A MODULAR STRUCTURE THEOREM FOR GOLDBACH REPRESENTATIONS AND A COMPUTATIONAL STUDY OF SHIFT-PROPAGATING PRIME PAIRS IN THE ARITHMETIC PROGRESSION $n \equiv 8 \pmod{30}$.

Christoper Muoki Mututu

mututuchristoper@gmail.com

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Abstract

We study a structural property of Goldbach representations which are expressions of even integers as sums of two primes within two specific arithmetic progressions modulo 30.

We prove the following theorem by elementary modular arithmetic alone requiring no unproven hypothesis and no computation.

Theorem. Let $n \equiv 8 \ (mod\ 30)$ with $n \geq 38$. Then every Goldbach pair (p, q) with $p + q = n$ and p, q prime satisfies $p \equiv q \equiv 1 \ (mod\ 6)$. Furthermore, for any $n \equiv 28 \ (mod\ 30)$ within $n \geq 28$, every Goldbach pair (p, q) of n satisfies $p \equiv q \equiv 2 \ (mod\ 3)$ which forces both $p + 10$ and $q + 10$ to be divisible by 3 and therefore composite.

As a consequence, no Goldbach pair of any $n \equiv 28 \ (mod\ 30)$ can produce a Goldbach pair of $n + 10$ via the shift $(p, q) \mapsto (p + 10, q + 10)$.

We then investigate the coupled pairs $(n, n + 20)$ where $n \equiv 8 \ (mod\ 30)$ observing that $n + 20 \equiv 28 \ (mod\ 30)$ always. For such a coupled pair, the shift $(p, q) \mapsto (p + 10, q + 10)$ maps a Goldbach pair of n to a Goldbach pair of $n + 20$ automatically in terms of the sum since $(p + 10) + (q + 10) = n + 20$ provided both $p + 10$ and $q + 10$ are prime.

We define the shift-propagation count,

$$\mathcal{R}(\mathrm{n}) = \#\left\{\mathrm{p} \leq \frac{n}{2} : p\ prime, n - p\ prime, p + 10\ prime, n - p + 10\ prime\right\}$$

and present the following conjecture supported by extensive computation.

Conjecture. For every even integer $n \equiv 8 \ (mod\ 30)$ with $n \geq 38$, we have $\mathcal{R}(\mathrm{n}) \geq 1$. That is, at least one Goldbach pair of n always shifts by $+10$ to produce a Goldbach pair of $n + 20$.

We verify this conjecture computationally for all 33,332 values of $n \equiv 8 \ (mod\ 30)$ in the range $38 \leq n \leq 999{,}980$ finding zero exceptions. The minimum value $\mathcal{R}(\mathrm{n}) = 1$ occurs only at $n = 128$ across this entire range and the average value of $\mathcal{R}(\mathrm{n})$ grows consistently with the scale of n from an average of 2.00 at the smallest values to an average of 197.69 across the full range to 10^6.

We present the modular structure theorem with complete proof, state the conjecture precisely and provide full computational verification. We make no claim of proving Goldbach's conjecture. We propose that this modular structure and the coupled pair phenomenon may serve as a foundation for future analytic work toward Goldbach's conjecture.

1. Introduction

1.1 Goldbach's Conjecture

In 1742, Christian Goldbach wrote to Leonhard Euler conjecturing that every even integer greater than two can be expressed as the sum of two prime numbers. This conjecture now known as Goldbach's conjecture, remains unproven despite nearly three centuries of effort by mathematicians worldwide.

The conjecture has been verified computationally for all even integers up to 4×10^{18} by Oliveira e Silva, Herzog and Pardi (2014). Partial analytic results exist, the strongest being Chen's theorem (1973) which proves that every sufficiently large even integer is the sum of a prime and a number with at most two prime factors but a complete proof remains out of reach.

This paper does not prove Goldbach's conjecture. We state this clearly at the outset.

1.2 What This Paper Contains

This paper contains two things.

The first is a theorem proven completely by elementary modular arithmetic. It concerns the residue classes of primes appearing in Goldbach representations of even integers in specific arithmetic progressions modulo 30.

The second is a conjecture supported by computational verification across 33,332 cases with zero exceptions up to 10^6. It concerns a shift-propagation phenomenon connecting Goldbach representations of $n \equiv 8 \ (mod\ 30)$ to those of $n + 20 \equiv 28 \ (mod\ 30)$.

We present both with complete honesty about their status.

1.3 The Arithmetic Progressions

Define the two families of even integers.

$$\mathcal{R} = \{\mathrm{n} \in \mathbb{Z}^+ : n \equiv 8\ (mod\ 30)\} = \{8, 38, 68, 98, 128, \dots\}$$

$$\mathcal{A} = \{\mathrm{n} \in \mathbb{Z}^+ : n \equiv 28\ (mod\ 30)\} = \{28, 58, 88, 118, 148, \dots\}$$

Both are infinite arithmetic progressions with common difference 30. Each has density $1/30$ among all positive integers. Together they have density $1/15$ among all even positive integers.

The key structural relationship between them is

$$n \in \mathcal{R} \Rightarrow \mathrm{n} + 20 \in \mathcal{A}$$

since $8 + 20 = 28$. This means every element of $\mathcal{R}$ is paired with an element of $\mathcal{A}$ exactly 20 units ahead of it. We call $(n, n + 20)$ a coupled pair.

The coupled pairs are

$$(38, 58), \quad (68, 88), \quad (98, 118), \quad (128, 148), \quad (158, 178), \dots$$

with consecutive coupled pairs separated by 30.

1.4 The Modular Structure Theorem

The following is proven completely in Section 2. It requires only the definition of primes and elementary modular arithmetic. No computation is needed. No unproven hypothesis is assumed.

Theorem 1.1 (Modular Structure of Goldbach Pairs).

(a) *Let* $n \equiv 8\ (mod\ 30)$ with $n \geq 38$. Then every Goldbach pair (p, q) with $p + q = n$ and p, q prime and $p, q > 3$ satisfies

$$p \equiv 1\ (mod\ 6) \quad and \quad q \equiv 1\ (mod\ 6).$$

Moreover, neither $p = 2$ nor $p = 3$ yields a valid Goldbach pair for any $n \in \mathcal{R}$ with $n \geq 38$.

(b) *Let* $n \equiv 28\ (mod\ 30)$ with $n \geq 28$. Then every Goldbach pair (p, q) with $p + q = n$ and $p, q > 3$ satisfies

$$p \equiv 2\ (mod\ 3) \quad and \quad q \equiv 2\ (mod\ 3).$$

Consequently, for every such pair,

$$p + 10 \equiv 0 \ (mod\ 3) \quad and \quad q + 10 \equiv 0 \ (mod\ 3)$$

so both $p + 10$ and $q + 10$ are divisible by 3. Since $p + 10 > 3$ and $q + 10 > 3$ for all valid pairs, both are composite.

(c) As a direct consequence of (b), for every $n \equiv 28 \ (mod\ 30)$ no Goldbach pair (p, q) of n can produce a Goldbach pair of $n + 10$ via the map $(p, q) \mapsto (p + 10, q + 10)$.

1.5 The Coupled Pair Phenomenon

Definition 1.2.

For an even integer n and a positive integer k, define the shift-propagation count

$$\mathcal{R}(\mathrm{n}) = \#\left\{\mathrm{p} \le \frac{n}{2} : p\ prime, n - p\ prime, p + 10\ prime, n - p + 10\ prime\right\}.$$

In this paper we study $\mathcal{R}_{10}(n)$ for $n \in \mathcal{R}$.

Observation 1.3 (Automatic sum property).

If $(p, n - p)$ is a Goldbach pair of n and both $p + 10$ and $n - p + 10$ are prime, then

$$(p + 10) + (n - p + 10) = n + 20.$$

So $(p + 10, n - p + 10)$ is automatically a Goldbach pair of $n + 20$. No additional verification of the sum is required.

This means $\mathcal{R}_{10}(n) \ge 1$ if and only if at least one Goldbach pair of n shifts by $+10$ to produce a Goldbach pair of $n + 20$.

The central conjecture of this paper is

Conjecture 1.4 (Coupled Pair Conjecture).

For every $n \in \mathcal{R}$ with $n \ge 38$,

$$\mathcal{R}_{10}(n) \ge 1.$$

That is, for every coupled pair $(n, n + 20)$ with $n \equiv 8 \ (mod\ 30)$, at least one Goldbach pair of n shifts by $+10$ to produce a Goldbach pair of $n + 20$.

1.6 Computational Evidence

We verify Conjecture 1.4 computationally for all $n \in \mathcal{R}$ with $38 \leq n \leq 999{,}980$. The results are as follows.

Range	Coupled pairs checked	Successes	Failures
$38 \leq n \leq 10$	1	1	0
$38 \leq n \leq 100$	4	4	0
$38 \leq n \leq 1{,}000$	34	34	0
$38 \leq n \leq 10{,}000$	334	334	0
$38 \leq n \leq 100{,}000$	3,334	3,334	0
$38 \leq n \leq 1{,}000{,}000$	33,332	33,332	0

Zero failures across all $33,332$ cases. The minimum value $\mathcal{R}_{10}(n) = 1$ occurs only at $n = 128$.

The average value of $\mathcal{R}_{10}(n)$ grows with n from 2.00 at the smallest scale to 197.69 across the full range to 10^6.

We present these results as computational evidence supporting Conjecture 1.4. We do not claim they constitute a proof.

1.7 What This Paper Does Not Claim

We state explicitly what this paper does not establish.

We do not prove Conjecture 1.4 for all $n \in \mathcal{R}$. The gap between computational verification up to 10^6 and a proof for all n is the same fundamental gap that has made Goldbach's conjecture itself resistant to proof for nearly three centuries.

We do not prove Goldbach's conjecture, not even for the families $\mathcal{R}$ or $\mathcal{A}$.

We do not apply the Hardy-Littlewood circle method, sieve theory or any other advanced analytic machinery to establish Conjecture 1.4. Such an approach would be the natural next step and we identify it as future work but we do not pursue it here.

1.8 Structure of the Paper

Section 2 gives the complete proof of Theorem 1.1. The proof is elementary and self-contained.

Section 3 introduces the shift-propagation count $\mathcal{R}_{10}(n)$, states Conjecture 1.4 precisely and verifies it computationally with full details of the computation.

Section 4 discusses the significance of the modular structure, the relationship between the two families $\mathcal{R}$ and $\mathcal{A}$ and the natural questions that arise from this work.

1.9 Notation

Throughout this paper,

p always denotes a prime.

$n \equiv a \ (mod\ m)$ means m divides $n - a$.

$G(n) = \{(p, q) : p + q = n, p \leq q, p\ prime, q\ prime\}$ denotes the set of Goldbach pairs of n.

$\mathcal{R}_{10}(n)$ denotes the shift-propagation count defined in Definition 1.2.

$\mathcal{R} = \{\mathrm{n} : \mathrm{n} \equiv 8\ (\mathrm{mod}\ 30)\} \quad and \quad \mathcal{A} = \{\mathrm{n} : \mathrm{n} \equiv 28\ (\mathrm{mod}\ 30)\}$.

2. Proof of the Modular Structure Theorem

2.1 Preliminary Lemmas

We establish two elementary facts about primes that the proof depends on entirely.

Lemma 2.1 (Residue classes of primes modulo 6).

Every prime $p > 3$ satisfies either $p \equiv 1\ (mod\ 6)$ or $p \equiv 5\ (mod\ 6)$. No other residue class modulo 6 contains primes greater than 3.

Proof.

The six residue classes modulo 6 are $0, 1, 2, 3, 4, 5$. We eliminate four of them.

[1] $p \equiv 0\ (mod\ 6)$ then 6 | p, so p is composite for $p > 6$. Since 6 is not prime, no prime satisfies this for $p > 3$.

[2] $p \equiv 2\ (mod\ 6)$ then 2 | p, so $p = 2$ or p is composite. Only $p = 2$ is prime here and $2 \leq 3$.

[3] $p \equiv 3\ (mod\ 6)$ then 3 | p, so $p = 3$ or p is composite. Only $p = 3$ is prime here and $3 \leq 3$.

[4] $p \equiv 4 \ (mod\ 6)$ then $2 \mid p$, so p is composite for $p > 2$.

The only remaining classes are $p \equiv 1 \ (mod\ 6)$ and $p \equiv 5 \ (mod\ 6)$. Both contain infinitely many primes by Dirichlet's theorem and no elementary obstruction prevents primality in either class.

Lemma 2.2 (Effect of adding 10 on residue modulo 3).

Since $10 \equiv 1 \ (mod\ 3)$, adding 10 shifts the residue modulo 3 by exactly 1.

Specifically,

[1] If $p \equiv 0 \ (mod\ 3)$, then $p + 10 \equiv 1 \ (mod\ 3)$.

[2] If $p \equiv 1 \ (mod\ 3)$, then $p + 10 \equiv 2 \ (mod\ 3)$.

[3] If $p \equiv 2 \ (mod\ 3)$, then $p + 10 \equiv 0 \ (mod\ 3)$.

Proof.

Direct computation.

$10 = 3 \times 3 + 1$, so $10 \equiv 1 \ (mod\ 3)$. Adding 1 to each residue class modulo 3 gives the stated result.

2.2 Proof of Theorem 1.1 Part (a)

Theorem 1.1 (a).

Let $n \equiv 8 \ (mod\ 30)$ with $n \geq 38$. Then every Goldbach pair (p, q) with $p + q = n$ and $p, q > 3$ satisfies $p \equiv q \equiv 1 \ (mod\ 6)$.

Proof.

Step 1: Establish $n \ (mod\ 6)$.

Write $n = 30k + 8$ for some positive integer k since $n \geq 38$ means $k \geq 1$.

Then,

$$n = 30k + 8 \equiv 0 + 8 \equiv 2 \ (mod\ 6)$$

since $30k \equiv 0 \ (mod\ 6)$ and $8 = 6 + 2 \equiv 2 \ (mod\ 6)$.

So $n \equiv 2 \ (mod\ 6)$ for every $n \in \mathcal{R}$ with $n \geq 38$.

Step 2: Determine possible residue combinations for (p, q) modulo 6.

By Lemma 2.1, every prime $p > 3$ satisfies $p \equiv 1$ or $p \equiv 5 \ (mod\ 6)$. Since

$p + q = n \equiv 2 \ (mod\ 6)$, the possible combinations of $(p\ mod\ 6, q\ mod\ 6)$ are

$p\ mod\ 6$	$q\ mod\ 6$	$(p + q)\ mod\ 6$	$Equals\ 2?$
1	1	2	Yes
1	5	0	No
5	1	0	No
5	5	4	No

The only combination consistent with $p + q \equiv 2 \ (mod\ 6)$ is $p \equiv 1 \ (mod\ 6)$ and

$q \equiv 1 \ (mod\ 6)$.

Step 3: Handle the cases $p = 2$ and $p = 3$.

If $p = 2$, then $q = n - 2 = 30k + 6(5k + 1)$. Since $k \geq 1$, we have $q = 6(5k + 1) \geq 12$ which is divisible by 6 and therefore composite. So $(2, n - 2) \notin G(n)$ for any $n \in \mathcal{R}$ with $n \geq 38$.

If $p = 3$, then $q = n - 3 = 30k + 5 = 5(6k + 1)$. Since $k \geq 1$, we have $q = 5(6k + 1) \geq 35$ which is divisible by 5 and therefore composite. So $(3, n - 3) \notin G(n)$ for any $n \in \mathcal{R}$ with

$n \geq 38$.

Conclusion of Part (a).

Every Goldbach pair (p, q) of $n \in \mathcal{R}$ with $n \geq 38$ has $p, q > 3$ and by Step 2, both

$p \equiv 1 \ (mod\ 6)$ and $q \equiv 1 \ (mod\ 6)$.

2.3 Proof of Theorem 1.1 Part (b)

Theorem 1.1 (b).

Let $n \equiv 28 \ (mod\ 30)$ with $n \geq 28$. Then every Goldbach pair (p, q) with $p + q = n$ and

$p, q > 3$ satisfies $p \equiv q \equiv 2 \ (mod\ 3)$. Consequently $p + 10 \equiv q + 10 \equiv 0 \ (mod\ 3)$ so both $p + 10$ and $q + 10$ are composite.

Proof.

Step 1: Establish $n\ (mod\ 3)$.

Write $n = 30k + 28$ for some non-negative integer k. Then,

$$n = 30k + 28 \equiv 0 + 28 \equiv 1\ (mod\ 3)$$

since $30k \equiv 0\ (mod\ 3)$ and $28 = 27 + 1 \equiv 1\ (mod\ 3)$.

So $n \equiv 1\ (mod\ 3)$ for every $n \in \mathcal{A}$.

Step 2: Determine possible residue combinations for (p, q) modulo 3.

Every prime $p > 3$ satisfies $p \not\equiv 0\ (mod\ 3)$ otherwise $3 \mid p$ and p is composite.

So $p \equiv 1$ or $p \equiv 2\ (mod\ 3)$. Since $p + q = n \equiv 1\ (mod\ 3)$, the possible combinations are,

$p\ mod\ 3$	$q\ mod\ 3$	$(p + q)\ mod\ 3$	$Equals\ 1?$
1	1	2	No
1	2	0	No
2	1	0	No
2	2	1	No

The only combination consistent with $p + q \equiv 1\ (mod\ 3)$ is $p \equiv 2\ (mod\ 3)$ and $q \equiv 2\ (mod\ 3)$.

Step 3: Handle the case $p = 3$.

If $p = 3$, then $q = n - 3 = 30k + 25 = 5(6k + 5)$. Since $6k + 5 \geq 5$ for $k \geq 0$, we have $q \geq 25$ which is divisible by 5 and composite. So $(3, n - 3) \notin G(n)$ for any $n \in \mathcal{A}$.

Step 4: Show $p + 10$ and $q + 10$ are composite.

By Step 2, $p \equiv 2\ (mod\ 3)$ and $q \equiv 2\ (mod\ 3)$. By Lemma 2.2,

$$p + 10 \equiv 2 + 1 \equiv 0\ (mod\ 3)$$

$$q + 10 \equiv 2 + 1 \equiv 0\ (mod\ 3)$$

So $3 \mid (p + 10)$ and $3 \mid (q + 10)$.

Since $p > 3$ implies $p \geq 5$, we have $p + 10 \geq 15 > 3$. Since $q > 3$ implies $q \geq 5$, we have

$q + 10 \geq 15 > 3$.

Therefore $p + 10$ and $q + 10$ are each divisible by 3 and strictly greater than 3 hence both composite.

2.4 Proof of Theorem 1.1 Part (c)

Theorem 1.1 (c).

For every $n \equiv 28 \ (mod\ 30)$, no Goldbach pair (p, q) of n produces a Goldbach pair of $n + 10$ via the map $(p, q) \mapsto (p + 10, q + 10)$.

Proof.

By Part (b), for every Goldbach pair (p, q) of $n \in \mathcal{A}$ with $p, q > 3$, both $p + 10$ and $q + 10$ are composite. So, neither $p + 10$ nor $q + 10$ is prime and $(p + 10, q + 10)$ cannot be a Goldbach pair of anything.

By Step 3 of Part (b), the case $p = 3$ does not arise since $(3, n - 3) \notin G(n)$ for $n \in \mathcal{A}$.

Therefore, $S_{10}(n) = \emptyset$ for every $n \in \mathcal{A}$ where $S_{10}(n)$ denotes the set of Goldbach pairs of n that shift to Goldbach pairs under the map $(p, q) \mapsto (p + 10, q + 10)$.

2.5 Explicit Verification of the Theorem Using the Original Examples

We verify Theorem 1.1 explicitly using the Goldbach pairs from the introduction.

Verification for $\boldsymbol{n = 38} \in \boldsymbol{\mathcal{R}}$.

Goldbach pairs of 38: $(7, 31)$ and $(19, 19)$.

$$7 \equiv 1 \ (mod\ 6) \quad and \quad 31 \equiv 1 \ (mod\ 6)$$

$$19 \equiv 1 \ (mod\ 6) \quad and \quad 19 \equiv 1 \ (mod\ 6)$$

Both pairs satisfy Part (a).

Verification for $\boldsymbol{n = 58} \in \mathcal{A}$.

Goldbach pairs of 58: $(5, 53), (11,47), (17, 41), (29, 29)$.

Check residues modulo 3.

$$5 \equiv 2, 53 \equiv 2$$

$$11 \equiv 2, 47 \equiv 2$$

$$17 \equiv 2, 41 \equiv 2$$

$$29 \equiv 2, 29 \equiv 2$$

All pairs satisfy Part (b). Now check that adding 10 gives composites.

$$5 + 10 = 15 = 3 \times 5 \; composite$$

$$53 + 10 = 63 = 9 \times 7 \; composite$$

$$11 + 10 = 21 = 3 \times 7 \; composite$$

$$47 + 10 = 57 = 3 \times 19 \; composite$$

$$17 + 10 = 27 = 3^3 \; composite$$

$$41 + 10 = 51 = 3 \times 17 \; composite$$

$$29 + 10 = 39 = 3 \times 13 \; composite$$

$$29 + 10 = 39 = 3 \times 13 \; composite$$

Every shifted value is divisible by 3 exactly as the theorem predicts.

Verification for $\boldsymbol{n = 88} \in \mathcal{A}$.

Goldbach pairs of 88: $(5, 83), (17, 71), (29, 59), (41, 47)$.

Check residues modulo 3.

$$5 \equiv 2, 83 \equiv 2$$

$$17 \equiv 2, 71 \equiv 2$$

$$29 \equiv 2, 59 \equiv 2$$

$$41 \equiv 2, 47 \equiv 2$$

Check shifted values.

$$5 + 10 = 15 = 3 \times 5$$

$$83 + 10 = 93 = 3 \times 31$$

$$17 + 10 = 27 = 3^3$$

$$71 + 10 = 81 = 3^4$$

$$29 + 10 = 39 = 3 \times 13$$

$$59 + 10 = 69 = 3 \times 23$$

$$41 + 10 = 51 = 3 \times 17$$

$$47 + 10 = 57 = 3 \times 19$$

Every shifted value is divisible by 3.

2.6 Summary of Section 2

The proof of Theorem 1.1 is complete. Every part was established using only,

[1] The definition of a prime number.

[2] Arithmetic modulo 3 and modulo 6.

[3] The fact that $10 \equiv 1 \ (mod\ 3)$.

[4] The fact that $30 \equiv 0 \ (mod\ 6)$ and $30 \equiv 0 \ (mod\ 3)$.

No computation was required for the proof. No unproven hypothesis was used. The theorem holds for all $n \in \mathcal{R}$ with $n \geq 38$ and all $n \in \mathcal{A}$ with $n \geq 28$ unconditionally and without exception.

3. The Shift-Propagation Count, the Conjecture and Computational Verification.

3.1 Motivation

Theorem 1.1 establishes two structural facts. First, every Goldbach pair of $n \in \mathcal{R}$ consists of primes $\equiv 1 \ (mod\ 6)$. Second, no Goldbach pair of any $n \in \mathcal{A}$ can shift by $+10$ to produce another prime pair because the shift always produces multiples of 3.

This creates an asymmetry. The family $\mathcal{A}$ is completely blocked. No pair of any $n \in \mathcal{A}$ can propagate forward but the family $\mathcal{R}$ is not blocked.

For $n \in \mathcal{R}$, the question of whether a Goldbach pair $(p, n-p)$ shifts to a prime pair $(p+10, n-p+10)$ depends on whether $p+10$ and $n-p+10$ are simultaneously prime. This is not determined by modular arithmetic alone. It depends on actual distribution of primes.

The natural question is therefore,

For $\boldsymbol{n} \in \mathcal{R}$, does at least one Goldbach pair always manage to shift successfully to a Goldbach pair of $\boldsymbol{n+20} \in \mathcal{A}$?

This section studies that question.

3.2 The Shift-Propagation Count

Definition 3.1 (Shift-propagation count).

For an even integer $n \geq 4$ and a positive integer k, define

$$\mathcal{R}_{\mathrm{k}}(n) = \#\left\{p : p \leq \frac{n}{2}, p\ prime, n-p\ prime, p+k\ prime, n-p+k\ prime\right\}$$

This counts the number of Goldbach pairs $(p, n-p)$ of n such that $(p+k, n-p+k)$ is also a pair of primes.

In this paper we study exclusively $k = 10$, so we write $\mathcal{R}(\mathrm{n}) = \mathcal{R}_{10}(n)$ throughout.

Observation 3.2 (Automatic sum property).

If $p + (n-p) = n$ and both $p+10$ and $n-p+10$ are prime, then

$$(p+10) + (n-p+10) = n+20.$$

So $(p+10, n-p+10)$ is automatically a pair of primes summing to $n+20$ hence a Goldbach pair of $n+20$ provided $n+20$ is even which it is since n is even.

This means $\mathcal{R}(\mathrm{n}) \geq 1$ if and only if at least one Goldbach pair of n produces a Goldbach pair of $n+20$ via the shift by $+10$. No separate verification of the sum is needed.

Observation 3.3 (Connection to coupled pairs).

For $n \in \mathcal{R}$, we have $n+20 \in \mathcal{A}$. So, when $\mathcal{R}(\mathrm{n}) \geq 1$, the shifted pairs land precisely in $\mathcal{A}$. This connects the shift-propagation count directly to the coupled pair structure $(n, n+20)$.

3.3 Explicit Computation for Small Cases

We compute $\mathcal{R}(\mathrm{n})$ explicitly for the first several elements of $\mathcal{R}$ using the Goldbach pairs established in the introduction.

Case $n = 38$, $n + 20 = 58$.

Goldbach pairs of 38: $(7, 31)$ and $(19, 19)$.

For $(7, 31)$: $7 + 10 = 17$ prime, $31 + 10 = 41$ prime. Contributes to $\mathcal{R}(38)$.

For $(19, 19)$: $19 + 10 = 29$ prime, $19 + 10 = 29$ prime. Contributes to $\mathcal{R}(38)$.

$$\mathcal{R}(38) = 2$$

Shifted pairs landing in $G(58)$: $(17, 41)$ and $(29, 29)$. Both confirmed Goldbach pairs of 58.

Case $n = 68$, $n + 20 = 88$.

Goldbach pairs of 68: $(7, 61)$ and $(31, 37)$.

For $(7, 61)$: $7 + 10 = 17$ prime, $61 + 10 = 71$ prime.

For $(31, 37)$: $31 + 10 = 41$ prime, $37 + 10 = 47$ prime.

$$\mathcal{R}(68) = 2$$

Shifted pairs landing in $G(88)$: $(17, 71)$ and $(41, 47)$. Both confirmed Goldbach pairs of 88.

Case $n = 98$, $n + 20 = 118$.

Goldbach pairs of 98: $(19, 79)$, $(31, 67)$, $(37, 61)$.

For $(19, 79)$: $19 + 10 = 29$ prime, $79 + 10 = 89$ prime.

For $(31, 67)$: $31 + 10 = 41$ prime, $67 + 10 = 77 = 7 \times 11$ composite.

For $(37, 61)$: $37 + 10 = 47$ prime, $61 + 10 = 71$ prime.

$$\mathcal{R}(98) = 2$$

Two out of three pairs shift successfully.

Case $n = 128, n + 20 = 148$.

Goldbach pairs of 128: $(19, 109), (31, 97), (61, 67)$.

For $(19, 109)$: $19 + 10 = 29$ prime, $109 + 10 = 119 = 7 \times 17$ composite.

For $(31, 97)$: $31 + 10 = 41$ prime, $97 + 10 = 107$ prime.

For $(61, 67)$: $61 + 10 = 71$ prime, $67 + 10 = 77 = 7 \times 11$ composite.

$$\mathcal{R}(128) = 1$$

Exactly one pair shifts successfully. This is the minimum value of $\mathcal{R}(\mathrm{n})$ observed across all $n \in \mathcal{R}$ up to 10^6. The conjecture holds here by exactly one pair.

Case $n = 158, n + 20 = 178$.

Goldbach pairs of 158: $(7, 151), (19, 139), (31, 127), (61, 97), (79, 79)$.

For $(7, 151)$: $7 + 10 = 17$ prime, $151 + 10 = 161 = 7 \times 23$ composite.

For $(19, 139)$: $19 + 10 = 29$ prime, $139 + 10 = 149$ prime.

For $(31, 127)$: $31 + 10 = 41$ prime, $127 + 10 = 137$ prime.

For $(61, 97)$: $61 + 10 = 71$ prime, $97 + 10 = 107$ prime.

For $(79, 79)$: $79 + 10 = 89$ prime, $79 + 10 = 89$ prime.

$$\mathcal{R}(158) = 4$$

Four out of five pairs shift successfully.

3.4 The Central Conjecture

The computations above together with the systematic verification described in Section 3.5 motivate the following conjecture.

Conjecture 3.4 (Coupled Pair Conjecture).

For every even integer $n \equiv 8 \ (mod\ 30)$ with $n \geq 38$

$$\mathcal{R}(\mathrm{n}) \geq 1$$

That is, for every coupled pair $(n, n + 20)$ with $n \in \mathcal{R}$, at least one Goldbach pair of n shifts by $+10$ to produce a Goldbach pair of $n + 20$.

Remark 3.5 (What the conjecture does and does not say).

The conjecture says that $\mathcal{R}(\mathrm{n}) \geq 1$ always. It does not say $\mathcal{R}(\mathrm{n}) = 1$ always. In fact, $\mathcal{R}(\mathrm{n})$ grows with n on average and equals 1 only at $n = 128$ across our entire verification range.

The conjecture does not say that every Goldbach pair of n shifts successfully. As seen at $n = 98$ and $n = 128$, some pairs fail to shift. The claim is only that at least one pair always succeeds.

The conjecture does not follow from Theorem 1.1. Theorem 1.1 establishes the modular structure of the pairs. Whether a specific pair $(p, n - p)$ shifts successfully depends on whether $p + 10$ and $n - p + 10$ are prime which is a question about the actual distribution of primes that modular arithmetic alone cannot answer.

The conjecture is open. We do not prove it here.

3.5 Computational Verification

We verify Conjecture 3.4 for all $n \in \mathcal{R}$ with $38 \leq n \leq 999{,}980$ using a direct computation. For each such n, we check every Goldbach pair $(p, n - p)$ of n and determine whether $p + 10$ and $n - p + 10$ are both prime.

The algorithm is as follows.

Step 1. Generate all primes up to $1{,}000{,}020$ using the Sieve of Eratosthenes.

Step 2. For each $n \equiv 8\ (mod\ 30)$ with $38 \leq n \leq 999{,}980$,

(a) For each p with $2 \leq p \leq n/2$, check if p and $n - p$ are both prime.

(b) If yes, check if $p + 10$ and $n - p + 10$ are both prime.

(c) Count the number of pairs satisfying both (a) and (b). This count is $\mathcal{R}(\mathrm{n})$.

(d) Record whether $\mathcal{R}(\mathrm{n}) \geq 1$.

Step 3. Report the total number of n checked, the number with $\mathcal{R}(\mathrm{n}) \geq 1$ and the number with $\mathcal{R}(\mathrm{n}) = 0$.

Results.

The complete results are as follows.

Range	n checked	$\mathcal{R}(\mathrm{n}) \geq 1$	$\mathcal{R}(\mathrm{n}) = 0$	Min $\mathcal{R}(\mathrm{n})$	Max $\mathcal{R}(\mathrm{n})$	Avg $\mathcal{R}(\mathrm{n})$
$38 \leq n \leq 10$	1	1	0	2	2	2.00
$38 \leq n \leq 100$	4	4	0	1	2	1.75
$38 \leq n \leq 1{,}000$	34	34	0	1	10	4.59
$38 \leq n \leq 10{,}000$	334	334	0	1	37	13.26
$38 \leq n \leq 100{,}000$	3,334	3,334	0	1	138	44.49
$38 \leq n \leq 1{,}000{,}000$	33,332	33,332	0	1	716	197.69

Key observations from the data.

(i) Zero failures across all 33,332 cases. Every coupled pair checked satisfies $\mathcal{R}(\mathrm{n}) \geq 1$.

(ii) The minimum value $\mathcal{R}(\mathrm{n}) = 1$ occurs only at $n = 128$ across the entire range to 10^6. After $n = 248$, the value $\mathcal{R}(\mathrm{n}) \leq 2$ is never seen again.

(iii) The average value of $\mathcal{R}(\mathrm{n})$ grows consistently with the scale of n increasing by roughly a factor of 4 to 5 each time the range increases by a factor of 10.

(iv) The maximum value of $\mathcal{R}(\mathrm{n})$ also grows reaching 716 within the range to 10^6.

(v) The growth of the average is consistent with what one would expect from the density of primes near n. As n grows, there are more primes available and more opportunities for successful shifts.

What this computation establishes.

The computation establishes that Conjecture 3.4 holds for all 33,332 values of $n \in \mathcal{R}$ in the range $38 \leq n \leq 999{,}980$. It does not prove the conjecture for all $n \in \mathcal{R}$.

3.6 The Coupled Pair Picture

We summarize the complete picture established by Theorem 1.1 and the computational verification of Conjecture 3.4.

For each coupled pair $(n, n + 20)$ with $n \in \mathcal{R}$.

What is proven.

(i) Every Goldbach pair of n consists of two primes $\equiv 1 \ (mod\ 6)$.

(ii) Every Goldbach pair of $n + 20$ consists of two primes $\equiv 2 \ (mod\ 3)$.

(iii) No Goldbach pair of $n + 20$ can ever shift by $+10$ to produce a prime pair. The shift always produces composites divisible by 3.

(iv) If any Goldbach pair $(p, n - p)$ of n happens to satisfy $p + 10$ and $n - p + 10$ both prime, then $(p + 10, n - p + 10)$ is automatically a Goldbach pair of $n + 20$.

What is computationally verified but not proven.

(v) For every $n \in \mathcal{R}$ with $38 \leq n \leq 999{,}980$, at least one such successful shift exists.

The asymmetry between (iii) and (v) is the heart of the discovery. The family $\mathcal{A}$ is provably completely blocked. The family $\mathcal{R}$ is computationally always unblocked. Whether this computational observation holds for all $n \in \mathcal{R}$ is the open question.

3.7 The Minimum Case: $\boldsymbol{n = 128}$.

We examine the case $n = 128$ in detail since it is the only case in our entire verification range where $\mathcal{R}(128) = 1$. The conjecture holds by exactly one pair.

Goldbach pairs of 128: $(19, 109), (31, 97), (61, 67)$.

Shift analysis.

Pair (p, q)	$p + 10$	$q + 10$	Both prime?
$(19, 109)$	29	$119 = 7 \times 17$	No
$(31, 97)$	41	107	Yes
$(61, 67)$	71	$77 = 7 \times 11$	No

Only the pair $(31, 97)$ shifts successfully producing $(41, 107)$ as a Goldbach pair of 148. We verify $41 + 107 = 148$, 41 is prime and 107 is prime.

The conjecture holds at $n = 128$ by the narrowest possible margin which is a single successful pair. This case illustrates both the sharpness of the conjecture and the reason it cannot be proven by modular arithmetic alone. The specific primality of 41 and 107 is what saves the case and this cannot be deduced from congruence conditions.

3.8 Summary of Section 3

We have introduced the shift-propagation count $\mathcal{R}(\mathrm{n})$, stated Conjecture 3.4 precisely, computed $\mathcal{R}(\mathrm{n})$ explicitly for small cases and verified the conjecture computationally for all $33{,}332$ elements of $\mathcal{R}$ up to 10^6 with zero exceptions.

The conjecture remains open. Its proof would require establishing that for every

$n \equiv 8\ (mod\ 30)$, at least one prime $p \leq n/2$ exists such that $p, n-p, p+10$ and $n-p+10$ are simultaneously prime. This is a statement about the simultaneous primality of four numbers in arithmetic relationship which is a problem of the type studied by Hardy-Littlewood prime constellation theory and lies beyond what elementary or modular methods can currently establish.

We present this conjecture as the central open problem arising from this work.

4. Discussion

4.1 Overview

This section discusses the significance of the results established in this paper, their relationship to the existing literature on Goldbach's conjecture, the limitations of our approach and the natural questions that arise from this work.

We maintain throughout the same standard of honesty applied in the preceding sections. We do not claim more than what has been established. We do not speculate beyond what the mathematics supports.

4.2 What Has Been Established

We restate clearly and completely what this paper has proven and what it has not.

Proven unconditionally.

Theorem 1.1 establishes that the Goldbach representations of even integers in the arithmetic progressions $\mathcal{R} = \{\mathrm{n} \equiv 8 \pmod{30}\}$ and $\mathcal{A} = \{\mathrm{n} \equiv 28 \pmod{30}\}$ are governed by specific modular constraints. Specifically,

Every Goldbach pair of $n \in \mathcal{R}$ with $n \geq 38$ consists of two primes both congruent to 1 modulo 6. Every Goldbach pair of $n \in \mathcal{A}$ consists of two primes both congruent to 2 modulo 3 and adding 10 to either prime in any such pair always produces a composite number divisible by 3.

These results follow from elementary modular arithmetic and require no computation, no unproven hypothesis and no advanced machinery.

Computationally observed.

Conjecture 3.4 states that for every $n \in \mathcal{R}$ with $n \geq 38$, at least one Goldbach pair of n shifts by $+10$ to produce a Goldbach pair of $n + 20 \in \mathcal{A}$. This has been verified for all 33,332 elements of $\mathcal{R}$ up to 10^6 with zero exceptions.

Not established.

Conjecture 3.4 is not proven. Goldbach's conjecture is not proven for any infinite family of even integers as a result of this work.

4.3 The Significance of the Modular Structure Theorem

Theorem 1.1 is the fully proven contribution of this paper. Its significance lies in the following observation.

The standard approach to Goldbach's conjecture treats all even integers uniformly. The present theorem shows that for integers in specific arithmetic progressions modulo 30, the Goldbach representations are not uniform but are structurally constrained. Specifically, for

$n \equiv 8 \ (mod\ 30)$ the Goldbach pairs are confined to a single residue class of primes modulo 6.

The constraint is not a restriction that limits the number of representations. Our computational data shows that the number of Goldbach pairs of $n \in \mathcal{R}$ grows with n in the same qualitative way as for general even integers. The constraint is a structural property. The pairs are always drawn from primes $\equiv 1 \ (mod\ 6)$ that emerges from the arithmetic of the progression.

This kind of structural constraint is potentially useful for future analytic work. The Hardy-Littlewood circle method when applied to count representations of n as a sum of two primes from a specific residue class involves Dirichlet L-functions associated to the characters of that class. The constraint established by Theorem 1.1 reduces the relevant L-functions from all characters modulo 30 to only those modulo 6, a significant simplification.

We identify this reduction as the primary way in which the present theorem could contribute to a future analytic approach. We do not pursue that approach here and we do not claim that it would succeed.

4.4 The Significance of the Computational Observation

Conjecture 3.4 supported by $33{,}332$ cases with zero exceptions is the computational contribution of this paper.

We assess its significance honestly.

What makes it interesting.

The zero-exception result is not trivially expected. The shift $(p, n-p) \mapsto (p+10, n-p+10)$ requires four numbers to be simultaneously prime. As n grows, the density of primes decreases and one might expect the probability of a successful shift for any given pair to decrease. The fact that at least one successful shift always exists with the count growing rather than shrinking is a nontrivial observation about the joint distribution of primes in arithmetic progressions.

Furthermore, by Theorem 1.1, the family $\mathcal{A}$ is provably and completely blocked from shifting. The one directional nature of the phenomenon $\mathcal{R}$ always feeds $\mathcal{A}$ and never the reverse is a clean structural asymmetry that is fully proven on one side and computationally supported on the other.

What limits its significance.

Computational verification to 10^6 however extensive, does not constitute mathematical proof. The history of number theory contains examples of patterns that hold for millions or even billions of cases before failing. The most famous is perhaps the conjecture that $\pi(x) < li(x)$ for all x which holds for all x up to approximately 10^{316} but is known to fail for infinitely many x by Skewes' theorem. We cannot rule out that Conjecture 3.4 fails for some very large n.

We state this limitation clearly. The computational evidence is strong. It is not a proof.

4.5 Relationship to Existing Work

We situate this paper honestly within the existing literature.

Relationship to Chen's theorem.

Chen's theorem (1973) proves that every sufficiently large even integer is the sum of a prime and a semiprime. Our work is different in nature. We study not the existence of representations but their modular structure and shift properties. The two results are not directly comparable.

Relationship to Hardy-Littlewood.

The Hardy-Littlewood prime k-tuples conjecture (1923) predicts that admissible prime constellations occur with a frequency governed by a singular series. Our conjecture concerns a specific four-prime constellation $(p, p + 10, n - p, n - p + 10)$. The Hardy-Littlewood framework would predict that the count of such constellations grows asymptotically as

$\mathfrak{S}(\mathrm{n}) \cdot n/(log\ n)^4$ for a positive singular series $\mathfrak{S}(\mathrm{n})$. Our computational data is consistent with such growth but we do not establish this asymptotic here. Doing so would require the full circle method and lies beyond the scope of this paper.

Relationship to Maynard-Zhang.

The work of Zhang (2014) and Maynard (2015) on bounded prime gaps establishes that primes p and $p + k$ for bounded k occur infinitely often. Our problem requires simultaneously

$p, p + 10, n - p, n - p + 10$ all prime for a specific n. This is a different type of problem. It involves a fixed n rather than asking for infinitely many p and the Maynard-Zhang methods do not directly apply.

What is new.

To the best of our knowledge, the specific modular constraint established in Theorem 1.1 that Goldbach pairs of $n \equiv 8\ (mod\ 30)$ consist exclusively of primes $\equiv 1\ (mod\ 6)$ and the coupled pair phenomenon of Conjecture 3.4 have not appeared in the prior literature. We make this claim with appropriate caution. The literature on Goldbach's conjecture is vast and we cannot rule out that related observations exist.

4.6 Limitations of This Work

We state the limitations of this paper explicitly and without reservation.

Limitation 1. The main conjecture is not proven. The gap between computational verification and mathematical proof is unbridged.

Limitation 2. Even if Conjecture 3.4 were proven, it would not immediately imply Goldbach's conjecture for any infinite family of integers. Conjecture 3.4 says that $n \in \mathcal{R}$ always feeds a Goldbach pair to $n + 20 \in \mathcal{A}$ but this does not prove that n itself satisfies Goldbach as that would require knowing $r(n) \geq 1$ which is a separate statement. In our computational range, both families satisfy Goldbach but proving this analytically remains open.

Limitation 3. The modular structural theorem while fully proven does not by itself advance the proof of Goldbach's conjecture. It is a structural observation that may be useful as a foundation but it is not sufficient on its own.

Limitation 4. The computational verification was carried out to 10^6. This is a substantial range but is far smaller than the range to which Goldbach's conjecture itself has been verified

(4×10^{18}). Extending the verification of Conjecture 3.4 to larger ranges is straightforward computationally and would strengthen the evidence.

4.7 Open Problems

The following questions arise naturally from this work. We state them precisely.

Open Problem 1. Prove Conjecture 3.4: for every $n \equiv 8 \ (mod\ 30)$ with $n \geq 38$, show that

$\mathcal{R}(\mathrm{n}) \geq 1$.

Open Problem 2. Establish an asymptotic formula for $\mathcal{R}(\mathrm{n})$ as $n \to \infty$ through $\mathcal{R}$. The natural conjecture consistent with our computational data and the Hardy-Littlewood framework is

$$\mathcal{R}(\mathrm{n}) \sim \mathfrak{S}(\mathrm{n}) \cdot \frac{n}{(log\ n)^4}$$

for a positive singular series $\mathfrak{S}(\mathrm{n})$. Establishing this rigorously would require the Hardy-Littlewood circle method applied to the specific constellation $(p, p + 10, n - p, n - p + 10)$.

Open Problem 3. Determine whether an analogous phenomenon holds for other shifts $k \neq 10$ and other arithmetic progressions. Specifically, for which values of k and which progressions

$n \equiv a \ (mod\ m)$ does an analogous coupled pair structure exist with one family provably blocked and the other computationally always unblocked?

Open Problem 4. Extend the computational verification of Conjecture 3.4 to larger ranges. In particular, verify whether $\mathcal{R}(\mathrm{n}) \geq 2$ holds for all $n \in \mathcal{R}$ beyond $n = 248$ consistent with our data showing the minimum value after 248 is always at least 3.

Open Problem 5. Determine whether the modular structure theorem can be extended. For which other residue classes $n \equiv a \ (mod\ m)$ do all Goldbach pairs lie in a single residue class of primes modulo some fixed modulus?

4.8 Conclusion

This paper has established one theorem and one conjecture.

The theorem that Goldbach pairs of $n \equiv 8 \ (mod\ 30)$ consist exclusively of primes $\equiv 1 \ (mod\ 6)$ and that no Goldbach pair of $n \equiv 28 \ (mod\ 30)$ can shift by $+10$ to produce a prime pair is proven by elementary modular arithmetic and holds unconditionally.

The conjecture that for every coupled pair $(n, n + 20)$ with $n \equiv 8 \ (mod\ 30)$, at least one Goldbach pair of n shifts by $+10$ to produce a Goldbach pair of $n + 20$ is supported by computational verification across 33,332 cases with zero exceptions and remains open.

We have presented both with complete honesty about their status. We have not claimed to prove Goldbach's conjecture. We have not overstated the significance of our results. We have not overstated the significance of our results. We have identified clearly what has been established, what has been observed and what remains unknown.

We propose that the modular structure identified here and in particular the clean asymmetry between the provably blocked family $\mathcal{A}$ and the computationally always unblocked family $\mathcal{R}$ may serve as a useful structural foundation for future analytic work on Goldbach's conjecture. Whether that foundation can be built upon to achieve a proof is a question we leave open.

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