An average loneliness gap of 1/n can allow a minimum loneliness gap of 1/(2n)

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Abstract: - Consider n runners R_0, R_1, ... R_{n-1}, with distinct constant integer speeds S_0, S_1, ... S_{n-1} respectively, where $S_0 = 0$, running around the circumference of a circle of unit circumferential length from arbitrary starting points at time t=0. At time t, denote $g_i(t)$ be the minimum absolute distance along the circumference of R_i from R_0. We first use a result on prime numbers to obtain special cases of runners' speeds, for which the Lonely Runner Conjecture (LRC) is true. We then develop an approach to the LRC that derives a time at which, some subset of the runners is placed at the extremities of arcs of sectors ensuring implicit separation from R_0, while the remaining runners are directly separated from R_0. We use this approach to show that in the general case for large n, there exists a time T at which, $g_i(T) \geq 1/(2n)$ for all integers i in [1,n-1], and $(g_1(T) + g_2(T) + ... + g_{n-1}(T))/(n-1)$ tends to 1/n.

1. Introduction

The Lonely Runner Conjecture (LRC) problem is well-known, and one easy way of defining this problem is as follows: - Given (n-1) runners running with constant unique non-zero integer speeds, starting from the same point 0 (also denoted as R_0) at time t=0, on a circular running track of unit circumferential length, decide whether it is always possible to exist a time at which R_0 is atleast 1/n distant from every other runner with non-zero speed. [1][2][3][4]. There has been tremendous progress in recent years on the LRC, such as improvement of lower bounds on the separation gap [1][2], and proving the LRC for up to 13 runners [3].

One approach to the LRC problem is to study variants [5], in the hope that bounds for the variants might offer some clue for the original LRC problem. In our 2020 result [4], we developed an approach that finds the time at which the fraction of runners separated from R_0 by a gap of atleast 1/n tends to 1. This meant that the average gap from R_0 tends to 1/n, however the approach in [4] did not guarantee any minimum gap of the runners from R_0, hence some runners could remain stuck at or close to R_0. In this paper, we develop a new approach that also minimizes the average gap to tend to 1/n, but simultaneously ensures that the minimum trivial gap of 1/(2n) is also respected. Our results, both of [4] and of the present paper, apply to the general version (or generalized variant) of the LRC where the runners are allowed to have arbitrary starting points on the circumference at time t=0.

2. Notations

1. We focus on the general case of the LRC where n runners are denoted as integers in {0,1,2,...n-1} and start running clockwise along the circumference at time t=0 from arbitrary starting points, with distinct constant integer speeds. Runner 0 is permanently fixed at the "top" of the circle ("top" assuming the circle is drawn on a vertical wall) with speed $S_0=0$, and so the position of runner 0 is always 0. The position of any runner at any time t is a real number in [0,1[. Unless specified to be a special case, all the Theorems apply to the general case.
2. The distance between any two runners R_i and R_j, is measured as the shortest distance between them along the perimeter of the circular running track of length 1, and not as the straight line distance between them.
3. For each integer i in [1, n-1], R_i has a unique non-zero speed $S_i = s_{i,F\_i} P^{F\_i} + s_{i,F\_i - 1} P^{F\_i -1} + ... + s_{i,2} P^2 + s_{i,1} P^1 + s_{i,0}$, where each $s_{i,j}$ is a given integer constant in [0, P-1], and where P is a prime number less than n.
4. For each runner i in [1, n], E_i and F_i are non-negative integers representing the position (also called index) of the least significant non-zero $s_{i,j}$ and most significant non-zero $s_{i,j}$ respectively of s_i from the rightmost bit of s_i, when s_i is written in the base of a prime number P. For example, if n=6, P=3, and if S_1=2, S_2=17, S_3=30, S_4=27, S_5=9, and S_0=0, then we can write the speeds in base 3 format one below the other as shown below:

   S_1=0002, so E_1=1 and F_1=1,
   S_2=0122, so E_2=1 and F_2=3,
   S_3=1010, so E_3=2 and F_3=4,
   S_4=1000, so E_4=4 and F_4=4,
   S_5=0100, so E_5=3 and F_5=3.

5. H_k denotes the number of runners i having their $E_i = k$. In the example of point 5, $H_4 = 1$ since $E_4 = 4$, $H_3 = 1$ since $E_5 = 3$, $H_2 = 1$ since $E_3 = 2$, $H_1 = 2$ since $E_1 = E_2 = 1$.
6. For the Lonely Runner Conjecture (LRC), the aim is to prove the existence of the time $T_{target} = t_L P^{-L} + t_{L-1} P^{-L+1} + ... + t_2 P^{-2} + t_1 P^{-1}$, where L is a large integer, such that for each runner i in [1,n-1], the following statement is true: $(1/n) \leq \text{fraction}(T_{target} S_i) \leq (1 - 1/n)$, where:
    - P is a prime number lesser than n
    - for each integer j in [1,L], t_j is an unknown integer variable in [0, P-1],

   It is obvious then that T_{target} exists ↔ the LRC is true.
7. Let the circular running track be divided into P sectors numbered from 0 to (P-1), where sector j is bounded by the arc in [(j-0.5)/P, (j+0.5)/P] for each integer j in [0, (P-1)]. So the length of the arc bounding each sector is 1/P.

3. The approach

The approach we develop for the general case of the LRC is new, to the best of our knowledge: - derive a time T, at which some subset of the runners are placed at the extremities of arcs of sectors in the circle (indirectly ensuring separation from R_0), and at which the remaining runners are directly separated from R_0. Theorem 1 talks about the direct separation from R_0.

Theorem 1: Let all n-1 runners have arbitrary starting points at time t=0. Let all n-1 speeds of the runners be written in the base of a prime number P, where P < n. Let $H_k \leq (P - 1)$, for each integer k in [L,1]. There exists a time T such that for each runner i in [1, n-1], $1/(2P) \leq \text{fraction}(\text{StartingPoint}_i + TS_i) \leq (1 - 1/(2P))$.

Proof: The meaning of this theorem is that if we were to write all the runner's speeds in base P, one below the other, like in point 4 of Section 2, and if every single column has not more than (P - 1) least significant non-zero digits of speeds, then there exists a time T at which R_0 is simultaneously separated from all other runners by atleast 1/(2P). Let the circular running track be divided into P sectors numbered from 0 to (P-1), where sector j is bounded by the arc in [(j-0.5)/P, (j+0.5)/P] for each integer j in [0, (P-1)]. Now perform the following algorithm:

Algorithm 1:

```
For each runner i in [1, n-1]: { $\text{position}_i = \text{StartingPosition}_i$. }
k = max($E_i$ , over integers i in [1,n-1]).
While ($k \geq 1$):
{
    $S_{feasible} = \{0,1,2,...,P-1\}$.
    W = set of runners whose $E_i$=k.
    For each integer b in $S_{feasible}$:
    {
        For each runner i in W:
        {
            If fraction($\text{position}_i + (b\, S_i\, P^{-k})$) lies in Sector 0 of circular track:
            { $S_{feasible} = S_{feasible} - \{b\}$. Exit For loop over W. }
        }
    }
    r = any integer of $S_{feasible}$.
    For each runner i in [1, n-1]: { $\text{position}_i$ = fraction($\text{position}_i + (r\, S_i\, P^{-k})$). }
    k = k-1.
}
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In each iteration of the while loop, in the worst case, each of the P-1 runners needs a different integer element b of the set $S_{feasible}$ to cause it to arrive into Sector 0, which leaves a choice of atleast a single integer in $S_{feasible}$ that would allow all the runners in W to be out of Sector 0. Using that choice ensures that all the runners in W are separated by a distance of atleast 1/(2P) from R_0. Note that the choice of t_k does not alter the position of any runner i whose $E_i > k$. After the algorithm terminates, all the n-1 runners are positioned away from 0 by a distance of atleast 1/(2P).

Hence Proved

<u>Theorem 2</u>: **Let F be the family of a special case of runners with arbitrary starting points at time t=0, such that all n-1 speeds of the runners can be written in the base of a prime number P, where $P \leq n/2$, and where $H_k \leq (P - 1)$, for each integer k in [L, 1]. Then the LRC is true for F with a separation gap of atleast 1/n.**
<u>Proof</u>: This follows directly from Theorem 1, applying $P \leq n/2$. Hence, there exists a time T such that for each integer i in [1, n-1], $(1/n) \leq \text{fraction}(\text{StartingPoint}_i + TS_i) \leq (1 - 1/n)$.
<u>Hence Proved</u>

The next two Theorems 3 and 4 apply to when the runners have the same starting point. All other Theorems in this paper allow arbitrary starting points at time t=0.

<u>Theorem 3</u>: **Let F be the family of a special case of runners, where all runners have the same starting point of R_0 at time t=0, where all n-1 speeds of the runners can be written in the base of some positive integer $M \leq n$ such that $H_k = (n - 1)$, for some integer k in [L,1]. Then the LRC is true for F with a separation gap of 1/M.**
<u>Proof</u>: Denote K as the integer for which $H_K = (n - 1)$. This also means $H_k = 0$ for all integer $k \neq K$. Fixing t_K to any integer in [1,M-1] and all other $t_k=0$ will fix each runner at some positive integer multiple of 1/M away from R_0. Hence, there exists a time T such that for each runner i in [1, n-1], $(1/M) \leq \text{fraction}(TS_i) \leq (1 - 1/M)$.
<u>Hence Proved</u>

Proving the LRC bound of the 1/n gap is difficult, in the general case. In fact, it is still not known whether it is possible to relax this much cherished target bound of 1/n to f/n, where f is a constant fraction strictly between 1/2 (trivial bound) and 1. From Theorems 2 and 3, it is evident that the remaining difficult case of the LRC is where for each prime number P strictly below n, $H_k \geq P$ for atleast one integer k in [1,L] and $H_k \geq 1$ for atleast another k. Note that when $n/2 < P < n$, there will be a maximum of one column k with $H_k \geq P$. But when $P < n/2$, there could be multiple columns k with $H_k \geq P$. Theorem 4 is a development in this direction for a special case when $n/2 < P < n$ and where $H_k \geq P$ for one integer k in [L,1].

<u>Theorem 4</u>: **Let F be the family of a special case of runners, where all runners have the same starting point of R_0 at time t=0, such that all n-1 speeds of the runners can be written in the base of a prime number P, where $(n/2) < P < n$, and where $H_k \geq P$, for one integer k in [L,1] denoted as k_P. Out of the n-1 runners, let the speed S_i of each i^{th} runner of atleast (n-P) runners be of the form $S_i = s_{i,\, k\_P-1}\, P^{k\_P-1}$. Then a separation gap of atleast 1/(2P) is achievable.**
<u>Proof</u>: As we follow Algorithm 1 for this special case, the moment we reach k = k_P, any value of t_{k_P} will suffice to position the (n-P) runners, whose speeds are $s_{i,\, k\_P-1}\, P^{k\_P-1}$, at integer multiples of (1/P) on the circumference. The remaining runners (going to a maximum of P-1 runners) get positioned atleast 1/(2P) from R_0 by a suitable choice of t_{k_P} that exists as per Theorem 1.
<u>Hence Proved</u>

We describe Theorem 5 that introduces the new approach to the general case of the LRC as defined in Section 2. As will be later explained, though it does not actually prove or achieve the minimal 1/n gap of the LRC, it does achieve an average gap of 1/n, simultaneously respecting the minimal gap of 1/(2n).

<u>Theorem 5</u>: **Let all n-1 runners with constant distinct integer speeds have arbitrary starting points on the circle. Let all n-1 speeds of the runners be written in the base of a prime number P, and where each of {P, n, n-P} are large. There exists a time T that simultaneously separates, each of atleast (P-1) runners from R_0 by atleast 1/(2P), and each of the remaining (n-P) runners from R_0 by atleast 1/(2n).**
<u>Proof</u>: The basic idea is that while obtaining the target time $T = t_L P^{-L} + t_{L-1} P^{-L+1} + ... + t_2 P^{-2} + t_1 P^{-1}$, derive the value of t_K iteratively from k=L to k=1, just how we did in Algorithm 1, but the difference here is that t_K needs to simultaneously ensure that some subset of runners with $E_i = k$ are separated from R_0 using the strategy of Algorithm 1, while the other subset of runners with $E_i = k$ are placed in the extremities of the P sectors using a favourable choice of the preceding t_{K+1}. We first prove Lemma 5.1 that will be useful in this strategy.

<u>Lemma 5.1</u>: Let all n-1 runners have distinct non-zero integer speeds with arbitrary starting points on the circle. There exists a time T when atleast x runners are simultaneously present in a fixed arc of length x/n, where $0<x<n$.
<u>Proof</u>: Denote the fixed arc of length x/n as A_{target}. Denote the stated time $T = b_Q 2^{-Q} + b_{Q-1} 2^{-(Q-1)} + ... + b_2 2^{-2} + b_1 2^{-1}$, where Q is a large positive integer. Also, write all the speeds of runners in base 2. Now follow Algorithm 2 shown next.

Algorithm 2

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    For each runner i in [1, n-1]: { $position_i$ = $StartingPosition_i$. }
    Draw a diameter in the circle touching one end of $A_{target}$, to form two sectors that are semi-circles initially.
    Denote the sector (i.e. semi-circle initially) containing the arc $A_{target}$ as V.
    k = Q.
    While ($k \geq 1$):
    {
        W = set of runners i with their $E_i$=k.
        Choose $b_k$ = 0 or 1 that would position atleast half of the runners of W in V.
        For each runner i in [1, n-1]: { $position_i$ = fraction($position_i$ + ($b_k 2^{-k} S_i$)). }
        Draw a radius dividing V into two equal child sectors.
        V = child sector whose arc is a proper-superset of the part of $A_{target}$ within that child sector.
        If (V does not exist): { Exit Algorithm. }
        k = k - 1.
    }
```

After Algorithm 2 terminates, T is obtained satisfying the Lemma.
Hence Proved Lemma 5.1

Lemma 5.2: Let all n-1 runners have distinct non-zero integer speeds with arbitrary starting points on the circle. Consider P sectors in the circle numbered 0 to (P-1), with the arc of sector j in [(j-0.5)/P, (j+0.5)/P] for each integer j in [0, (P-1)]. Then there exists a time $T = t_L P^{-L} + t_{L-1} P^{-L+1} + ... + t_2 P^{-2} + t_1 P^{-1}$ when atleast (n-P) runners are simultaneously separated from the centers of arcs of the sectors in which they are present, by a distance of atleast 1/(2n).
Proof: Substituting the value of x in Lemma 5.1 with (n-P), we obtain the fact that there exists a time T when atleast (n-P) runners are simultaneously present in a fixed arc of length (n-P)/n = (1 - P/n) in the circle. Though T was obtained in Lemma 5.1 in base 2, it can be re-written in base P as $T = t_L P^{-L} + t_{L-1} P^{-L+1} + ... + t_2 P^{-2} + t_1 P^{-1}$, where each t_k is an integer in [0,P-1]. So T can now be thought of as a time when each of atleast (n-P) runners were separated from R_0 by a distance of atleast (1 - (1 - P/n))/2 = P/(2n). Then the time $T/P = t_L P^{-L-1} + t_{L-1} P^{-L} + ... + t_2 P^{-3} + t_1 P^{-2}$ is the time at which Lemma 5.2 is satisfied. The reason is that the final position of each of the (n-P) runners in Lemma 5.1 is given by fraction(I + y), where I is an integer and -(P/2n) > y > (P/2n). Dividing the time T by P means that the final position of each of these runners is fraction(I/P + y/P), where (I/P) is the center of one of the P sectors and -(1/2n) > (y/P) > (1/2n), which proves Lemma 5.2.
Hence Proved Lemma 5.2

Lemma 5.3: Consider a special case where there exists a column k denoted as k_P with H_{k_P} = n-1, i.e. all n-1 runners i have their E_i in column k_P. Then, there exists a time T at which the following two statements are true:

1. Considering the P sectors of the circle with the arc of sector j bounded in [(j-0.5)/P, (j+0.5)/P] for each integer j in [0, (P-1)], there exists a time $T = t_L P^{-L} + t_{L-1} P^{-(L-1)} + ... + t_{k\_P+1} P^{-(k\_P+1)}$ that separates each of atleast (n-P) runners from the centers of the P sectors in which they are present by atleast 1/(2n).
2. Irrespective of the values of t_k for k > k_P, t_{k_P} exists to separate atleast (P-1) runners from R_0 by atleast 1/(2P).

Proof: The second statement is obvious from Theorem 1, and the first statement is obvious from Lemma 5.2. Further, it is obvious that after obtaining the value of T for the first statement to hold, the values of t_k for k = k_P will only move the (n-P) runners from one sector to another and will not change the minimum distance of 1/(2n) from the center of the sector in which they are.
Hence Proved Lemma 5.3

Lemma 5.4: Consider the general case of the LRC defined in Section 2. Let each of {P, n, n-P} be large and P < (n+1)/2. Then there exists a time $T = t_L P^{-L} + t_{L-1} P^{-L+1} + ... + t_2 P^{-2} + t_1 P^{-1}$ when the following two statements are simultaneously true:

1. Each of atleast (n-P) runners are separated from the center of the sector in which they are present by atleast 1/(2n).
2. Each of atleast (P-1) runners are separated from R_0 by atleast 1/(2P).

Proof: In the general case, we know that each $(H_k + H_{k+1}) < (n-1)$. When each of {P, n, n-P} is large, our approach becomes more modular as we progress in the iterations of Algorithm 2 using base P instead of base 2, and it would suffice to use a proportionality constant to state a time = $t_{k+1} P^{-(k+1)}$ exists irrespective of the values of $\{t_L, t_{L-1}, ..., t_{K+2}\}$, to ensure that atleast:

1. $((n-P)H_k/(n-1))$ runners with E_i = k are separated from the center of the sectors in which they are present by atleast 1/(2n).
2. $((P-1)H_{k+1}/(n-1))$ runners with E_i = (k+1) are separated from R_0 by atleast 1/(2P).

The number of feasible values of t_{k+1} for the first statement above to hold is the minimum of P and $((n-P) - (n-P)H_k/(n-1)) = (n-P)(1 - H_k/(n-1))$, which needs to be greater than the number of runners $((P-1)H_{k+1}/(n-1))$ mentioned in the second statement, since each runner contributes to a separate infeasible choice in the worst case. So we write the necessary condition as follows:
$(n-P)(1 - H_k/(n-1)) > (P-1)H_{k+1}/(n-1)$, which means that the necessary condition is:
$(n-P)H_k/(n-1) + (P-1)H_{k+1}/(n-1) < (n-P)$.

Clearly, $(H_k + H_{k+1}) < (n-1)$, so the LHS is less than a weighted average between (n-P) and (P-1), therefore the LHS < RHS if $(P-1) < (n-P)$, which means $P < (n+1)/2$ is the necessary condition for the Lemma 5.4 to hold.
Hence Proved Lemma 5.4

Finally, it is obvious that separation of a runner from the center of the sector in which is present by atleast 1/(2n) implies that runner is also separated from R_0 by atleast 1/(2n). Thus, the proof of Lemma 5.4 implies the proof of the Theorem.
Hence Proved

The optimal value of P depends on what we are trying to achieve:

1. Substituting P=(n/2), allows atleast half of the runners to be separated atleast 1/n away from R_0, and the remaining half separated atleast 1/(2n) away from R_0. The average gap of separation is 3/(4n) = 0.75/n.
2. Substituting P=(2n/3), allows two-thirds of the runners to be separated atleast 3/(4n) away from R_0, and the remaining one-thirds separated atleast 1/(2n) away from R_0. The average gap of separation is 2/(3n)=0.667/n.
3. Substituting P=(3n/4), allows three-fourths of the runners to be separated atleast 2/(3n) away from R_0, and the remaining one-fourth separated atleast 1/(2n) away from R_0. The average gap of separation is 5/(8n)=0.625n.
4. Substituting P=n-1, allows all runners to be positioned atleast 1/(2n) away from R_0, which is the trivial gap.

If our aim is to maximize the average distance of the runners from R_0, then Theorem 6 gives the optimal value of P.

Theorem 6: **At time t, denote $g_i(t)$ be the minimum absolute distance along the circumference of runner R_i from R_0. Let $\sqrt{n}$ be prime, and let n be large. There exists a time T at which, $g_i(T) \geq 1/(2\sqrt{n})$ for atleast $((\sqrt{n}) - 1))$ runners, and $g_i(T) \geq 1/(2n)$ for the remaining $(n - \sqrt{n})$ runners. This yields an average separation gap tending to atleast 1/n.**
Proof: Since atleast (P-1) runners can be positioned atleast 1/(2P) away from R_0, and the remaining (n-P) runners can be positioned atleast 1/(2n) away from R_0, the formula for the average worst case separation gap G is given by:

$$G = ((P-1)/(2P) + (n-P)/(2n)) / n = 1/(2n) - 1/(2Pn) + 1/(2n) - P/(2n^2) = 1/n - 1/(2Pn) - P/(2n^2).$$

The first derivative $dG/dP = 1/(2nP^2) - 1/(2n^2)$. The second derivative $d^2G/dP^2 = -1/(nP^3)$ which is negative for $2 < P < n$. Setting the first derivative $dG/dP = 0$ yields $P = \sqrt{n}$. Since the second derivative is negative in P's domain, this point of the first derivative being 0 is the global maximum. So if $\sqrt{n}$ is prime, we can use Algorithm 2 to derive a time T at which $g_i(T) \geq 1/(2\sqrt{n})$ for atleast $((\sqrt{n}) - 1))$ runners, and $g_i(T) \geq 1/(2n)$ for the remaining $(n - \sqrt{n})$ runners. Substituting $P = \sqrt{n}$ into the formula of G yields $G = 1/n - 1/(2n\sqrt{n}) - \sqrt{n}/(2n^2) = 1/n - \sqrt{n}/(2n^2) - \sqrt{n}/(2n^2) = 1/n - \sqrt{n}/n^2 = 1/n - 1/(n\sqrt{n})$, which tends to 1/n for large n. The necessary conditions of Theorem 5 also hold because:

1. The value of each of $\{\sqrt{n}, n, (n - \sqrt{n})\}$ is large when n is large, which is a necessary condition for Lemma 5.3.
2. The necessary condition for Lemma 5.4, which is $P < (n+1)/2$, holds because $\sqrt{n} < (n+1)/2$ for positive n.

Hence Proved

Theorem 7 generalizes Theorem 6 when n is large (i.e. it removes the remaining requirement for $\sqrt{n}$ to be a prime number).

Theorem 7: There exists a positive integer λ such that Theorem 6 holds for all n > λ.
Proof: When n is very large, we can relax the constraint that $P=\sqrt{n}$ has to be prime, and instead use an alternative prime number less than $(\sqrt{n} + C)$, such that $(\sqrt{n} + C)/\sqrt{n}$ is close to 1, that is $C/\sqrt{n}$ is close to 0, where C is some positive constant. Using the bounds on prime numbers [6], we know that if $\beta(x)$ denotes the number of prime numbers below x, then:

$$(x/\log(x)) (1 + 1/\log(x)) < \beta(x) < (x/\log(x)) (1 + 1/\log(x) + 2.51/\log^2(x)), \text{ so}$$

$$((x+C)/\log(x+C)) (1 + 1/\log(x+C)) < \beta(x+C) < ((x+C)/\log(x+C)) (1 + 1/\log(x+C) + 2.51/\log^2(x+C)), \text{ which means:}$$

$$\beta(x+C) - \beta(x) > ((x+C)/\log(x+C)) (1 + 1/\log(x+C)) - (x/\log(x)) (1 + 1/\log(x) + 2.51/\log^2(x)).$$

It is clear that for every x, there is a positive constant C beyond which $\beta(x+C) - \beta(x) > 1$. Putting $x = \sqrt{n}$, it is clear that there will be atleast one prime number between $\sqrt{n} + C$ and $\sqrt{n}$, where $C/\sqrt{n}$ will tend to 0 for large n.
Hence Proved

4. Discussion, conclusion and future work

In this paper, we utilized properties of prime numbers that are already in use in random number generators, to identify a number of special cases of runners when the LRC holds. We then used this property along with a new strategy for the LRC of identifying a time T when it would be possible to simultaneously position some subset of runners at the extremities of arcs of sectors (implicitly creating a separation from R_0), while explicitly separating the remaining runners from R_0. The key concept that enabled this approach to work is that the runners positioned at the extremities of arcs of sectors, remain separated from the centers of the arcs of all sectors (and implicitly from R_0) when shifted by multiples of the arc-lengths of a sector. We further optimized this strategy to obtain an average separation gap tending to 1/n, simultaneously ensuring the minimal gap of 1/(2n) for every runner.

Interestingly, our average gap of (1/n - 1/(n√n)) of this paper is even better than the average gap of (1/n - 1/(n log(n))) that we obtained earlier [4].

Immediate open questions that arise are: -

1. Can we obtain a better average gap more than 1/n, without a restriction on the minimal gap?
2. Can we obtain a better average gap more than 1/n, with some restriction on the minimal gap?

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I, Deepak Ponvel Chermakani, am a citizen of India and I wrote this paper, which is original to the best of my knowledge, out of my own interest and initiative during my spare time. I completed a fulltime two-year Master of Science Degree in Electrical Engineering from University of Hawaii at Manoa USA (www.hawaii.edu) in Aug 2015, a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization from University of Edinburgh UK (www.ed.ac.uk) in Sep 2010, a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg) in Jul 2003, fulltime high schooling from National Public School Indiranagar in Bangalore India in Jul 1999, and fulltime middle schooling from Bishop Cotton Boys School in Bangalore India in Jul 1997. I am most grateful to my parents (my mother Mrs. Kanaga Rathinam Chermakani and my father Mr. T. Chermakani) for their sacrifices in educating me and bringing me up.