SOME NEW COLLATZ-LIKE SEQUENCES: IF ODD ADD d, IF EVEN DIVIDE BY 2

Abstract. In this paper, we describe what seems to be a new Collatz-like ("if odd/if even") function, and propose some related conjectures. For any arbitrary positive number, x, iterative operations can be made such that, when even, x is divided by two, and when odd, it is added to odd integer, d. It appears that when $x = 1$, after sufficient iterations, the sequence always reaches 1, creating a loop. The iterative function can be stated as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x + d & \text{if } x \text{ is odd.} \end{cases}$$

Introduction

In 1937 Lothar Collatz proposed that, for any arbitrary positive number, n, iterative operations can be made such that, when even, n is divided by two, and when odd, it is multiplied by three and added to one, and that when this process is sufficiently repeated, the sequence will always reach 1. This iterative function is normally stated as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

So far all known sequences always ends in the loop $1 \to 4 \to 2 \to 1$. A counterexample sequence would either continue *ad infinitum* without converging to 1, or it would end in another loop (not ending in 1). According to Paul Erdős "Mathematics may not be ready for such problems." Jeffrey Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics."

Here, we propose a similar function: that for any positive integer, x, if even is divided by two, and if odd is added to d, and that when this process is sufficiently repeated, the sequence will always reach 1. The general iterative function can be stated as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x + d & \text{if } x \text{ is odd.} \end{cases}$$

We also consider the special case for $d = 3^n$, whose iterative function is as follows:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x + 3^n & \text{if } x \text{ is odd.} \end{cases}$$

These functions resemble the Collatz Conjecture in many ways, but they are too significantly different to automatically assume that one is a corollary of the other.

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THE GENERAL CASE FOR $d + x$.
We begin by observing how when x is added to any odd number d, such that:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ d + x & \text{if } x \text{ is odd.} \end{cases}$$

We observe that when $x_0 = 1$ the sequence always returns to 1. However, only in certain cases does the sequence include all the odd numbers $< d$. We will call these Type A cases.

Type A cases.
For example, let $d = 5$:
1 6 3 16 8 4 2 1.
Odd numbers: 1 3 1 (2 odd steps).

Or let $d = 11$:
1 12 6 3 14 7 18 9 20 10 5 16 8 4 2 1.
Odd numbers: 1 3 7 9 5 1 (5 odd steps).

Or let $d = 13$:
1 14 7 20 10 5 18 9 22 11 24 12 6 3 16 8 4 2 1.
Odd numbers: 1 7 5 9 11 3 1 (6 odd steps).

Or let $d = 19$:
1 20 10 5 24 12 6 3 22 11 30 15 34 17 36 18 9 28 14 7 26 13 32 16 8 4 2 1.
Odd numbers: 1 5 3 11 15 17 9 7 13 1 (9 odd steps).

Conjectures for Type A cases.
Conjecture 1: when this process is sufficiently repeated, the sequence will always return to 1.
Conjecture 2: we conjecture that d is a prime with a primitive root of 2 (cf. A001122) iff the following conditions apply:
a) the total number of steps to reach 1 is $\frac{3*(d-1)}{2}$;
b) the number of odd steps in a loop is always equal to $\frac{(d-1)}{2}$;
c) the peak odd value reached is d–2; and
d) every odd number $< d$ and every even number up $< 2d$ is included, then

Type B cases.
When d is not a prime with a primitive root of 2, then it seems that the resulting sequence do not include all the odd numbers $< d$. For example, let $d = 9$:
1 10 5 14 7 16 8 4 2 1.
Odd numbers: 1 5 7 1 (3 odd steps).

Or when $d = 41$, then 1, 42, 21, 62, 31, 72, 36, 18, 9, 50, 25, 66, 33, 74, 37, 78, 39, 80, 40, 20, 10, 5, 46, 23, 64, 32, 16, 8, 4, 2, 1.

Notice that this sequence does not include 3, 11 or 17, for example. But when $x_0 = 3$, then all the omitted values are included:

3, 44, 22, 11, 52, 26, 13, 54, 27, 68, 34, 17, 58, 29, 70, 35, 76, 38, 19, 60, 30, 15, 56, 28, 14, 7, 48, 24, 12, 6, 3...

THE SPECIAL CASE FOR $3^n + x$.
Here we show three loops for $n = 3, 4, 5$ starting with 1 (with peak odd numbers in bold):

Loop 1a. Let $n = 3, x = 1$:
1 10 5 14 **7** 16 8 4 2 1 (9 steps).
Odd numbers: 1 5 7 1 (3 odd steps).

Loop 1b. Let $n = 4, x = 1$:
1 28 14 7 34 17 44 22 11 38 19 46 23 50 **25** 52 26 13 40 20 10 5 32 16 8 4 2 1 (27 steps).
Odd numbers: 1 7 17 11 19 23 25 13 5 1 (9 odd steps).

Loop 1c. Let $n = 5, x = 1$:
1 82 41 122 61 142 71 152 76 38 19 100 50 25 106 53 134 67 148 74 37 118 59 140 70 35 116 58 29 110 55 136 68 34 17 98 49 130 65 146 73 154 77 158 **79** 160 80 40 20 10 5 86 43 124 62 31 112 56 28 14 7 88 44 22 11 92 46 23 104 52 26 13 94 47 128 64 32 16 8 4 2 1 (81 steps).
Odd numbers: 1 41 61 71 19 25 53 67 37 59 35 29 55 17 49 65 73 77 79 5 43 31 7 11 23 13 47 1 (27 odd steps).

The formula for the toral number of steps to 1 seems to be 3^n, and the total odd steps 3^{n-1}. The peak value seems to be equal to $3^{n-1} - 2$.

Conjectured Questions. All this begs some questions:

Q.1 Does each sequence include all the odd numbers congruent to $\pm$ 1 mod 6 up to $(3^n - 1)$.

Q.2 Is the total number of steps always 3^{n-1}?

Q.3 Is the number of odd steps always 3^{n-2}?

Q.4 Does every sequence reach an odd peak after $5 \cdot 3^{n-1} - 1$ steps?

LOOPS WHERE x IS SUBTRACTED.
We also consider when x is subtracted from 3^n, such that:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3^n - x & \text{if } x \text{ is odd.} \end{cases}$$

In this case, the odd numbers do not appear to exceed $\frac{3^n-1}{2}$. Here we show three loops for $n = 3, 4, 5$ starting with 1 (with peak odd numbers in bold):

Loop 2a. Let $n = 3, x = 1$:
1 26 **13** 14 7 20 10 5 22 11 16 8 4 2 1 (14 steps).
Odd numbers: 1 13 7 5 11 1 (5 odd steps).

Loop 2b. Let $n = 4, x = 1$:
1 80 40 20 10 5 76 38 19 62 31 50 25 56 28 14 7 74 **37** 44 22 11 70 35 46 23 58 29 52 26 13 68 34 17 64 32 16 8 4 2 1 (40 steps).
Odd numbers: 1 5 19 31 25 7 37 11 35 23 29 13 17 1 (13 odd steps).

Loop 2c. Let $n = 5, x = 1$:
1 242 121 122 61 182 91 152 76 38 19 224 112 56 28 14 7 236 118 59 184 92 46 23 220 110 55 188 94 47 196 98 49 194 97 146 73 170 85 158 79 164 82 41 202 101 142 71 172 86 43 200 100 50 25 218 109 134 67 176 88 44 22 11 232 116 58 29 214 107 136 68 34 17 226 113 130 65 178 89 154 77 166 83 160 80 40 20 10 5 238 **119** 124 62 31 212 106 53 190 95 148 74 37 206 103 140 70 35 208 104 52 26 13 230 115 128 64 32 16 8 4 2 1 (123 steps).
Odd numbers: 1 121 61 91 19 7 59 23 55 47 49 97 73 85 79 41 101 71 43 25 109 67 11 29 107 17 113 65 89 77 83 5 119 31 53 95 37 103 35 13 115 1 (41 odd steps).

Conjectured Questions. Similar questions can be asked:

Q.1 Does each sequence include all the odd numbers congruent to ± 1 mod 6 up to $\frac{(3^n-1)}{2}$?

Q.2 Does every sequence reach an odd peak at an odd number such that its sum with the subsequent number is equal to 3^n?

Q.3 Is the total number of steps always $\frac{(3^n-(-1)^n)}{2^a}$?

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