 The central logarithmic coefficient sequence

Write

$$\log G(z) = \sum_{m=1}^{\infty} a_m z^{2m} \tag{6.1}$$

near $z = 0$. The evenness of G ensures that only even powers occur. Define

$$p_m = -m a_m, \qquad m \geq 1. \tag{6.2}$$

This is the central sequence of the framework.

Now

$$F(t) = G(i\sqrt{t}), \tag{6.3}$$

so

$$\log F(t) = \sum_{m=1}^{\infty} (-1)^m a_m t^m. \tag{6.4}$$

If the positive logarithmic representation holds, then

$$\int_0^R \log(1+tx)\, d\mu(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\int_0^R x^m\, d\mu(x) \right) t^m. \tag{6.5}$$

Comparison of coefficients gives the exact moment identity

$$p_m = \int_0^R x^m\, d\mu(x), \qquad m \geq 1. \tag{6.6}$$

Thus the analytic criterion is equivalent, at the level of Taylor coefficients, to the assertion that $\{p_m\}_{m\geq 1}$ is a compact positive moment sequence. This sequence is defined without reference to the unknown zero locations, which is essential for non-circularity.

7 Compact moment criterion

Theorem 7.1 (Compact moment criterion)**.** *Let*

$$p_m = -m[z^{2m}] \log \frac{\Xi(z)}{\Xi(0)}. \tag{7.1}$$

Suppose there exist $R > 0$ and a positive measure μ on $[0, R]$ such that

$$p_m = \int_0^R x^m\, d\mu(x), \qquad m \geq 1. \tag{7.2}$$

Then RH is true.

Proof. Define

$$\Psi_0(t) = \int_0^R \log(1+tx)\, d\mu(x). \tag{7.3}$$

The Taylor expansion of Ψ_0 at the origin is

$$\Psi_0(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m t^m. \tag{7.4}$$

By the definition of p_m,

$$\log F(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m t^m. \tag{7.5}$$

Therefore $\log F(t) = \Psi_0(t)$ near $t = 0$. Both sides analytically continue on their common domain. Hence

$$\frac{F'(t)}{F(t)} = \int_0^R \frac{x}{1+tx}\, d\mu(x). \tag{7.6}$$

The right-hand side has singularities only on the negative real axis. Thus the zeros of F

lie on $(-\infty, 0)$. The equivalence from Section 3 gives RH. □

This is the principal theorem of the framework. It converts RH into a compact moment positivity theorem for central logarithmic derivatives of Ξ.

8 Hausdorff finite-difference formulation

Let

$$q_n = p_{n+1}, \qquad n \geq 0. \tag{8.1}$$

If

$$p_m = \int_0^R x^m \, d\mu(x), \tag{8.2}$$

then

$$q_n = \int_0^R x^{n+1} \, d\mu(x) = \int_0^R x^n \, d\sigma(x), \tag{8.3}$$

where

$$d\sigma(x) = x \, d\mu(x). \tag{8.4}$$

Thus $\{q_n\}$ is a Hausdorff moment sequence on $[0, R]$.

After scaling,

$$b_n = \frac{q_n}{R^n}, \tag{8.5}$$

the sequence $\{b_n\}$ is a Hausdorff moment sequence on $[0, 1]$. By Hausdorff's finite-difference criterion [9, 10],

$$(-1)^k \Delta^k b_n \geq 0, \qquad n, k \geq 0, \tag{8.6}$$

where

$$\Delta^k b_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} b_{n+j}. \tag{8.7}$$

Equivalently, the RH-sufficient finite-difference inequalities are

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} R^{k-j} p_{n+j+1} \geq 0, \qquad n, k \geq 0. \tag{8.8}$$

The first levels are

$$p_{n+1} \geq 0, \tag{8.9}$$

$$Rp_{n+1} - p_{n+2} \geq 0, \tag{8.10}$$

$$R^2 p_{n+1} - 2Rp_{n+2} + p_{n+3} \geq 0, \tag{8.11}$$

and

$$R^3 p_{n+1} - 3R^2 p_{n+2} + 3Rp_{n+3} - p_{n+4} \geq 0. \tag{8.12}$$

Every inequality is finite and concerns only a finite block of central coefficients.

9 Hankel matrix formulation

Moment positivity also has a Hankel-matrix formulation. The shifted sequence $q_n = p_{n+1}$ is a positive moment sequence only if

$$H_N^{(0)} = \big[q_{i+j}\big]_{i,j=0}^{N} \geq 0 \tag{9.1}$$

for every N. Since the support is contained in $[0, R]$, one also needs

$$H_N^{(1)} = R\big[q_{i+j}\big]_{i,j=0}^{N} - \big[q_{i+j+1}\big]_{i,j=0}^{N} \geq 0. \tag{9.2}$$

In terms of p_m, these conditions become

$$\big[p_{i+j+1}\big]_{i,j=0}^{N} \geq 0, \tag{9.3}$$

and

$$R\big[p_{i+j+1}\big]_{i,j=0}^{N} - \big[p_{i+j+2}\big]_{i,j=0}^{N} \geq 0. \tag{9.4}$$

The first nontrivial Hankel determinant is

$$p_1 p_3 - p_2^2 \geq 0. \tag{9.5}$$

The next compact-support determinant is

$$(Rp_1 - p_2)(Rp_3 - p_4) - (Rp_2 - p_3)^2 \geq 0. \tag{9.6}$$

These are not numerical decorations. They are finite shadows of the Stieltjes measure that would force the zeros onto the critical line.

10 Derivative-only formulation at the central point

The coefficient p_m can be written directly through central logarithmic derivatives of ξ. Since

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right), \tag{10.1}$$

we have

$$\frac{d^{2m}}{dz^{2m}} \log \Xi(z)\bigg|_{z=0} = (-1)^m \frac{d^{2m}}{ds^{2m}} \log \xi(s)\bigg|_{s=1/2}. \tag{10.2}$$

Therefore

$$p_m = -\frac{m(-1)^m}{(2m)!} \frac{d^{2m}}{ds^{2m}} \log \xi(s)\bigg|_{s=1/2}. \tag{10.3}$$

The first sign condition $p_m \geq 0$ is thus

$$(-1)^{m+1} \frac{d^{2m}}{ds^{2m}} \log \xi(s)\bigg|_{s=1/2} \geq 0. \tag{10.4}$$

The full finite-difference condition becomes

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}R^{k-j}\left[-\frac{(n+j+1)(-1)^{n+j+1}}{(2n+2j+2)!}\,\frac{d^{2n+2j+2}}{ds^{2n+2j+2}}\log\xi(s)\bigg|_{s=1/2}\right]\geq 0. \tag{10.5}$$

This is the most explicit local version of the criterion. It involves only derivatives at the symmetry point $s=1/2$, and it does not assume any information about the zero set.

11 Stieltjes logarithmic derivative

The compact moment representation implies that

$$\frac{F'(t)}{F(t)}=\int_0^R\frac{x}{1+tx}\,d\mu(x). \tag{11.1}$$

Equivalently, with the substitution $y=1/x$, this is a Stieltjes transform:

$$\frac{F'(t)}{F(t)}=\int_{1/R}^{\infty}\frac{d\nu(y)}{t+y}, \tag{11.2}$$

where ν is the push-forward of $x\,d\mu(x)$ under $y=1/x$. Thus the logarithmic derivative is Stieltjes.

Conversely, if

$$\frac{F'(t)}{F(t)} \tag{11.3}$$

is Stieltjes and is the actual meromorphic logarithmic derivative of the entire function F, then the poles of F'/F lie on the negative real axis. These poles are precisely the zeros of F. Hence

$$\frac{F'(t)}{F(t)}\ \text{Stieltjes}\quad\Longrightarrow\quad\text{RH}. \tag{11.4}$$

A practical analytic target is therefore

$$\frac{d}{dt}\log\frac{\xi(1/2-\sqrt{t})}{\xi(1/2)}\quad\text{is Stieltjes on }(0,\infty). \tag{11.5}$$

This statement is very close in strength to RH, but it is a real-variable transform property rather than a direct zero-location claim.

12 Complete monotonicity tests

A Stieltjes function is completely monotone. Hence a necessary hierarchy is

$$(-1)^n\frac{d^n}{dt^n}\left(\frac{F'(t)}{F(t)}\right)\geq 0,\qquad t>0,\quad n\geq 0. \tag{12.1}$$

The first condition is

$$\frac{F'(t)}{F(t)}>0. \tag{12.2}$$

The second is

$$\frac{d}{dt}\left(\frac{F'(t)}{F(t)}\right) \le 0. \tag{12.3}$$

Using the kernel representation, set

$$M(r) = \int_0^\infty \Phi(u)\cosh(ru)\,du, \qquad r = \sqrt{t}. \tag{12.4}$$

Then

$$F(t) = \frac{M(\sqrt{t})}{M(0)} \tag{12.5}$$

and

$$\frac{F'(t)}{F(t)} = \frac{1}{2r}\frac{M'(r)}{M(r)}. \tag{12.6}$$

Let

$$L(r) = \log M(r). \tag{12.7}$$

Then

$$\frac{F'(t)}{F(t)} = \frac{L'(r)}{2r}. \tag{12.8}$$

The first nontrivial complete-monotonicity inequality becomes

$$L'(r) - rL''(r) \ge 0. \tag{12.9}$$

Equivalently,

$$M(r)M'(r) + rM'(r)^2 - rM(r)M''(r) \ge 0. \tag{12.10}$$

This is the first global analytic inequality that the Riemann kernel must satisfy in this route.

13 Kernel and cumulant interpretation

Normalize the Riemann kernel into an even probability law

$$d\nu(u) = \frac{\Phi(|u|)}{2\int_0^\infty \Phi(v)\,dv}\,du. \tag{13.1}$$

Then

$$G(z) = \int_{\mathbb{R}} e^{izu}\,d\nu(u). \tag{13.2}$$

Let $X \sim \nu$. Then

$$F(t) = G(i\sqrt{t}) = \mathbb{E}\left(e^{\sqrt{t}X}\right) = \mathbb{E}\left(\cosh(\sqrt{t}X)\right). \tag{13.3}$$

Writing the even cumulants of X as κ_{2m}, one has

$$\log G(z) = \sum_{m=1}^{\infty} \frac{(-1)^m \kappa_{2m}}{(2m)!} z^{2m}. \tag{13.4}$$

Consequently

$$p_m = \frac{m(-1)^{m+1}\kappa_{2m}}{(2m)!}. \tag{13.5}$$

The compact moment criterion therefore becomes a theorem about alternating even cumulants of the Riemann-kernel probability law:

$$\left\{\frac{m(-1)^{m+1}\kappa_{2m}}{(2m)!}\right\}_{m\geq 1} \quad \text{is a compact positive moment sequence.} \tag{13.6}$$

This interpretation is useful because it makes clear that ordinary positivity of the kernel is not enough. The required property is a strong cumulant-level positivity.

14 Operator and determinant interpretation

If $\{p_m\}$ is a compact positive moment sequence, it is natural to seek a positive trace-class operator L such that

$$p_m = \operatorname{Tr}(L^m), \qquad m \geq 1. \tag{14.1}$$

Then

$$\log\det(I + tL) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} t^m \operatorname{Tr}(L^m). \tag{14.2}$$

Hence

$$F(t) = \det(I + tL) \tag{14.3}$$

would imply the compact-moment representation and RH.

One possible concrete operator route is to use a weighted Hankel operator generated by the Riemann kernel:

$$(K_{\Phi,w} f)(x) = \int_0^{\infty} \Phi(x+y) f(y) w(y)\, dy \tag{14.4}$$

on

$$L^2((0,\infty), w(x)\, dx). \tag{14.5}$$

If $K_{\Phi,w}$ is self-adjoint and Hilbert–Schmidt, then

$$L = c^2 K_{\Phi,w}^2 \tag{14.6}$$

is positive trace class. The determinant target becomes

$$\frac{\Xi(z)}{\Xi(0)} = \det(I - z^2 c^2 K_{\Phi,w}^2). \tag{14.7}$$

Equivalently,

$$\frac{\Xi(i\sqrt{t})}{\Xi(0)} = \det(I + tc^2 K_{\Phi,w}^2). \tag{14.8}$$

The trace identities required for this determinant are

$$c^{2m}\,\mathrm{Tr}(K_{\Phi,w}^{2m}) = p_m, \qquad m \ge 1. \tag{14.9}$$

The first identity fixes the scale:

$$c^2 = \frac{p_1}{\mathrm{Tr}(K_{\Phi,w}^2)}. \tag{14.10}$$

All higher identities are nontrivial constraints. This operator formulation is not needed for the abstract moment criterion, but it gives a constructive path toward the representing measure.

15 Jacobi operator and continued fractions

If $q_n = p_{n+1}$ is a compact moment sequence, then there is a positive measure σ on $[0, R]$ such that

$$q_n = \int_0^R x^n\, d\sigma(x). \tag{15.1}$$

The orthonormal polynomials for σ satisfy a three-term recurrence

$$xP_n(x) = \alpha_{n+1}P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x), \tag{15.2}$$

with

$$\alpha_n > 0, \qquad 0 \le \beta_n \le R. \tag{15.3}$$

The corresponding Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & \alpha_1 & 0 & \cdots \\ \alpha_1 & \beta_1 & \alpha_2 & \cdots \\ 0 & \alpha_2 & \beta_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{15.4}$$

is self-adjoint and positive with spectrum in $[0, R]$. Its resolvent gives the Stieltjes transform:

$$\int_0^R \frac{d\sigma(x)}{1+tx} = \left\langle (I + tJ)^{-1} e_0, e_0 \right\rangle. \tag{15.5}$$

This leads to a continued-fraction representation

$$\int_0^R \frac{d\sigma(x)}{1+tx} = \cfrac{q_0}{1 + \beta_0 t - \cfrac{\alpha_1^2 t^2}{1+\beta_1 t - \cfrac{\alpha_2^2 t^2}{1+\beta_2 t - \cdots}}}. \tag{15.6}$$

Thus another route to the criterion is to prove positivity of the Hankel determinants that define the Jacobi coefficients and then prove convergence of the corresponding continued fraction to F'/F. This connects the framework to the classical theory of continued fractions and moment problems [11, 12, 13].

16 Finite real-rooted approximants

The compact moment framework also gives a finite approximation scheme. Let J_N be the $N \times N$ Jacobi truncation associated with the moment sequence. Its eigenvalues satisfy

$$0 \le x_{1,N} \le x_{2,N} \le \cdots \le x_{N,N} \le R. \tag{16.1}$$

Define

$$F_N(t) = \prod_{j=1}^{N} (1 + t x_{j,N}) \tag{16.2}$$

and

$$G_N(z) = F_N(-z^2). \tag{16.3}$$

Then

$$G_N(z) = \prod_{j=1}^{N} (1 - z^2 x_{j,N}), \tag{16.4}$$

so every G_N has only real zeros. If

$$G_N \longrightarrow G \tag{16.5}$$

locally uniformly, then G belongs to the Laguerre–Pólya class [14, 15, 16, 17], and RH follows. The compact moment representation gives this convergence by identifying the logarithmic derivative of the limit. This provides a second route from central moment positivity to real-rootedness.

17 Non-circularity of the framework

The construction is non-circular because the sequence p_m is defined by central derivatives:

$$p_m = -m[z^{2m}] \log \frac{\Xi(z)}{\Xi(0)}. \tag{17.1}$$

No zero ordinate is used in this definition. Under RH one would have the identity

$$p_m = \sum_{\gamma > 0} m_\gamma \gamma^{-2m}, \tag{17.2}$$

but that formula is not used as a definition. The proof objective is precisely to obtain a positive moment representation from the analytic structure of Ξ, not from the assumed location of its zeros.

The framework also avoids the common analytic traps in RH attempts. It does not use

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{17.3}$$

inside the critical strip, where the Euler product is not valid. It does not infer RH from the symmetry

$$\xi(s) = \xi(1-s), \tag{17.4}$$

because symmetry alone allows off-line zeros in reflected pairs. It does not replace ζ by a modified function. The object remains the classical ξ, and the final target is an explicit positivity theorem for its central logarithmic derivatives.

18 The core positivity problem

The framework has one remaining mathematical core.

Conjecture 18.1 (Central compact-moment positivity)**.** *Let*

$$p_m = -m[z^{2m}] \log \frac{\Xi(z)}{\Xi(0)}. \tag{18.1}$$

There exists $R > 0$ such that

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} R^{k-j} p_{n+j+1} \geq 0 \tag{18.2}$$

for every $n, k \geq 0$.

By the Hausdorff moment theorem, the conjecture is equivalent to the existence of a positive measure σ on $[0, R]$ such that

$$p_{n+1} = \int_0^R x^n \, d\sigma(x), \qquad n \geq 0. \tag{18.3}$$

Equivalently, it is equivalent to the existence of a positive measure μ on $[0, R]$ such that

$$p_m = \int_0^R x^m \, d\mu(x), \qquad m \geq 1. \tag{18.4}$$

If this conjecture is proved, the Complete-Bernstein criterion proves RH. Thus the open core of the framework is not vague. It is a precise family of finite inequalities.

19 Suggested proof strategy for the core positivity theorem

The first layer is the sign pattern

$$p_m \geq 0, \qquad m \geq 1. \tag{19.1}$$

The second layer is Hankel log-convexity,

$$p_m p_{m+2} - p_{m+1}^2 \geq 0, \qquad m \geq 1. \tag{19.2}$$

The third layer is bounded support, which can be approached by proving

$$\sup_{m\geq 1} \frac{p_{m+1}}{p_m} < \infty. \tag{19.3}$$

Together with the higher finite differences, these conditions point toward compact moment positivity.

A possible analytic route is to express p_m through the Riemann kernel cumulants. With the normalized kernel measure $d\nu$,

$$G(z) = \int_{\mathbb{R}} e^{izu}\, d\nu(u), \tag{19.4}$$

and

$$p_m = \frac{m(-1)^{m+1}}{(2m)!} \kappa_{2m}. \tag{19.5}$$

The problem becomes a theorem about the alternating even cumulants of the Riemann-kernel distribution. A successful proof would likely require a hidden total-positivity or infinite-divisibility property of the theta kernel.

A second route is operator-theoretic. One seeks a positive trace-class operator L constructed from the Riemann kernel such that

$$p_m = \mathrm{Tr}(L^m). \tag{19.6}$$

Then

$$F(t) = \det(I + tL) \tag{19.7}$$

and RH follows. The difficulty is the trace identity, not the final implication.

20 Conclusion

The framework developed here reduces RH to a complete-Bernstein moment problem. The transformation

$$F(t) = \frac{\Xi(i\sqrt{t})}{\Xi(0)} \tag{20.1}$$

converts the critical-line problem into a negative-real-zero problem. The logarithmic transform

$$\Psi(t) = \log F(t) \tag{20.2}$$

is the central object. If Ψ is a complete Bernstein function, or equivalently if Ψ' is Stieltjes, then all zeros of F lie on $(-\infty, 0)$, all zeros of Ξ are real, and RH follows.

The coefficient form of the same condition is

$$p_m = -m[z^{2m}] \log \frac{\Xi(z)}{\Xi(0)}. \tag{20.3}$$

The decisive theorem to prove is that $\{p_m\}_{m\geq 1}$ is a compact positive moment sequence.

Equivalently, for some $R > 0$,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} R^{k-j} p_{n+j+1} \geq 0 \tag{20.4}$$

for all $n, k \geq 0$. This is the final mathematical core of the framework.

The result is a non-circular reduction rather than a completed proof. It identifies an explicit, finite-at-each-level family of inequalities whose proof would imply RH. The value of the framework is that it replaces informal spectral analogy by a concrete central-derivative positivity problem.

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