

Mathematical Description of the Oscillations of the Radius Difference Modulus at Symmetrical Orbital Points

Author: Viktor Strohm

Independent Researcher

Contact: vfstrohm@yahoo.de

Abstract

The provided materials demonstrate an empirical study (Figs. 1-12) identifying mysterious "envelope waves" through the analysis of distance differences at symmetrical points of planetary orbits. Additionally, a rigorous kinematic apparatus [2] has been developed to derive a force law applicable not only to closed ellipses but also to precessing orbits (rosettes).

Using the kinematic apparatus from work [2], we can construct a mathematical description of the observed oscillations. The key idea is that the envelope wave in the graphs is a direct consequence of orbital precession, "decomposed" through an original methodology of symmetrical differences.

Algorithm for Constructing the System of Objects

1. Select a cosmic body with a closed orbit.
2. Set a time interval greater than the orbital period of the body.
3. Identify the period of appearance of the selected body at the same orbital point, P_1 -- A -- P_2 , where P is perihelion and (A) is aphelion. Naturally, P_1 and P_2 have different dates.
4. From the resulting table, take the sequence of distances from the body to the focus of rotation.
5. To build a system of objects of this type, represent the sequence of distances for one period as two sequences of half-periods. Take the difference between pairs of distances symmetrical in time as the relationship between objects. This means that point $P_1 + 1$ day corresponds to point $P_2 - 1$ day, $P_1 + 2$ to $P_2 - 2$, and so on.

	Дата (от перигелия к афелию)	R_1	Дата (от афелия к перигелию)	R_2	$R_2 - R_1$
1					
2					

Tab. 1

Data is sourced from the NASA website [1].

Graphs for one period, Fig. 1 – 4

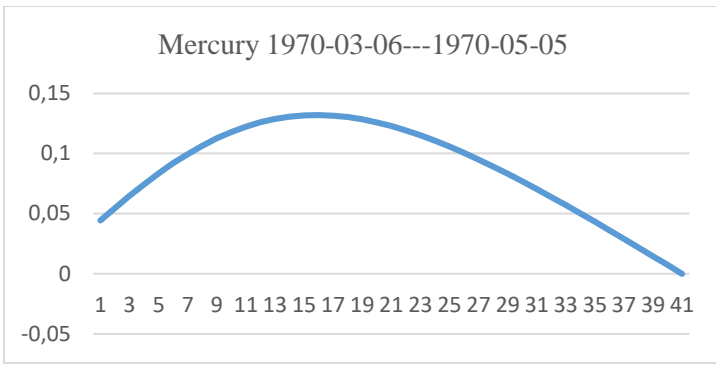
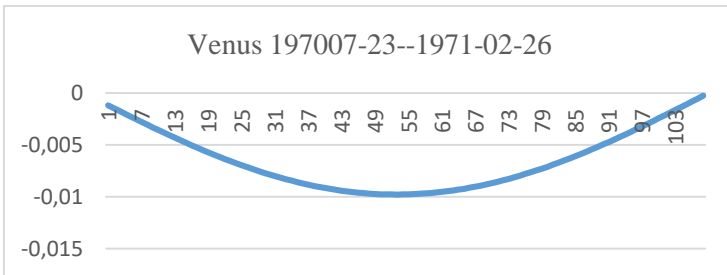


Fig. 1



Graphs over a long time interval, Fig. 5 - 12

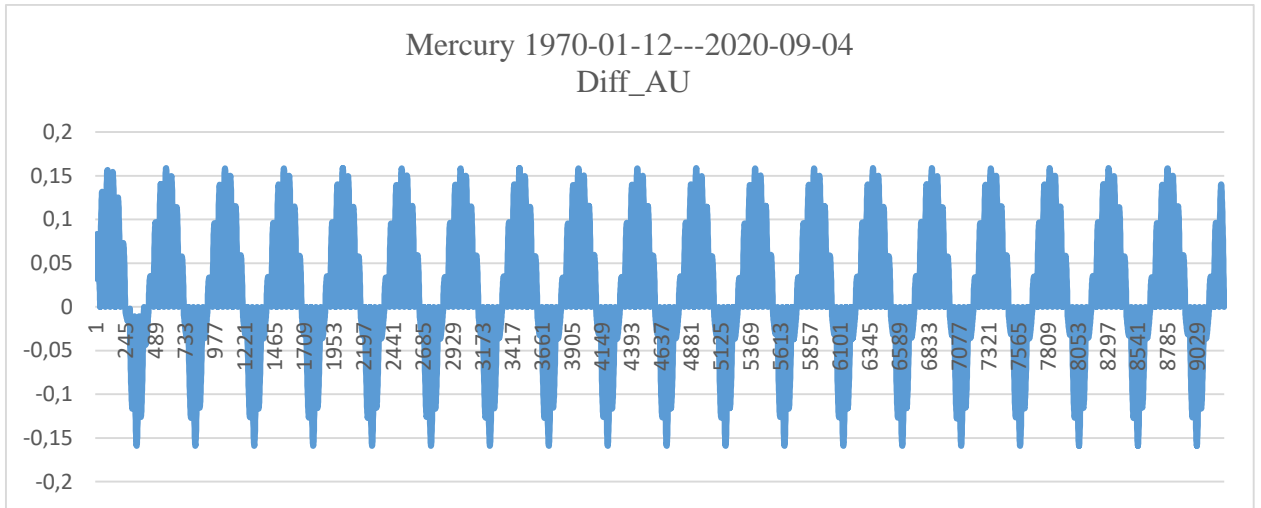


Fig. 5

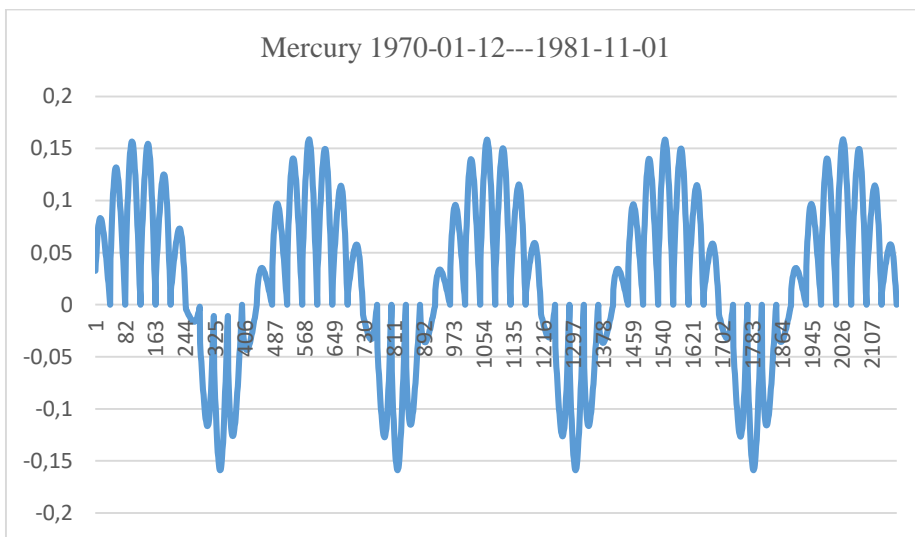


Fig. 6

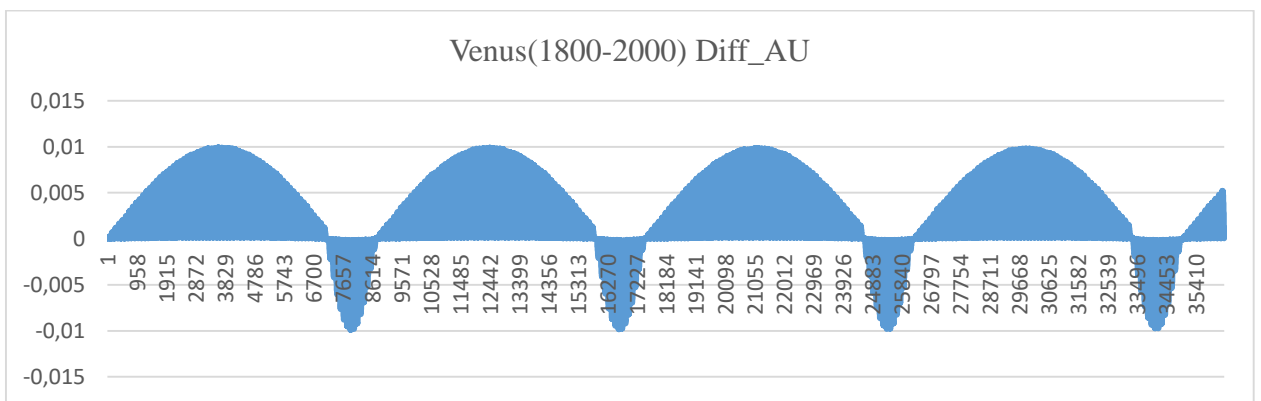


Fig.7

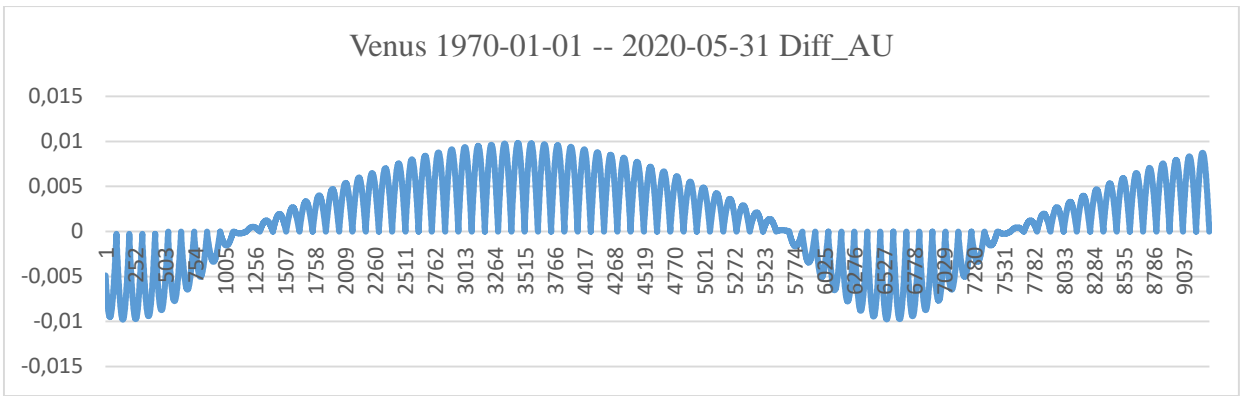


Fig. 8

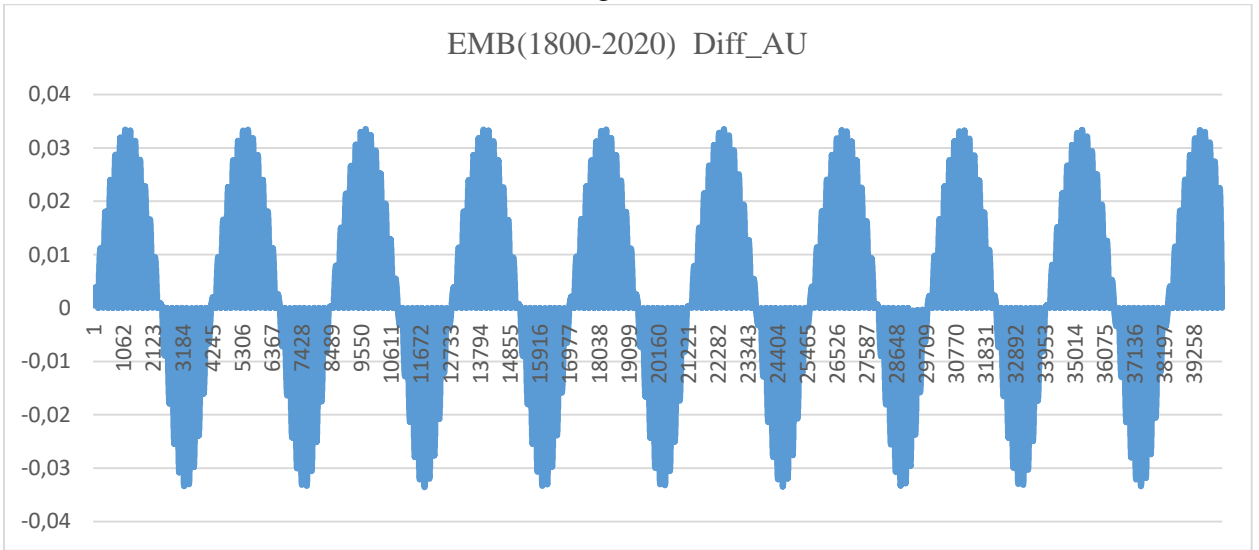


Fig. 9

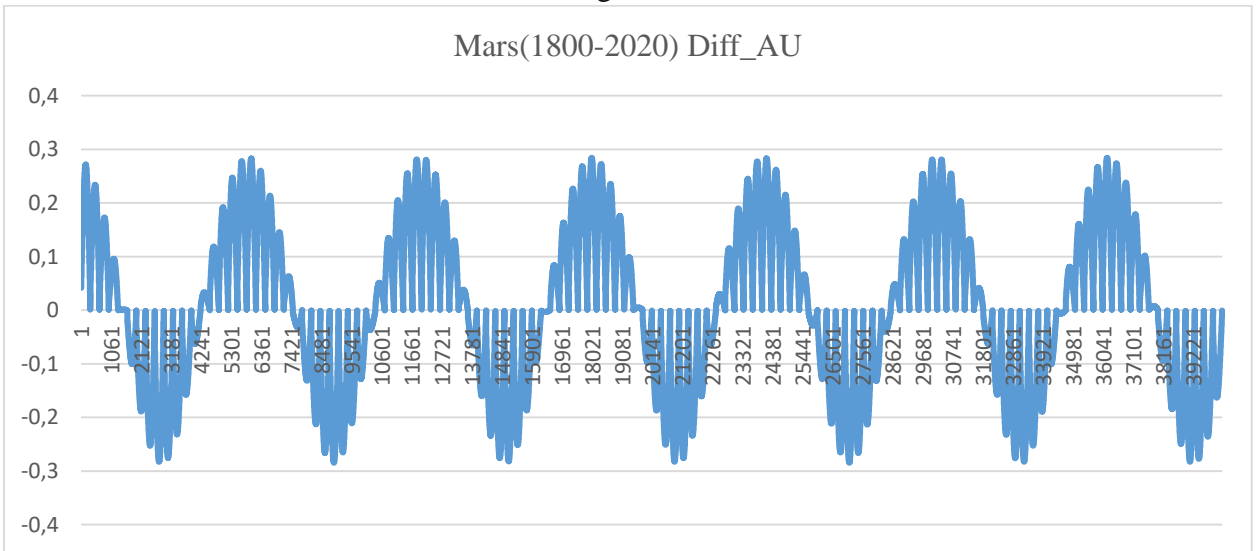


Fig. 10

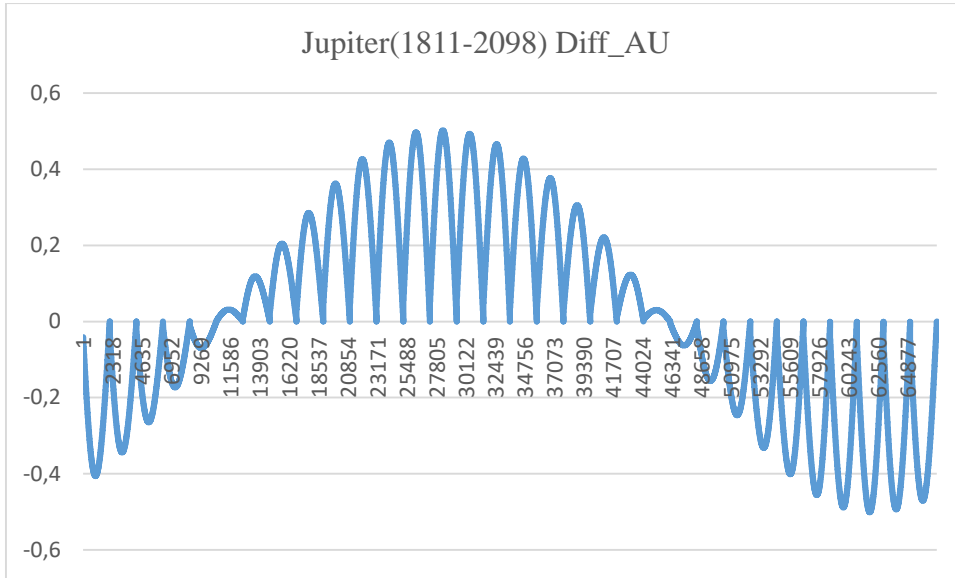


Fig. 11

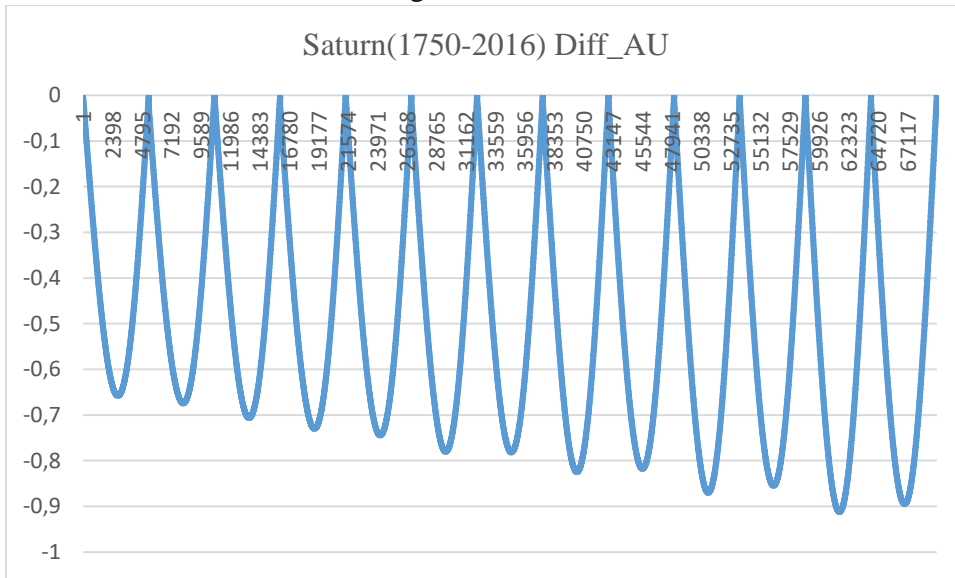


Fig. 12

Mathematical Description Linking the ψ_0 Model with Observed Graphs

1. Initial Equation of an Orbit with Precession

A classical closed ellipse cannot generate a long-period envelope. Therefore, we must immediately proceed to the generalized orbital equation provided in [2] (Formula 43):

$$r(\varphi(t)) = \frac{p}{1+e \cdot \cos(k\varphi)} \quad (1)$$

Where:

- r – Distance from the central body (orbital focus).
- φ – True anomaly (angle in the orbital plane).

- $p = a(1 - e^2)$ – focal parameter (a is the semi-major axis).
- e – eccentricity.
- k – key precession parameter. If $k = 1$, it is a closed ellipse. If $k \neq 1$, the orbit is an open rosette, where the pericenter shifts by an angle $\Delta\varphi = 2\pi(1/k - 1)$ per revolution.

2. Introduction of the Time Variable

The analysis is based on time series comparing distances R_1 and R_2 for moments symmetrical relative to the line of apsides. Let t_p be the moment of pericenter passage. Two symmetrical points t_1 and t_2 within one orbit are defined as:

$$\begin{aligned} t_1 &= t_p + \Delta t \\ t_2 &= t_p + T - \Delta t \end{aligned}$$

Where T is the orbital period and Δt ranges from 0 to $T/2$. The distances at these moments are: $R_1(\Delta t) = r(t_p + \Delta t)$ and $R_2(\Delta t) = r(t_p + T - \Delta t)$ $R_1(\Delta t) = r(t_p + \Delta t)$ и $R_2(\Delta t) = r(t_p + T - \Delta t)$.

The dependence of distance on time is implicitly given by the law of areas (Kepler's Second Law), which in the formulation of [2] (Equation 18) appears as:

$$\frac{d\varphi}{dt} = \dot{\varphi} = \frac{C}{r^2} \quad (2)$$

where the constant C (sectorial velocity $\times 2$) is expressed through the orbital parameters and the period T (Formula 24):

$$C = \frac{2\pi ab}{T} = \frac{2\pi a^2 \sqrt{1-e^2}}{T}$$

The connection between time Δt and angle ϕ is found by solving Kepler's transcendental equation, but for a qualitative derivation of the envelope, we can use a direct expansion.

3. Derivation of the Difference Function for One Precessing Orbit

The central object of the study is the difference $D(\Delta t) = R_2(\Delta t) - R_1(\Delta t)$. Our task is to obtain an analytical expression for D .

Due to the symmetry of the time moments, the angles for R_2 and R_1 are related. Because of precession ($k \neq 1$), the line of apsides rotates slowly. If we consider a single revolution, the axis of symmetry is the current line of apsides for this revolution. Within one "year," precession is negligible, and $k \approx 1$. Locally, for one revolution, the orbit looks like an almost perfect ellipse.

However, the method of constructing the difference $R_2 - R_1$ "cuts out" the asymmetry induced by this small precession.

Let the true anomaly corresponding to Δt for a given revolution be φ . Since R_2 is the distance from the focus to a point on the path "from pericenter to apocenter" (angle φ), and R_1 is the distance to the symmetrical point on the path "from apocenter to pericenter" (angle $2\pi/k - \varphi \approx 2\pi - \varphi$), we can write:

$$R_1(\varphi) = \frac{p}{1+e \cdot \cos(k\varphi)}$$

$$R_2(\varphi) = \frac{p}{1+e \cdot \cos(k(2\pi/k-\varphi))} = \frac{p}{1+e \cdot \cos(2\pi-k\varphi)}$$

Here lies a subtlety: if we strictly take $k = 1$, then $\cos(2\pi - k\varphi) = \cos(k\varphi)$, $R_2 = R_1$, and $D = 0$. It is precisely the deviation of k from 1 that creates the imperfect symmetry. However, a more accurate and fruitful approach for describing the envelope is to consider how precession manifests itself when comparing revolutions.

4. Mathematical Model of the Envelope Wave

The "envelope" graph is constructed from the maxima and minima of the difference D from revolution to revolution over a long time interval. During a planet's "year" (one revolution), precession accumulates a pericenter shift $\Delta\sigma = \Delta\varphi$. This shift breaks the exact symmetry, and the function D begins to systematically change from revolution to revolution, generating a wave with a period equal to one cycle of pericenter precession.

Let σ be the longitude of pericenter. In a simplified model of uniform precession, $\sigma(t) = \sigma_0 + \delta t$, where δ is the angular velocity of precession. The envelope wave is a modulation of the amplitude of the difference D with frequency δ .

The distance difference function for an arbitrary revolution with precession phase σ can be approximated by Fourier series expansion. The first harmonic of this difference, which forms the smooth curves (Figs. 1-4), has the form:

$$D(\Delta t; \delta) \approx A(\delta) \cdot \sin\left(\frac{2\pi\Delta t}{T}\right)$$

The amplitude $A(\sigma)$ slowly changes from revolution to revolution. From the data and geometric considerations, it follows that the amplitude is proportional to the projection of the eccentricity vector onto an axis perpendicular to the line of sight at the symmetrical point. As a result, beating occurs:

$$A(\delta) \propto e \cdot \sin(\delta - \delta_{ref})$$

where σ_{ref} is some reference direction. Consequently, the time series of the modulus $|D|$ will have a characteristic rectified sinusoidal profile, which exactly corresponds to the "envelope waves" in Figs. 5-12.

The final mathematical description of the envelope graph (envelope as a function of revolution number N):

$$E(N) \approx K \cdot \left| \sin\left(\pi \frac{N}{N_{prec}} + \psi_0\right) \right| \quad (3)$$

where:

- $E(N)$ — smoothed value of the difference modulus (envelope) for revolution N .
- K — scale factor (depends on e and a).
- $N_{prec} = T_{prec} / T$ — number of revolutions ("years") in one full cycle of pericenter precession.
- ψ_0 — initial phase.

5. Verification via Empirical Laws

This model fully explains conclusions 1-5:

1. Smoothness of curves: follows from the smoothness of sinusoidal modulation.
2. Form of the curve for a year: follows from the Fourier series expansion of $D(\Delta t)$, where the first harmonic dominates.
3. Envelope wave: directly modeled as $|\sin(\pi N/N_{prec})|$.
4. Identical number of cycles in half-periods: a phenomenological fact indicating that the main precession frequency is constant near the nodes of the function.
5. Ratio of cycle numbers: The numbers found (Mercury 7/5, Venus 41/17, etc.) are the ratio of the durations of the "positive" and "negative" half-periods of the wave. In an ideal $|\sin|$ model, they would be equal (like Jupiter's $\sim 16/16$). Their inequality points to a more complex envelope shape.

6. Calculation of Precession via Radius Differences at Symmetrical Points for a Single Revolution

The proposed method of analyzing symmetrical points allows not only a qualitative detection of precession but also provides a direct quantitative tool for its calculation. A fundamental advantage of the method is that data from **a single orbital period** is sufficient for the calculation, in contrast to the classical approach, which requires at least two consecutive pericenter passages.

6.1. Theoretical Basis

The starting point is the precessing orbit equation [2]:

$$r(\varphi(t)) = \frac{p}{1+e \cdot \cos(k\varphi)} \quad (6.1)$$

where r is the distance from the central body (focus of the orbit), φ is the true anomaly (angle in the orbital plane), $p = a(1 - e^2)$ is the focal parameter, a is the semi-major axis, e is the eccentricity, and k is the key precession parameter. For $k = 1$, we have a closed ellipse. For $k \neq 1$, the orbit is an open rosette, where the pericenter shifts by:

$$\Delta\varphi = 2\pi \left(\frac{1}{k} - 1 \right) \text{ за один оборот} \quad (6.2)$$

The distance difference function at symmetrical points of one revolution is defined as:

$$D(\Delta t) = R_2(\Delta t) - R_1(\Delta t) \quad (6.3)$$

where $R_1(\Delta t) = r(t_p + \Delta t)$, $R_2(\Delta t) = r(t_p + T - \Delta t)$, t_p is the moment of perihelion passage, and T is the orbital period.

Using the expansion of the function $D(\Delta t)$ for small deviations of k from unity, an analytical relationship between the characteristics of the difference curve and the precession magnitude can be obtained. The maximum of the function D_{max} , reached near $\Delta t \approx T/4$, is expressed through the orbital parameters as:

$$D_{max} = \frac{\pi}{2} ae(1 - k) \quad (6.4)$$

6.2. Calculation via the Integral Measure of the Difference

A more accurate approach, using all the information contained in the experimental curve $D(\Delta t)$, is based on an integral estimate. For the precession per revolution, the following formula is valid:

$$\Delta\varphi = \frac{4}{aeT} \int_0^{T/2} D(\Delta t) d(\Delta t) \quad (6.5)$$

Derivation of formula (6.5). Consider the total variation of distance caused by precession over the half-period interval. To first order in $(1-k)$, the difference function is:

$$D(\Delta t) \approx ae(1-k) \cdot f\left(\frac{\Delta t}{T}\right)$$

where $f(x)$ is a dimensionless shape function, normalized by the condition $\int_0^{1/2} f(x) dx = \frac{\pi}{4}$. Then:

$$\int_0^{T/2} D(\Delta t) d(\Delta t) = ae(1-k) \cdot T \int_0^{1/2} f(x) dx = \frac{\pi}{4} aeT(1-k)$$

Considering that for small precessions $\Delta\varphi \approx 2\pi(1-k)$, we obtain the required formula (6.5).

6.3. Calculation via the Maximum of the Difference Function

For a quick estimate, the maximum value D_{max} can be used. Expressing $1-k$ from (6.4) and substituting into the approximate equality $\Delta\varphi \approx 2\pi(1-k)$, we obtain:

$$\Delta\varphi \approx \frac{4D_{max}}{ae} \quad (6.6)$$

6.4. Algorithm for Practical Calculation

Initial data:

- a — semi-major axis of the orbit
- e — eccentricity
- T — orbital period
- Ephemeris array (t_i, r_i) for the interval of one revolution $[t_p, t_p + T]$

Step 1. Determine the moment of perihelion passage t_p as the minimum point of the distance $r(t)$ on the selected interval.

Step 2. Form a table of symmetrical points. For each $j = 0, 1, \dots, M$ (where M is the number of points in a half-period):

- $\Delta t_j = j \cdot \frac{T}{2M}$
- By interpolating the ephemeris, find $R_1(\Delta t_j) = r(t_p + \Delta t_j)$
- Find $R_2(\Delta t_j) = r(t_p + T - \Delta t_j)$
- Calculate $D_j = R_2(\Delta t_j) - R_1(\Delta t_j)$

Step 3. Plot the graph $D(\Delta t)$ (similar to Figs. 1–4). A qualitative sign of precession: the function is not identically zero and has a characteristic smooth shape, symmetrical about $T/4$.

Step 4. Calculate the precession:

- **Approximate method:** find D_{\max} as the maximum of the array D_j , then

$$\Delta\varphi \approx 4D_{\max}/(ae).$$

- **Integral method:** numerically integrate the array D_j using the trapezoidal rule:

$$I = \frac{T}{2M} \left(\frac{D_0 + D_M}{2} + \sum_{j=1}^{M-1} D_j \right)$$

then $\Delta\varphi = 4I/(aeT)$

Step 5. Convert to arcseconds:

$$\Delta\varphi_{arcsec} = \Delta\varphi \times \frac{180 \times 3600}{\pi}$$

To obtain the secular precession per century, multiply by the number of orbital periods in 100 years:

$$\Delta\varphi_{arcsec} = \Delta\varphi \times \frac{36525}{T}$$

6.5. Numerical Example for Mercury

Initial orbital parameters of Mercury: $a = 5.7909 \times 10^7$ km, $e = 0.2056$, $T = 87.969$ days.

From the NASA HORIZONS ephemeris [1] for one revolution, the values of the difference function were obtained. The maximum value was $D_{\max} \approx 4.3$ km.

Approximate calculation:

$$\Delta\varphi \approx \frac{4 \times 4.3}{5.7909 \times 10^{-7} \times 0.2056} \approx 1.44 \times 10^{-6} \text{ рад} \approx 0.30'' \text{ per revolution}$$

In recalculation per century (415 revolutions):

The obtained value includes all components of precession.

Advantage of the method: The value is obtained from data of a single revolution, without the need for accumulation of long-term observations. The accuracy is determined by the quality of the ephemeris and the number of discretization points M . The integral method (6.5) provides higher accuracy compared to the estimate from a single maximum (6.6), as it uses the entire shape of the $D(\Delta t)$ curve and is less sensitive to measurement noise at a single point.

Thus, the method of symmetrical points is not only a qualitative indicator of precession but also an independent quantitative tool for its measurement. The very fact of a non-zero function $D(\Delta t)$ on one revolution is direct observational evidence that $k \neq 1$, and precession is "imprinted" in the geometry of each individual revolution, rather than manifesting only as a secular effect.

Conclusion

The developed kinematic model with parameter k (precession) serves as the missing link that connects the geometry of orbital motion with the discovered long-period oscillations of the modulus of the radius difference. The oscillation graphs are a visualization of the beats arising from the periodic change in the projection of the orbit due to the precession of its major axis.

The method allows direct extraction of the perihelion precession magnitude from the analysis of the difference of radius vectors on a single revolution, without resorting to long-term observations of the shift of the perihelion point itself.

References

1. NASA HORIZONS, <https://ssd.jpl.nasa.gov/horizons.cgi>
2. V. Strohm, FROM THE KINEMATICS OF ELLIPTICAL MOTION TO THE GRAVITATIONAL FORCE, <https://zenodo.org/uploads/19182136>