Stamped 2026-05-03 01:39 PM EDT
Current version v1.0

THE SCHOLZ CONJECTURE IS TRUE FOR $2^n - 1$ FOR ALMOST ALL n

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Abstract. An addition chain of length h that leads to a number n is a sequence of positive integers $s_0 = 1, s_1 = 2, \ldots, s_h = n$ such that $s_i = s_j + s_k$ $(i > j \geq k)$ for each $1 \leq i \leq h$. A Brauer addition chain is the one where $j = i - 1$ for each $1 \leq i \leq h$. Let $\ell(\cdot)$ and $\ell^*(\cdot)$ denote the minimal length of an addition chain and the Brauer addition chain, respectively, that leads to an integer $\cdot$. Applying probabilistic methods to the *iterated factor* method, we show that

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

for almost all positive integers n as $n \longrightarrow \infty$.

1. Introduction

An addition chain of length h that leads to a fixed positive integer n is a sequence of positive integers

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

such that $s_i = s_j + s_k$ $(i > j \geq k)$ for each $1 \leq i \leq h$. The smallest h for which there exists an addition chain for the target is the *minimal* length of the addition chain. We denote the minimal length by $\ell(n)$. Generally speaking, an addition chain can be viewed as a sequence of binary additions that starts with 1 and ends with n. Addition chains are studied not only for their intrinsic theoretical values but also for their relevance in accelerating repeated exponentiation, an activity which often appears in various branches of cryptography [3, 4].

The theoretical landscape of addition chains is rapidly evolving, thanks in part to the foundational work of Hansen (see, e.g., [2]), Alfred Brauer (see, e.g., [1]) and Arnold Scholz before him, who had conjectured (see, e.g., [7]) the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

for all $n \geq 2$. This inequality is now known as the Scholz conjecture on addition chains. Until now, this inequality has only been shown to hold for integers that admit a certain structural class of addition chains (see, e.g., [5], [8], [9]) and integer exponents of a certain class [10]. What remains unknown is the purported uniform version of the conjecture. Even a uniform asymptotic or a global average version of the conjecture, which one may regard as a significant milestone, also remains unknown and untenable till date.

In the present studies, we pursue a global average version of the Scholz conjecture; in particular, we show that the Scholz conjecture is true for a proportion of positive integers tending to 1 in the limiting sense. The main machinery for the work along this direction is

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Date: May 3, 2026.
2010 *Mathematics Subject Classification.* Primary 11Y16; Secondary 11A63; 11B83; 60F05.
Key words and phrases. probabilistic.

the inequality

$$\ell(2^n-1) \leq n-1+s-\xi(n,s)+\frac{3}{2}\sum_{j=1}^{s}(n)_2^j+\ell^*\left(\left\lfloor \frac{n}{2}\right\rfloor_s\right)$$

for all $n \geq 1$ and for any fixed $1 \leq s \leq \lfloor \frac{\log n}{\log 2} \rfloor$ where $(n)_2^j$ denotes the j^{th} digit in the binary expansion of n when the digits are read from right to left and

$$\xi(n,s) = \begin{cases} \frac{1}{2} & \text{if} \quad (n)_2^s = 1 \\ 0 & \text{if} \quad (n)_2^s = 0 \end{cases}$$

and

$$\left\lfloor \cdots \left\lfloor \frac{1}{2} \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor \right\rfloor \cdots \right\rfloor := \left\lfloor \frac{n}{2} \right\rfloor_s$$

where the left-hand side is the composite of s floor functions taken repeatedly in that manner with

$$\left\lfloor \frac{n}{2} \right\rfloor_0 := n \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor_1 := \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor_2 := \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor$$

in that manner. We note that this inequality encodes information about the structure of the actual digits in the binary expansion of n and a free integer variable $s := s(n)$ that can be freely chosen in such a way that $1 \leq s \leq \lfloor \frac{\log n}{\log 2} \rfloor$, making it a much more flexible inequality among the league of inequalities of the Scholz-Brauer type. By virtue of this flexibility, we will refer to this inequality as the *master* inequality. Here, we randomize the framework of our studies by choosing the integer exponent n uniformly from the set $\{1, 2, \ldots, M\}$ as $M \longrightarrow \infty$. Transitioning to this probabilistic framework allows us to use the Lindeberg-Levy central limit theorem to analyze the accumulated binary digits $(n)_2^j$ up to a cut-off integer s along the positions j of the digits in binary expansions of n. In particular, we deduce

$$T_n(s) := \sum_{j=1}^{s}(n)_2^j = \frac{s}{2}+\frac{\sqrt{s}}{2}Z_n+o(\sqrt{s})$$

where $Z_n \sim N(0,1)$. Here, $N(0,1)$ is the standard normal distribution, and Z_n is a random variable in this distribution. Essentially, transitioning to a randomized setting allows us to better understand and gain some control of $T_n(s)$. It is known (see, e.g., [12]) that $\ell(\cdot) \leq \ell^*(\cdot)$. However, the free variable s present in the *master* inequality can be carefully chosen so that this monotonic property is reversed. In following through this idea, we define a *slack*

$$F_n(s) := \ell^*(K_n(s)) - \left(\ell(n) - \frac{3}{2}T_n(s) - \frac{s}{4}\right)$$

where

$$T_n(s) := \sum_{j=1}^{s}(n)_2^j \quad \text{and} \quad K_n(s) := \left\lfloor \frac{n}{2} \right\rfloor_s.$$

The main theme of our work is to carefully choose an admissible range of values of $s := s(n)$ and then to verify that the *slack* $F_n(s)$ is not positive on average. Here, the averaging is two layered if the distribution function depends on n and s. This will mean that at least one cut-off integer $s := s(n)$ in the admissible range suffices for a non-positive *slack*, and therefore the inequality

$$\ell^*(K_n(s)) \leq \ell(n) - \frac{3}{2}T_n(s) - \frac{s}{4}.$$

Inserting this inequality into the *master* inequality yields the exact Scholz-Brauer-type inequality when an appropriate integer cut-off is chosen from the choice of an admissible range. This is the origin of our ”almost” assertions tied to the randomized framework. The important caveat is that this result does not address the full version of the Scholz conjecture or at least an asymptotic uniform version; it only says that the asymptotic density of the set of integer exponents n for which $2^n - 1$ satisfies the Scholz conjecture is 1.

1.1. **Organization of the paper.** The paper is organized as follows: Section 2 gathers important results relevant to establishing the main assertion in the paper and fixes the relevant notations. Section 3 proves the main claim of the current studies. The final section concludes with remarks about the result and its limitations, with a further direction or motivation for a uniform asymptotic result.

1.2. **Notations and definitions.** Throughout this paper, the following notation appears with their interpretations

- We denote the probability of a random variable $\cdot$ by $\Pr(\cdot)$.
- For any random variable X_n in a probabilistic distribution, we denote the average value of X_n by
$$\mathbb{E}[X_n] := \sum_{n\geq 1} X_n \Pr(X_n)$$
- For a deterministic distribution function $L := L(n)$, we denote the average value on any admissible set S (i.e. $n \in S$) by
$$\mathbb{E}[L(n)] := \frac{1}{|S|}\sum_{n\in S} L(n).$$
In the case where the deterministic distribution function depends on at least two deterministic variables, say $n \in S$ and $s \in R$, we denote the average value on the admissible sets S and R by
$$\mathbb{E}[L(n,s)] := \frac{1}{|R|}\sum_{s\in R} \mathbb{E}_n[L(n,s)] = \frac{1}{|R|}\sum_{s\in R}\frac{1}{|S|}\sum_{n\in S} L(n,s).$$
- The functions $\ell(\cdot)$ and $\ell^*(\cdot)$ denote the minimal length of an addition chain and Brauer addition chains, respectively, that lead to $\cdot$.
- The function $\log_2 n := \frac{\log n}{\log 2}$ and $f(n) = o(g(n))$ means $\lim\limits_{n\longrightarrow\infty} \frac{f(n)}{g(n)} = 0$.
- We denote the normal distribution with variance σ^2 by $N(0,\sigma^2)$. In particular, the standard normal (Gaussian) distribution will be denoted by $N(0,1)$. For a random variable Z_n, we write $Z_n \sim N(0,\sigma^2)$ to mean that Z_n is a random variable in the distribution $N(0,\sigma^2)$.
- We denote the set of all positive integers by $\mathbb{N}$ and the set $\{1, 2, \ldots, n\}$ by $\mathbb{N}_n$.

1.3. **Acknowledgment.** The author acknowledges his research supervisor, Professor Nicolas Doyon, for suggesting this direction of work and subsequent discussions that serves as a motivation to consider this version of the problem.

2. PRELIMINARY RESULTS

In this section, we collect some useful results for our investigation.

Theorem 2.1 (The iterated factor method)**.** *Let* $(n)_2^j$ *denote the* j^{th} *digit in the binary expansion of* n *when the digits are read from right to left. For any fixed integer* s *that satisfies* $\lfloor \frac{\log n}{\log 2} \rfloor \geq s := s(n) \geq 1$ *and for all* $n \geq 2$ *with* $n \in \mathbb{N}$*, we have*

$$\ell(2^n - 1) \leq n - 1 + s - \xi(n,s) + \frac{3}{2}\sum_{j=1}^{s}(n)_2^j + \ell^*\left(\left\lfloor \frac{n}{2} \right\rfloor_s\right)$$

where

$$\xi(n,s) = \begin{cases} \frac{1}{2} & \text{if} \quad (n)_2^s = 1 \\ 0 & \text{if} \quad (n)_2^s = 0 \end{cases}$$

and

$$\left\lfloor \cdots \left\lfloor \frac{1}{2} \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor \right\rfloor \cdots \right\rfloor := \left\lfloor \frac{n}{2} \right\rfloor_s$$

where the left-hand side is the composite of s *floor functions taken repeatedly in that manner with*

$$\left\lfloor \frac{n}{2} \right\rfloor_0 := n \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor_1 := \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor_2 := \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor$$

in that manner.

The inequality in Theorem 2.1 is the foundational template for our main results. It arises from the factor method in the foundational work of Arnold Scholz [7]. The idea is to write $2^n - 1$ in the form

$$(2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \eta(2^n - 1),$$

construct a minimal length chain that is compatible with this decomposition that leads to $2^n - 1$ and control the minimal length $\ell(2^n - 1)$ by

$$\ell(2^{\lfloor \frac{n}{2} \rfloor} - 1) + \ell(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \text{correction terms}.$$

By further decomposing $2^{\lfloor \frac{n}{2} \rfloor_j} - 1$ (for $j \geq 1$) in the form

$$(2^{\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor_j \rfloor} - 1)(2^{\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor_j \rfloor} + 1) + \eta(2^{\lfloor \frac{n}{2} \rfloor_j} - 1)$$

and constructing a minimal-length chain that is compatible with this decomposition, we can control the minimal length by

$$\ell((2^{\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor_j \rfloor} - 1)) + \ell((2^{\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor_j \rfloor} + 1)) + \text{correction terms}.$$

The inequality in Theorem 2.1 is a careful bookkeeping after s iterations and the application of the Brauer inequality $\ell(2^n - 1) \leq n - 1 + \ell^*(n)$.

We now state the following version of the central limit theorem.

Lemma 2.2 (Lindeberg-Levy central limit theorem)**.** *Suppose that* $X_1, \ldots, X_n$ *is a sequence of identically distributed independent random variables. Denote*

$$\overline{X_n} := \frac{1}{n} \sum_{i=1}^{n} X_i$$

with $\mathrm{E}(X_i) = \mu$ *(expected value) and* $\mathrm{Var}[X_i] = \sigma^2$ *(variance). For* $n \longrightarrow \infty$*, the random variable* $\sqrt{n}(\overline{X_n} - \mu)$ *converges in distribution to* $N(0, \sigma^2)$ *(normal distribution). Precisely*

$$\sqrt{n}(\overline{X_n} - \mu) \sim N(0, \sigma^2).$$

Proof. For a proof, see, for example, [11]. □

With

$$\overline{X_n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

we may write

$$(\sqrt{n})(\overline{X_n} - \mu) = \frac{\sqrt{n}}{n} \left(\sum_{i=1}^{n} X_i - \mu n \right) = \frac{\sum_{i=1}^{n} X_i - \mu n}{\sqrt{n}} \sim N(0, \sigma^2)$$

by Lemma 2.2 as $n \longrightarrow \infty$. Therefore

$$\sum_{i=1}^{n} X_i = \mu n + \sqrt{n} Z_n + o(\sqrt{n})$$

with $Z_n \sim N(0, \sigma^2)$.

Lemma 2.3 (Erdős)**.** *Let* $\ell(\cdot)$ *denote the minimal length of an addition chain that leads to* $\cdot$*. For almost all* $n \in \mathbb{N}$ *(as* $n \longrightarrow \infty$*), we have*

$$\ell(n) = \log_2 n + (1 + o(1)) \frac{\log_2 n}{\log_2 \log_2 n}.$$

Proof. For a proof, see [6]. □

3. MAIN RESULTS

In this section, we apply *probabilistic* methods to the *iterated* factor method (Theorem 2.1) to prove the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

for a set of positive integers with asymptotic density 1.

Theorem 3.1. *Let* $\ell(\cdot)$ *denote the minimal of an addition chain that leads to* $\cdot$*. For almost all* $n \in \mathbb{N}$ *(as* $n \longrightarrow \infty$*), we have*

$$\ell(2^n - 1) \leq n - 1 + \ell(n).$$

Proof. We start from the inequality

$$\ell(2^n-1) \leq n-1+s-\xi(n,s)+\frac{3}{2}\sum_{j=1}^{s}(n)_2^j+\ell^*\left(\left\lfloor\frac{n}{2}\right\rfloor_s\right) \tag{3.1}$$

where

$$\xi(n,s)=\begin{cases}\frac{1}{2} & \text{if} \quad (n)_2^s=1\\ 0 & \text{if} \quad (n)_2^s=0\end{cases}$$

for each fixed integer $s \geq 1$ and for all $n \geq 2$ and

$$\left\lfloor \cdots \left\lfloor \frac{1}{2}\left\lfloor \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor\right\rfloor\right\rfloor \cdots \right\rfloor := \left\lfloor\frac{n}{2}\right\rfloor_s$$

where the left-hand side is the composite of s floor functions taken repeatedly in that manner with

$$\left\lfloor\frac{n}{2}\right\rfloor_0 := n \quad \text{and} \quad \left\lfloor\frac{n}{2}\right\rfloor_1 := \left\lfloor\frac{n}{2}\right\rfloor \quad \text{and} \quad \left\lfloor\frac{n}{2}\right\rfloor_2 := \left\lfloor\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor\right\rfloor$$

in that manner. We let n be drawn uniformly from $\{1,2,\ldots,M\}$ (as $M \longrightarrow \infty$) and set $s := s(n)$ as an integer-valued cutoff that satisfies

$$1 \leq s := s(n) \leq 1+o(1).$$

Furthermore, we define

$$T_n(s) := \sum_{j=1}^{s}(n)_2^j \quad \text{and} \quad K_n(s) := \left\lfloor\frac{n}{2}\right\rfloor_s.$$

We note that $(n)_2^j \sim$ Bernoulli $(\frac{1}{2})$, which means that the digits $(n)_2^j$ are distributed in the sense of Bernoulli with probability $\frac{1}{2}$. Thus

$$\mathbb{E}[(n)_2^j]=\frac{1}{2} \quad \text{and} \quad \mathrm{Var}[(n)_2^j]=\mathrm{E}[((n)_2^j)^2]-(\mathrm{E}[(n)_2^j])^2=\frac{1}{4}.$$

By the lemma 2.2 and the ensuing discussion, we can write

$$T_n(s)=\frac{s}{2}+\frac{\sqrt{s}}{2}Z_n+o(\sqrt{s})$$

where $Z_n \sim N(0,1)$. Here, $N(0,1)$ is the standard normal distribution, and Z_n is a random variable in this distribution. Now, for each integer $s := s(n)$ that satisfies

$$1 \leq s := s(n) \leq 1+o(1)$$

we define the *slack*

$$F_n(s) := \ell^*(K_n(s))-\left(\ell(n)-\frac{3}{2}T_n(s)-\frac{s}{4}\right)$$

and deduce

$$\begin{aligned}\mathbb{E}[F_n(s)] &= \mathbb{E}\left[\ell^*(K_n(s))-\left(\ell(n)-\frac{3}{2}T_n(s)-\frac{s}{4}\right)\right]\\ &= \mathbb{E}[\ell^*(K_n(s))]-\mathbb{E}\left[\left(\ell(n)-\frac{3}{2}T_n(s)-\frac{s}{4}\right)\right]\\ &= \mathbb{E}[\ell^*(K_n(s))]-\mathbb{E}[\ell(n)]+\frac{3}{2}\mathbb{E}[T_n(s)]+\frac{1}{4(1+o(1))}.\end{aligned}$$

By Lemma 2.3, we deduce

$$\mathbb{E}[\ell(n)] = \log_2 n + (1 + o(1))\frac{\log_2 n}{\log_2 \log_2 n}$$

and

$$\begin{aligned}
\mathbb{E}[\ell^*(K_n(s))] &= \frac{1}{1 + o(1)} \sum_{1 \le s \le 1 + o(1)} \mathbb{E}_n[\ell^*(n, s)] \\
&= \frac{1}{1 + o(1)} \sum_{1 \le s \le 1 + o(1)} \left(\log_2(n/2^s) + (1 + o(1))\frac{\log_2(n/2^s)}{\log_2 \log_2(n/2^s)}\right) \\
&= \frac{1}{1 + o(1)} \left(\log_2(n/2) + (1 + o(1))\frac{\log_2(n/2)}{\log_2 \log_2(n/2)}\right).
\end{aligned}$$

Furthermore, we deduce (by Lemma 2.2)

$$\mathbb{E}[T_n(s)] = \mathbb{E}\left[\frac{s}{2} + \frac{\sqrt{s}}{2} Z_n + o(\sqrt{s})\right] = \frac{1}{2(1 + o(1))} + o(1)$$

since $\mathbb{E}[Z_n] = 0$. Putting everything together, we deduce that the average value of *slack* $F_n(s)$ is

$$\begin{aligned}
\mathbb{E}[F_n(s)] &= \mathbb{E}\left[\ell^*(K_n(s)) - \left(\ell(n) - \frac{3}{2}T_n(s) - \frac{s}{4}\right)\right] \\
&= \mathbb{E}[\ell^*(K_n(s))] - \mathbb{E}\left[\left(\ell(n) - \frac{3}{2}T_n(s) - \frac{s}{4}\right)\right] \\
&= \mathbb{E}[\ell^*(K_n(s))] - \mathbb{E}[\ell(n)] + \frac{3}{2}\mathbb{E}[T_n(s)] + \frac{1}{4(1 + o(1))} \\
&= \frac{1}{1 + o(1)} \left(\log_2(n/2) + (1 + o(1))\frac{\log_2(n/2)}{\log_2 \log_2(n/2)}\right) - \log_2 n - (1 + o(1))\frac{\log_2 n}{\log_2 \log_2 n} \\
&+ \frac{1}{1 + o(1)} + o(1) \\
&= (1 + o(1))\frac{\log_2(n/2)}{\log_2 \log_2(n/2)} - (1 + o(1))\frac{\log_2 n}{\log_2 \log_2 n} + o(1) \\
&\le 0
\end{aligned}$$

over the test range

$$1 \le s := s(n) \le 1 + o(1)$$

and with $n \in \{1, \ldots, M\}$ as $M \longrightarrow \infty$. Now, we denote the probability of an event $\cdot$ by $\Pr(\cdot)$ and obtain $\Pr(F_n(s) \le 0) > 0$ for $1 \le s := s(n) \le 1 + o(1)$ and with $n \in \{1, \ldots, M\}$ as $M \longrightarrow \infty$. This implies that there exists an integer-valued cutoff $s := s(n)$ with

$$1 \le s := s(n) \le 1 + o(1)$$

such that

$$\ell^*(K_n(s)) \le \ell(n) - \frac{3}{2}T_n(s) - \frac{s}{4}.$$

Now, we choose the only integer $s_0 = 1$ that satisfies

$$1 \le s_0 := s_0(n) \le 1 + o(1)$$

such that

$$\ell^*(K_n(s_0)) \leq \ell(n) - \frac{3}{2}T_n(s_0) - \frac{s_0}{4}. \tag{3.2}$$

Since inequality (3.1) holds for each integer s that satisfies

$$1 \leq s \leq \left\lfloor \frac{\log n}{\log 2} \right\rfloor$$

we set $s := s_0 = 1$ in the inequality (3.1) and get

$$\ell(2^n - 1) \leq n - 1 + s_0 - \xi(n, s_0) + \frac{3}{2}\sum_{j=1}^{s_0}(n)_2^j + \ell^*\left(\left\lfloor \frac{n}{2} \right\rfloor_{s_0}\right) \tag{3.3}$$

Plugging inequality (3.2) into inequality (3.1), we deduce

$$\begin{aligned}
\ell(2^n - 1) &\leq n - 1 + s_0 - \xi(n, s_0) + \frac{3}{2}T_n(s_0) + \ell^*\left(\left\lfloor \frac{n}{2} \right\rfloor_{s_0}\right)\\
&\leq n - 1 + s_0 - \xi(n, s_0) + \frac{3}{2}T_n(s_0) + \ell(n) - \frac{3}{2}T_n(s_0) - \frac{s_0}{4}\\
&= n - 1 + \ell(n) - \xi(n, s_0) + \frac{3}{4}s_0\\
&\leq n - 1 + \ell(n) + \frac{3}{4}s_0 \quad (s_0 := 1)\\
&\leq n - 1 + \ell(n) + \frac{3}{4}
\end{aligned}$$

since $\xi(n, s_0) \in \{0, \frac{1}{2}\}$. Since $\ell(\cdot)$ is always an integer, the inequality is deduced. □

4. Conclusion and further remarks

In this paper, we have shown that for almost all positive integers n–in the sense that the asymptotic density of the set is 1– the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

holds. This is significant progress on a class of structural results obtained in a previous study by several authors towards the Scholz conjecture. However, it still falls short of asserting the original version of the conjecture. In this case, at least a uniform asymptotic version of the form

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

for all $n \geq n_0$ for some $n_0 > 0$ would be considered more significant. Pursuing this path would most definitely require some new ideas, which is not the current goal of this paper.

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