A Complete Proof of the Rational Distance Problem for the Unit Square

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Abstract

Abstract: This paper studies the following classical geometric problem: does there exist a point inside the unit square whose distances to all four vertices are rational? We first prove that if such a point exists, its coordinates must be rational. Through a scaling transformation, the original problem is equivalently reduced to a Diophantine problem involving an integer square with integer coordinates and integer distances. Based on the parity alignment of common legs, we discuss three cases and derive contradictions using the parameterization of primitive Pythagorean triples and parity analysis. Combined with known results for boundary cases, we prove that no such point exists inside the unit square.

Keywords: rational distance problem, square problem, Pythagorean triples, Diophantine geometry **MSC2020:** 11D25, 51M04

1 Introduction

In 1976, C.W. Dodge posed the following "square problem" in *Mathematical Magazine* [1]: does there exist a point inside the unit square whose distances to all four vertices are rational? Several special cases have been settled: no such point exists on the diagonals, the medians, or the edges [2, 4, 5]. Berry [6] proved that no point on the circumcircle can have three rational distances to the vertices. The problem is also listed in Guy's *Unsolved Problems in Number Theory* [7] and in Brass, Moser, and Pach's *Research Problems in Discrete Geometry* [3].

Let $P(x, y)$ be a point inside an integer square of side length z (after scaling), with vertices $A(0, 0)$, $B(0, z)$, $C(z, z)$, $D(z, 0)$, satisfying $\gcd(x, y, z) = 1$. The distances from P to the four sides are x, $z-y$, $z-x$, y, respectively (see Figure 1). Four integer right triangles are formed:

$$x^2+y^2 = PA^2, \quad (z-x)^2+y^2 = PD^2, \quad x^2+(z-y)^2 = PB^2, \quad (z-x)^2+(z-y)^2 = PC^2. \tag{1}$$

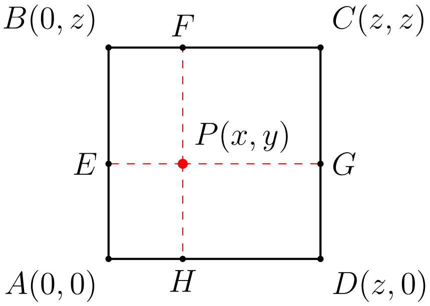

Figure 1: Integer square $ABCD$ of side length z with interior point $P(x,y)$

2 Preliminaries

2.1 Rationality of coordinates

Lemma 2.1. *If a point inside the unit square has rational distances to all four vertices, then its coordinates are rational.*

Proof. From $PA^2 = x^2 + y^2 \in \mathbb{Q}$ and $PB^2 = x^2 + (1-y)^2 \in \mathbb{Q}$, subtracting gives $1 - 2y \in \mathbb{Q}$, hence $y \in \mathbb{Q}$. Similarly $x \in \mathbb{Q}$. □

By Lemma 2.1, any candidate point must have rational coordinates. Write $x = p/q$ and $y = r/s$ in lowest terms, and let $z = \operatorname{lcm}(q, s)$. Scaling by z yields an integer square with integer distances. The problem becomes: does there exist an integer $z > 0$ and $0 < x, y < z$ such that $P(x, y)$ has integer distances to all four vertices of the square? If a solution exists, dividing by $g = \gcd(x, y, z)$ yields a **primitive solution** with $\gcd(x, y, z) = 1$. Henceforth we always assume this condition.

2.2 Primitive Pythagorean triples

Lemma 2.2 (Parameterization of primitive triples)**.** *Let positive integers a, b, c satisfy $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = 1$. Then there exist coprime positive integers $m > n$ of opposite parity such that $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ (or a, b interchanged).*

Corollary 2.1. *In an integer right triangle, the even leg is always a multiple of 4.*

Proof. For non-primitive triples, the sides are $k(m^2 - n^2)$, $k \cdot 2mn$, $k(m^2 + n^2)$. Since $2mn \equiv 0 \pmod 4$, multiplying by k preserves this property. □

Corollary 2.2. *An integer right triangle cannot have both legs odd.*

Proof. If both legs were odd, the hypotenuse squared would be $\equiv 1 + 1 = 2 \pmod 4$, but squares are only $\equiv 0$ or $1 \pmod 4$. □

3 Contradiction for interior points

3.1 Three alignment cases

The four equations (1) correspond to four integer right triangles. The common leg y is shared by $\triangle PHA$ and $\triangle PHD$; the common leg x is shared by $\triangle PEB$ and $\triangle PEA$. Each Pythagorean triple has an odd leg $a = m^2 - n^2$ and an even leg $b = 2mn$ ($\equiv 0 \pmod 4$). Let k_i be positive integer scaling factors with $\gcd(k_1, k_2) = 1$.

Case A (even–even): $y = k_1 \cdot 2m_1n_1 = k_2 \cdot 2m_2n_2$.

Case B (even–odd): $y = k_1 \cdot 2m_1n_1 = k_2(m_2^2 - n_2^2)$.

Case C (odd–odd): $y = k_1(m_1^2 - n_1^2) = k_2(m_2^2 - n_2^2)$.

3.2 Case B: even–odd alignment

Proposition 3.1. *The even–odd alignment cannot occur.*

Proof. Let $y = k_1b_1 = k_2a_2$ with b_1 even and a_2 odd. Then y is even, so k_2 must be even and k_1 odd (since $\gcd(k_1, k_2) = 1$). Now $x = k_1a_1$ is odd, $z - x = k_2b_2$ is even, hence z is odd and $PF = z - y$ is odd. In $\triangle PFB$, both legs PF and x are odd, contradicting Corollary 2.2. ☐

3.3 Case A: even–even alignment

Theorem 3.1. *The even–even alignment cannot occur.*

Proof. Let $y = k_1b_1 = k_2b_2$ where $b_i = 2m_in_i$ are even legs. By Corollary 2.1, $b_i \equiv 0 \pmod 4$, so $y \equiv 0 \pmod 4$.

Write $y = 4Y$ with $Y \geq 1$ (the case $Y = 0$ corresponds to a point on the boundary, already excluded). From $\triangle PHA$ we have $x^2 + 16Y^2 = PA^2$, so

$$(PA - x)(PA + x) = 16Y^2. \tag{2}$$

Since $x = k_1a_1$ with k_1 odd and a_1 odd, x is odd. From (2), $PA^2 = x^2 + 16Y^2$ is odd, hence PA is odd. Write

$$PA - x = 2s, \qquad PA + x = 2t,$$

then s, t are positive integers with $st = 4Y^2$. Since $x = t - s$ is odd, s and t have opposite parity.

As $st = 4Y^2 \equiv 0 \pmod 4$, the even factor must be a multiple of 4. Without loss of generality let s be odd and $t = 4r$, then

$$x = t - s = 4r - s, \qquad rs = Y^2. \tag{3}$$

Applying the same reasoning to $\triangle PHD$: from $(z - x)^2 + 16Y^2 = PD^2$, both $z - x$ and PD are odd, so

$$z - x = 4r' - s', \qquad r's' = Y^2, \tag{4}$$

where s' is odd and r' is a positive integer.

From (3) and (4):

$$4r - s = x = z - (z - x) = z - (4r' - s'),$$

hence $4(r + r') - (s + s') = z$.

Let $g = \gcd(r, r')$, and write $r = gm$, $r' = gn$ with $\gcd(m, n) = 1$. From $rs = r's' = Y^2$ we get $ms = ns'$. Since $\gcd(m, n) = 1$, we have $n \mid s$ and $m \mid s'$. Write $s = nv$ and $s' = mv$ for some positive integer v, so $Y^2 = gmnv$.

From $4r - s = 4r' - s'$:

$$4gm - nv = 4gn - mv \implies 4g(m - n) = -v(m - n).$$

If $m \neq n$, then $4g = -v < 0$, contradicting $g, v > 0$. Hence $m = n = 1$ (as $\gcd(m, n) = 1$), giving $r = r' = g$ and $s = s' = v$.

Substituting into (3) and (4) yields $x = 4g - v = z - x$, so $z = 2x$. Thus P lies on the median $x = z/2$, contradicting Theorem 2 of Ji [4]. $\square$

3.4 Case C: odd–odd alignment

Theorem 3.2. *The odd–odd alignment cannot occur.*

Proof. Let $y = k_1 a_1 = k_2 a_2$ with y odd, so k_1, k_2 are odd. Then $x = k_1 b_1$ is even, $z - x = k_2 b_2$ is even, so z is even and $PF = z - y$ is odd.

In the horizontal direction, x is the common leg of $\triangle PEB$ and $\triangle PEA$. We have $EB = PF$ odd, $EA = y$ odd, and x even. Thus x forms an **even–even** alignment across these two triangles. By Theorem 3.1, the even–even alignment is impossible, so the odd–odd alignment is also impossible. $\square$

Theorem 3.3. *There exist no integers $z > 0$ and $0 < x, y < z$ such that $P(x, y)$ has integer distances to all four vertices of the square.*

Proof. By Proposition 3.1, Theorem 3.1, and Theorem 3.2, all three alignment cases are impossible. $\square$

4 Boundary cases and conclusion

Boundary cases have been excluded by prior work: on edges [2], on diagonals [3], on medians [4], and on the circumcircle [6].

Combining Lemma 2.1 and Theorem 3.3:

There does not exist a point inside (or on the boundary of) the unit square whose distances to all four vertices are rational.

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单位正方形有理距离问题的完整证明

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摘要

摘要：本文研究如下经典几何问题：单位正方形内是否存在一点，使其到四个顶点的距离均为有理数？首先证明：若存在这样的点，则其坐标必为有理数。通过缩放变换，将原问题等价转化为整数正方形、整数坐标、整数距离的丢番图问题。基于公共直角边的奇偶性对齐方式分三种情形讨论，利用本原勾股数的参数化表示及模 4 分析导出矛盾。结合已有边界情形的结论，证明单位正方形内不存在具有上述性质的点。

关键词：有理距离问题；正方形问题；勾股数；丢番图几何

1　引言

1976 年，C.W. Dodge 在《数学杂志》上提出了如下“正方形问题”[1]：单位正方形内是否存在一点，使其到四个顶点的距离均为有理数？已有研究证明了若干特例：若点位于正方形的对角线、中位线或边上，则不存在这样的点 [2, 4, 5]。Berry[6] 证明了外接圆上不存在三个有理距离。该问题在 Guy[7] 以及 Brass 等 [3] 的专著中均有收录。

设 $P(x, y)$ 为边长 z 的整数正方形内一点（缩放后），顶点 $A(0, 0)$、$B(0, z)$、$C(z, z)$、$D(z, 0)$，满足 $\gcd(x, y, z) = 1$。P 到四边的距离分别为 x、$z - y$、$z - x$、y（见图 1）。四个整边直角三角形满足：

$$x^2 + y^2 = PA^2,\ (z - x)^2 + y^2 = PD^2,\ x^2 + (z - y)^2 = PB^2,\ (z - x)^2 + (z - y)^2 = PC^2. \quad (1)$$

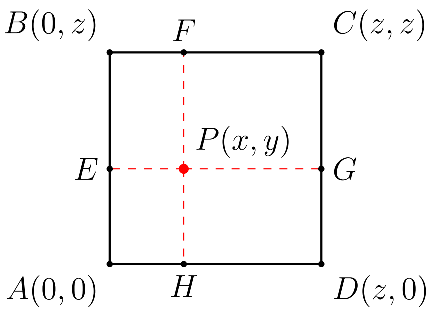

图 1: 边长为 z 的整数正方形 $ABCD$ 及内点 $P(x,y)$

2　预备知识

2.1　坐标的有理性

引理 2.1. 若单位正方形内一点到四个顶点的距离均为有理数，则其坐标为有理数。

证明. 由 $PA^2 = x^2 + y^2 \in \mathbb{Q}$ 和 $PB^2 = x^2 + (1-y)^2 \in \mathbb{Q}$，相减得 $1 - 2y \in \mathbb{Q}$，故 $y \in \mathbb{Q}$。同理 $x \in \mathbb{Q}$。 □

由引理 2.1，候选点必具有有理坐标。设 $x = p/q$、$y = r/s$ 为既约分数，$z = \mathrm{lcm}(q,s)$。缩放 z 倍后边长为整数，距离也变为整数。问题等价于：是否存在整数 $z > 0$ 及 $0 < x, y < z$，使 $P(x,y)$ 到四顶点的距离均为整数？若存在解，除以 $g = \gcd(x,y,z)$ 可得满足 $\gcd(x,y,z) = 1$ 的**本原解**。以下总设此条件成立。

2.2　本原勾股数

引理 2.2 (本原勾股数参数化). 设正整数 a, b, c 满足 $a^2 + b^2 = c^2$ 且 $\gcd(a,b,c) = 1$。则存在互质正整数 $m > n$ 一奇一偶，使 $a = m^2 - n^2$，$b = 2mn$，$c = m^2 + n^2$（或 a, b 互换）。

推论 2.1. 在整边直角三角形中，偶数直角边必是 4 的倍数。

证明. 非本原时三边为 $k(m^2 - n^2)$、$k \cdot 2mn$、$k(m^2 + n^2)$。$2mn \equiv 0 \pmod 4$，乘以 k 后仍为 4 的倍数。 □

推论 2.2. 整边直角三角形的两条直角边不能同为奇数。

证明. 若均为奇数，则斜边平方 $\equiv 1+1 = 2 \pmod 4$，但完全平方数模 4 只能是 0 或 1。 □

3　内部点的矛盾

3.1　三种对齐方式

四个方程 (1) 对应四个整边直角三角形。公共直角边 y 是 $\triangle PHA$ 与 $\triangle PHD$ 的公共边，x 是 $\triangle PEB$ 与 $\triangle PEA$ 的公共边。每个勾股三元组有奇数边 $a = m^2 - n^2$（奇）和偶数边 $b = 2mn$（$\equiv 0 \pmod 4$）。设 k_i 为正整数缩放因子，$\gcd(k_1, k_2) = 1$。

情形 A（偶–偶）：$y = k_1 \cdot 2m_1n_1 = k_2 \cdot 2m_2n_2$。

情形 B（偶–奇）：$y = k_1 \cdot 2m_1n_1 = k_2(m_2^2 - n_2^2)$。

情形 C（奇–奇）：$y = k_1(m_1^2 - n_1^2) = k_2(m_2^2 - n_2^2)$。

3.2　情形 B：偶–奇对齐

命题 3.1. 偶–奇对齐不可能成立。

证明. 设 $y = k_1b_1 = k_2a_2$，b_1 偶，a_2 奇。则 k_2 偶，k_1 奇。$x = k_1a_1$ 奇，$z - x = k_2b_2$ 偶，z 奇，$PF = z - y$ 奇。$\triangle PFB$ 中两直角边 PF 与 x 同为奇数，由推论 2.2不可能。　□

3.3　情形 A：偶–偶对齐

定理 3.1. 偶–偶对齐不可能成立。

证明. 设 $y = k_1b_1 = k_2b_2$，其中 $b_i = 2m_in_i$ 为偶数边。由推论 2.1，$b_i \equiv 0 \pmod 4$，故 $y \equiv 0 \pmod 4$。

令 $y = 4Y$ ($Y \geq 1, Y = 0$ 对应点在边上，已排除)。由 $\triangle PHA$ 的方程 $x^2 + 16Y^2 = PA^2$，有

$$(PA - x)(PA + x) = 16Y^2. \tag{2}$$

因 $x = k_1a_1$（k_1 奇，a_1 奇），x 为奇数。由 (2)，$PA^2 = x^2 + 16Y^2$ 为奇数，故 PA 亦为奇数。设

$$PA - x = 2s, \qquad PA + x = 2t,$$

则 s, t 为正整数，且 $st = 4Y^2$。由 $x = t - s$ 为奇数，s 与 t 一奇一偶。

因 $st = 4Y^2 \equiv 0 \pmod 4$，其中偶者必为 4 的倍数。不失一般性设 s 奇、$t = 4r$，则

$$x = t - s = 4r - s, \qquad rs = Y^2. \tag{3}$$

对 $\triangle PHD$ 作同样推导：由 $(z - x)^2 + 16Y^2 = PD^2$，$z - x$ 与 PD 同为奇数，同理得

$$z - x = 4r' - s', \qquad r's' = Y^2, \tag{4}$$

其中 s' 奇，r' 为正整数。

由 (3) 和 (4)：

$$4r - s = x = z - (z - x) = z - (4r' - s'),$$

即 $4(r + r') - (s + s') = z$。

设 $g = \gcd(r, r')$，写 $r = gm$，$r' = gn$，$\gcd(m, n) = 1$。由 $rs = r's' = Y^2$ 得 $ms = ns'$。因 $\gcd(m, n) = 1$，有 $n \mid s$，$m \mid s'$。设 $s = nv$，$s' = mv$（v 为正整数），则 $Y^2 = gmnv$。

由 $4r - s = 4r' - s'$：

$$4gm - nv = 4gn - mv \;\Rightarrow\; 4g(m - n) = -v(m - n).$$

若 $m \neq n$，则 $4g = -v < 0$，与 $g, v > 0$ 矛盾。故 $m = n = 1$（因 $\gcd(m, n) = 1$），即 $r = r' = g$，$s = s' = v$。

代入 (3) 和 (4)得 $x = 4g - v = z - x$，故 $z = 2x$。点 P 位于中位线 $x = z/2$ 上，已被 Ji [4] 的定理 2 排除。 □

3.4 情形 C：奇–奇对齐

定理 3.2. 奇–奇对齐不可能成立。

证明. 设 $y = k_1 a_1 = k_2 a_2$, y 奇, k_1, k_2 奇。则 $x = k_1 b_1$ 偶, $z - x = k_2 b_2$ 偶, z 偶, $PF = z - y$ 奇。

在水平方向上，x 是 $\triangle PEB$ 与 $\triangle PEA$ 的公共直角边。$EB = PF$ 奇，$EA = y$ 奇，x 偶。故 x 在这两个三角形上形成**偶–偶**对齐。由定理 3.1，偶–偶对齐不可能，故奇–奇对齐亦不可能。 □

定理 3.3. 不存在整数 $z > 0$ 及 $0 < x, y < z$，使 $P(x, y)$ 到正方形四顶点的距离均为整数。

证明. 由命题 3.1、定理 3.1、定理 3.2，三种对齐方式均不可能。 □

4 边界情形与结论

边界情形已被排除：边上 [2]、对角线上 [3]、中位线上 [4]、外接圆上 [6]。

综合引理 2.1和定理 3.3：

单位正方形内（含边界）不存在一点，使其到四个顶点的距离均为有理数。

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