AN IDENTITY WITH SYMMETRY IN BASIC HYPERGEOMETRIC SERIES AND ITS APPLICATIONS

EDIGLES GUEDES

Abstract. This paper determines a symmetry relation between basic hypergeometric series that has escaped the scrutiny of other mathematicians. We demonstrate that, for $|q| < 1$, $|a| < 1$, $|b| < 1$ and $|z| < 1$, the following identity holds

$$\sum_{n=0}^{\infty} \frac{b^n z^n}{(aq;aq)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^n q^n}{(bz;aq)_{n+1}}.$$

As a direct application of this identity, we derive a double-sum symmetry and present a particular case as an exercise. These results contribute to the understanding of hidden symmetries in q-series. Moreover, it may be useful in the study of basic hypergeometric functions and q-analogues of special functions.

For God so loved the world, that he gave his only begotten Son, that whosoever believeth in him should not perish, but have everlasting life.

John 3:16

1. Introduction

The theory of basic hypergeometric series occupies a central position in the field of quantum calculus. Since the pioneering works of Heine, Ramanujan, Rogers and others, numerous identities, transformations, infinite products and summation formulas have been discovered. Many of these identities reveal deep structural symmetries that are not immediately apparent, but that can be proved mathematically.

In this paper, we present a new symmetry involving infinite series expressed in terms of q-Pochhammer symbols. More precisely, we show that two seemingly different q-series are equal under suitable convergence conditions. This symmetry is interesting because it interchanges the roles of the parameters in a non-trivial way, suggesting an underlying duality in the structure of these series.

The main result (Theorem 1) is proved using classical techniques of q-series, namely: the q-binomial theorem, manipulation of q-Pochhammer symbols, and the generating function for q-binomial coefficients. As a consequence, we obtain a corresponding symmetry for a double series (Corollary 2), which demonstrates how the basic identity can be extended to more complex summations. A particular case of the main theorem is also presented as an exercise for the reader.

2. A Symmetry in q-Series

Theorem 1. *Let $|q| < 1$, $|a| < 1$, $|b| < 1$ and $|z| < 1$, then*

$$\sum_{n=0}^{\infty} \frac{b^n z^n}{(aq;aq)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^n q^n}{(bz;aq)_{n+1}}. \tag{2.1}$$

Proof. Define $p = aq$ and $X = bz$. The conditions $|q| < 1$, $|a| < 1$, $|b| < 1$, $|z| < 1$ guarantee that $|p| < 1$ and $|X| < 1$. Therefore, we can rewrite (2.1) as follows

$$\sum_{n=0}^{\infty} \frac{X^n}{(p;p)_{n+1}} = \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \tag{2.2}$$

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Let's analyze the right-hand side (RHS) of (2.2)

$$\text{RHS} = \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \tag{2.3}$$

But before that, let's do the series expansion to $1/(X;p)_{n+1}$, using the q-binomial theorem [1]

$$\frac{1}{(X;p)_{n+1}} = \sum_{k=0}^{\infty} \frac{(p^{n+1};p)_k}{(p;p)_k} X^k, \quad |X| < 1. \tag{2.4}$$

Insert the expansion (2.4) into the sum in (2.3)

$$\begin{aligned} \text{RHS} &= \sum_{n=0}^{\infty} p^n \sum_{k=0}^{\infty} \frac{(p^{n+1};p)_k}{(p;p)_k} X^k \\ &= \sum_{k=0}^{\infty} \frac{X^k}{(p;p)_k} \sum_{n=0}^{\infty} p^n \left(p^{n+1};p\right)_k. \end{aligned} \tag{2.5}$$

Now, we evaluate the inner sum in (2.5). First, let's write the q-Pochhammer symbol as quotient

$$\left(p^{n+1};p\right)_k = \frac{(p;p)_{n+k}}{(p;p)_n}, \tag{2.6}$$

where $(p;p)_m = \prod_{j=1}^{m} \left(1-p^j\right)$. Apply (2.6) to the inner sum in (2.5), which becomes

$$\begin{aligned} \sum_{n=0}^{\infty} p^n \left(p^{n+1};p\right)_k &= \sum_{n=0}^{\infty} p^n \frac{(p;p)_{n+k}}{(p;p)_n} \\ &= (p;p)_k \sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_p p^n, \end{aligned} \tag{2.7}$$

with the q-binomial coefficient $\begin{bmatrix} n+k \\ n \end{bmatrix}_p = \frac{(p;p)_{n+k}}{(p;p)_n)_n (p;p)_k}$.

On the other hand, a standard q-series identity is given by

$$\sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_q z^n = \frac{1}{(z;q)_{k+1}}. \tag{2.8}$$

Set $z = p$ and $q = p$ in (2.8) and obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_p p^n &= \frac{1}{(p;p)_{k+1}} \\ &= \frac{1}{(p;p)_k \left(1-p^{k+1}\right)}. \end{aligned} \tag{2.9}$$

From (2.7) and (2.9), the inner sum is

$$\begin{aligned} \sum_{n=0}^{\infty} p^n \left(p^{n+1};p\right)_k &= (p;p)_k \sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_p p^n \\ &= (p;p)_k \cdot \frac{1}{(p;p)_k \left(1-p^{k+1}\right)} \\ &= \frac{1}{1-p^{k+1}}. \end{aligned} \tag{2.10}$$

Replace (2.10) in (2.5)

$$\begin{aligned} \text{RHS} &= \sum_{k=0}^{\infty} \frac{X^k}{(p;p)_k} \cdot \frac{1}{1-p^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{X^k}{\left(1-p^{k+1}\right) (p;p)_k}. \end{aligned} \tag{2.11}$$

Apply the identity $(p;p)_{k+1} = \left(1-p^{k+1}\right)\ (p;p)_k$ to the right-hand side of (2.11), and obtain exactly the left-hand side (LHS) of (2.2), as follows

$$\begin{aligned} \text{RHS} &= \sum_{k=0}^{\infty} \frac{X^k}{\left(1-p^{k+1}\right)\ (p;p)_k} \\ &= \sum_{k=0}^{\infty} \frac{X^k}{(p;p)_{k+1}} \\ &= \text{LHS}. \end{aligned}$$

Note that the absolute convergence of the double series, guaranteed by the conditions imposed in the theorem statement, allows for the implicit rearrangement of the sums. Finally, this concludes the proof. □

The corollary below is a direct application of the symmetric identity that we proved earlier.

Corollary 2. *Let $|q| < 1$, $|a| < 1$ and $|z| < 1$, then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^n z^n q^k}{\left(q^{k+1};q^{k+1}\right)_{n+1}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{(k+1)n+k}}{\left(az;q^{k+1}\right)_{n+1}}. \tag{2.12}$$

Proof. Let $p = q^{k+1}$ and $X = az$. The conditions $|q| < 1$, $|a| < 1$ and $|z| < 1$ guarantee that $|p| < 1$ and $|X| < 1$. Apply the relevant substitutions and swap the summations with each other, both on the left and right sides, in (2.12), finding

$$\begin{aligned} \text{LHS} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^n z^n q^k}{\left(q^{k+1};q^{k+1}\right)_{n+1}} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{X^n q^k}{(p;p)_{n+1}} \\ &= \sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{X^n}{(p;p)_{n+1}} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \text{RHS} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{(k+1)n+k}}{\left(az;q^{k+1}\right)_{n+1}} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{(k+1)n} q^k}{\left(az;q^{k+1}\right)_{n+1}} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{p^n q^k}{(X;p)_{n+1}} \\ &= \sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \end{aligned} \tag{2.14}$$

From (2.12), (2.13) and (2.14), we obtain the identity

$$\sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{X^n}{(p;p)_{n+1}} = \sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \tag{2.15}$$

On the other hand, due to Theorem 1, for $|p| < 1$ and $|X| < 1$, the following identity is valid

$$\sum_{n=0}^{\infty} \frac{X^n}{(p;p)_{n+1}} = \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \tag{2.16}$$

Note that, with $p = q^{k+1}$ and $X = az$, the inner sums (in n) are identical for each fixed k. Now, multiply both sides of (2.16) by q^k and sum them over k from 0 to infinity, obtaining

$$\sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{X^n}{(p;p)_{n+1}} = \sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty} \frac{p^n}{(X;p)_{n+1}}. \tag{2.17}$$

By comparing the identity in (2.17) with the identity in (2.15), we conclude that both are identical. Therefore, the identity proposed in (2.12) follows, namely,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^n z^n q^k}{\left(q^{k+1};q^{k+1}\right)_{n+1}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{(k+1)n+k}}{\left(az;q^{k+1}\right)_{n+1}}.$$

Observe that the absolute convergence of the double series, guaranteed by the conditions imposed in the corollary statement, allows for the implicit rearrangement of the sums. Lastly, this concludes the proof. □

Exercise 3. Let $|q| < 1$and $|y| < 1$, then

$$\sum_{n=0}^{\infty} \frac{y^n}{(q;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^n}{(y;q)_{n+1}}.$$

References

[1] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*. Number 96 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2nd edition, 2004.