The speed limit for a rotating object

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Abstract.
This paper presents a new kinematic analysis of relativistic rotating objects within the framework of special relativity. By examining the Lorentz contraction of concentric circumferences on a spinning disk, we show that when the edge speed approaches the speed of light, the outer circumference may paradoxically become shorter than the inner one. To eliminate this geometric inconsistency, we derive a strict upper speed limit for rotating bodies: the linear velocity of any point must not exceed the speed of light divided by the square root of 2. This constraint arises purely from special relativistic kinematics, independent of material strength or mechanical properties, offering a novel resolution to the Ehrenfest paradox complementary to existing approaches.
Keywords: Special relativity; rotating rigid body; Ehrenfest paradox; Lorentz contraction; kinematic speed limit

Introduction

The relativistic paradox of rotating rigid bodies was first formulated by Ehrenfest in 1909[1] and has remained a foundational thought experiment in special relativity. The Ehrenfest paradox arises when considering a rigid disk rotating at relativistic speed: special relativity predicts Lorentz contraction along the tangential circumference, while the radial dimension—being transverse to the motion — remains unchanged. This creates a fundamental geometric conflict: the disk's circumference would appear to contract while its radius stays constant, violating the Euclidean relation S = 2πr. For more than a century, this contradiction has sparked extensive debate, with proposed resolutions falling into two main categories: those invoking material stresses and deformations, and those employing non-Euclidean geometry within general relativity. [2-7] To date, no single consensus resolution has been universally adopted. [8]

This work revisits the rotating disk from a purely kinematic perspective within special relativity, independent of material properties or general relativistic effects. By analyzing concentric circumferences at different radii, we identify a new paradox: at sufficiently high rotation rates, Lorentz contraction can cause the outer circumference to become shorter than the inner one—a physically impossible geometric inversion. Through rigorous relativistic kinematic analysis, we derive a strict upper bound on the linear speed of any point on a rotating object: $v < c/\sqrt{2}$. This limit is determined solely by special relativity and applies universally to all rotating bodies, regardless of mechanical strength.

Analysis and Arguments

Consider a rigid disk rotating about its central axis with angular frequency ω. For two concentric circles at radii r_1 and r_2 ($r_2 > r_1$), the rest circumferences are $S_1 = 2\pi r_1$ and $S_2 = 2\pi r_2$. Under rotation, the tangential speeds are $v_1 = r_1\omega$ and $v_2 = r_2\omega$, respectively. According to special relativity, length contraction occurs only along the direction of motion, yielding the contracted circumferences:

$$S_1' = S_1\sqrt{1 - v_1^2/c^2} = 2\pi r_1\sqrt{1 - r_1^2\omega^2 / c^2}$$

$$S_2' = S_2 \sqrt{1 - v_2^2/c^2} = 2\pi r_2\sqrt{1 - r_2^2\omega^2 / c^2}$$

As rotation speed increases, $v_2 > v_1$, so the outer circumference contracts more strongly. This can lead to the paradoxical condition $S_2' < S_1'$ despite $r_2 > r_1$.
Deriving the threshold for this inversion:

$$S_2' - S_1' = 2\pi \left(r_2\sqrt{1 - r_2^2\omega^2 / c^2} - r_1\sqrt{1 - r_1^2\omega^2 / c^2}\right)$$

$S_2' - S_1' < 0$ means,

$$r_2\sqrt{1 - r_2^2\omega^2 / c^2} < r_1\sqrt{1 - r_1^2\omega^2 / c^2}$$

Squaring both sides and simplifying (assuming $r_2 > r_1 > 0$) gives:

$$r_2^2 \ (1 - r_2^2\omega^2 / c^2) < r_1^2 \ (1 - r_1^2\omega^2 / c^2)$$

Rearranging terms:

$$r_2^2 - r_1^2 < (r_2^4 - r_1^4)\frac{\omega 2}{c^2} = (r_2^2 - r_1^2)(r_2^2 + r_1^2) \frac{\omega 2}{c^2}$$

$$1 < (r_2^2 + r_1^2) \frac{\omega 2}{c^2}$$

$$(r_2^2 + r_1^2)\omega^2 > c^2$$

Factoring and canceling $r_2^2 - r_1^2$:

$$1 < (r_2^2 + r_1^2) \frac{\omega 2}{c^2}$$

$$(r_2^2 + r_1^2)\omega^2 = v_2^2 + v_1^2 > c^2$$

Thus, the paradox arises when:

$$v_2^2 + v_1^2 > c^2$$

To prevent this physically inconsistent state, we require:

$$v_2^2 + v_1^2 < c^2$$

In the limit where $r_1 \approx r_2$ (adjacent concentric circles), this reduces to:

$$2v^2 < c^2 \rightarrow v < c/\sqrt{2}$$

This establishes a universal kinematic speed limit for all points on any rotating object: linear velocity must not exceed $c/\sqrt{2}$ (approximately 0.707c). This constraint is purely kinematic and does not depend on material strength, rigidity, or structural integrity.

Discussion

Traditional resolutions of the Ehrenfest paradox rely on either material stresses (which break perfect rigidity and allow deformation) or general relativity (which treats rotating frames as non-inertial and uses curved non-Euclidean spacetime).[4, 6, 8] By contrast, this paper provides a purely special-relativistic kinematic resolution that requires neither material failure nor general relativity. The key insight is that geometric consistency—preserving the ordering $r_2 > r_1 \rightarrow S_2' > S_1'$ — imposes a stricter speed limit than the universal light barrier c.

The threshold $v < c/\sqrt{2}$ ensures that Lorentz contraction cannot invert the circumferential order of concentric circles. This limit is fundamental to relativistic rotational kinematics and applies to all rotating bodies, from hypothetical rigid disks to astrophysical objects such as neutron stars, pulsars, and compact relativistic rotors. Unlike constraints imposed by material strength, this kinematic limit cannot be overcome by engineering or material design; it is a direct consequence of special relativity.

This analysis complements existing resolutions rather than contradicting them. While general relativity correctly describes rotating frames as non-inertial, and material effects do limit real rotating bodies, our result reveals an independent kinematic barrier that exists even for idealized perfectly rigid bodies. Thus, the Ehrenfest paradox is resolved not only by deformations or general relativity but also by a fundamental kinematic speed limit inherent to special relativity.

Conclusions

1. Rotating objects in special relativity are governed by a stricter speed limit $v < c/\sqrt{2}$, lower than the universal light-speed limit c.
2. Violating this limit leads to a physically impossible kinematic contradiction: under Lorentz contraction, the outer circumference of a rotating disk becomes shorter than the inner one, violating basic geometric ordering.
3. This rotational speed bound is determined **exclusively by kinematic constraints** of special relativity, with no dependence on the mechanical strength or material properties of the object
4. The derived limit resolves the Ehrenfest paradox from a pure special-relativistic perspective, without invoking general relativistic spacetime curvature or material stress arguments, and applies universally to all rotating bodies.

References

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